

Interval Recourse Linear Programming for Resources and Environmental Systems Management under Uncertainty

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ABSTRACT. Interval recourse linear programming (*IRLP*) is proposed for mitigating constraint violation problems in resources and environmental systems management (*REM*) under interval uncertainties. Based on a review of interval linear programming (*ILP*) and its significances to *REM*, two linear programming sub-models are employed to initialize a decision space in *IRLP*. The causes of constraint violation are examined based on identification of a violation criterion. Contraction ratios are defined after revelation of violation ranges of constraints. As a recourse measure to resolve constraint violation problems, another two linear programming sub-models are constructed given a series of contraction ratios. A hypercube decision space where infeasible solutions are excluded is obtained. A post-optimality analysis is conducted to deal with the barriers for applying the *IRLP* approach to address real-world *REM* problems under interval uncertainties. A selected *REM* problem is introduced to demonstrate the procedures and effectiveness of the *IRLP* approach. Comparisons with existing *ILP* methods reveal that the *IRLP* approach is effective at avoiding constraint violation, reproducing the largest decision space which does not include infeasible solutions, and enhancing the reliability of decision support for *REM*.

Keywords: resources and environmental management; interval linear programming; constraint violation

1. Introduction

It is common that properties of elements or influencing factors of resources and environmental management (*REM*) systems are of interval uncertainties. Effective reflection of this system complexity in the decision support process is desired for assuring reliability of suggested management schemes and avoiding penalties to socio-economic and eco-environmental profits. One of representative approaches for achieving it is interval linear programming (*ILP*). In the past decades, a number of *ILP* models which were mixtures of linear programming models and interval-set coefficients were developed (e.g., Soyster, 1973; Moore, 1979; Steuer, 1981; Huang et al., 1992; Sengupta and Pal, 2000; Sengupta et al., 2001; Hladik, 2012). Among them, the first *ILP* model that could reflect multiplicities of decision alternatives under interval uncertainties was developed by Huang et al. (1992). Due to higher robustness than others in terms of approaching realities, it has been applied to many real-world *REM* problems such as waste management (Chang et al., 1997; Pires et al., 2011; Wu et al., 2015), water resources alloca-

tion (Maqsood et al., 2005; Li et al., 2008; Lv et al., 2010; Nikoo et al., 2012), energy systems management (Lin et al., 2005; Cai et al., 2009; Dong et al., 2012; Li et al., 2014), and water quality management (Zeng et al., 2000; Guo et al., 2003; Jin et al., 2004; Qin et al., 2009; Nikoo et al., 2012; Hu et al., 2013; Liu et al., 2015). However, the effectiveness of the *ILP* method (Huang et al., 1992) is dependent with solution methods.

Previously, many methods (e.g., Charnes et al., 1977; Ishibuchi and Tanaka, 1990; Huang et al., 1992; Tong, 1994; Inuiguchi and Sakawa, 1995, 1997; Chanas and Kuchta, 1996; Sengupta and Pal, 2000; Chinneck and Ramadan, 2000; Ben-Tal and Nemirovski, 2002; Inuiguchi et al., 2003; Bertsimas and Sim, 2004; Ben-Tal et al., 2009; Bertsimas and Brown, 2011; Gabrel et al., 2014) were developed for solving the *ILP* model (Huang et al., 1992). For instance, an *ILP* model could be equivalent to a robust linear programming (*RLP*) problem; a robust solution of which performances remained relatively unchanged when exposed to uncertain conditions (Beyer and Sendhoff, 2007) could be obtained from a robust counterpart of the *RLP* model. Tong (1994), Levin (1994) and Chinneck and Ramadan (2000) proposed methods that could obtain the most optimistic decision scheme from a best-case sub-model and the most conservative one from a worst-case sub-model. These methods were challenged by lack of abundant alternatives for decision making under interval uncertainties. To mitigate this challenge, Huang et al. (1992) proposed a two-step approach (*TSA*) in which a continuous decision space could be provided for decision makers based on two correlated linear programming sub-mo-

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dels (denoted as *ILP1* and *ILP2*).

Zhou et al. (2008) pointed out that constraints might be violated by solutions in the decision space for *TSA*. Accordingly, a modified interval linear programming (*MILP*) method, a three-step (*ThS*) method and a robust two-step (*RTS*) method were contributed by Zhou et al. (2008), Huang and Cao (2011) and Fan and Huang (2012), respectively, to deal with such a difficulty and eliminate unexpected infeasibilities. In *MILP*, an additional constraint was coupled with constraints of sub-model *ILP2* for avoiding violation of constraints. In the *ThS* method, another sub-model was constructed to constrict the initial decision space obtained from sub-models *ILP1* and *ILP2*. Parameters constricting ratios of decision variables were introduced to reflect constricting ranges of initialized interval-set solutions. They were optimized through the third sub-model subject to avoidance of constraint violation. In the *RTS* method, sub-model *ILP2* that focused on conservative schemes was solved prior to sub-model *ILP1*. A group of additional constraints was incorporated into the original sub-model *ILP1* for avoiding occurrence of constraint violation.

However, the constraint-violation problem of the *TSA* approach is not effectively resolved yet. Methods of *MILP*, *ThS* and *RTS* cannot reproduce the largest decision space which does not include infeasible solutions. Part of decision alternatives are neglected due to ineffectiveness of these methods. As one of representative strengthens of the *TSA* approach, a hypercube decision space that covers most alternative schemes for *ILP* problems under interval uncertainties can be identified. Failures to remain such strengthen as well as others may reduce reliability of optimization efforts, neglect desired schemes for decision makers of diverse preferences, and sacrifice potential optimality of real-world *REM* systems.

Therefore, an interval recourse linear programming (*IRLP*) approach will be proposed in this study for mitigating constraint violation and enhancing reliability of decision support for *REM* under interval uncertainties. In Section 2, a review of *ILP* models, the *TSA* approach and their significances to *REM* problems will be conducted. As the first stage of *IRLP*, two linear programming sub-models *ILP1* and *ILP2* in *TSA* will be employed to initialize a decision space. Causes of constraint violation will be examined based on identification of a violation criterion in Section 3. Contraction ratios will be defined after revelation of violation ranges of constraints. As a recourse measure to constraint violation problems, two linear programming sub-models (*ILP3* and *ILP4*) will be constructed in Section 4 given a series of contraction ratios. A hypercube decision space where infeasible solutions are excluded through sub-models *ILP3* and *ILP4* will be obtained for decision makers. Post-optimality analysis in Section 5 will help enhance adaptability of the *IRLP* approach to interval linear *REM* problems of various complexities. In Section 6, an *REM* problem that can be characterized as an *ILP* model will be introduced to demonstrate procedures and effectiveness of the proposed *IRLP* approach. Strengthens of the *IRLP* approach for *REM* problems will be revealed from comparisons with existing *ILP* methods in Section 7. Its shortcomings and potential solution strategies will also be examined.

2. Review of Interval Linear Programming

2.1. Characterization of *REM* systems

Properties of elements or influencing factors in real-world *REM* systems may be of uncertainties due to ineffectiveness of estimation techniques, inaccuracy of data monitoring, vagueness of subjective evaluation, unpredictability of system noises, or other potential causes. One of possible cases is that the only reliable estimation of these properties is a series of fluctuation ranges of which the distributional information is unknown. For example, rates of nitrogen use in Canada ranged from 25 to 225 kg/ha in 2000 (<http://www.fao.org>), and the growth of energy demand in Canada will range from 0.3 to 1.4% per year as estimated by the National Energy Board (<http://www.neb-one.gc.ca/index-eng.html>).

These properties of interval uncertainties can be expressed as interval sets (Huang et al., 1992) that are defined as closed and bounded sets of real numbers. One representative characteristic is that the distributional information is unknown for any real number in interval sets. Reflection of these properties in other forms, e.g., real numbers, random variables, fuzzy sets or their combinations, would decrease robustness of the constructed programming model. Simplification of interval sets into constants may lead to loss of valuable information (i.e., boundaries of properties). Other uncertainty analysis methods such as stochastic analysis and fuzzy set theory can barely reflect these properties. Artificially assumed information that does not exist would be adulterated into the optimization process if interval uncertainties are interpreted as random variables with probability density functions, fuzzy sets with membership functions, or more complex forms such as random fuzzy sets or fuzzy random variables.

Coupling interval sets with linear programming models leads to a range of interval linear programming (*ILP*) models (Steuer, 1981; Huang et al., 1992; Sengupta and Pal, 2000; Sengupta et al., 2001; Hladik, 2012). Coefficients in both the objective function and constraints of *ILP* models can be interval sets. It is of low reliability that a deterministic solution is provided for decision making under interval uncertainties. The solution can hardly reflect the trade-off between system optimality and constraint-violation risks. A solution which is a set of interval sets is desired for *ILP* problems. Accordingly, a generalized *ILP* model is an *LP* model where both coefficients and decision variables are interval sets. The first generalized *ILP* model is proposed by Huang et al. (1992) based on interval analysis. According to (Huang et al., 1992), an *ILP* model can be formulated as:

$$\text{Max } F^\pm = C^\pm X^\pm \tag{1-1}$$

subject to

$$A^\pm X^\pm \leq b^\pm \tag{1-2}$$

$$X^\pm \geq 0 \tag{1-3}$$

where $X^\pm = \{x_j^\pm\}^{n \times 1}$, $C^\pm = \{c_j^\pm\}^{1 \times n}$, $A^\pm = \{a_{ij}^\pm\}^{m \times n}$, $b^\pm = \{b_i^\pm\}^{m \times 1}$;

positive integers n and m are numbers of decision variables $\{x_j^\pm\}$ and constraints (1-2), respectively; for any $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$, coefficients c_j^\pm , a_{ij}^\pm and b_i^\pm are interval sets of which values range from a real-valued lower bound to a real-valued upper bound without distribution information (Huang et al., 1992); the lower and upper bounds of interval sets are denoted as symbols $-$ and $+$, respectively; decision variables $\{x_j^\pm | j = 1, 2, \dots, n\}$ as well as the objective function value (F^\pm) are also interval sets due to interval uncertainties of coefficients; $\mathbf{0} = \{0\}^{m \times 1}$; and the programming objective is to maximize the value range of objective function F^\pm .

2.2. Principle of ILP Solution Algorithm

Suppose $s^\pm = [s^-, s^+]$ be any interval set. Any real value belonging to s^\pm , e.g., $s \in [s^-, s^+]$, is defined as a *whitened value* of s^\pm (Huang et al., 1992). Any interval-set coefficient (e.g., c_j^\pm , a_{ij}^\pm or b_i^\pm) in ILP model (1) can be whitened as any real value within its boundaries. For any combination of whitened values of interval-set coefficients (abbreviated as whitened coefficients), there might be an optimal solution that can optimize objective function (1-1) subject to whitened constraints. Infiniteness of parameter combinations leads to that of optimal solutions. All optimal solutions constitute a decision space which is denoted as *DILPMax*. For any solution in *DILPMax*, it is optimal for at least one combination of whitened coefficients. All decision alternatives for management under interval uncertainties are located in *DILPMax*. However, *DILPMax* is an irregular space which is hardly intuitive for decision makers. There are complicated interactive relationships among best solutions of all decision variables. In contrast, a hypercube decision space in which alternatives of any decision variable fluctuate within a range and are independent with that of other decision variables is of higher applicability for engineering practices.

Potential methods are to contract or extend *DILPMax* to a hypercube space. Let the hypercube space obtained from two operations be denoted as *DILPHL* and *DILPHU*, respectively. We have $DILPHL \subseteq DILPMax$ and $DILPMax \subseteq DILPHU$. Let the difference between decision spaces *DILPMax* and *DILPHU* be denoted as *DILPMU*. For any solution in *DILPMU*, there does not exist a combination of whitened coefficients for which it is the optimal solution. Equivalently, it may be a conservative solution that correspond to over-satisfaction of constraints and waste of optimality. It may also be a radical solution that implies violation of constraints and potentials of penalties. Local optimality is disabled for solutions in *DILPMU*. They should be excluded from the final decision space. In comparison, any solution in decision space *DILPHL* remains being optimal for at least one combination of whitened coefficients. Contraction of *DILPMax* to a hypercube space is more desired for real-world *REM* problems.

