Interval extension of Bézier curve

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Abstract: - By extending definition interval of the classical Bernstein basis functions to be dynamic, a class of Bernstein basis functions with a shape parameter is constructed in this work. The new basis functions are simple extension of the classical Bernstein basis functions. Then the corresponding Bézier-like curve is generated on base of the introduced basis functions. The new curve not only has most properties of the classical Bézier curve, but also can be adjusted by altering value of the shape parameter when the control points are fixed. Because the proposed curve is a polynomial model of the same degree and having most properties of the classical Bézier curve, it has more advantages than some existing similar models.

Key- words: Bernstein basis functions; Bézier curve; the same degree; shape adjustment; shape parameter.

1 Introduction

As an important geometric modeling tool, Bézier curve has been widely used in Computer Aided Geometric Design (CAGD) and Computer Aided Design (CAD). However, when the control points are given, shape of the classical cubic Bézier curve cannot be changed. With the development of geometric design industry, shapes of curves often need to be changed freely. For relieving the default of the classical Bézier curve, the Bézier-like curves with shape parameters have been paid more and more attention by many researchers.

Because the Bézier curve can be naturally defined after the basis functions are determined. Therefore, constructing the basis functions with shape parameters becomes the most effective way for establishing Bézier-like curves with shape parameters. At present, in order to introduce shape parameters to the basis functions of Bézier curve, the commonly used method has two kinds. One is to construct non-polynomial basis functions with shape parameters based on trigonometric or hyperbolic functions, such as [1-6]. Another is to construct the high-degree polynomial basis functions with shape parameters by increasing the degree of the classical Bernstein basis functions, such as [7-11]. Although the Bézier-like curves generated by those methods can effectively realize shape adjustment by altering values of the shape parameters, the structure complexity is thereupon increased. The polynomial Bézier-like curve with multiple parameters of the same degree [12] was a practical method, but the curve did not have the strict symmetry that the classical Bézier curve has. Although the polynomial Bézier-like curve of degree n with n-1 shape parameters [13] satisfied the same properties with the classical Bézier curve, value range of the shape parameters of different order curve are diverse from each other would cause users with confusion.

Is there a simpler method for constructing basis functions describing Bézier-like curve with shape parameters that has most properties of the classical Bézier curve? Aiming this problem, the main purpose of this work is to present a simple method for constructing Bernstein basis functions with a shape parameter of the same degree. A class of cubic Bernstein basis functions with a shape parameter α, named cubic α-Bernstein basis functions, is constructed through extending definition interval of the classical cubic Bernstein basis functions from

$[0,1]$ to $[0,\alpha]$ $(0<\alpha\le 1)$. On base of the cubic α-Bernstein basis functions, the α-Bernstein basis functions of degree n $(n\ge 4)$ are generated by recursion property of the classical Bernstein basis functions. Then the corresponding Bézier-like curve, named α-Bézier curve, is naturally defined on base of the α-Bernstein basis functions. The proposed α-Bézier curve has most properties of the classical Bézier curve, and its shape can be adjusted by altering value of the shape parameter when the control points are fixed.

The rest of this paper is organized as follows. In Section 2, the α-Bernstein basis functions are constructed, and some properties of the basis functions are given. In Section 3, the corresponding α-Bézier curve is defined. Some properties of the curve, effects of the shape parameter on the curve and continuity of the curve are discussed. A short conclusion is given in Section 4.

2 The *α*-Bernstein basis functions

2.1 Construction of the basis functions

Generally, the classical Bernstein basis functions can be expressed as follows [14],

$$B_{n,i}(t)=\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i,$$

where $0\le t\le 1$, $i=0,1,2,\cdots,n$.

The classical Bernstein basis functions have the following properties,

(a) Nonnegativity: $B_{n,i}(t)\ge 0\quad (i=0,1,2,\cdots,n)$.

(b) Normalization: $\sum_{i=0}^{n}B_{n,i}(t)\equiv 1$.

(c) Symmetry:

$$B_{n,i}(t)=B_{n,n-i}(1-t)\quad (i=0,1,2,\cdots,n).$$

(d) Properties at the endpoints:

$$B_{n,i}(0)=\begin{cases}1, & i=0\\ 0, & i\ne 0\end{cases},\quad B_{n,i}(1)=\begin{cases}1, & i=n\\ 0, & i\ne n\end{cases},$$

$$B'_{n,i}(0)=\begin{cases}-n, & i=0\\ n, & i=1\\ 0, & i\ne 0,1\end{cases},\quad B'_{n,i}(1)=\begin{cases}n, & i=n\\ -n, & i=n-1\\ 0, & i\ne n-1,n\end{cases}.$$

Besides, the classical Bernstein basis functions have the recursion property as follows,

$$B_{n,i}(t)=(1-t)B_{n-1,i}(t)+tB_{n-1,i-1}(t),$$

where $0\le t\le 1$, $i=0,1,2,\cdots,n$, and setting

$$B_{n-1,-1}(t)=B_{n-1,n}(t)\equiv 0.$$

In order to construct a class of Bernstein basis functions with a shape parameter α, a simple ideal is to extend definition interval of the classical Bernstein basis functions from $[0,1]$ to $[0,\alpha]$ $(0<\alpha\le 1)$. Inspired by the recursion property of the classical Bernstein basis functions, the basis functions of degree $n\,(n\ge 4)$ with a shape parameter α can be generated on base of the cubic basis functions. Therefore, the cubic Bernstein basis functions with a shape parameter α are firstly constructed as below.

