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Brânzei, R.; Dall'Aglio, M.; Tijs, S.H.

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By Rodica Brânzei, Marco Dall'Aglio, Stef Tijs

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Rodica Branzei

University "Alexandru Ioan Cuza", Iaşi, Romania

branzeir@info.uaic.ro

Marco Dall'Aglio

Università "G. d'Annunzio", Pescara, Italy

marco.dallaglio@unich.it

Stef Tijs

Tilburg University, The Netherlands

S.H.Tijs@uvt.nl

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Abstract

Interval bankruptcy problems arise in situations where an estate has to be liquidated among a fixed number of creditors and uncertainty about the amounts of the estate and the claims is modeled by intervals. We extend in the interval setting the classic results by Curiel, Maschler and Tijs [Bankruptcy games, Zeitschrift für Operations Research, 31 (1987), A 143 – A 159] that characterize division rules which are solutions of the cooperative bankruptcy game.

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JEL Classification: C71.

1   Introduction

Bankruptcy problems provide a simple and effective mathematical model to describe situations where an estate has to be divided among a fixed number

of individuals (creditors or players) who advance claims with total value too large to be compensated by the value of the estate. The foundations for these models are set in the works of O'Neill [8] and Aumann and Maschler [2]. These authors analyze the seemingly mysterious solutions for specific instances of a bankruptcy problem prescribed in the Babylonian Talmud and find that the answers given by the ancient book are in fact solutions of a cooperative game, called the bankruptcy game, played by the creditors. Curiel, Maschler and Tijs [6] consolidate the links between bankruptcy problems and cooperative game theory by studying the whole class of division rules for bankruptcy problems which are solutions of the corresponding bankruptcy game. They provide a characterization of such rules by means of a truncation property: the solution based on the bankruptcy game are those, and only those, that ignore claims which are higher than the whole estate, and reduce them to the value of the estate. The same work also defines a simple characterization for a division rule to provide allocations which belong to the core of the bankruptcy game.

The bankruptcy problem studied in those pioneering works requires an exact knowledge of all the terms of a bankruptcy problem. We allow instead for a certain degree of uncertainty on the problem data. This may be the result of limited knowledge of the bankruptcy problem by the creditors, who, perhaps, will be able to determine the exact amount of the terms only at a later time. Uncertainty takes here the form of interval uncertainty: the estate and the claims are expressed as intervals, making up an interval bankruptcy problem.

Our aim is to extend the general result by Curiel, Maschler and Tijs regarding bankruptcy problems with classical (or exact) data to the interval setting. Can we characterize interval division rules which are solutions of interval bankruptcy games? Special care is placed on the definitions of the entities and of the operations in the new environment. In particular we provide two definitions for an interval bankruptcy game. The first proposal, defined as the range of the classical bankruptcy games as the data for the bankruptcy problem span the intervals is effective only under additional assumptions. Another proposal for an interval bankruptcy game in which the estate is fixed at its upper bound allows us to replicate the characterization theorem of Curiel, Maschler and Tijs (Theorem 5 in [6]).

The remainder of the paper is organized as follows: Section 2 reviews the definitions and the results of interest in the classical setting, Section 3 introduces interval bankruptcy problems and their truncation properties. Sections

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4 and 5 deal, respectively, with two proposals for an interval bankruptcy game. In both contexts we will seek to extend the characterizations theorem contained in [6].

2   The classical setting

All the results in this section are taken from [6].

Consider the following basic elements of a bankruptcy problem:

The claimants (players) $N = \{1, 2, \ldots, n\}$.

The estate $E \in \Re_+$.

The claims $d = (d_1, d_2, \ldots, d_n) \in \Re_+^n$.

Basic assumptions $d_1 \leq d_2 \leq \cdots \leq d_n$ and $E \leq \sum_{i \in N} d_i$.

A division rule $f(E, d) = (f_1(E, d), f_2(E, d), \ldots, f_n(E, d))$ such that

$$d_i \geq f_i(E, d) \geq 0 \text{ for every } i \in N \quad \text{and} \quad \sum_{i \in N} f_i(E, d) = E.$$

The bankruptcy game $v_{E,d}(S) = (E - \sum_{i \in N \setminus S} d_i)_+$.

The truncated bankruptcy problem $(E, d \wedge \hat{E})$ where

$$d \wedge \hat{E} = (d_1 \wedge E, d_2 \wedge E, \ldots, d_n \wedge E)$$

and $\hat{E} = (E, \ldots, E) \in \Re_+^n$.

We denote by BR^N the set of all bankruptcy problems (E, d) with claimants N.