As the first attempt to obtain a hypercube decision space for ILP problems through contraction of *DILPMax*, Huang et al. (1992) developed a two-step approach (*TSA*). Prior to presentation of formulations of *TSA* in the next sub-section, multiple assumptions on which reliability of the *TSA* approach relies and which are inherent with most real-world ILP systems

are summarized as follows. These assumptions are also required for the proposed *IRLP* approach.

Assumption 1: Zero is not included in all interval coefficients such as $\{c_j^\pm | j = 1, 2, \dots, n\}$, $\{a_{ij}^\pm | i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ and $\{b_i^\pm | i = 1, 2, \dots, m\}$ unless it is the lower or upper bound. Namely, the product of bounds of any interval coefficient is not less than zero.

Assumption 2: Values of any decision variable are non-negative. It holds for all interval sets that the lower bound is not greater than the upper bound. Thus, we can have both the lower bound and the upper bound of interval-valued decision variable x_j^\pm ($j = 1, 2, \dots, n$) are not less than zero as stated by constraints (1-3).

Assumption 3: Let the set of all interval coefficients in ILP model (1) be denoted as *IC*, i.e., $\mathbf{IC} = \{c_j | c_j^- \leq c_j \leq c_j^+; j = 1, 2, \dots, n\} \cup \{a_{ij} | a_{ij}^- \leq a_{ij} \leq a_{ij}^+; i = 1, 2, \dots, m; j = 1, 2, \dots, n\} \cup \{b_i | b_i^- \leq b_i \leq b_i^+; i = 1, 2, \dots, m\}$. For any two different elements in *IC*, they would fluctuate within corresponding interval bounds independently.

Assumption 4: There is at least one optimal solution for any combination of all coefficients in ILP model (1) as the fluctuation of coefficients within interval bounds. This is equivalent to the existence of at least one feasible solution for any combination of coefficients $\{a_{ij}^\pm | i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ and $\{b_i^\pm | i = 1, 2, \dots, m\}$ in constraints (1-2) subject to constraints (1-3).

Assumption 5: It holds for interval coefficients in the objective function (1-1) that $c_j^\pm \geq 0$ (for $j = 1, 2, \dots, k$) and $c_j^\pm < 0$ (for $j = k + 1, k + 2, \dots, n$) where $k \in \{1, 2, \dots, n\}$. This could be achieved by reordering decision variables and correspondingly coefficients if this assumption cannot be directly satisfied.

2.3. Initialization of Decision Space

Based on analysis of interactions among decision variables, coefficients and the objective function under interval uncertainties, two interactive LP sub-models, as an equivalence of the initial ILP model, are constructed on the basis of assumptions 1 to 5. The first sub-model (denoted as *ILPI*) can be formulated as:

$$\max F^+ = \sum_{j \in J_1} (c_j^+ \cdot x_j^+) + \sum_{j \in J_2} (c_j^+ \cdot x_j^-) \tag{2-1}$$

subject to

$$\begin{aligned} & \sum_{j \in J_1} (\text{sign}(a_{ij}^\pm) \cdot |a_{ij}^\pm|^- \cdot x_j^+) \\ & + \sum_{j \in J_2} (\text{sign}(a_{ij}^\pm) \cdot |a_{ij}^\pm|^+ \cdot x_j^-) \\ & \leq b_i^+ \text{ for any } i \in \mathbf{I} \end{aligned} \tag{2-2}$$

$$x_j^+ \geq 0 \text{ for any } j \in J_1 \tag{2-3}$$

$$x_j^- \geq 0 \text{ for any } j \in J_2 \tag{2-4}$$

where symbols $-$ and $+$ denote the lower and upper bounds of interval sets, respectively; $\mathbf{I} = \{1, 2, \dots, m\}$, $J_1 = \{1, 2, \dots, k\}$,

$J_2 = \{k + 1, k + 2, \dots, n\}$, and $J = \{1, 2, \dots, n\}$; sign function $sign(a_{ij}^\pm)$ equals to 1 if and only if $a_{ij}^+ \geq a_{ij}^- \geq 0$, and equals to -1 if and only if $a_{ij}^- < a_{ij}^+ < 0$; $|a_{ij}^\pm|$ is the absolute value of interval coefficient a_{ij}^\pm ($i \in I$; $j \in J$); $|a_{ij}^\pm| = [a_{ij}^-, a_{ij}^+]$ if $a_{ij}^+ \geq a_{ij}^- \geq 0$, and $|a_{ij}^\pm| = [-1 \cdot a_{ij}^+, -1 \cdot a_{ij}^-]$ if $0 \geq a_{ij}^+ \geq a_{ij}^-$; decision variables are upper bounds of the first k interval decision variables in *ILP* model (1), i.e., x_j^+ ($j \in J_1$), as well as lower bounds of the latter $(n - k)$ ones, i.e., x_j^- ($j \in J_2$).

Model *ILP1* is a crisp linear programming model. All coefficients in either the objective function (2-1) or constraints (2-2) are constants instead of interval sets. The simplex method (Dantzig, 1963), as an efficient algorithm for linear programming problems, is employed to solve *LP* model *ILP1*. Let the optimum values of decision variables be denoted as x_{jopt}^+ ($j \in J_1$) and x_{jopt}^- ($j \in J_2$). The corresponding objective function value (denoted as F_{opt}^+) can be obtained by $F_{opt}^+ = \sum_{j \in J_1} (c_j^+ \cdot x_{jopt}^+) + \sum_{j \in J_2} (c_j^- \cdot x_{jopt}^-)$. Solutions of model *ILP1* are preferred by optimistic decision makers who pursue higher system optimality while taking higher constraint-violation risks. Based on solutions x_{jopt}^+ ($j \in J_1$) and x_{jopt}^- ($j \in J_2$), the second sub-model (denoted as *ILP2*) can be formulated as:

$$\max F^- = \sum_{j \in J_1} (c_j^- \cdot x_j^-) + \sum_{j \in J_2} (c_j^+ \cdot x_j^+) \quad (3-1)$$

subject to

$$\sum_{j \in J_1} (sign(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^-) + \sum_{j \in J_2} (sign(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^+) \leq b_i^- \text{ for any } i \in I \quad (3-2)$$

$$x_{jopt}^+ \geq x_j^- \geq 0 \text{ for any } j \in J_1 \quad (3-3)$$

$$x_j^+ \geq x_{jopt}^- \text{ for any } j \in J_2 \quad (3-4)$$

where symbols $-$ and $+$, sign functions $sign(a_{ij}^\pm)$, and absolute values of interval coefficients (i.e., $|a_{ij}^\pm|$) are defined as that in model *ILP1*; decision variables are lower bounds of the first k interval decision variables in *ILP* model (1), i.e., x_j^- ($j \in J_1$), as well as upper bounds of the latter $n - k$ ones, i.e., x_j^+ ($j \in J_2$).

The connection between models *ILP1* and *ILP2* is reflected by constraints (3-3) and (3-4). For any interval-valued decision variable x_j^\pm ($j \in J$), the lower bound (x_j^-) cannot be greater than the upper one (x_j^+) although they are obtained from two separate models. This connection is necessary for guaranteeing continuity of the initial decision space. Model *ILP2* is also a deterministic linear programming model without interval coefficients. The simplex method (Dantzig, 1963) is employed again for obtaining the optimum values of decision variables. Let them be denoted as x_{jopt}^- ($j \in J_1$) and x_{jopt}^+ ($j \in J_2$). Let the resulting objective function value be denoted as F_{opt}^- . It equals to $\sum_{j \in J_1} (c_j^- \cdot x_{jopt}^-) + \sum_{j \in J_2} (c_j^+ \cdot x_{jopt}^+)$. These solutions represent lower system optimality and lower constraint-violation risks, thus they are pursued by conservative decision makers.

Coupling optimal solutions of sub-models *ILP1* and *ILP2* leads to the optimum interval-set solution (denoted as x_{jopt}^\pm where $j \in J$) of decision variables. For any $j \in J$, we can have x_{jopt}^\pm

$= [x_{jopt}^-, x_{jopt}^+]$ where x_{jopt}^- and x_{jopt}^+ are obtained from interactive *LP* models *ILP1* and *ILP2*. For example, $x_{1opt}^\pm = [x_{1opt}^-, x_{1opt}^+]$ where x_{1opt}^- and x_{1opt}^+ are obtained from *ILP2* and *ILP1*, respectively. For *ILP* model (1), the initial decision space obtained from models *ILP1* and *ILP2* is $\{(x_1, x_2, \dots, x_n) \mid x_j \in [x_{jopt}^-, x_{jopt}^+]$ for $j \in J\}$. Let it be denoted as *IDS*.

2.4. Significance of ILP

Due to high reliability on providing desired decision schemes for *ILP* problems, the *ILP* model and the *TSA* approach have been extended to planning of a large number of resources and environmental programming problems under interval uncertainties and other system complexities. Among them, representative ones included, but not limited to, the followings:

(a) Waste management (Huang et al., 1992; Chang et al., 1997; Maqsood and Huang, 2003; Huang et al., 2005a; Huang et al., 2005b; Pires et al., 2011; Sun et al., 2013; Wu et al., 2015; Xu et al., 2014),

(b) Water resources allocation (Chang et al., 1997; Wu et al., 1997; Huang and Loucks, 2000; Maqsood et al., 2005; Li et al., 2008; Lv et al., 2010; Zhang et al., 2011; Nikoo et al., 2012; Miao et al., 2014),

(c) Energy systems management (Lin et al., 2005; Cai et al., 2009; Dong et al., 2012; Li et al., 2014; Hu et al., 2014),

(d) Air quality control (Huang, 1992; Liu et al., 2003; Li et al., 2006; Lu et al., 2008; Qin et al., 2010; Wang and Huang, 2013; Shao et al., 2014),

(e) Flood mitigation (Xia et al., 2001),

(f) Water quality management (Huang, 1996; Zeng et al., 2000; Guo et al., 2003; Jin et al., 2004; Qin et al., 2009; Nikoo et al., 2012; Hu et al., 2013; Liu et al., 2015),

(g) Eutrophication control (Huang et al., 2012),

(h) Land-use planning (Huang et al., 1998; Han et al., 2013; Xu et al., 2014; Ni et al., 2014; Zhang et al., 2014),

(i) Resources management (Xia et al., 2001; Nasiri and Huang, 2007; Li et al., 2011; You et al., 2014; Wang et al., 2014; Niu et al., 2014),

(j) Carbon dioxide trading planning (Li et al., 2011; Chen et al., 2013; Zhu et al., 2015), and

(k) Transportation planning (Nasiri et al., 2009; Yan et al., 2010; Tan et al., 2011).

The initial decision space (*IDS*) is an n -dimension hypercube. It is supposed that there is at least one combination of whitened coefficients, i.e., an element $\{C, A, b\}$ of set $\{\{c_j\}^{1 \times n}, \{a_{ij}\}^{m \times n}, \{b_i\}^{m \times 1}\} \mid c_j^- \leq c_j \leq c_j^+ \text{ for } j \in J; a_{ij}^- \leq a_{ij} \leq a_{ij}^+ \text{ for } i \in I \text{ and } j \in J; b_i^- \leq b_i \leq b_i^+ \text{ for } i \in I\}$, such that X_0 is the optimal solution for $\max\{CX \mid AX \leq b; X \geq 0\}$ where X_0 is any combination of decision variables belonging to *IDS*. This can be defined as *continuity of the initial decision space*. In comparison with discrete interval linear programming (*DILP*) methods (e.g., robust linear programming) where only one or two discrete decision schemes could be obtained, more alternatives can be provided by the initial continuous decision space for deci-

sion making under interval uncertainties. One-sidedness of a single decision scheme in *DILP* can be avoided through introduction of interval-set decision variables and proposition of the *TSA* approach. This is helpful for increasing reliabilities of *ILP* methods for real-world *ILP* problems in which decision makers are diversified in preferences. In addition, evaluation of application efficiencies indicates that the *ILP* model is capable of effectively reflecting interval uncertainties in *ILP* problems, incorporating them into the optimization process, and enhancing robustness of related modeling efforts.