Suppose the cubic basis functions are expressed as follows,

$$\begin{bmatrix} f_{3,0}(t) & f_{3,1}(t) & f_{3,2}(t) & f_{3,3}(t)\end{bmatrix}=\begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix}M \quad (1)$$

where $0\le t\le\alpha$, $0<\alpha\le 1$, and M is an undetermined 4×4 matrix.

Derivation calculus to Eq. (1), then

$$\begin{bmatrix} f'_{3,0}(t) & f'_{3,1}(t) & f'_{3,2}(t) & f'_{3,3}(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 2t & 3t^2\end{bmatrix}M \quad (2)$$

Because the cubic basis functions are hoped to satisfy the same properties with the classical cubic Bernstein basis functions at the end point, therefore, let $t=0$ and $t=\alpha$ in Eq. (1) and Eq. (2) respectively, then

$$\begin{cases}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}M\\ \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & \alpha & \alpha^2 & \alpha^3\end{bmatrix}M\\ \begin{bmatrix}-3 & 3 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}M\\ \begin{bmatrix}0 & 0 & -3 & 3\end{bmatrix}=\begin{bmatrix}0 & 1 & 2\alpha & 3\alpha^2\end{bmatrix}M\end{cases} \quad (3)$$

From Eq. (3), then

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -3 & 3 & 0 & 0\\ 0 & 0 & -3 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & \alpha & \alpha^2 & \alpha^3\\ 0 & 1 & 0 & 0\\ 0 & 1 & 2\alpha & 3\alpha^2\end{bmatrix}M \quad (4)$$

Solving Eq. (4), then

$$M=\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ \frac{6\alpha-3}{\alpha^2} & -\frac{6}{\alpha} & \frac{3}{\alpha} & \frac{3-3\alpha}{\alpha^2}\\ \frac{2-3\alpha}{\alpha^3} & \frac{3}{\alpha^2} & -\frac{3}{\alpha^2} & \frac{3\alpha-2}{\alpha^3}\end{bmatrix} \quad (5)$$

Taking Eq. (5) to Eq. (1), the cubic basis functions can be expressed as follows,

$$\begin{cases} f_{3,0}(t) = 1 - 3t + \dfrac{6\alpha - 3}{\alpha^2}t^2 + \dfrac{2 - 3\alpha}{\alpha^3}t^3 \\ f_{3,1}(t) = 3t - \dfrac{6}{\alpha}t^2 + \dfrac{3}{\alpha^2}t^3 \\ f_{3,2}(t) = \dfrac{3}{\alpha}t^2 - \dfrac{3}{\alpha^2}t^3 \\ f_{3,3}(t) = \dfrac{3 - 3\alpha}{\alpha^2}t^2 + \dfrac{3\alpha - 2}{\alpha^3}t^3 \end{cases} \tag{6}$$

where $0 \le t \le \alpha$, $0 < \alpha \le 1$.

The cubic basis functions expressed as Eq. (6) can be reparametrized to the basis functions by $b_{3,i}(u) = f_{3,i}(\alpha u)$ $(i = 0,1,2,3)$, then $b_{3,i}(u)$ $(i = 0,1,2,3)$ is defined on a fixed interval $u \in [0,1]$ which can be defined as follows.

Definition 1 The following four functions of *u* are called the cubic Bernstein basis functions with a shape parameter *α* (cubic *α*-Bernstein basis functions for short),

$$\begin{cases} b_{3,0}(u) = 1 - 3\alpha u + 3(2\alpha - 1)u^2 + (2 - 3\alpha)u^3 \\ b_{3,1}(u) = 3\alpha u - 6\alpha u^2 + 3\alpha u^3 \\ b_{3,2}(u) = 3\alpha u^2 - 3\alpha u^3 \\ b_{3,3}(u) = 3(1 - \alpha)u^2 + (3\alpha - 2)u^3 \end{cases} \tag{7}$$

where $0 \le u \le 1$, $0 < \alpha \le 1$.

Eq. (7) can be rewritten as follows,

$$\begin{cases} b_{3,0}(u) = (1-u)^2\left[(1-u) + 3(1-\alpha)u\right] \\ b_{3,1}(u) = 3\alpha u(1-u)^2 \\ b_{3,2}(u) = 3\alpha u^2(1-u) \\ b_{3,3}(u) = u^2\left[u + 3(1-\alpha)(1-u)\right] \end{cases} \tag{8}$$

On base of the cubic *α*-Bernstein basis functions, the *α*-Bernstein basis functions of degree $n\,(n \ge 4)$ can be generated according to the recursion property of the classical Bernstein basis functions. Then the *α*-Bernstein basis functions of degree $n\,(n \ge 4)$ can be defined as follows.

Definition 2 The following functions of *u* are called the *α*-Bernstein basis functions of degree $n\,(n \ge 4)$,

$$b_{n,i}(u) = (1-u)b_{n-1,i}(u) + ub_{n-1,i-1}(u) \tag{9}$$

where $0 \le u \le 1$, $0 < \alpha \le 1$, $i = 0,1,2,\cdots,n$, and setting $b_{n-1,-1}(u) = b_{n-1,n}(u) \equiv 0$.