Definition 2.1. A division rule f for a bankruptcy problem is a game theoretic division rule if there is a solution concept g for cooperative games such that
$$f(E, d) = g(v_{E,d}) \text{ for every } (E, d) \in BR^N.$$

We denote with BRG^N the set of all bankruptcy games $(N, v_{E,d})$.

Theorem 2.1. *([6], Theorem 5) A division rule f for bankruptcy problems is a game theoretic division rule if and only if $f(E, d) = f(E, d \wedge E)$.*

3   The interval setting

The basic notions for an interval bankruptcy problem are:

A set of claimants (players) $N = \{1, 2, \ldots, n\}$.

An interval estate $[E] = [\underline{E}, \overline{E}]$.

A set of interval claims $[d] = ([\underline{d}_1, \overline{d}_1], [\underline{d}_2, \overline{d}_2], \ldots, [\underline{d}_n, \overline{d}_n])$.

Basic assumption

$$\overline{E} \leq \sum_{i \in N} \underline{d}_i. \tag{1}$$

We denote by I^N the set of n-dimensional vectors of closed and bounded intervals, and by IBR^N the family of all interval bankruptcy problems with claimant set N.

Definition 3.1. An interval bankruptcy rule determines, for each interval estate $[E]$ and each set of interval claims $[d]$ a set of interval rewards

$$\mathcal{F}([E], [d]) = (\mathcal{F}_1([E], [d]), \mathcal{F}_2([E], [d]), \ldots, \mathcal{F}_n([E], [d])) \in I^N$$

which are:

Reasonable, i.e. $[0, 0] \preceq \mathcal{F}_i([E], [d]) \preceq [\underline{d}_i, \overline{d}_i]$ for each $i \in N$,

Efficient, i.e. $\sum_{i \in N} \mathcal{F}_i([E], [d]) = [E]$.

We now define two interval bankruptcy rules based on a classical bankruptcy rule f which satisfies reasonable monotonicity assumptions. Denote with d_{-i} the set of all claims but the claim of the i-th player, i.e. $d_{-i} = \{d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n\}$. Our f will satisfy the following:

Assumption 3.1. For every $i \in N$, the component f_i of the classical bankruptcy rule f is weakly increasing in E and d_i, while it is weakly decreasing in each d_j, $j \in N \setminus \{i\}$.

In the appendix we show that the most important bankruptcy rules verify Assumption 3.1.

First of all, consider the interval bankruptcy rule based on f as

$$\mathcal{F}(f; [E], [d]) = (\mathcal{F}_i(f; [E], [d]))_{i \in N}$$

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where

$$\mathcal{F}_i(f; [E], [d]) = [f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}), f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i})] \qquad \text{for every } i \in N . \qquad (2)$$

We say that an interval with a given property is *tight* if each proper subset of the same interval *does not* satisfy the same property.

Proposition 3.1. *Suppose f satisfies Assumption 3.1, then*

(i) *For each $i \in N$, $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$, we have*

$$f_i(\tilde{E}, \tilde{d}) \in \mathcal{F}_i(f; [E], [d])$$

and, for all $i \in N$, the interval $\mathcal{F}_i(f; [E], [d])$ is tight.

(ii) *$\mathcal{F}(f, \cdot, \cdot)$ is efficient and reasonable.*

Proof. To prove *(i)* consider the following chain of inequalities valid by Assumption 3.1 for each $i \in N$, $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$:

$$f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) \le f_i(\underline{E}, \underline{d}_i, \tilde{d}_{-i}) \le f_i(\tilde{E}, \tilde{d}) \le f_i(\overline{E}, \overline{d}_i, \tilde{d}_{-i}) \le f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) .$$

Since the extremes are attained, they define the smallest interval with this property.

To prove *(ii)* simply note that the classical bankruptcy rule f is reasonable, and therefore, for each $i \in N$,

$$\underline{d}_i \ge f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) \ge 0 \quad ; \quad \overline{d}_i \ge f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) \ge 0,$$

and efficient, so

$$\sum_{i \in N} f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) = \underline{E} \quad ; \quad \sum_{i \in N} f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) = \overline{E}.$$

$\square$

Next, we focus on truncation properties for interval claims. Any claim that exceeds the highest possible estate $\overline{E}$ may be considered excessive. Accordingly, we truncate all claims with respect to this single value. Denote

$$[d] \wedge \overline{E} = \left([\underline{d}_i, \overline{d}_i] \wedge [\overline{E}, \overline{E}]\right)_{i \in N} = \left([\underline{d}_i \wedge \overline{E}, \overline{d}_i \wedge \overline{E}]\right)_{i \in N}$$

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Definition 3.2. The *truncated* interval bankruptcy rule based on f is given by

$$\mathcal{F}^t(f; [E], [d]) = \mathcal{F}(f; [E], [d] \wedge \overline{E}). \tag{3}$$

The truncated interval rule plays an important role when the underlying classical division rule f is game theoretic.