3. Violation Analysis

3.1. Violation Criterion and Ranges

As stated above, two optimal solutions are obtained from the simplex method on two correlated *LP* sub-models. The initial decision space (i.e., *IDS*) is constructed by coupling them together. In addition to the two solutions, infinite solutions are included in *IDS*. It is possible that constraints of either sub-model are violated by solutions in *IDS*, of which the occurrence likeliness varies with sub-models *ILP1* and *ILP2*. Sub-model *ILP1* aims to obtain an optimistic solution of *ILP* model (1) under continuity of the initial decision space and intervalness of system properties. The solution is preferred by decision makers who pursue high system profits under high risks of constraint violation. In contrast, sub-model *ILP2* is developed for identifying a scheme desired for conservative decision makers who prefer to mitigate constraint violation risks at the cost of high system profits. A few of solutions in *IDS* may violate constraints of sub-model *ILP1*, which may lead to high penalties because of optimism of *ILP1*. As for sub-model *ILP2*, constraint violation may hold for more solutions in *IDS*, and the average penalty is lower than sub-model *ILP1* due to conservativeness of sub-model *ILP2*. High penalties from constraint violation of sub-model *ILP1* are usually unaffordable in real-world management problems under interval uncertainties. Therefore, the *criterion of constraint violation* of *ILP* model (1) is identified as violation of constraints of sub-model *ILP1* in this study. The criterion can be others such as violation of constraints that correspond to the highest system optimality or security, which will be examined in future studies.

Assumption 6: The criterion of constraint violation of *ILP* model (1) is violation of constraints of sub-model *ILP1* for all solutions in the initial decision space.

Let the left-hand side of sub-model *ILP1* (2) be abbreviated as $(A_i^\pm X^\pm)_C$, i.e., $(A_i^\pm X^\pm)_C = \sum_{j \in J_1} (\text{sign}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j) + \sum_{j \in J_2} (\text{sign}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j)$ for any $i \in I$. Inequalities (2-2) are equivalent to $(A_i^\pm X^\pm)_C \leq b_i^+$ for any. Let *SIGp*() and *SIGN*() be sign functions of coefficients a_{ij}^\pm ($i \in I$ and $j \in J$). For any $i \in I$ and $j \in J$, *SIGp*(a_{ij}^\pm) = 1 and *SIGN*(a_{ij}^\pm) = 0 if $a_{ij}^\pm \geq 0$, and *SIGp*(a_{ij}^\pm) = 0 and *SIGN*(a_{ij}^\pm) = -1 if $a_{ij}^\pm < 0$. Inequalities (2-2) are equivalent with $\sum_{j \in J_1} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^+) + \sum_{j \in J_1} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^+) + \sum_{j \in J_2} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^-) + \sum_{j \in J_2} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_j^-) \leq b_i^+$ for any $i \in I$. With fluctuation of decision variables within initial decision space *IDS*, i.e., $\{(x_1, x_2, \dots, x_n) \mid x_j \in [x_{j\text{opt}}^-, x_{j\text{opt}}^+]\}$ for $j \in J$, the left-hand side of inequalities (2-2) can be maximized

at $\sum_{j \in J_1} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-)$ where $i \in I$. Namely, $\text{Max} \{ \sum_{j \in J_1} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) \mid (x_1, x_2, \dots, x_n) \in \text{IDS} \} = \sum_{j \in J_1} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-)$ for any $i \in I$. Therefore, violation of the i th ($i \in I$) constraint of model *ILP1* is equivalent with $\sum_{j \in J_1} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{ij}^\pm) \cdot |a_{ij}^\pm| \cdot x_{j\text{opt}}^-) > b_i^+$.

Let violated constraints be denoted as a set $IV = \{i(1), i(2), \dots, i(V)\}$ where $V \in V$ and $V = \{1, 2, \dots, m\}$. For any $v \in V$, the $i(v)$ th constraint in the first sub-model (2) is violated. Equivalently, we have $\sum_{j \in J_1} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^-) > b_{i(v)}^+$ for any $v \in V$.

Definition 1: The *violation range* of the $i(v)$ th constraint, denoted as $VR_{i(v)}$, is the range of the right-hand coefficient (i.e., $b_{i(v)}^+$) being violated by the maximum value of $(A_i^\pm X^\pm)_C$. It equals to $VR_{i(v)} = \sum_{j \in J_1} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}}^-) - b_{i(v)}^+$ for any $v \in V$.

3.2. Contraction Ranges

To mitigate the problem of constraint violation in the initial decision space, violation range $VR_{i(v)}$ should be contracted as zero for any $v \in V$. This can be achieved through contraction of four groups of decision variables, i.e.: i) the upper bound of the first k interval decision variables with non-negative technical coefficients (i.e., $x_{j\text{opt}}^+$ where $j \in J_1$ and $a_{i(v)j}^\pm \geq 0$); ii) the lower bound of the first k interval decision variables with negative technical coefficients (i.e., $x_{j\text{opt}}^-$ where $j \in J_1$ and $a_{i(v)j}^\pm < 0$); iii) the upper bound of the last $n - k$ interval decision variables with non-negative technical coefficients (i.e., $x_{j\text{opt}}^+$ where $j \in J_2$ and $a_{i(v)j}^\pm \geq 0$); and iv) the lower bound of the last $n - k$ interval decision variables with negative coefficients (i.e., $x_{j\text{opt}}^-$ where $j \in J_2$ and $a_{i(v)j}^\pm < 0$). For the convenience of expression in the followings, the first and fourth groups of decision variables are denoted as *type-II decision variables*, and the other two groups as *type-I decision variables*.

Let solutions after contraction be denoted as $x_{j\text{opt}^*}^\pm$ ($j \in J$). The $i(v)$ th ($v \in V$) constraint in model *ILP1* is not violated if and only if $\sum_{j \in J_1} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}^*}^+) + \sum_{j \in J_1} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}^*}^-) + \sum_{j \in J_2} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}^*}^+) + \sum_{j \in J_2} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot x_{j\text{opt}^*}^-) \leq b_{i(v)}^+$. Therefore, the overall *contraction range* of the second and third groups of decision variables in $(A_{i(v)}^\pm X^\pm)_C$, denoted as $SR_{i(v)1}$, equals to $\sum_{j \in J_1} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot (x_{j\text{opt}}^- - x_{j\text{opt}^*}^-)) + \sum_{j \in J_2} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot (x_{j\text{opt}}^+ - x_{j\text{opt}^*}^+))$. The maximum value of $SR_{i(v)1}$, i.e., the *type-I maximum contraction range* of $(A_{i(v)}^\pm X^\pm)_C$ (abbreviated as $MSR_{1i(v)}$), is $MSR_{1i(v)} = \sum_{j \in J_1} (\text{SIGN}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot (x_{j\text{opt}}^- - x_{j\text{opt}^*}^-)) + \sum_{j \in J_2} (\text{SIGp}(a_{i(v)j}^\pm) \cdot |a_{i(v)j}^\pm| \cdot (x_{j\text{opt}}^+ - x_{j\text{opt}^*}^+))$. Similarly, the overall *contraction range* of the first and fourth groups of decision variables in $(A_{i(v)}^\pm X^\pm)_C$, deno-

ted as $SR_{i(v)2}$, equals to $\sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*}))$. Its maximum value is $\sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*}))$, which is named as the *type-II maximum contraction range* of $(A_{i(v)}^\pm X^\pm)_C$ and abbreviated as $MSR_{2i(v)}$.

Proposition 1:

- (a) $VR_{i(v)} \leq MSR_{1i(v)}$ for any $v \in V$;
- (b) $VR_{i(v)} = MSR_{1i(v)}$ if and only if $(A_{i(v)}^\pm X^\pm)_C = b_{i(v)}^+$ ($v \in V$);
- (c) $SR_{i(v)1} + SR_{i(v)2} \geq VR_{i(v)}$ for any $v \in V$;
- (d) $SR_{i(v)1} \leq MSR_{1i(v)}$ for any $v \in V$; and
- (e) $SR_{i(v)2} \leq MSR_{2i(v)}$ for any $v \in V$.

Proof:

(a) For any $v \in V$, $MSR_{1i(v)} - VR_{i(v)} = \{ \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) - \{ \sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+}) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) - b_{i(v)}^+ \}$ that can be simplified as $b_{i(v)}^+ - \sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+}) - \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-}) - \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) - \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*})$. Since the $i(v)$ th constraint is violated for initialized solutions, we have $\sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+}) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) \leq b_{i(v)}^+$. Equivalently, $VR_{i(v)} \leq MSR_{1i(v)}$ for any $v \in V$.

(b) For any $v \in V$, it is equivalent with $VR_{i(v)} = MSR_{1i(v)}$ that $\sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+}) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) = b_{i(v)}^+$. From formulations of $(A_{i(v)}^\pm X^\pm)_C$, we can have that $(A_{i(v)}^\pm X^\pm)_C = b_{i(v)}^+$.

(c) For any $v \in V$, $SR_{i(v)1} + SR_{i(v)2} - VR_{i(v)} = \{ \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) \} + \{ \sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*})) \} - \{ \sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+}) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) - b_{i(v)}^+ \}$ which is equivalent with $b_{i(v)}^+ - [\sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*})]$. From the definition of solutions after contraction (i.e., $\{x_{jopt^*}^\pm\}^{1 \times n}$), we have $\sum_{j \in J_1} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_2} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) + \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^*}) \leq b_{i(v)}^+$ for any $v \in V$. Thus, $SR_{i(v)1} + SR_{i(v)2} \geq VR_{i(v)}$ holds for any $v \in V$.

(d) For any $v \in V$, $MSR_{1i(v)} - SR_{i(v)1} = \{ \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) \} - \{ \sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^*})) \}$ which is equivalent with $\sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^*} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^*} - x_{jopt^*}))$. Since $x_{jopt^*} \in [x_{jopt^-}, x_{jopt^+}]$ as well as $x_{jopt^*} \in [x_{jopt^-}, x_{jopt^+}]$ for any $j \in J$, $x_{jopt^*} \leq x_{jopt^+}$ and $x_{jopt^*} \geq x_{jopt^-}$. Based on definitions of sign functions $SIGP()$ and $SIGN()$, we have $\sum_{j \in J_1} (SIGN(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^*} - x_{jopt^*})) + \sum_{j \in J_2} (SIGP(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^*} - x_{jopt^*})) \geq 0$. Accordingly, $SR_{i(v)1} \leq$

$MSR_{1i(v)}$ for any $v \in V$.

(e) Similarly, inequality $SR_{i(v)2} \leq MSR_{2i(v)}$ can be deduced from the relationship between x_{jopt^\pm} and x_{jopt^*} and definitions of sign functions.

3.3. Contraction Ratios

For any violated constraint ($v \in V$), the overall contraction range of four groups of decision variables, i.e., $SR_{i(v)1} + SR_{i(v)2}$, should be greater than or equal to violation range $VR_{i(v)}$. It is implied that the violation problem of any single constraint ($v \in V$) can be mitigated by infinite alternatives. There is a tradeoff between contraction ranges $SR_{i(v)1}$ and $SR_{i(v)2}$ for any violated interval-coefficient constraint. The number of possible contraction schemes for all violated constraints further increases with V exponentially. Therefore, an index named as *contraction ratio* is proposed to eliminate differences of violation ranges between violated constraints. It is defined as normalization of contraction ranges.

Definition 2: For any $v \in V$, the *contraction ratio* of *type-I decision variables* is $SRA_{i(v)1} = SR_{i(v)1} / VR_{i(v)}$, while that of *type-II decision variables* is $SRA_{i(v)2} = SR_{i(v)2} / VR_{i(v)}$.