By simple deduction, expression of the *α*-Bernstein basis functions of degree n $(n \ge 4)$ can be got from Eq. (8) and Eq. (9). For example, when $n = 4$, the quartic *α*-Bernstein basis functions can be expressed as follows,

$$\begin{cases} b_{4,0}(u) = (1-u)^3\left[(1-u) + 3(1-\alpha)u\right] \\ b_{4,1}(u) = u(1-u)^2\left[(1+3\alpha)(1-u) + 3(1-\alpha)u\right] \\ b_{4,2}(u) = 6\alpha u^2(1-u)^2 \\ b_{4,3}(u) = u^2(1-u)\left[(1+3\alpha)u + 3(1-\alpha)(1-u)\right] \\ b_{4,4}(u) = u^3\left[u + 3(1-\alpha)(1-u)\right] \end{cases} \tag{10}$$

where $0 \le u \le 1$, $0 < \alpha \le 1$.

2.2 Properties of the basis functions

For the sake of convenience, the *α*-Bernstein basis functions of degree $n\,(n \ge 3)$ are called *α*-Bernstein basis functions for short in the following discussion.

From the construction process of the *α*-Bernstein basis functions, some properties of the basis functions can be obtained as follows.

Theorem 1 The *α*-Bernstein basis functions defined as Eq. (8) and Eq. (9) have the following properties,

(a) Non-negativity: $b_{n,i}(u) \ge 0$ $(i = 0,1,2,\cdots,n)$.

(b) Normalization: $\sum_{i=0}^{n} b_{n,i}(u) \equiv 1$.

(c) Symmetry:

$$b_{n,i}(u) = b_{n,n-i}(1-u) \quad (i = 0,1,2,\cdots,n).$$

(d) Properties at the endpoints:

$$b_{n,i}(0) = \begin{cases} 1, & i = 0 \\ 0, & i \ne 0 \end{cases}, \quad b_{n,i}(1) = \begin{cases} 1, & i = n \\ 0, & i \ne n \end{cases},$$

$$b'_{n,i}(0) = \begin{cases} -(n-3+3\alpha), & i = 0 \\ n-3+3\alpha, & i = 1 \\ 0, & i \ne 0,1 \end{cases},$$

$$b'_{n,i}(1) = \begin{cases} n-3+3\alpha, & i = n \\ -(n-3+3\alpha), & i = n-1 \\ 0, & i \ne n-1,n \end{cases}.$$

Proof Mathematical induction is used to prove this theorem.

(a) When $n = 3$, from Eq. (8), $b_{3,i}(u) \ge 0$ $(i = 0,1,2,3)$ follow obviously because $1 - u \ge 0$ and $1 - \alpha \ge 0$. Suppose that the *α*-Bernstein basis functions satisfy non-negative for $n = m$. When $n = m + 1$, from Eq. (9), then

$$b_{m+1,i}(u) = (1-u)b_{m,i}(u) + ub_{m,i-1}(u)$$
$$(i = 0,1,2,\cdots,m+1)$$

By the inductive hypothesis and the fact that $1 - u \ge 0$, $u \ge 0$, it is obviously that $b_{m+1,i}(u) \ge 0$

$(i = 0,1,2,\cdots,m+1)$.

(b) When $n = 3$, from Eq. (7), it is easy to conclude that $\sum_{i=0}^{3} b_{3,i}(u) \equiv 1$. Suppose that the α-Bernstein basis functions satisfy normalized for $n = m$. When $n = m+1$, by the inductive hypothesis and Eq. (9), then

$$\sum_{i=0}^{m+1} b_{m+1,i}(u) = (1-u)\sum_{i=0}^{m} b_{m,i}(u) + u\sum_{i=1}^{m} b_{m,i-1}(u) = 1-u+u \equiv 0 .$$

(c) When $n = 3$, the cubic α-Bernstein basis functions satisfy symmetry can be obtained by simple deduction from Eq. (8). Suppose that the α-Bernstein basis functions are symmetrical for $n = m$. When $n = m+1$, by the inductive hypothesis and Eq. (9), then

$$\begin{aligned} b_{m+1,i}(1-u) &= ub_{m,i}(1-u) + (1-u)b_{m,i-1}(1-u) \\ &= ub_{m,m-i}(u) + (1-u)b_{m,m-i+1}(u) = b_{m+1,m-i+1}(u) . \end{aligned}$$

(d) When $n = 3$, from Eq. (7), then

$$\begin{cases} b'_{3,0}(u) = -3\alpha + 6(2\alpha-1)u + 3(2-3\alpha)u^2 \\ b'_{3,1}(u) = 3\alpha - 12\alpha u + 9\alpha u^2 \\ b'_{3,2}(u) = 6\alpha u - 9\alpha u^2 \\ b'_{3,3}(u) = 6(1-\alpha)u + 3(3\alpha-2)u^2 \end{cases} \tag{11}$$

By simple deduction from Eq. (7) and Eq. (11), then

$$b_{3,i}(0) = \begin{cases} 1, & i=0 \\ 0, & i \neq 0 \end{cases}, \quad b_{3,i}(1) = \begin{cases} 1, & i=3 \\ 0, & i \neq 3 \end{cases},$$

$$b'_{3,i}(0) = \begin{cases} -3\alpha, & i=0 \\ 3\alpha, & i=1 \\ 0, & i=2,3 \end{cases}, \quad b'_{3,i}(1) = \begin{cases} 3\alpha, & i=3 \\ -3\alpha, & i=2 \\ 0, & i=0,1 \end{cases}.$$

Suppose that the α-Bernstein basis functions hold the properties at the endpoints for $n = m$. When $n = m+1$, by the inductive hypothesis and Eq. (9), then

$$\begin{cases} b_{m+1,i}(0) = b_{m,i}(0) = \begin{cases} 1, & i=0 \\ 0, & i \neq 0 \end{cases} \\ b_{m+1,i}(1) = b_{m,i-1}(1) = \begin{cases} 1, & i=m+1 \\ 0, & i \neq m+1 \end{cases} \end{cases} \tag{12}$$