Proposition 3.2. *Suppose that f is a game theoretic division rule satisfying Assumption 3.1. Then*

(i) *The interval bankruptcy rule coincides with the truncated bankruptcy rule, i.e.*

$$\mathcal{F}^t(f; [E], [d]) = \mathcal{F}(f; [E], [d]) \; ; \tag{4}$$

(ii) *For each $i \in N$, $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$, we have*

$$f_i(\tilde{E}, \tilde{d}) \in \mathcal{F}_i^t(f; [E], [d])$$

and, for all $i \in N$, the interval $\mathcal{F}_i^t(f, [E], [d])$ is tight;

(iii) $\mathcal{F}^t(f; \cdot, \cdot)$ *is efficient and reasonable.*

Proof. To prove *(i)*, note that, since f is game theoretic

$$\mathcal{F}(f; [E], [d]) = \left([f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}), f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i})] \right)_{i \in N} =$$
$$\left([f_i(\underline{E}, \underline{d}_i \wedge \underline{E}, \overline{d}_{-i} \wedge \underline{E}), f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E})] \right)_{i \in N} =$$
$$\left([f_i(\underline{E}, \underline{d}_i \wedge \overline{E}, \overline{d}_{-i} \wedge \overline{E}), f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E})] \right)_{i \in N} =$$
$$\mathcal{F}(f; [E], [d] \wedge \overline{E}) = \mathcal{F}^t(f; [E], [d]).$$

The last but one inequality is justified by the fact that, for any $x \in \mathbb{R}$

$$(x \wedge \overline{E}) \wedge \underline{E} = x \wedge \underline{E}.$$

To show *(ii)* consider the following chain of inequalities valid for each $i \in N$, $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$:

$$f_i(\underline{E}, \underline{d}_i \wedge \overline{E}, \overline{d}_{-i} \wedge \overline{E}) = f_i(\underline{E}, \underline{d}_i \wedge \underline{E}, \overline{d}_{-i} \wedge \underline{E}) = f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) \leq$$
$$f_i(\tilde{E}, \tilde{d}) \leq f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) = f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E}) \; . \tag{5}$$

The inequalities in the middle derive from the monotonicity property of f, while the fact that f is a game theoretic rule and Theorem 2.1 explain the equality signs at the extremes. Once again, the extremes are attained.

Regarding *(iii)*, we have

$$\underline{d}_i \geq f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) = f_i(\underline{E}, \underline{d}_i \wedge \underline{E}, \overline{d}_{-i} \wedge \underline{E}) = f_i(\underline{E}, \underline{d}_i \wedge \overline{E}, \overline{d}_{-i} \wedge \overline{E}) \geq 0 \ ,$$
$$\overline{d}_i \geq f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) = f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E}) \geq 0 \ ,$$

while

$$\sum_{i \in N} f_i(\underline{E}, \underline{d}_i \wedge \overline{E}, \overline{d}_{-i} \wedge \overline{E}) = \sum_{i \in N} f_i(\underline{E}, \underline{d}_i \wedge \underline{E}, \overline{d}_{-i} \wedge \underline{E}) = \sum_{i \in N} f_i(\underline{E}, \underline{d}_i, \overline{d}_{-i}) = \underline{E} \ ,$$
$$\sum_{i \in N} f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E}) = \sum_{i \in N} f_i(\overline{E}, \overline{d}_i, \underline{d}_{-i}) = \overline{E} \ .$$

$\square$

4 Interval Bankruptcy Games and Game Theoretic Rules

We now extend the notion of bankruptcy game to the specific interval setting we are considering. An interval bankruptcy game has already been defined in [5] for the case where only the claims are expressed in the form of intervals, while the estate is exact.

Definition 4.1. The *interval bankruptcy game* for the interval estate $[E]$ and interval claims $[d]$ is defined, for each $S \subset N$, by

$$w_{[E],[d]}(S) = [v_{\underline{E},\overline{d}}(S), v_{\overline{E},\underline{d}}(S)] =$$
$$\left[\left(\underline{E} - \sum_{i \in N \setminus S} \overline{d}_i\right)_+ , \left(\overline{E} - \sum_{i \in N \setminus S} \underline{d}_i\right)_+\right] . \quad (6)$$

For each $S \subset N$, the interval is delimited by what is left to coalition S in the worst and in the best possible situation, respectively, after the players outside S have been compensated with their full claim. Note that $w_{[E],[d]}$ coincides with the interval bankruptcy game defined in [5] when $[E]$ shrinks to a single value.