The contraction ratio can be defined as other forms. Definition 2 is aimed to intuitively reflect the relationship between parameter values (0/1) and contributions (zero/full) of contraction range for violation range and to avoid complicated sub-models. This is helpful for facilitating engineering practices. Besides, the value range of contraction ratios $r_{i(v)}$ ($v \in V$) is a closed interval $[1 - MSR_{2i(v)} / VR_{i(v)}, 1]$ if $MSR_{2i(v)} \leq VR_{i(v)}$; otherwise, it equals to $[0, 1]$. According to the definition of *contraction ratios*, we can have the following corollary.

Corollary 1: If any violated constraint $v \in V$ is satisfied after contraction, then

- (a) $0 \leq SRA_{i(v)1} \leq 1$ if $(A_{i(v)}^\pm X^\pm)_C = b_{i(v)}^+$;
- (b) $0 \leq SRA_{i(v)2} \leq MSR_{2i(v)} / VR_{i(v)}$; and
- (c) $SR_{i(v)2} \geq VR_{i(v)} \cdot (1 - r)$ if $SRA_{i(v)1} \geq r$ ($r \in [0, 1]$).

Proof: Straightforward from definitions of parameters such as $SRA_{i(v)1}$, $SRA_{i(v)2}$, $MSR_{1i(v)}$, $MSR_{2i(v)}$, $VR_{i(v)}$, $SR_{i(v)1}$ and $SR_{i(v)2}$ and their relationships as stated in proposition 1.

4. Contraction of Initial Decision Space

4.1. Contraction of Optimistic Solutions

Given a set of minimum contraction ratios $r_{i(v)}$ for violated constraints ($v \in V$), the satisfaction of violated constraints after contraction is equivalent with the following inequalities: (a) $SRA_{i(v)1} \geq r_{i(v)}$ for any $v \in V$, (b) $SRA_{i(v)2} \geq 1 - r_{i(v)}$ for any $v \in V$, (c) $x_{jopt^\pm} \in x_{jopt^*}$ for any $j \in J$, and (d) $x_{jopt^*} \leq x_{jopt^+}$ for any $j \in J$. A linear programming sub-model (denoted as *ILP3*) that is aimed to achieve contraction of *type-I decision variables* is formulated as:

$$\max F^{*+} = \sum_{j \in J_1} (c_j^+ \cdot x_j^{*+}) + \sum_{j \in J_2} (c_j^+ \cdot x_j^{*-}) \tag{4-1}$$

subject to

$$SRA_{i(v)2} \geq 1 - r_{i(v)} \text{ for any } v \in V \quad (4-2)$$

$$x_{j^*}^+ \in x_{jopt}^{\pm} \text{ for any } j \in J_1 \quad (4-3)$$

$$x_{j^*}^- \in x_{jopt}^{\pm} \text{ for any } j \in J_2 \quad (4-4)$$

where $x_{j^*}^+$ ($j \in J_1$) and $x_{j^*}^-$ ($j \in J_2$) are decision variables; $SRA_{i(v)2}$ ($v \in V$) are contraction ratios for any violated constraint; x_{jopt}^{\pm} ($j \in J$) are interval-set solutions obtained from sub-models *ILP1* and *ILP2*. Based on formulations of contraction ratios, contraction ranges and maximum contraction ranges, sub-model *ILP3* can be re-formulated as:

$$\max F_{*}^+ = \sum_{j \in J_1} (c_j^+ \cdot x_{j^*}^+) + \sum_{j \in J_2} (c_j^+ \cdot x_{j^*}^-) \quad (5-1)$$

subject to

$$\begin{aligned} & \sum_{j \in J_1} (SIGp(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot (x_{j^*}^+ - x_{jopt}^+ \cdot r_{i(v)})) \\ & + \sum_{j \in J_1} (SIGN(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot x_{jopt}^- \cdot (1 - r_{i(v)})) \\ & + \sum_{j \in J_2} (SIGp(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot x_{jopt}^+ \cdot (1 - r_{i(v)})) \\ & + \sum_{j \in J_2} (SIGN(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot (x_{j^*}^- - x_{jopt}^- \cdot r_{i(v)})) \\ & \leq b_{i(v)}^+ \cdot (1 - r_{i(v)}) \text{ for any } v \in V \end{aligned} \quad (5-2)$$

$$x_{j^*}^+ \geq x_{jopt}^- \text{ and } x_{j^*}^+ \leq x_{jopt}^+ \text{ for any } j \in J_1 \quad (5-3)$$

$$x_{j^*}^- \geq x_{jopt}^- \text{ and } x_{j^*}^- \leq x_{jopt}^+ \text{ for any } j \in J_2. \quad (5-4)$$

Model (5) is a linear programming model where all coefficients are deterministic rather than interval sets. The simplex method (Dantzig, 1963) is employed again for solving *LP* model (5). Let the obtained solutions be denoted as $x_{jopt^{*+}}$ ($j \in J_1$) and $x_{jopt^{*-}}$ ($j \in J_2$). The objective function (5-1), denoted as $F_{opt^{*+}}$, equals to $F_{opt^{*+}} = \sum_{j \in J_1} (c_j^+ \cdot x_{jopt^{*+}}) + \sum_{j \in J_2} (c_j^+ \cdot x_{jopt^{*-}})$. Accordingly, the optimistic solution is contracted from x_{jopt}^+ ($j \in J_1$) and x_{jopt}^- ($j \in J_2$) to $x_{jopt^{*+}}$ ($j \in J_1$) and $x_{jopt^{*-}}$ ($j \in J_2$). The maximum objective function value under optimism decreases from $\sum_{j \in J_1} (c_j^+ \cdot x_{jopt}^+) + \sum_{j \in J_2} (c_j^+ \cdot x_{jopt}^-)$ to $\sum_{j \in J_1} (c_j^+ \cdot x_{jopt^{*+}}) + \sum_{j \in J_2} (c_j^+ \cdot x_{jopt^{*-}})$ for mitigating violation of constraints (2-2) to (2-4).

4.2. Contraction of Conservative Solutions

Based on solutions from the third sub-model, another sub-model (denoted as *ILP4*) for contracting conservative solutions obtained from the second sub-model (*ILP2*) can be formulated as follows:

$$\min F_{*}^- = \sum_{j \in J_1} (c_j^- \cdot x_{j^*}^-) + \sum_{j \in J_2} (c_j^- \cdot x_{j^*}^+) \quad (6-1)$$

subject to

$$SRA_{i(v)1} \geq r_{i(v)} \text{ for any } v \in V \quad (6-2)$$

$$x_{j^*}^- \in x_{jopt}^{\pm} \text{ for any } j \in J_1 \quad (6-3)$$

$$x_{j^*}^+ \in x_{jopt}^{\pm} \text{ for any } j \in J_2 \quad (6-4)$$

$$x_{j^*}^- \leq x_{jopt^{*+}} \text{ for any } j \in J_1 \quad (6-5)$$

$$x_{j^*}^+ \geq x_{jopt^{*-}} \text{ for any } j \in J_2 \quad (6-6)$$

where $x_{j^*}^-$ ($j \in J_1$) and $x_{j^*}^+$ ($j \in J_2$) are decision variables and $SRA_{i(v)1}$ ($v \in V$) are contraction ratios for any violated constraint. Equivalently, sub-model *ILP4* can be formulated as:

$$\min F_{*}^- = \sum_{j \in J_1} (c_j^- \cdot x_{j^*}^-) + \sum_{j \in J_2} (c_j^- \cdot x_{j^*}^+) \quad (7-1)$$

subject to

$$\begin{aligned} & \sum_{j \in J_1} (SIGp(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot x_{jopt}^+ \cdot r_{i(v)}) \\ & + \sum_{j \in J_1} (SIGN(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot (x_{jopt}^- \cdot r_{i(v)} - x_{jopt}^- + x_{j^*}^-)) \\ & + \sum_{j \in J_2} (SIGp(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot (x_{jopt}^+ \cdot r_{i(v)} - x_{jopt}^+ + x_{j^*}^+)) \\ & + \sum_{j \in J_2} (SIGN(a_{i(v)j^{\pm}}) \cdot |a_{i(v)j^{\pm}}|^{\pm} \cdot x_{jopt}^- \cdot r_{i(v)}) \\ & \leq b_{i(v)}^+ \cdot r_{i(v)} \text{ for any } v \in V \end{aligned} \quad (7-2)$$

$$x_{jopt}^- \leq x_{j^*}^- \leq x_{jopt^{*+}} \text{ for any } j \in J_1 \quad (7-3)$$

$$x_{jopt}^+ \geq x_{j^*}^+ \geq x_{jopt^{*-}} \text{ for any } j \in J_2. \quad (7-4)$$

The simplex method (Dantzig, 1963) is employed again to solve *LP* model (7). Let the obtained solutions be denoted as $x_{jopt^{*-}}$ ($j \in J_1$) and $x_{jopt^{*+}}$ ($j \in J_2$). The objective function ($F_{opt^{*-}}$) equals to $\sum_{j \in J_1} (c_j^- \cdot x_{jopt^{*-}}) + \sum_{j \in J_2} (c_j^- \cdot x_{jopt^{*+}})$. As a result, the conservative solution is contracted from x_{jopt}^- ($j \in J_1$) and x_{jopt}^+ ($j \in J_2$) to $x_{jopt^{*-}}$ ($j \in J_1$) and $x_{jopt^{*+}}$ ($j \in J_2$). As another cost of mitigating constraint violation of *ILP* models, the maximum objective function value under conservative conditions decreases from $\sum_{j \in J_1} (c_j^- \cdot x_{jopt}^-) + \sum_{j \in J_2} (c_j^- \cdot x_{jopt}^+)$ to $\sum_{j \in J_1} (c_j^- \cdot x_{jopt^{*-}}) + \sum_{j \in J_2} (c_j^- \cdot x_{jopt^{*+}})$.

4.3. Finalization of Decision Space

Corresponding to a given set of contraction ratios, interval-set solutions are obtained through combining crisp solutions from the third and fourth linear programming sub-models, i.e., $\{x_{jopt}^{\pm} | x_{jopt^{*+}} \leq x_{jopt}^{\pm} \leq x_{jopt^{*-}}; j \in J\}$. The range of objective function $F_{opt^{*+}}$ is a closed interval $[F_{opt^{*-}}, F_{opt^{*+}}]$. Let the final decision space after contraction be denoted as *CDS*. We can have $CDS = \{x_{jopt}^* | x_{jopt}^* \in [x_{jopt^{*-}}, x_{jopt^{*+}}]; j \in J\}$.