From Eq. (9), then

$$b'_{m+1,i}(0) = -b_{m,i}(0) + b'_{m,i}(0) + b_{m,i-1}(0) \tag{13}$$

$$b'_{m+1,i}(1) = -b_{m,i}(1) + b_{m,i-1}(1) + b'_{m,i-1}(1) \tag{14}$$

By the inductive hypothesis and Eq. (12), the following conclusions can be got from Eq. (13),

(i) If $i = 0$, then

$$\begin{aligned} b'_{m+1,0}(0) &= -b_{m,0}(0) + b'_{m,0}(0) \\ &= -m+2-3\alpha = -((m+1)-3+3\alpha) . \end{aligned}$$

(ii) If $i = 1$, then

$$\begin{aligned} b'_{m+1,1}(0) &= -b_{m,1}(0) + b'_{m,1}(0) + b_{m,0}(0) \\ &= m-2+3\alpha = (m+1)-3+3\alpha . \end{aligned}$$

(iii) If $i \neq 0,1$, then $b'_{m+1,i}(0) = 0$.

Similarly, the following conclusions can be got from Eq. (14),

(i) If $i = m+1$, then $b'_{m+1,m+1}(1) = (m+1)-3+3\alpha$.

(ii) If $i = m$, then $b'_{m+1,m}(1) = -((m+1)-3+3\alpha)$.

(iii) If $i \neq m, m+1$, then $b'_{m+1,i}(1) = 0$. □

Theorem 1 shows that the α-Bernstein basis functions have most properties of the classical Bernstein basis functions. Particularly, the α-Bernstein basis functions would degenerate to the classical Bernstein basis functions for $\alpha = 1$. Hence, the α-Bernstein basis functions are simple extensions of the classical Bernstein basis functions.

3 α-Bézier curve

3.1 Definition and properties of the curve

On base of α-Bernstein basis functions, the corresponding Bézier-like curve can be naturally defined as follows.

Definition 3 Given control points $\boldsymbol{P}_i$ $(i = 0,1,2,\cdots,n)$ in R^2 or R^3, for $0 \le u \le 1$, $0 < \alpha \le 1$,

$$\boldsymbol{r}_n(u) = \sum_{i=0}^{n} b_{n,i}(u)\boldsymbol{P}_i \tag{15}$$

is called α-Bézier curve, where $b_{n,i}(u)$ $(i = 0,1,2,\cdots,n; n \ge 3)$ are the α-Bernstein basis functions expressed as Eq. (8) and Eq. (9).

From Theorem 1, the α-Bézier curve defined as Eq. (15) has the following properties,

(a) Terminal properties: From the properties at the endpoints of the α-Bernstein basis functions, then

$$\boldsymbol{r}_n(0) = \boldsymbol{P}_0, \quad \boldsymbol{r}_n(1) = \boldsymbol{P}_n;$$
$$\boldsymbol{r}'_n(0) = (n-3+3\alpha)(\boldsymbol{P}_1 - \boldsymbol{P}_0),$$
$$\boldsymbol{r}'_n(1) = (n-3+3\alpha)(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}).$$

Hence, the α-Bézier curve interpolates the first and the end control points and tangent to the first and the end edges of the control polygon.

(b) Symmetry: From the symmetry of the α-Bernstein basis functions, then

$$\boldsymbol{r}_n(u;\boldsymbol{P}_0,\boldsymbol{P}_1,\boldsymbol{P}_2,\cdots,\boldsymbol{P}_n)=\sum_{i=0}^{n}b_{n,i}(u)\boldsymbol{P}_i$$
$$=\sum_{i=0}^{n}b_{n,n-i}(1-u)\boldsymbol{P}_i=\sum_{j=0}^{n}b_{n,j}(1-u)\boldsymbol{P}_{n-j}$$
$$=\boldsymbol{r}_n(1-u;\boldsymbol{P}_n,\boldsymbol{P}_{n-1},\boldsymbol{P}_{n-2},\cdots,\boldsymbol{P}_0)\,.$$

Hence, both $\boldsymbol{P}_i$ $(i=0,1,2,\cdots,n)$ and $\boldsymbol{P}_{n-i}$ $(i=0,1,2,\cdots,n)$ define the same α-Bézier curve in a different parameterization for the same shape parameter $\alpha\,(0<\alpha\le 1)$.

(c) Geometric invariant and affine invariance: Due to parametric form of the α-Bézier curve, the location and shape of the curve depend only on the control points $\boldsymbol{P}_i$ $(i=0,1,2,\cdots,n)$ and the shape parameter α, regardless of the choice of coordinate system, *i.e.*, the shape of the curve will keep unchanged after rotation and coordinate translation. In addition, after implementing affine transformation to the control points, the new curve will correspond to the same affine transformation curve.

(d) Convex hull property: Because the α-Bernstein basis functions are nonnegative and sum to one, the α-Bézier curve lies inside its control polygons span by the control points $\boldsymbol{P}_i$ $(i=0,1,2,\cdots,n)$.

It is clear that the α-Bézier curve has most properties of the classical Bézier curve. Particularly, the α-Bézier curve would degenerate to the classical Bézier curve for $\alpha=1$. Hence, the α-Bézier curve is an extension of the classical Bézier curve.

Remark 1 In contrast with some existing similar models, the α-Bézier curve presented in this work has the following characteristic,

(a) In contrast with the non-polynomial Bézier-like curves with shape parameters (such as [1-6]), the α-Bézier curve is polynomial. Hence, structure of the α-Bézier curve is simpler than those models based on the non-polynomial basis functions.