We denote by $IBRG^N$ the family of all interval bankruptcy games with claimant set N. We now show that the interval given in the definition is proper and every classical bankruptcy game originating from the estate $[E]$ and claims $[d]$ is a selection of the interval bankruptcy game as defined in [1].

Proposition 4.1. *For each $S \subset N$, each $\tilde{E} \in [E]$ and each $\tilde{d} \in [d]$*

$$v_{\tilde{E},\tilde{d}}(S) \in w_{[E],[d]}(S) \qquad \text{for each } S \subset N \text{ ,}$$

and each interval $w_{[E],[d]}(S)$ is tight.

Proof. Simply note that, for each $S \subset N$, each $\tilde{E} \in [E]$ and each $\tilde{d} \in [d]$,

$$\underline{E} - \sum_{i \in N \setminus S} \overline{d}_i \le \tilde{E} - \sum_{i \in N \setminus S} \tilde{d}_i \le \overline{E} - \sum_{i \in N \setminus S} \underline{d}_i$$

and the chain of inequalities remains valid if we take the positive parts. Therefore

$$\underline{w}_{[E],[d]}(S) \le v_{\tilde{E},\tilde{d}}(S) \le \overline{w}_{[E],[d]}(S).$$

$\square$

Having extended the notions of division rule and bankruptcy game, respectively, to the interval setting, we may hope that an analogous of Theorem 2.1 would hold, namely that the coincidence between a division rule and its truncated form is a necessary and sufficient condition for the rule to be a solution concept for interval bankruptcy games. The following counterexample, however, highlights a situation where condition (4) holds, but the division rule cannot be based on the interval bankruptcy game. Therefore, a straightforward extension of Theorem 3 is not possible.

Example 4.1. Compare the following two situations with two claimants.

Situation a $[\underline{E}, \overline{E}]_a = [6, 8]$, $[\underline{d}_1, \overline{d}_1]_a = [6, 7]$ and $[\underline{d}_2, \overline{d}_2]_a = [2, 3]$

Situation b $[\underline{E}, \overline{E}]_b = [6, 8]$, $[\underline{d}_1, \overline{d}_1]_b = [6, 7.5]$ and $[\underline{d}_2, \overline{d}_2]_b = [2, 3]$

If we consider f to be the contested garment (CG) rule, the truncation property (4) holds by *(iii)* in Proposition 3.2. However

$$\mathcal{F}(CG; [E]_a, [d]_a) = ([4.5, 6.5], [1, 2.5]) \neq$$
$$([4.5, 6.75], [1, 2.5]) = \mathcal{F}(CG; [E]_b, [d]_b) \quad (7)$$

On the other hand, the two interval games $w_{[E]_a,[d]_a}$ and $w_{[E]_b,[d]_b}$ coincide, since

$$w_{[E]_a,[d]_a}(\{1\}) = w_{[E]_b,[d]_b}(\{1\}) = [3,6]$$
$$w_{[E]_a,[d]_a}(\{2\}) = w_{[E]_b,[d]_b}(\{2\}) = [0,2]$$
$$w_{[E]_a,[d]_a}(\{1,2\}) = w_{[E]_b,[d]_b}(\{1,2\}) = [0,0]$$

In conclusion we cannot express $\mathcal{F}(CG;\cdot,\cdot)$ as a solution concept of w.

The trouble seems to be lying in the given definition of interval bankruptcy game, which is not able to deliver information about the upper claims $\overline{d}_i$ when these are enclosed between the lower and the upper estate. We overcome this problem by considering another interval game based entirely on the upper estate $\overline{E}$.

Definition 4.2. The *upper estate interval bankruptcy game* (UEIBG) $w^u_{[E],[d]}$ for the interval bankruptcy situation $([E],[d])$ is defined as

$$w^u_{[E],[d]}(N) = [\underline{E},\overline{E}] \qquad \text{and, for } S \subsetneqq N \tag{8}$$

$$w^u_{[E],[d]}(S) = \left[v_{\overline{E},\overline{d}}(S), v_{\overline{E},\underline{d}}(S)\right] = \left[(\overline{E} - \sum_{i \in N \setminus S} \overline{d}_i)_+, (\overline{E} - \sum_{i \in N \setminus S} \underline{d}_i)_+\right]. \tag{9}$$

We verify that the definition is correct, and the set of classical bankruptcy games with parameters $\tilde{E}$ and $\tilde{d}$ in the ranges $[E]$ and $[d]$, respectively, can be described in terms of this interval game.