The decision space (*CDS*) is projected as an interval (i.e., $[x_{jopt}^-, x_{jopt}^+]$) on the j th ($j \in J$) dimension. In engineering practices, this interval can provide decision makers with an initial estimation of desired ranges of the j th decision (i.e., the j th decision variable x_j) in a visualized way. As stated in section 2, the largest continuous decision space (*DILPMax*) of *ILP* models is an irregular multi-dimension space. It is not easily visualized by decision makers especially when a number of decision variables are involved. In comparison with *CDS*, less valuable information can be provided from *DILPMax* for decision makers even when only a few of decision variables are involved. Another potential substitute of *IDS* and *DILPMax* is the hypercube space that can cover *DILPMax* and of which the projection on any dimension is an interval. Let it be denoted as *IDSMMax*. However, decision space *IDSMMax* may involve many absolutely infeasible solutions that are infeasible for any combination of coefficients. The stability of real-world management systems

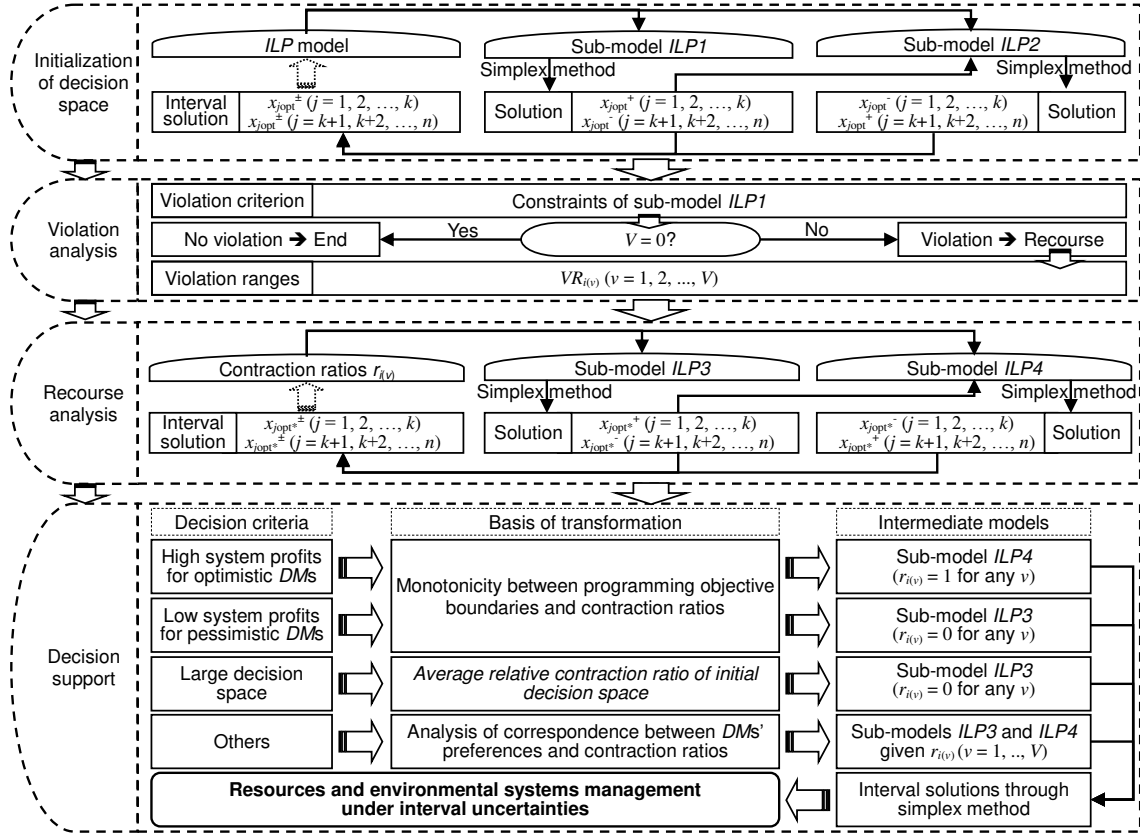


Figure 1. Structure and procedures of the IRLP approach for ILP models.

corresponding to ILP models is decreased if management schemes are determined on the basis of *IDSMax*.

The decision space after contraction provides a potential to analyze the tradeoff between system optimality and constraint violation of ILP models. The existence of this tradeoff is due to interval uncertainties of inexact coefficients. Decisions approaching $(x_{1opt}^+, x_{2opt}^+, \dots, x_{kopt}^+, x_{(k+1)opt}^-, x_{(k+2)opt}^-, \dots, x_{nopt}^-)$ in *CDS* would lead to high system profits while taking high chances of constraint violation as the fluctuation of interval coefficients. If decisions around $(x_{1opt}^-, x_{2opt}^-, \dots, x_{kopt}^-, x_{(k+1)opt}^+, x_{(k+2)opt}^+, \dots, x_{nopt}^+)$ in *CDS* are executed the likeliness of constraint violation would decrease at the cost of decreased system profits. A quantitative analysis of this tradeoff may be enabled through Fourier amplitude sensitivity test (Saltelli and Bolado, 1998), factorial analysis (Jaccard, 1998), multi-criteria assessment (Linkov et al., 2006), or other post-optimality techniques in the future studies. The most desired deterministic decision scheme which may be located at a balanced point in *CDS* will be provided for decision makers for whom acceptance levels of system optimality and constraint violation are varied.

To sum up, the proposed IRLP approach consists of three steps as presented in the Figure 1. An initial interval-set solution of ILP model (1) is obtained through two correlated linear programming sub-models. Violation analysis is conducted to identify violated constraints. Given a set of constraint contraction ratios, two linear programming sub-models are formulated. The

simplex method is employed twice for obtaining interval-set solutions as a contraction of initialized solutions.

5. Post-Optimality Analysis

5.1. Analysis of Interactions

As a resource measure to eradicate constraint violation problems of initial solutions from sub-models ILP1 and ILP2, sub-models ILP3 and ILP4 are proposed in the IRLP approach. The two sub-models can avoid high penalties resulting from violation of constraints in sub-model ILP1 while remaining optimality of decision alternatives after resource. This capability relies on designs of sub-models ILP3 and ILP4. Constraints of both sub-models reflect the target of mitigating constraint violation through contracting the initial decision space, and objective functions aim to maximize the final decision space after contraction and avoid neglecting potential decision alternatives and resultant system optimality. It should be noted that formulations of the two sub-models can be in other forms. For instance, sub-models ILP3 and ILP4 can be constructed as:

$$\min F^{*-} = \sum_{j \in J_1} (c_j^- \cdot x_j^{*-}) + \sum_{j \in J_2} (c_j^- \cdot x_j^{*+}) \quad (8-1)$$

subject to

$$\sum_{j \in J_1} (SIGp(a_{i(v)}^{\pm}) \cdot |a_{i(v)}^{\pm}| \cdot x_{jopt}^{\pm} \cdot r_{i(v)})$$

$$\begin{aligned}
 & + \sum_{j \in J_1} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} \cdot r_{i(v)} - x_{jopt^-} + x_{j^*}^-)) \\
 & + \sum_{j \in J_2} (SIGp(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} \cdot r_{i(v)} - x_{jopt^+} + x_{j^*}^+)) \quad (8-2) \\
 & + \sum_{j \in J_2} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-} \cdot r_{i(v)}) \\
 & \leq b_{i(v)^+} \cdot r_{i(v)} \text{ for any } v \in V
 \end{aligned}$$

$$x_{jopt^-} \leq x_{j^*}^- \leq x_{jopt^+} \text{ for any } j \in J_1 \quad (8-3)$$

$$x_{jopt^+} \geq x_{j^*}^+ \geq x_{jopt^-} \text{ for any } j \in J_2 \quad (8-4)$$

and

$$max F^{*+} = \sum_{j \in J_1} (c_j^+ \cdot x_{j^*}^+) + \sum_{j \in J_2} (c_j^+ \cdot x_{j^*}^-) \quad (9-1)$$

subject to

$$\begin{aligned}
 & \sum_{j \in J_1} (SIGp(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{jopt^+} \cdot r_{i(v)})) \\
 & + \sum_{j \in J_1} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^-} \cdot (1 - r_{i(v)})) \\
 & + \sum_{j \in J_2} (SIGp(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot x_{jopt^+} \cdot (1 - r_{i(v)})) \quad (9-2) \\
 & + \sum_{j \in J_2} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{j^*}^- - x_{jopt^-} \cdot r_{i(v)})) \\
 & \leq b_{i(v)^+} \cdot (1 - r_{i(v)}) \text{ for any } v \in V
 \end{aligned}$$

$$x_{jopt^*}^- \leq x_{j^*}^+ \leq x_{jopt^+} \text{ for any } j \in J_1 \quad (9-3)$$

$$x_{jopt^-} \leq x_{j^*}^- \leq x_{jopt^*}^+ \text{ for any } j \in J_2 \quad (9-4)$$

respectively, where $x_{1opt^*}^-$, $x_{2opt^*}^-$, ..., $x_{kopt^*}^-$, $x_{(k+1)opt^*}^+$, $x_{(k+2)opt^*}^+$, ..., and $x_{nopt^*}^+$ are optimal solutions of LP model (8) through the simplex method (Dantzig, 1963). The above formulations of sub-models *ILP3* and *ILP4* can also effectively resolve constraint violation problems of the *TSA* approach through contraction of the initial decision space. It is possible that solutions of models (8) and (9) are identical with those of models (5) and (7). On the other hand, it may hold in some cases that optimal values of objective functions F^{*-} and F^{*+} in models (8) and (9) are decreased in comparison with $F_{opt^*}^-$ and $F_{opt^*}^+$ in models (7) and (5). Potentially high system profits may be neglected although the constraint violation risk is decreased simultaneously. Results of models (8) and (9) are relatively conservative. Formulating sub-models *ILP3* and *ILP4* as models (8) and (9) is preferable for decision makers who do not pursue the highest system profit. Based on the assumption that provision of the highest system profit is desired for most decision makers, sub-models *ILP3* and *ILP4* are formulated as models (5) and (7) in this study, respectively.

Proposition 2: If $SR_{i(v)1} = VR_{i(v)}$ for any $v \in V$, then $F_{opt^*}^+ = F_{opt^+}$; if $VR_{i(v)} \leq MSR_{2i(v)}$ and $SR_{i(v)2} = VR_{i(v)}$ for any $v \in V$, then $F_{opt^*}^- = F_{opt^-}$.

Proof: Since the formulation of $SRA_{i(v)1}$, we have $SRA_{i(v)1} = 1$ if $SR_{i(v)1} = VR_{i(v)}$ for any $v \in V$. From $SRA_{i(v)1} + SRA_{i(v)2} \geq 1$, we have $SRA_{i(v)2} \geq 0$. Sub-model *ILP3* is simplified as $Max F^{*+} = \sum_{j \in J_1} (c_j^+ \cdot x_{j^*}^+) + \sum_{j \in J_2} (c_j^+ \cdot x_{j^*}^-)$ subject to $SRA_{i(v)2} \geq 0$ for any $v \in V$, $x_{j^*}^+ \in x_{jopt^+}$ for any $j \in J_1$, and $x_{j^*}^- \in x_{jopt^-}$ for any $j \in J_2$. It is equivalent with maximizing $\{F^{*+} = \sum_{j \in J_1} (c_j^+ \cdot x_{j^*}^+) + \sum_{j \in J_2} (c_j^+ \cdot x_{j^*}^-) \mid x_{j^*}^+ \in x_{jopt^+}$ for any $j \in J_1$; $x_{j^*}^- \in x_{jopt^-}$ for any $j \in J_2\}$ due to non-negativity of $SRA_{i(v)2}$ according to its definition. The optimal solution of the simplified crisp model is $\{x_{1opt^+}, x_{2opt^+}, \dots, x_{kopt^+}, x_{(k+1)opt^-}, x_{(k+2)opt^-}, \dots, x_{nopt^-}\}$ and we can have $F_{opt^*}^+ = \sum_{j \in J_1}$

$(c_j^+ \cdot x_{jopt^+}) + \sum_{j \in J_2} (c_j^+ \cdot x_{jopt^-})$. From the formulation of F_{opt^+} , we have $F_{opt^*}^+ = F_{opt^+}$. Similarly, sub-model *ILP4* is equivalent with minimizing $\{F^{*-} = \sum_{j \in J_1} (c_j^- \cdot x_{j^*}^-) + \sum_{j \in J_2} (c_j^- \cdot x_{j^*}^+) \mid SRA_{i(v)1} \geq 0$ for any $v \in V$; $x_{jopt^-} \leq x_{j^*}^- \leq x_{jopt^*}$ for any $j \in J_1$; $x_{jopt^+} \geq x_{j^*}^+ \geq x_{jopt^*}$ for any $j \in J_2\}$ if $VR_{i(v)} \leq MSR_{2i(v)}$ and $SR_{i(v)2} = VR_{i(v)}$. Thus, $F_{opt^*}^- = \sum_{j \in J_1} (c_j^- \cdot x_{jopt^-}) + \sum_{j \in J_2} (c_j^- \cdot x_{jopt^+})$ and $F_{opt^*}^- = F_{opt^-}$.

Proposition 3: Let $r_{i(v)1}$ and $r_{i(v)2}$ be two contraction ratios given for the v th violated constraint. If $r_{i(v)1} \leq r_{i(v)2}$, then $F_{opt1^*}^- \leq F_{opt2^*}^-$ and $F_{opt1^*}^+ \leq F_{opt2^*}^+$ where $F_{opt^*}^-$ and $F_{opt^*}^+$ ($s = 1$ or 2) are the lower and upper bounds of optimal objective function values under contraction ratio $r_{i(v)s}$, respectively.