(b) In contrast with the high-degree Bézier-like curves with shape parameters (such as [7-11]), the α-Bézier curve is still a polynomial of the same degree. Hence, formula complexity of the α-Bézier curve is simpler than those models constructed by increasing the degree of the Bernstein basis functions.

(c) In contract with the Bézier-like curve with multiple shape parameters of the same degree [12], the α-Bézier curve satisfies strict symmetry that the classical Bézier curve has. Hence, the α-Bézier curve is more suitable in practical engineering than the models that did not satisfy strict symmetry.

(d) In contract with the Bézier-like curve of degree n with n-1 shape parameters [13], value range of the shape parameter of the α-Bézier curve is fixed, which makes more use-friendly operation for users.

3.2 Effects of the shape parameter on the curve

For fixed control points $\boldsymbol{P}_i$ $(i=0,1,2,\cdots,n)$, shape of the classical Bézier curve cannot be changed, while shape of the α-Bézier curve can be adjusted by altering value of the shape parameter α $(0<\alpha\le 1)$.

In order to discuss effects of the shape parameter α on the α-Bézier curve, a lemma is given firstly as follows.

Lemma 1 The α-Bernstein basis functions defined as Eq. (8) and Eq. (9) satisfy that

(a) $b_{n,0}\left(\frac{1}{2}\right)=b_{n,n}\left(\frac{1}{2}\right)=\frac{4-3\alpha}{2^n}$ $(n\ge 3)$.

(b) There exist constants c_n such that

$$b_{n,1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}=b_{n,n-1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}=c_n\cdot\frac{4-3\alpha}{2^n}$$
$$(n\ge 3).$$

(c) There exist constants $k_{n,i}$ such that

$$b_{n,i}\left(\frac{1}{2}\right)-\frac{n-2}{2^{n-2}}=b_{n,n-i}\left(\frac{1}{2}\right)-\frac{n-2}{2^{n-2}}=k_{n,i}\cdot\frac{4-3\alpha}{2^n}$$
$$(i=2,3,\cdots,n-2;n\ge 4).$$

Proof From the symmetry of the α-Bernstein basis functions, then

$$b_{n,0}\left(\frac{1}{2}\right)=b_{n,n}\left(\frac{1}{2}\right)\ (n\ge 3),$$
$$b_{n,1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}=b_{n,n-1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}\ (n\ge 3),$$
$$b_{n,i}\left(\frac{1}{2}\right)-\frac{n-2}{2^{n-2}}=b_{n,n-i}\left(\frac{1}{2}\right)-\frac{n-2}{2^{n-2}}$$
$$(i=2,3,\cdots,n-2;n\ge 4).$$

Hence, only the other half of every equation needed to be proved. Mathematical induction is used to prove.

(a) When $n=3$, from Eq. (8), it is easy to conclude that $b_{3,0}\left(\frac{1}{2}\right)=\frac{4-3\alpha}{8}$. Suppose that

$b_{m,0}\left(\frac{1}{2}\right)=\frac{4-3\alpha}{2^m}$ for $n=m$. When $n=m+1$, by the inductive hypothesis and Eq. (9), then

$$b_{m+1,0}\left(\frac{1}{2}\right)=\frac{1}{2}b_{m,0}\left(\frac{1}{2}\right)=\frac{4-3\alpha}{2^{m+1}}.$$

(b) When $n=3$, from Eq. (8), it is easy to conclude that $b_{3,1}\left(\frac{1}{2}\right)-\frac{1}{2}=c_1\cdot\frac{4-3\alpha}{8}$, where $c_1=-1$. For $n=m$, suppose that there exist constants c_m such that $b_{m,1}\left(\frac{1}{2}\right)-\frac{1}{2^{m-2}}=c_m\cdot\frac{4-3\alpha}{2^m}$. When $n=m+1$, by the inductive hypothesis and Eq. (9), then

$$\begin{aligned}b_{m+1,1}\left(\frac{1}{2}\right)-\frac{1}{2^{m-1}}&=\frac{1}{2}b_{m,1}\left(\frac{1}{2}\right)+\frac{1}{2}b_{m,0}\left(\frac{1}{2}\right)-\frac{1}{2^{m-1}}\\&=\frac{1}{2}\left(\frac{1}{2^{m-2}}+c_m\cdot\frac{4-3\alpha}{2^m}\right)+\frac{1}{2}\cdot\frac{4-3\alpha}{2^m}-\frac{1}{2^{m-1}}\\&=c_{m+1}\cdot\frac{4-3\alpha}{2^{m+1}}\end{aligned}$$

where $c_{m+1}=c_m+1$.