Proposition 4.2. *For each $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$,*

$$v_{\tilde{E},\tilde{d}}(N) \in w^u_{[E],[d]}(N) \qquad \text{and, for each } S \subsetneqq N,$$
$$v_{\tilde{E},\tilde{d}}(S) \in \left[(\underline{w}^u_{[E],[d]}(S) - \overline{w}^u_{[E],[d]}(N) + \underline{w}^u_{[E],[d]}(N))_+, \overline{w}^u_{[E],[d]}(S)\right] \tag{10}$$

Proof. We need to show that, for $S \subsetneqq N$,

$$(\underline{w}^u_{[E],[d]}(S) - \overline{w}^u_{[E],[d]}(N) + \underline{w}^u_{[E],[d]}(N))_+ = (\underline{E} - \sum_{i \in N \setminus S} \overline{d}_i)_+. \tag{11}$$

Simply note that we can write the first term as

$$\left((\overline{E} - \sum_{i \in N \setminus S} \overline{d}_i)_+ - \overline{E} + \underline{E}\right)_+.$$

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If $\sum_{i \in N \setminus S} \overline{d}_i \leq \overline{E}$ this coincides with the right-hand side in (11). Otherwise, $\sum_{i \in N \setminus S} \overline{d}_i > \overline{E}$, and then

$$\left((\overline{E} - \sum_{i \in N \setminus S} \overline{d}_i)_+ - \overline{E} + \underline{E} \right)_+ = (\underline{E} - \overline{E})_+ = 0$$

which, again coincides with the right-hand side in (11) in the present situation. We proved already in Proposition 4.1 that for each $\tilde{E} \in [E]$ and $\tilde{d} \in [d]$,

$$(\underline{E} - \sum_{i \in N \setminus S} \overline{d}_i)_+ \leq v_{\tilde{E},\tilde{d}} \leq (\overline{E} - \sum_{i \in N \setminus S} \underline{d}_i)_+ \,,$$

and, therefore, (10) holds. $\square$

Just as in the previous section, we consider a truncated form.

Definition 4.3. The *truncated upper estate interval bankruptcy game* (TUEIBG) is defined as $w^u_{[E],[d] \wedge \overline{E}}$. More in detail:

$$w^u_{[E],[d] \wedge \overline{E}}(N) = w^u_{[E],[d]}(N) = [\underline{E}, \overline{E}] \qquad \text{and, for } S \subsetneq N \,,$$

$$w^u_{[E],[d] \wedge \overline{E}}(S) = \left[v_{\overline{E},\overline{d} \wedge \overline{E}}(S), v_{\overline{E},\underline{d} \wedge \overline{E}}(S) \right] = \left[(\overline{E} - \sum_{i \in N \setminus S} (\overline{d}_i \wedge \overline{E}))_+, (\overline{E} - \sum_{i \in N \setminus S} (\underline{d}_i \wedge \overline{E}))_+ \right] .$$

As in the previous case, the two games coincide.

Proposition 4.3. *The games UEIBG and TUEIBG coincide, i.e.* $w^u_{[E],[d] \wedge \overline{E}}(S) = w^u_{[E],[d]}(S)$ *for all* $S \subset N$.

Proof. We need to prove that

$$\left(\overline{E} - \sum_{i \in N \setminus S} (\overline{d}_i \wedge \overline{E}) \right)_+ = \left(\overline{E} - \sum_{i \in N \setminus S} \overline{d}_i \right)_+ ; \qquad (12)$$

$$\left(\overline{E} - \sum_{i \in N \setminus S} (\underline{d}_i \wedge \overline{E}) \right)_+ = \left(\overline{E} - \sum_{i \in N \setminus S} \underline{d}_i \right)_+ . \qquad (13)$$

We distinguish three cases.

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Case 1 $\overline{d}_i \leq \overline{E}$ for every $i \in N \setminus S$. In this case, no truncation occurs and both sides of (12) and (13) are identical.