Proof: Denote the feasible region of sub-model *ILP3* as $FR3_1$ and $FR3_2$ corresponding to $r_{i(v)1}$ and $r_{i(v)2}$, respectively. We can have $FR3_1 = \{(x_1, x_2, \dots, x_k, x_{(k+1)}, x_{(k+2)}, \dots, x_n) \mid SRA_{i(v)2} \geq 1 - r_{i(v)1}$ for any $v \in V$; $x_j \in x_{jopt^+}$ for any $j \in J_1$; $x_j \in x_{jopt^-}$ for any $j \in J_2\}$ and $FR3_2 = \{(x_1, x_2, \dots, x_k, x_{(k+1)}, x_{(k+2)}, \dots, x_n) \mid SRA_{i(v)2} \geq 1 - r_{i(v)2}$ for any $v \in V$; $x_j \in x_{jopt^+}$ for any $j \in J_1$; $x_j \in x_{jopt^-}$ for any $j \in J_2\}$. Suppose $\{x_{11^*}^+, x_{21^*}^+, \dots, x_{k1^*}^+, x_{(k+1)1^*}^-, x_{(k+2)1^*}^-, \dots, x_{n1^*}^-\}$ is any feasible solution of sub-model *ILP3* under contraction ratios $r_{i(v)1}$. Let it be abbreviated as X_{1^*} . Equivalently, $X_{1^*} \in FR3_1$. As for constraint (5-2), we have $\sum_{j \in J_1} (SIGp(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{j1^*}^+)) + \sum_{j \in J_2} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{j1^*}^-)) \geq VR_{i(v)} - VR_{i(v)} \cdot r_{i(v)1}$. Since $0 \leq r_{i(v)1} \leq r_{i(v)2} \leq 1$ and $VR_{i(v)} \geq 0$, we have $VR_{i(v)} - VR_{i(v)} \cdot r_{i(v)1} \geq VR_{i(v)} - VR_{i(v)} \cdot r_{i(v)2}$. Accordingly, $\sum_{j \in J_1} (SIGp(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^+} - x_{j1^*}^+)) + \sum_{j \in J_2} (SIGn(a_{i(v)j^\pm}) \cdot |a_{i(v)j^\pm}| \cdot (x_{jopt^-} - x_{j1^*}^-)) \geq VR_{i(v)} - VR_{i(v)} \cdot r_{i(v)2}$. We can have $X_{1^*} \in FR3_2$ since the only difference of constraints between feasible regions $FR3_1$ and $FR3_2$ is constraint (5-2). Thus, $FR3_1 \subseteq FR3_2$. Accordingly, we can have $F_{opt1^*}^+ \leq F_{opt2^*}^+$ because of the enlarged feasible region under contraction ratio $r_{i(v)2}$. Similarly, we can have the feasible region of sub-model *ILP4* is shrunk as the contraction ratio increases from $r_{i(v)1}$ to $r_{i(v)2}$. Thus, the optimality of objective function (6-1) or (7-1) is reduced. Accordingly, we have $F_{opt1^*}^- \leq F_{opt2^*}^-$.

5.2. Tradeoff Analysis

In the proposed *IRLP* approach, potential violation of constraints of sub-model *ILP1* is mitigated through contracting optimistic decisions from sub-model *ILP1* or conservative decision from sub-model *ILP2*. Contraction of either group of decisions is helpful for coping with constraint-violation problems. There is a tradeoff between contraction ranges of two groups of decisions. The tradeoff is reflected as values of contraction ratios. Under higher contraction ratios, mitigation of constraint violation is contributed more by contraction of conservative decisions. Namely, more conservative decisions in the initial decision space are eradicated, and optimistic decisions are remained. This strategy is preferable for optimistic decision makers other than conservative ones. In contrast, as contraction ratios decrease from one to zero, contraction ranges of optimistic decisions climb while those of conservative decisions drop. Mitigation of constraint violation is achieved more by contraction of optimistic decisions. Retention of more conservative decisions can help conservative decision makers identify a more desired management scheme although this is beyond expectations of optimistic decision makers. The *IRLP* approach can be simp-

lified under particular scenarios due to diversity of preferences of decision makers.

For the most optimistic decision makers, contraction ratio $r_{i(v)}$ approaches 1 for all violated constraints. In this case, there are only three sub-models, i.e., *ILP1*, *ILP2* and *ILP4*. In this case, sub-model *ILP4* can be simplified as:

$$\min F^{*-} = \sum_{j \in J_1} (c_j^- \cdot x_j^{*-}) + \sum_{j \in J_2} (c_j^- \cdot x_j^{*+}) \quad (10-1)$$

subject to

$$\begin{aligned} & \sum_{j \in J_1} (SIGp(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_{jopt}^+) \\ & + \sum_{j \in J_1} (SIGn(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot (x_{jopt}^- - x_{jopt}^- + x_j^{*-})) \\ & + \sum_{j \in J_2} (SIGp(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot (x_{jopt}^+ - x_{jopt}^+ + x_j^{*+})) \\ & + \sum_{j \in J_2} (SIGn(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_{jopt}^+) \\ & \leq b_{i(v)}^+ \text{ for any } v \in V \end{aligned} \quad (10-2)$$

$$x_{jopt}^- \leq x_j^{*-} \leq x_{jopt}^+ \text{ for any } j \in J_1 \quad (10-3)$$

$$x_{jopt}^+ \geq x_j^{*+} \geq x_{jopt}^- \text{ for any } j \in J_2 \quad (10-4)$$

where x_j^{*-} ($j \in J_1$) and x_j^{*+} ($j \in J_2$) are decision variables and x_{jopt}^{\pm} ($j \in J$) are initialized interval-set solutions obtained from sub-models *ILP1* and *ILP2*. However, the initial decision space composed of x_{jopt}^{\pm} ($j \in J$) may be wasted in some cases. Assume the first k interval-set coefficients in the $i(v)$ th ($v \in V$) constraint, i.e., $a_{i(v)1}^{\pm}$, $a_{i(v)2}^{\pm}$, ... and $a_{i(v)k}^{\pm}$, are negative; the last $n - k$ interval-set coefficients, i.e., $a_{i(v)(k+1)}^{\pm}$, $a_{i(v)(k+2)}^{\pm}$, ... and $a_{i(v)n}^{\pm}$, are nonnegative; and $(A_{i(v)}^{\pm} X^{\pm})_C \leq b_{i(v)}^+$. We have $SIGp(a_{i(v)j}^{\pm}) = 0$ and $SIGN(a_{i(v)j}^{\pm}) = -1$ hold for any $j \in J_1$, and $SIGp(a_{i(v)j}^{\pm}) = 1$ and $SIGN(a_{i(v)j}^{\pm}) = 0$ hold for any $j \in J_2$. The final interval-set solution ($IFS_{1i(v)}^{\pm}$) after contraction through the first method is degenerated into a vector of real numbers, i.e., $(x_{1opt}^+, x_{2opt}^+, \dots, x_{kopt}^+, x_{(k+1)opt}^-, x_{(k+2)opt}^-, \dots, x_{nopt}^-)^T$. The degenerated solution corresponds to the upper bound of optimal objective-function values, i.e., $F^+ = \sum_{j \in J_1} (c_j^+ \cdot x_j^+) + \sum_{j \in J_2} (c_j^+ \cdot x_j^-)$. It can only be provided for optimistic decision makers other than conservative ones, since undertaking high constraint-violation risks is necessary.

On the other hand, the most conservative decision makers may expect that contraction ratio $r_{i(v)}$ equals to 0 for any violated constraint $v \in V$. In this case, sub-model *ILP4* would be eliminated. There would be just sub-models *ILP1*, *ILP2* and *ILP3*. Sub-model *ILP3* is equivalent to the following linear programming model:

$$\max F^{*+} = \sum_{j \in J_1} (c_j^+ \cdot x_j^{*+}) + \sum_{j \in J_2} (c_j^+ \cdot x_j^{*-}) \quad (11-1)$$

subject to

$$\begin{aligned} & \sum_{j \in J_1} (SIGp(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_j^{*+}) \\ & + \sum_{j \in J_1} (SIGn(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_{jopt}^-) \\ & + \sum_{j \in J_2} (SIGp(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_{jopt}^-) \\ & + \sum_{j \in J_2} (SIGn(a_{i(v)j}^{\pm}) \cdot |a_{i(v)j}^{\pm}| \cdot x_j^{*-}) \\ & \leq b_{i(v)}^+ \text{ for any } v \in V \end{aligned} \quad (11-2)$$

$$x_j^{*+} \geq x_{jopt}^- \text{ and } x_j^{*-} \leq x_{jopt}^+ \text{ for any } j \in J_1 \quad (11-3)$$

$$x_j^{*-} \geq x_{jopt}^- \text{ and } x_j^{*+} \leq x_{jopt}^+ \text{ for any } j \in J_2 \quad (11-4)$$

where x_j^{*+} ($j \in J_1$) and x_j^{*-} ($j \in J_2$) are decision variables and x_{jopt}^{\pm} ($j \in J$) are initialized interval-set solutions obtained from sub-models *ILP1* and *ILP2*. The potential challenge is that most of the initial decision space may be cut. In particular cases, it would be degenerated into a point in an n -dimensional Euclidean space, i.e., $(x_{1opt}^-, x_{2opt}^-, \dots, x_{kopt}^-, x_{(k+1)opt}^+, x_{(k+2)opt}^+, \dots, x_{nopt}^+)^T$. The degenerated solution which corresponds to the lower bound of optimal objective-function values $F^- = \sum_{j \in J_1} (c_j^- \cdot x_j^-) + \sum_{j \in J_2} (c_j^- \cdot x_j^+)$, is preferred for conservative decision makers but not for optimistic ones. The expectation of high system profits can hardly be satisfied. The interval-set solution (i.e., $IFS_{2i(v)}^{\pm}$) is conditionally feasible. It is feasible for *ILP* model (1) and can mitigate the violation problem of constraints (2-2) if and only if $MSR_{2i(v)} \geq VR_{i(v)}$.

5.3. Dimensional Analysis of Decision Space

It is also of possibility that decision makers prefer to a large decision space that includes as many alternatives as possible for management under interval uncertainties. This relies on proposition of an index for quantifying the size of the decision space. Multiple conventional geometric indexes are available, e.g., $\prod X = \prod_{j=1}^n (x_{jopt}^{*+} - x_{jopt}^{*-})$ or $\sum X = \sum_{j=1}^n (x_{jopt}^{*+} - x_{jopt}^{*-})$, but they are ineffective for the proposed *IRLP* approach. The j th interval-set solution, i.e., x_{jopt}^{\pm} ($j \in J$), may be contracted to a real number instead of an interval set, i.e., $x_{jopt}^{*+} = x_{jopt}^{*-}$. Accordingly, index $\prod X$ equals to zero no matter how much value ranges of other decision variables are after contraction. As for index $\sum X$, it can only quantify the absolute size of the final decision space. The connection between initial interval-set solutions, x_{jopt}^{\pm} ($j \in J$), and final ones, $x_{jopt}^{*\pm}$ ($j \in J$) may be removed if it is employed. In consideration of the process that final solutions $x_{jopt}^{*\pm}$ ($j \in J$) are obtained through contraction of the initial interval-set solutions (x_{jopt}^{\pm} , $j \in J$), the index of the *average relative contraction ratio of the initial decision space* is proposed for measuring the size of finalized decision space in this study. It is abbreviated as *DSR*. A higher value of *DSR* represents a broader decision space, and vice versa.

Definition 3: $DSR = \sum_{s=1}^S ((x_{j(s)opt}^{*+} - x_{j(s)opt}^{*-}) / (x_{j(s)opt}^+ - x_{j(s)opt}^-)) / S$ where $j(s)$ represents type-I decision variables that are interval sets rather than real numbers.