(c) When $n=4$, then $i=2$, from Eq. (10), it can conclude that

$$b_{4,2}\left(\frac{1}{2}\right)-\frac{2}{4}=\frac{3}{8}\alpha-\frac{1}{2}=k_{4,2}\cdot\frac{4-3\alpha}{2^4}$$

where $k_{4,2}=-2$. For $n=m$, suppose that there exist constants $k_{m,i}$ such that

$$b_{m,i}\left(\frac{1}{2}\right)-\frac{m-2}{2^{m-2}}=k_{m,i}\cdot\frac{4-3\alpha}{2^m}\ (i=2,3,\cdots,m-2).$$

When $n=m+1$, from Eq. (9), then

$$b_{m+1,i}\left(\frac{1}{2}\right)-\frac{m-1}{2^{m-1}}=\frac{1}{2}b_{m,i}\left(\frac{1}{2}\right)+\frac{1}{2}b_{m,i-1}\left(\frac{1}{2}\right)-\frac{m-1}{2^{m-1}}$$
$$(i=2,3,\cdots,m-1).$$

By the inductive hypothesis and the results have been proved in (b), then

$$\begin{aligned}b_{m+1,i}\left(\frac{1}{2}\right)-\frac{m-1}{2^{m-1}}&=\frac{1}{2}\left(\frac{m-2}{2^{m-2}}+k_{m,i}\cdot\frac{4-3\alpha}{2^m}\right)\\&+\frac{1}{2}\left(\frac{1}{2^{m-2}}+c_m\cdot\frac{4-3\alpha}{2^m}\right)-\frac{m-1}{2^{m-1}}=k_{m+1,i}\cdot\frac{4-3\alpha}{2^{m+1}}\end{aligned}$$

where $k_{m+1,i}=k_{m,i}+c_m$. □

On the base of Lemma 1, effects of the shape parameter α on α-Bézier curve approaching to its polygon can be shown as follows.

Theorem 2 For fixed control points $\boldsymbol{P}_i\ (i=0,1,2,\cdots,n)$, suppose $\boldsymbol{P}_i\ (i=1,2,\cdots,n-1)$ lie on the same side of edge $\boldsymbol{P}_0\boldsymbol{P}_n$. The α-Bézier curve defined as Eq. (15) approaches closer to its control polygon as the shape parameter α increases.

Proof When $\boldsymbol{P}_i\ (i=1,2,\cdots,n-1)$ lie on the same side of edge $\boldsymbol{P}_0\boldsymbol{P}_n$, let

$$\boldsymbol{P}^*=\frac{\boldsymbol{P}_1+(n-2)(\boldsymbol{P}_2+\boldsymbol{P}_3+\cdots+\boldsymbol{P}_{n-2})+\boldsymbol{P}_{n-1}}{2^{n-2}}\quad(16)$$

From Eq. (15) and Lemma 1, then

$$\begin{aligned}\boldsymbol{r}_n\left(\frac{1}{2}\right)-\boldsymbol{P}^*&=b_{n,0}\left(\frac{1}{2}\right)\boldsymbol{P}_0+\\&\left[b_{n,1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}\right]\boldsymbol{P}_1+\sum_{i=2}^{n-2}\left[b_{n,i}\left(\frac{1}{2}\right)-\frac{n-2}{2^{n-2}}\right]\boldsymbol{P}_i\\&+\left[b_{n,n-1}\left(\frac{1}{2}\right)-\frac{1}{2^{n-2}}\right]\boldsymbol{P}_{n-1}+b_{n,n}\left(\frac{1}{2}\right)\boldsymbol{P}_n\\&=\frac{4-3\alpha}{2^n}\left(\boldsymbol{P}_0+c_n\boldsymbol{P}_1+\sum_{i=2}^{n-2}k_{n,i}\boldsymbol{P}_i+c_n\boldsymbol{P}_{n-1}+\boldsymbol{P}_n\right)\end{aligned}\quad(17)$$

where $0<\alpha\le 1$, c_n and $k_{n,i}$ are constants.

Taking the norm in Eq. (17), then

$$\left\|\boldsymbol{r}_n\left(\frac{1}{2}\right)-\boldsymbol{P}^*\right\|=\frac{4-3\alpha}{2^n}\left\|\boldsymbol{P}_0+c_n\boldsymbol{P}_1+\sum_{i=2}^{n-2}k_{n,i}\boldsymbol{P}_i+c_n\boldsymbol{P}_{n-1}+\boldsymbol{P}_n\right\|\quad(18)$$

When control points $\boldsymbol{P}_i\ (i=0,1,2,\cdots,n)$ are fixed, $\left\|\boldsymbol{P}_0+c_n\boldsymbol{P}_1+\sum_{i=2}^{n-2}k_{n,i}\boldsymbol{P}_i+c_n\boldsymbol{P}_{n-1}+\boldsymbol{P}_n\right\|$ in Eq. (18) would keep unchanged. Since $\frac{4-3\alpha}{2^n}$ decreases as α increases, the α-Bézier curve defined as Eq. (15) approaches closer to its control polygon with the increase of α. □

Remark 2 For ease of understanding, set $\boldsymbol{P}_i^{(0)}=\boldsymbol{P}_i$, $\boldsymbol{P}_i^{(j+1)}=\frac{\boldsymbol{P}_i^{(j)}+\boldsymbol{P}_{i+1}^{(j)}}{2}$, then Eq. (16) can be rewritten as follows,

$$\boldsymbol{P}^*=\frac{\boldsymbol{P}_1^{(n-3)}+\boldsymbol{P}_2^{(n-3)}}{2}\quad(19)$$

By simple deduction, relations between $\boldsymbol{P}^*$ and $\boldsymbol{P}_i\ (i=0,1,2,\cdots,n)$ for α-Bézier curve of degree n $(n\ge 3)$ can be got from Eq. (19). For examples, when $n=3$, $\boldsymbol{P}^*=\frac{\boldsymbol{P}_1+\boldsymbol{P}_2}{2}$; when $n=4$, $\boldsymbol{P}^*=\frac{\frac{\boldsymbol{P}_1+\boldsymbol{P}_2}{2}+\frac{\boldsymbol{P}_2+\boldsymbol{P}_3}{2}}{2}$.