Case 2 $\underline{d}_i \leq \overline{E}$ for every $i \in N \setminus S$ and $\overline{d}_j > \overline{E}$ for some $j \in N \setminus S$. No truncation occurs for the lower bounds of the claims w.r.t. the estate, so (13) holds. Regarding (12), we have

$$\overline{E} - \sum_{i \in N \setminus S} (\overline{d}_i \wedge \overline{E}) \leq \overline{E} - \overline{d}_j \wedge \overline{E} = \overline{E} - \overline{E} = 0 \ ,$$

$$\overline{E} - \sum_{i \in N \setminus S} \overline{d}_i \leq \overline{E} - \overline{d}_j \leq 0 \ ,$$

and (12) holds, both terms being null.

Case 3 $\underline{d}_j > \overline{E}$ for some $j \in N \setminus S$. We have

$$\overline{E} - \sum_{i \in N \setminus S} (\underline{d}_i \wedge \overline{E}) \leq \overline{E} - \underline{d}_j \wedge \overline{E} = \overline{E} - \overline{E} = 0$$

and

$$\overline{E} - \sum_{i \in N \setminus S} \underline{d}_i \leq \overline{E} - \underline{d}_j \leq 0 \ .$$

Therefore, the positive parts of both terms are null, and (13) holds. Identity (12) holds because both terms are non negative and they are not greater than the corresponding terms in (13). Consequently, they are both null.

$\square$

We now consider those division rules which can be based on the interval games just introduced.

Definition 4.4. An interval division rule $\mathcal{F}_u$ is an *interval game theoretic rule* (IGTR) if there exists a solution concept $\mathcal{G}$ for interval cooperative games such that
$$\mathcal{F}_u([E], [d]) = \mathcal{G}(w^u_{[E],[d]}).$$

We are now able to state a full extension of Theorem 2.1 We begin with the following

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Theorem 4.4. *An interval division rule based on the classical bankruptcy rule f is game theoretic if and only if the rule coincides with the truncated form, i.e. (4) holds.*

Proof. Suppose that $\mathcal{F}(f; \cdot, \cdot)$ is game theoretic. Then, for any interval estate $[E]$ and claims $[d]$

$$\mathcal{F}(f; [E], [d]) = \mathcal{G}(w^u_{[E],[d]}) = \mathcal{G}(w^u_{[E],[d] \wedge \overline{E}}) =$$
$$\mathcal{F}(f; [E], [d] \wedge \overline{E}) = \mathcal{F}^t(f; [E], [d])$$

and, therefore (4) holds.

Conversely, suppose that (4) holds, and consider the following solution concept defined for each $w \in IG^N$ by

$$\mathcal{G}(w) = \left(\left[f_i(\underline{w}(N), \underline{M}^w_i + \underline{e}^w_i, \overline{M}^w_{-i} + \underline{e}^w_i), f_i(\overline{w}(N), \overline{M}^w_i + \overline{e}^w_i, \underline{M}^w_{-i} + \overline{e}^w_i) \right] \right)_{i \in N}$$

where, for any $i \in N$,

$$\overline{M}^w_i = \overline{w}(N) - \underline{w}(N - i); \qquad \underline{M}^w_i = \overline{w}(N) - \overline{w}(N - i);$$

$$\overline{e}^w_i = \frac{\left(\overline{w}(N) - \overline{M}^w_i - \sum_{j \neq i} \underline{M}^w_j \right)_+}{n};$$

$$\underline{e}^w_i = \frac{\left(\underline{w}(N) - \underline{M}^w_i - \sum_{j \neq i} \overline{M}^w_j \right)_+}{n}.$$

When an UEIBG replaces the generic game w, we obtain $\overline{w}(N) = \overline{E}$, $\underline{w}(N) = \underline{E}$, and, for every $i \in N$

$$\overline{M}^{w^u}_i = \overline{d}_i \wedge \overline{E}; \qquad \underline{M}^{w^u}_i = \underline{d}_i \wedge \overline{E}; \tag{14}$$

$$\overline{e}^{w^u}_i = 0; \qquad \underline{e}^{w^u}_i = 0. \tag{15}$$

To prove (14), note that, for any $i \in N$

$$\overline{M}^{w^u}_i = \overline{w}^u(N) - \underline{w}^u(N - i) = \overline{E} - (\overline{E} - \overline{d}_i)_+ = \overline{d}_i \wedge \overline{E}$$

and

$$\underline{M}^{w^u}_i = \overline{w}^u(N) - \overline{w}^u(N - i) = \overline{E} - (\overline{E} - \underline{d}_i)_+ = \underline{d}_i \wedge \overline{E}.$$

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Passing to the first equation in (15), we can write

$$\overline{E} - (\overline{d}_i \wedge \overline{E}) - \sum_{j \neq i}(\underline{d}_i \wedge \overline{E}) \leq \overline{E} - \sum_{i \in N}(\underline{d}_i \wedge \overline{E}).$$