It has been proved above that, when $SR_{i(v)1} = VR_{i(v)}$ for any $v \in V$, we can have $F_{opt}^{*+} = F_{opt}^+$. In this case, the contraction ratio equals to one for any v . Violation of constraints of sub-model *ILP1* is caused by interval-set solutions of type-I decision variables. Mitigation of this problem is achieved through contracting boundaries of these decision variables while remaining those of type-II decision variables. As a result, $DSR = 0$ when $r_{i(v)} = 1$ for any $v \in V$, which is verified through an example in the next section. This implies that higher contraction ratios are preferable for decision makers who expect few alternative management schemes. In contrast, others pursuing a large decision space desire as low contraction ratios as possible. Setting $r_{i(v)}$ as zero may be the most advisable option for them. In addition, due to complexities of real-world *ILP* problems, there may be many other situations such as varied preferences of contraction ranges for violated constraints. It is impossible to cover

all of them excluding representative ones as discussed above. Adaptation of the proposed *IRLP* approach to *ILP* problems under diverse complexities will be examined in the future studies.

6. Example of REM through IRLP

An interval-parameter linear *REM* problem will be provided to demonstrate advantages of the proposed *IRLP* method as well as the detailed procedures for the relevant algorithm. Such a problem can be formulated as an *ILP* model with an objective of economic revenue, and constraints of environmental loading capacity and resources recovery:

$$\max F^\pm = [50, 60] \cdot x_1^\pm - [70, 90] \cdot x_2^\pm \tag{12-1}$$

subject to

$$[4, 6] \cdot x_1^\pm + [1, 2] \cdot x_2^\pm \leq [140, 150] \tag{12-2}$$

$$[7, 10] \cdot x_2^\pm - [1, 2] \cdot x_1^\pm \geq [1, 2] \tag{12-3}$$

$$x_j^\pm \geq 0 \text{ for } j = 1 \text{ and } 2 \tag{12-4}$$

where x_1^\pm and x_2^\pm are two interval-set decision variables of which the lower and upper bounds are to be determined; F^\pm is a linear function of interval-set coefficients and decision variables. Assumptions 1 to 6 of the *IRLP* approach are satisfied, e.g., zero is not included in all interval-set coefficients and all decision variables are non-negative.

6.1. Initialization of Decision Space

Firstly, *ILP* model (12) is converted to two correlated linear programming sub-models. Since the objective (12-1) is to maximize value ranges of objective function F^\pm , the objective function of the first sub-model should be maximizing the upper bound of objective function F^\pm . Since $[50, 60] \geq 0$, $[-90, -70] < 0$ and $x_j^\pm \geq 0$ for any $j = 1$ and 2 , we can have the upper bound of F^\pm equals to $60 \cdot x_1^+ - 70 \cdot x_2^-$. Maximization of $60 \cdot x_1^+ - 70 \cdot x_2^-$ is equivalent with maximizing x_1^+ as well as minimizing x_2^- . Thus, decision variables in the first sub-model are the upper bound of x_1^\pm and the lower bound of x_2^\pm . According to formulations of sub-model *ILP1*, the first sub-model corresponding to model (12) can be formulated as:

$$\max F^+ = 60 \cdot x_1^+ - 70 \cdot x_2^- \tag{13-1}$$

subject to

$$4 \cdot x_1^+ + 2 \cdot x_2^- \leq 150 \tag{13-2}$$

$$10 \cdot x_2^- - x_1^+ \geq 1 \tag{13-3}$$

$$x_1^+ \geq 0 \text{ and } x_2^- \geq 0. \tag{13-4}$$

As presented in Figure 2, the feasible region constrained by inequalities (13-2) to (13-4) is the parallel-lines-filled area that is under the line l_1 (i.e., $4 \cdot x_1^+ + 2 \cdot x_2^- = 150$), above the lines l_2 (i.e., $10 \cdot x_2^- - x_1^+ = 1$) and $x_2^- = 0$, at the right of axis $x_1^+ = 0$ and above the axis $x_2^- = 0$. Any combination of x_1^+ and x_2^- in

this area is feasible for all constraints in model (13). To avoid abnormally high penalties caused by constraint violations, the optimum solution should be located within this area. As shown in Figure 2, the objective function (i.e., line l_3) is maximized at point p_1 . Namely, the optimum solution of model (13) is point p_1 where $x_{1\text{opt}}^+ = 35.67$ and $x_{2\text{opt}}^- = 3.67$. The upper bound of objective function values (denoted as F_{opt}^+) equals to $60 \cdot x_{1\text{opt}}^+ - 70 \cdot x_{2\text{opt}}^-$ or 1883.33, which means the optimum value of objective function (13-1) can reach 1883.33 in the best case.

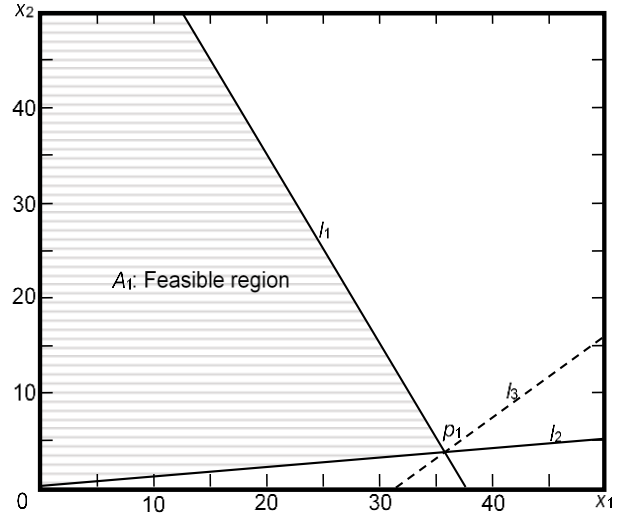


Figure 2. Feasible region and optimum solution of the sub-model (13).

The second sub-model is formulated based on solutions of model (13). In contrast to the first sub-model, the objective function of the second one is to maximize the lower bound of objective function (12-1), i.e., $\text{Max } F^- = 50 \cdot x_1^- - 90 \cdot x_2^+$. To achieve this, the lower bound of x_1^\pm should be minimized, while the upper bound of x_2^\pm should be maximized. Thus, decision variables of the second sub-model are the lower bound of x_1^\pm and the upper bound of x_2^\pm , which exactly counters to that in the first sub-model (13). In accordance with the formulation of model *ILP2*, the second sub-model is formulated as follows:

$$\max F^- = 50 \cdot x_1^- - 90 \cdot x_2^+ \tag{14-1}$$

subject to

$$6 \cdot x_1^- + x_2^+ \leq 140 \tag{14-2}$$

$$7 \cdot x_2^+ - 2 \cdot x_1^- \geq 2 \tag{14-3}$$

$$x_1^- \geq 0 \text{ and } x_2^+ \geq 0 \tag{14-4}$$

$$x_{1\text{opt}}^- \geq x_1^- \text{ and } x_2^+ \geq x_{2\text{opt}}^+ \tag{14-5}$$

The simplex method is employed again for solving deterministic linear programming model (14). The optimal solutions

are $x_{1opt}^- = 22.23$ and $x_{2opt}^+ = 6.64$. As shown in Figure 3, the feasible region constrained by inequalities (14-2) to (14-5) is the parallel-lines-filled area A_2 that is below the line l_4 (i.e., $6 \cdot x_1^- + x_2^+ = 140$), above the line l_5 (i.e., $7 \cdot x_2^+ - 2 \cdot x_1^- = 2$), at the right of axis $x_1^- = 0$, above the axis $x_2^+ = 0$, above the line l_7 (i.e., $x_2^+ = x_{2opt}^-$) and at the left of line l_8 (i.e., $x_{1opt}^+ = x_1^-$). The maximum value of objective function $F^- = 50 \cdot x_1^- - 90 \cdot x_2^+$ (i.e., line l_6) is obtained at the vertex of A_2 , i.e., point p_2 (22.23, 6.64). The optimum lower bound of objective function $F_{opt}^- = 514.09$ that can be reached at point p_2 .

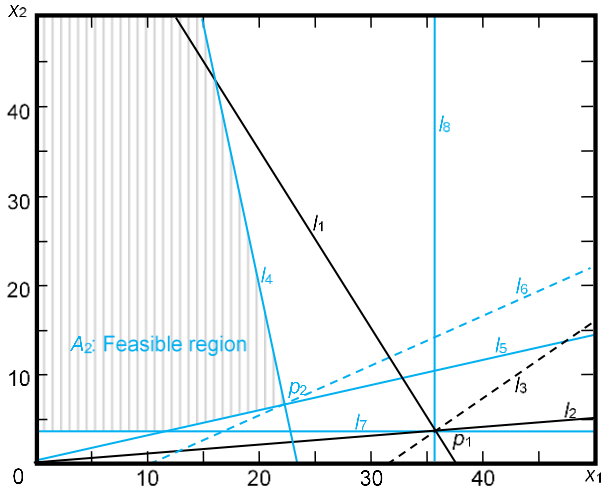


Figure 3. Feasible region and optimum solution of the sub-model (14).

As a combination of solutions from models (13) and (14), an initial interval-set solution of the original *ILP* model (12) is $x_{1opt}^\pm = [22.23, 35.67]$ and $x_{2opt}^\pm = [3.67, 6.64]$, leading to a rectangular decision space presented as the green grid area (D) in Figure 4. Correspondingly, the maximum value of objective function (12-1) ranges from the maximum objective value of (14-1) (i.e., 514.09) to that of (13-1) (i.e., 1883.33). This implies that any real value between the optimum objective bounds, i.e., [514.09, 1883.33] for *ILP* model (12), can be obtained for at least one point in initial decision space D . Besides, the recommended solution of any inexact decision variable is a continuous value range in which most of satisfactory decisions are located. Compared to other potential solution formats, e.g., complex formulas and discrete solutions, the interval decision space is more user-friendly and reliable for planners or engineers to guide programming problems especially large-scale ones of thousands of constraints and parameters.

6.2. Violation Analysis

Combining feasible regions of models (13) and (14) together, we can have three areas of different feasibilities.

(a) The first one (abbreviated as R_1) is the intersection of two feasible regions A_1 and A_2 , i.e., $R_1 = A_1 \cap A_2$. For any element $X(x_1, x_2)$ in area R_1 , we can have $X \in A_1$ and $X \in A_2$, and it is feasible for constraints of both models. Thus, R_1 is defined as the *absolutely feasible* region (Huang et al., 1992) of the

original *ILP* model (12). In Figure 4, it is presented as the field filled with grids.

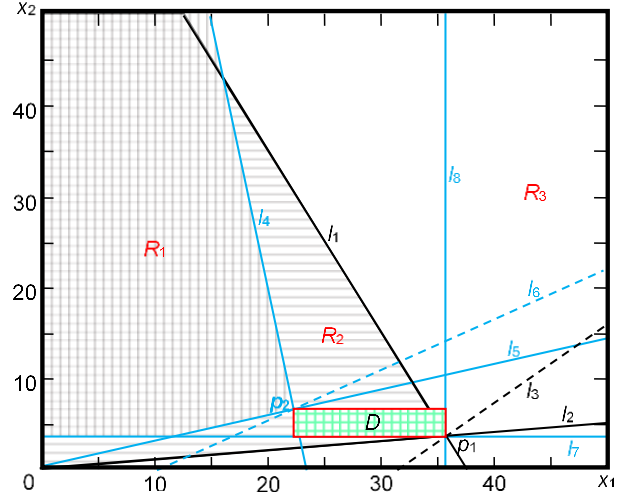


Figure 4. Interval-set solutions of *ILP* model (12).

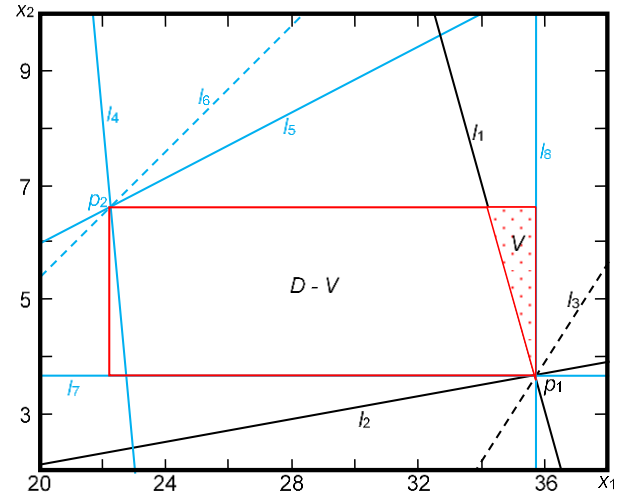


Figure 5. Constraint violation for *ILP* model (12).