When control points are fixed, effects of the shape parameter α on cubic α-Bézier curve and quartic α-Bézier curve is shown in Fig. 1 and Fig. 2

respectively, where value of the shape parameter α is set for $\alpha = 0.1, 0.2, \cdots, 0.9, 1$ respectively from outside to inside.

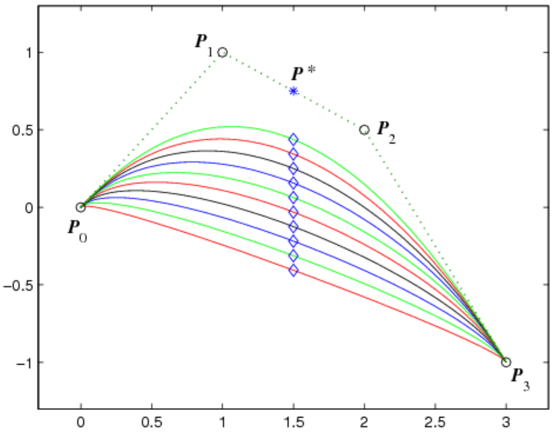

Fig.1 Cubic α-Bézier curve for different α

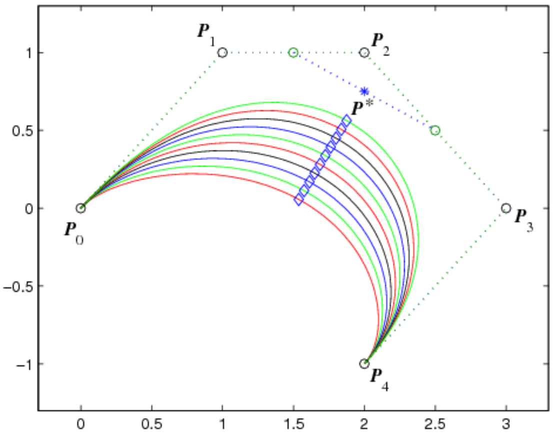

Fig. 2 Quartic α-Bézier curve for different α

3.3 Splicing of the curve

Given two segments of adjacent α-Bézier curves $r_{i,m}(t) = \sum_{j=0}^{m} b_{m,j}(t) P_{i,j}$ and $r_{i+1,n}(t) = \sum_{j=0}^{n} b_{n,j}(t) P_{i+1,j}$, where the former and the latter is a α-Bézier curve of degree $m\,(m \ge 3)$ and degree $n\,(n \ge 3)$ respectively. The shape parameter of $r_{i,m}(t)$ and $r_{i+1,n}(t)$ is α_i and α_{i+1} respectively.

Generally, $r_{i,m}(t)$ and $r_{i+1,n}(t)$ would satisfy G^1 continuous if

$$\begin{cases} r_{i,m}(1) = r_{i+1,n}(0) \\ r'_{i,m}(1) = \delta r'_{i+1,n}(0) \end{cases} \tag{20}$$

where δ is a given constant. Furthermore, $r_{i,m}(t)$ and $r_{i+1,n}(t)$ would satisfy C^1 continuous if $\delta = 1$ in Eq. (20). Then, the splicing conditions of $r_{i,m}(t)$ and $r_{i+1,n}(t)$ satisfying G^1 and C^1 continuous can be shown as follows.

Theorem 3 Two segments of adjacent α-Bézier curves $r_{i,m}(t)$ and $r_{i+1,n}(t)$ would satisfy G^1 continuous if $P_{i,m-1}$, $P_{i,m} = P_{i+1,0}$ and $P_{i+1,1}$ are collinear. Furthermore, $r_{i,m}(t)$ and $r_{i+1,n}(t)$ would satisfy C^1 continuous if $\frac{n-3+3\alpha_{i+1}}{m-3+3\alpha_i} = \lambda_i$, where λ_i is a given constant.

Proof By the terminal properties of the α-Bézier curve, then

$$\begin{cases} r_{i,m}(1) = P_{i,m} \\ r'_{i,m}(1) = (m-3+3\alpha_i)(P_{i,m} - P_{i,m-1}) \\ r_{i+1,n}(0) = P_{i+1,0} \\ r'_{i+1,n}(0) = (n-3+3\alpha_{i+1})(P_{i+1,1} - P_{i+1,0}) \end{cases} \tag{21}$$

If $P_{i,m-1}$, $P_{i,m} = P_{i+1,0}$ and $P_{i+1,1}$ are collinear, there would exit a constant λ_i such that

$$P_{i,m} - P_{i.m-1} = \lambda_i (P_{i+1,1} - P_{i+1,0}) \tag{22}$$

From Eq. (21) and Eq. (22), then

$$\begin{cases} r_{i,m}(1) = r_{i+1,n}(0) \\ r'_{i,m}(1) = \lambda_i \dfrac{m-3+3\alpha_i}{n-3+3\alpha_{i+1}} r'_{i+1,n}(0) \end{cases} \tag{23}$$

Eq. (23) shows that the two adjacent curves satisfy G^1 continuous.

Furthermore, if $\frac{n-3+3\alpha_{i+1}}{m-3+3\alpha_i} = \lambda_i$, *viz.*, $\lambda_i \frac{m-3+3\alpha_i}{n-3+3\alpha_{i+1}} = 1$, Eq. (23) would be rewritten as follows,

$$\begin{cases} r_{i,m}(1) = r_{i+1,n}(0) \\ r'_{i,m}(1) = r'_{i+1,n}(0) \end{cases} \tag{24}$$

Eq. (24) shows that the two adjacent curves satisfy C^1 continuous. □

Suppose a whole G^1 continuous curve is spliced by a number of α-Bézier curves of different degree. From Theorem 3, only shape of the ith curve segment would be locally adjusted if altering value of the shape parameter α_i, while shapes of the other curve segments would keep unchanged. When the shape parameters of all the curve segments are set for $\alpha_i = \alpha$, then shape of the whole G^1 continuous curve can be globally adjusted by altering value of the shape parameter α.