Now, if $\underline{d}_i \leq \overline{E}$ for any $i \in N$, then

$$\overline{E} - \sum_{i \in N}(\underline{d}_i \wedge \overline{E}) = \overline{E} - \sum_{i \in N}\underline{d}_i \leq 0$$

by the basic assumption on interval bankruptcy problems. Otherwise, $\underline{d}_j > \overline{E}$ for some $j \in N$, and

$$\overline{E} - \sum_{i \in N}(\underline{d}_i \wedge \overline{E}) \leq \overline{E} - (\underline{d}_j \wedge \overline{E}) = \overline{E} - \overline{E} = 0.$$

In both cases we deal with non positive quantities, and therefore $\overline{e}_i^{w^u} = 0$ for every $i \in N$.

To prove the second inequality in (15) simply note that

$$\underline{E} - (\underline{d}_i \wedge \overline{E}) - \sum_{j \neq i}(\overline{d}_i \wedge \overline{E}) \leq \overline{E} - \sum_{i \in N}(\underline{d}_i \wedge \overline{E}) \leq 0.$$

Therefore $\underline{e}_i^{w^u} = 0$ for every $i \in N$.

In conclusion, we have verified that

$$\mathcal{G}(w_{[E],[d]}^u) = \left([f_i(\underline{E}, \underline{d}_i \wedge \overline{E}, \overline{d}_{-i} \wedge \overline{E}), f_i(\overline{E}, \overline{d}_i \wedge \overline{E}, \underline{d}_{-i} \wedge \overline{E})]\right)_{i \in N} =$$
$$\mathcal{F}^t(f; [E], [d]),$$

and, since (4) holds,

$$\mathcal{F}(f; [E], [d]) = \mathcal{G}(w_{[E],[d]}^u).$$

Therefore, the division rule is game theoretic. $\square$

Example 4.2 (Example 4.1 continued). In this case

$$w_{[E]_a,[d]_a}^u(\{2\}) = [0.5, 2] \neq [1, 2] = w_{[E]_b,[d]_b}^u(\{2\}).$$

Therefore, relation (4) does not contradict the general result.

Corollary 4.5. *If f is game theoretic, then $\mathcal{F}(f; \cdot, \cdot)$ is game theoretic.*

Proof. If f is game theoretic, then (4) holds by Proposition 3.2. Apply Theorem 4.4. $\square$

A   Most bankruptcy rules satisfy Assumption 3.1

It can be checked that the most important bankruptcy rules verify assumption 3.1:

The proportional rule defined as

$$\text{PROP}_i(E, d) = \frac{d_i E}{\sum_{i \in N} d_i} = \frac{d_i E}{d_i + \sum_{j \neq i} d_j}.$$

Just take the partial derivatives:

$$\frac{\partial \text{PROP}_i}{\partial E} = \frac{d_i}{\sum_{i \in N} d_i} > 0;$$

$$\frac{\partial \text{PROP}_i}{\partial d_i} = \frac{(\sum_{j \neq i} d_j) E}{(\sum_{i \in N} d_i)^2} > 0;$$

$$\frac{\partial \text{PROP}_i}{\partial d_j} = \frac{-d_i E}{(\sum_{i \in N} d_i)^2} < 0.$$

The constrained equal awards rule defined as

$$\text{CEA}_i(E, d) = \min\{\alpha, d_i\}$$

$$\text{with } \alpha > 0 \text{ such that } \sum_{i \in N} \text{CEA}_i(E, d) = E.$$

Here it is more convenient to think in terms of the hydraulic rationing setting proposed by Kaminski [7]. In this context, a quantity E of water is poured into n communicating vessels, with heights given by the vector d, as illustrated in Figure 1[1].

The elements of the bankruptcy problem are in turn increased by a small quantity $\Delta > 0$.

- If E is increased, then the vessels of the players J such that $d_j > \alpha$ are increased by $\Delta/|\{j : d_j > \alpha\}|$ each[2].

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[1]In this picture, α has been replaced by a, and Δ by D

[2]Note the increase Δ is so small that no vessel gets filled up, and a similar precaution will be used in the sequel.
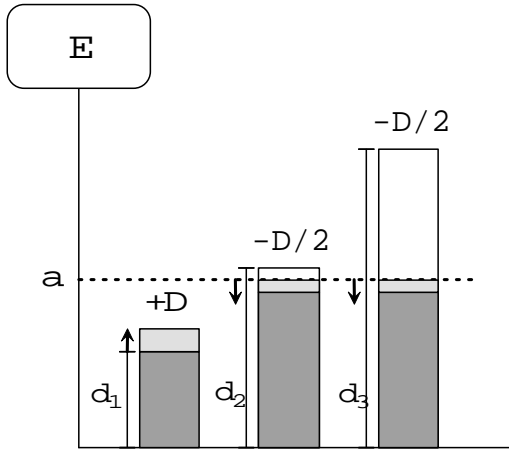
Figure 1: Illustration of the CEA rule and its monotonicity properties in the claims

- If $d_i \geq \alpha$ an increase in the claims of player i does not affect the reward of any player.