For choosing proper control points, local adjustment of the shape parameter α_2 on a whole G^1 continuous curve spliced by three segments of α-Bézier curves is shown in Fig. 3, where the first and the third curve segments are quartic, and the second curve segment is cubic. In Fig. 3, the shape parameters of the first and the third curve segments are set for $\alpha_1 = \alpha_3 = 0.5$, the shape parameters of the second curve segments is set for $\alpha_2 = 0.6$ (dotted lines), $\alpha_2 = 0.8$ (solid lines) and $\alpha_2 = 1.0$ (dashed lines) respectively.

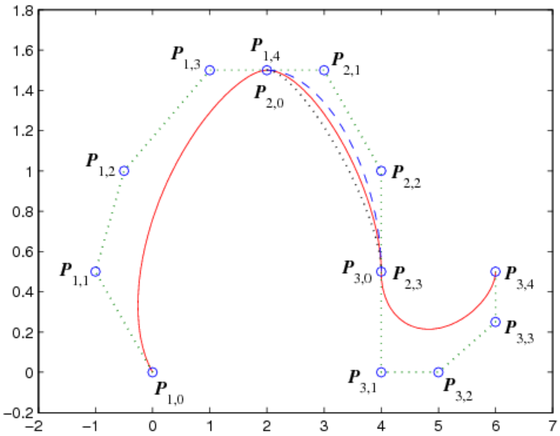

Fig. 3 Local adjustment of a whole G^1 continuous curve

For the same control points in Fig. 3, global adjustment of the shape parameter $\alpha_i = \alpha \ (i = 1, 2, 3)$ on the whole G^1 continuous curve is shown in Fig. 4, where the shape parameter α is set for $\alpha = 0.3$ (dotted lines), $\alpha = 0.6$ (solid lines) and $\alpha = 0.9$ (dashed lines) respectively.

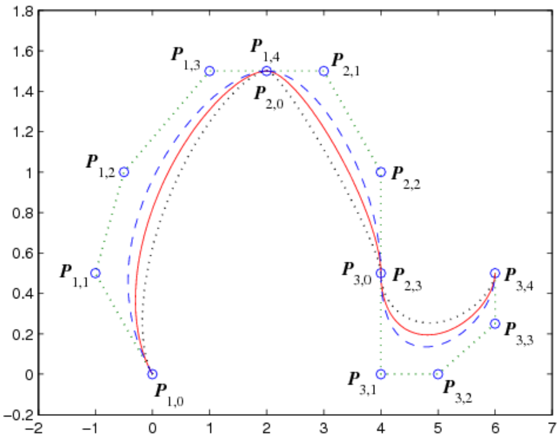

Fig. 4 Global adjustment of a whole G^1 continuous curve

Suppose a whole C^1 continuous curve is spliced by a number of α-Bézier curves of different degree. From Theorem 3, for the whole curve satisfying C^1 continuous, values of the shape parameters of the other curve segments would change if altering value of the shape parameter of the ith curve segment. Then, shape of the whole C^1 continuous curve would be globally adjusted.

For choosing proper control points, global adjustment of the shape parameter $\alpha_i = \alpha \ (i = 1, 2, 3)$ on a whole C^1 continuous curve spliced by three segments of cubic α-Bézier curves is shown in Fig. 5, where the shape parameter α is set for $\alpha = 0.6$ (dotted lines), $\alpha = 0.8$ (solid lines) and $\alpha = 1.0$ (dashed lines) respectively.

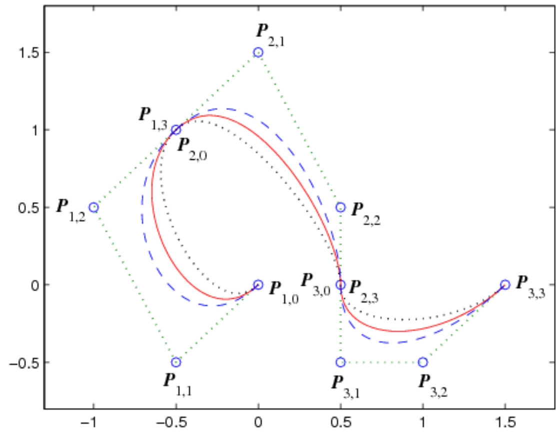

Fig. 5 Global adjustment of a whole C^1 continuous curve

4 Conclusion

The Bernstein basis functions with a shape parameter presented in this paper have the same properties to those of the classical Bernstein basis functions. The Bézier-like curve defined by the introduced basis functions not only has most properties of the classical Bézier curve, but also can be easily adjusted by altering value of the shape parameter. In construct with other similar models, the Bézier-like curve presented in this paper is still a polynomial model of the same degree. Hence, it has simpler structure. Because there is nearly no difference in structure between the proposed Bézier-like curve and the classical Bézier curve, it is no difficult to adopt the proposed Bézier-like curve to a CAD/CAM system that already uses the classical Bézier curve.

For practical applications of the proposed Bézier-like curve in geometric modeling, it is clear that some special algorithms need to be established. Furthermore, the corresponding Bézier-like surface also needs to be discussed. Some interesting results in this area will be presented in the following study.

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