- If $d_i < \alpha$, then an increase of player i's claims by Δ will increase her award by the same amount. The level α will decrease by $\Delta/|\{j : d_j \geq \alpha\}|$ for all players j with $d_j \geq \alpha$. All the other players will have their reward unaltered.

The constrained equal loss rule defined by

$$\mathrm{CEL}_i(E, d) = \max\{d_i - \beta, 0\}$$
$$\text{with } \beta \text{ such that } \sum_{i \in N} \mathrm{CEL}_i(E, d) = E.$$

Once again, we recur to a hydraulic representation (see Figure 2[3]), and increase each quantity by a small amount $\Delta > 0$.

- An increase in E will increase the reward of each player such i that $d_i \geq \beta$. The other rewards do not change.

- If $d_i < \beta$, then an increase in the claim of player i will produce no effect.

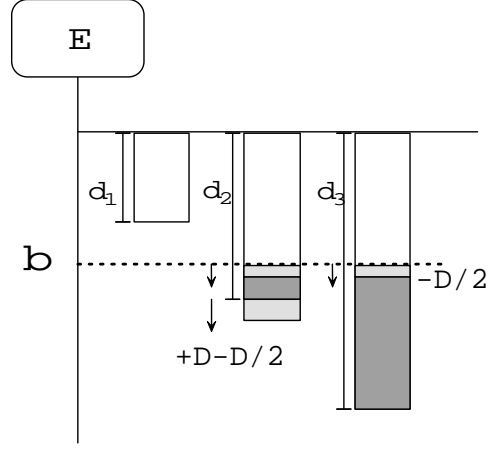[3]In this picture, β has been replaced by b, and Δ by D.

Figure 2: Illustration of the CEL rule and its monotonicity properties in the claims

- If $d_i \geq \beta$, an increase by Δ of player i's claim will decrease β by $\Delta/|\{j : d_j > \beta\}|$. Therefore, for the players j with $d_j > \beta$ the reward will decrease by the same amount. Player i's reward, instead, will increase since

$$\Delta - \frac{\Delta}{|\{j : d_j > \beta\}|} > 0,$$

All the other players will get the same as before.

The contested garment rule defined as follows. Define $E^* = \sum_{i \in N} d_i/2$.
If $E \leq E^*$ then

$$\mathrm{CG}_i(E, d) = \mathrm{CEA}_i(E, d/2) \tag{16}$$

where $d/2 = (d_1/2, d_2/2, \ldots, d_n/2)$. If $E > E^*$ then

$$\mathrm{CG}_i(E, d) = \mathrm{CEL}_i(E - E^*, d/2) + d_1/2. \tag{17}$$

If $E < E^*$ then (16) shares the properties of CEA, while if $E > E^*$ then (17) inherits the properties of CEL. Extra care is needed for the border case $E = E^*$ and we recur to the hydraulic representation (see Figure 3). If E is increased by Δ, then player n's reward will increase by the same amount and the other rewards remain unchanged. If any

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claim is increased by Δ, no reward changes (not even the reward of the player with the claim increased).
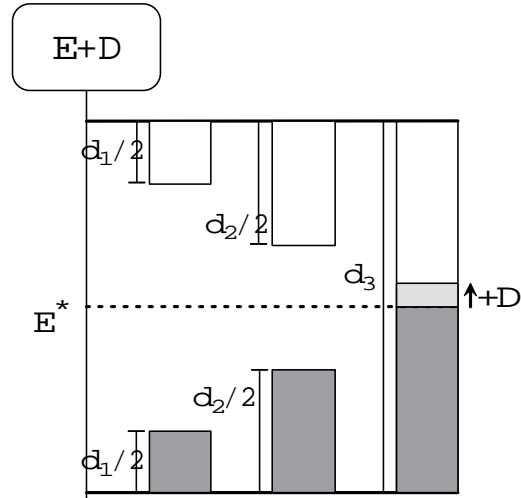

Figure 3: Illustration of the CG rule and its monotonicity properties in the estate when $E = E^*$

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