(b) The second one (abbreviated as R_2) is the difference between region R_1 and the union of two feasible regions A_1 and A_2 , i.e., $R_2 = (A_1 \cup A_2) - R_1$. It holds for any element $X(x_1, x_2)$ in the area R_2 that $X \notin R_1$ and $X \in A_1$ or A_2 . In other words, any element in region R_2 is feasible for constraints of either model (13) or (14), but not for both models. As defined in (Huang et al., 1992), any solution within R_2 is *softly feasible* for *ILP* model (12). In Figure 4, it is presented as the field filled with parallel lines.

(c) The third one (abbreviated as R_3) is the difference between the two-dimensional real space and the union of two feasible regions A_1 and A_2 , i.e., $R_3 = R - (A_1 \cup A_2)$ where $R = \{X(x_1, x_2) | x_j \text{ is a real number for } j = 1 \text{ and } 2\}$. For any element $(X(x_1, x_2))$ in region R_3 , it is infeasible for either model (13) or (14). Therefore, region R_3 is defined as the *absolutely infeasible* region (Huang et al., 1992). It is presented as the blank field without any pattern in Figure 4.

For *ILP* model (12), any point in the final rectangular decision space (*D*) obtained from *TSA* is supposed to be feasible for at least one combination of interval coefficients' whitened values. As presented in Figure 5, however, the top right corner (i.e., red region *V*) as a part of initial decision space *D* is located outside of softly feasible region *R*₂. Any point in region *V* belongs to the absolutely infeasible region (*R*₃), and violates line *l*₁ which corresponds to constraint (13-2) of the first sub-model, i.e., $4 \cdot x_1^+ + 2 \cdot x_2^- = 150$.

Given the initial decision space (*D*) obtained from *TSA*, the most desired *REM* scheme for decision makers may correspond to any point in space *D*. If it is located in area *V*, high penalties would be paid for violation of constraint (13-2). Therefore, it is assumed that such violation should be avoided for decision makers who prefer to maximize the net system profit, although its occurrence probability is not significantly high.

As fluctuation of decision variables within bounds x_{jopt}^- and x_{jopt}^+ ($j = 1, 2$), the maximum values of the left-hand side of constraints (13-2) and (13-3) in the first sub-model are $4 \cdot x_{1opt}^+ + 2 \cdot x_{2opt}^+ = 155.94$, and $x_{1opt}^+ - 10 \cdot x_{2opt}^- = -1$, respectively. Therefore, the first constraint (13-2) is violated. The set of indexes of violated constraints is {1} in which only one element is included. The violation range can be obtained from $155.94 - 150 = 5.94$.

6.3. Contraction of Initial Decision Space

Referring to the classification method of decision variables in violated constraints that was previously proposed in this study, the second and fourth groups of decision variables are empty, and x_1^\pm and x_2^\pm belong to the first and third groups, respectively. Let the contraction ratio r of x_2^+ be 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1, respectively. The third sub-model in the *IRLP* approach can be formulated as:

$$\max F^{*+} = 60 \cdot x_1^{*+} - 70 \cdot x_2^{*-} \tag{15-1}$$

subject to

$$4 \cdot (x_1^{*+} - x_{1opt}^+ \cdot r) + 2 \cdot x_{2opt}^+ \cdot (1 - r) \leq 150 \cdot (1 - r) \tag{15-2}$$

$$x_1^{*+} \in [22.23, 35.67] \tag{15-3}$$

$$x_2^{*-} \in [3.67, 6.64] \tag{15-4}$$

where the upper bound (x_1^{*+}) of x_{1opt}^\pm and the lower bound (x_2^{*-}) of x_{2opt}^\pm are bounds of decision variables that should be adjusted. The optimal solutions of linear programming model (15) corresponding to varied constraint contraction ratio r can be obtained from the simplex method. They are presented in table 1.

On the basis of solutions from the third sub-model, the fourth sub-model can be formulated as:

$$\min F^{*-} = 50 \cdot x_1^{*-} - 90 \cdot x_2^{*+} \tag{16-1}$$

subject to

$$4 \cdot x_{1opt}^+ \cdot r + 2 \cdot (x_2^{*+} - x_{2opt}^+ \cdot (1 - r)) \leq 150 \cdot r \tag{16-2}$$

$$x_{1opt}^- \leq x_1^{*-} \leq x_{1opt}^{*+} \tag{16-3}$$

$$x_{2opt}^+ \geq x_2^{*+} \geq x_{2opt}^{*-} \tag{16-4}$$

where the lower bound x_1^{*-} of x_{1opt}^\pm and the upper bound x_2^{*+} of x_{2opt}^\pm are bounds of decision variables that should be adjusted. The simplex method is employed again for solving linear programming model (16). The optimal solutions corresponding to varied contraction ratios are presented in Table 1. Combining solutions from the third and fourth sub-models together, we can have the final interval-set solutions under varied contraction ratios as presented in Figure 6. It is indicated that the proposed *IRLP* approach enables achieving the maximal decision space in which infeasible solutions of sub-model *ILP1* are excluded through sub-models *ILP3* and *ILP4*.

Table 1. Solutions of sub-models (15) and (16)

<i>r</i>	Solution of model (15)			Solution of model (16)		
	x_{1opt}^{*+}	x_{2opt}^{*-}	F_{opt}^{*+}	x_{1opt}^{*-}	x_{2opt}^{*+}	F_{opt}^{*-}
0	34.18	3.67	1794.24	22.23	6.64	514.09
0.1	34.33	3.67	1803.15	22.23	6.34	540.82
0.2	34.48	3.67	1812.06	22.23	6.04	567.55
0.3	34.63	3.67	1820.97	22.23	5.75	594.27
0.4	34.78	3.67	1829.88	22.23	5.45	621.00
0.5	34.92	3.67	1838.79	22.23	5.15	647.73
0.6	35.07	3.67	1847.70	22.23	4.86	674.46
0.7	35.22	3.67	1856.61	22.23	4.56	701.18
0.8	35.37	3.67	1865.52	22.23	4.26	727.91
0.9	35.52	3.67	1874.42	22.23	3.96	754.64
1	35.67	3.67	1883.33	22.23	3.67	781.36

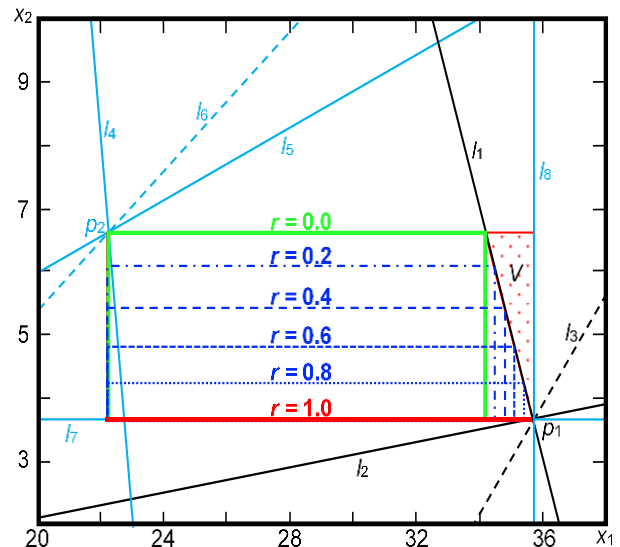


Figure 6. Solutions of *ILP* model (12) through the *IRLP* approach.

6.4. Post-optimality Analysis

Optimistic decision makers may prefer to high system profits that can be accomplished through setting contraction ratio as 1. Accordingly, the optimal decision ranges $x_{1opt}^\pm = [22.23,$

35.67] and $x_{2opt}^{\pm} = 3.67$. The maximum value range of objective function is $F_{opt}^{\pm} = [781.36, 1883.33]$. For conservative decision makers who expect low system profit and low constraint violation risk under uncertainties, the preferable solutions are corresponded by lowest contraction ratios. In detail, the optimal decision ranges $x_{1opt}^{\pm} = [22.23, 34.18]$ and $x_{2opt}^{\pm} = [3.67, 6.64]$. The maximum objective value range is $F_{opt}^{\pm} = [514.09, 1794.24]$. These solutions are also of high priority for decision makers who desire a large decision space, if the size of decision space is quantified by the proposed index *DSR*. As for other decision makers, they can be provided with interval-set solutions based on analysis of the correspondence between their preferences and constraint contraction ratios.

7. Discussions

7.1. Connection with TSA Method

The proposed *IRLP* approach can be taken as a supplement to the *TSA* method developed by Huang et al. (1992). The latter one is incorporated into the former one for initializing a decision space through sub-models *ILP1* and *ILP2*. Effectiveness of *TSA* at obtaining a continuous decision space is remained in *IRLP*. To mitigate constraint-violation problems of *TSA*, the cause of violation is revealed in the section of violation analysis based on the definition of violation criterion (i.e., constraints of sub-model *ILP1* of which violation may lead to high penalties to real-world *ILP* systems). Violation ranges of constraints are quantified, leading to proposition of contraction ranges and contraction ratios. A series of contraction ratios are provided by decision makers of diverse preferences for violated constraints of sub-model *ILP1*. As a recourse measure for constraint violation, the initial decision space is contracted through sub-models *ILP3* and *ILP4* that correspond to optimistic solutions and conservative ones, respectively. A continuous decision space is finalized by combining crisp solutions of two linear programming sub-models. It is located within the initial decision space while solutions causing violation of constraints of sub-model *ILP1* are removed. Contraction ratios can help distinguish preferences of decision makers, e.g., low values for ones who pursue conservative schemes or a large decision space. Combining solutions under all combinations of contraction ratios for violated constraints can facilitate decision makers identifying the maximal decision space which is the initial one excluding infeasible schemes. Therefore, the proposed *IRLP* approach enables mitigation of constraint-violation problems while remaining strengthens of *TSA* and adapting to decision makers of diverse preferences. The reliability of the optimization process for *ILP* problems is enhanced in comparison with the *TSA* method.

7.2. Comparison with Improved ILP Methods

The proposed *IRLP* approach is not the first one to examine constraint-violation problems of the *TSA* method. Previously, a modified interval linear programming (*MILP*) method, a three-step (*ThS*) method and a robust two-step (*RTS*) method were contributed by Zhou et al. (2008), Huang and Cao (2011) and Fan and Huang (2012), respectively. In *MILP*, an additional

constraint is coupled with constraints of sub-model *ILP2* for avoiding violation of constraints. As for *ILP* model (12), the additional constraint of sub-model (14) can be formulated as $x_2^+ - 2 \cdot 3.67 \leq 0$. The optimal solution of *TSA* is $x_{1opt}^{\pm} = [22.23, 35.67]$ and $x_{2opt}^{\pm} = [3.67, 6.64]$. It is identical with that of *MILP*, implying that constraint violation remains for *MILP*. In the *ThS* method, another sub-model is constructed to constrict the initial decision space obtained from first two sub-models *ILP1* and *ILP2*. Parameters *constricting ratios of decision variables* are introduced to reflect constricting ranges of initialized interval-set solutions. They are optimized through the third sub-model subject to avoidance of constraint violation. However, the sub-model may be a nonlinear programming model which is of limited applicability to large-scale *ILP* problems. Otherwise, it may lead to a great waste of the initial decision space in the process of constricting initialized solutions synchronously. In the *RTS* method, sub-model *ILP2* that focuses on conservative schemes is solved prior to sub-model *ILP1*. A group of additional constraints is incorporated into the original sub-model *ILP1* for avoiding occurrence of constraint violation. However, *RTS* is challenged by possible sacrifice of optimistic solutions that correspond to high system profits. *RTS* is only adaptive for conservative decision makers who concern schemes of relatively low risks. In contrast, the *IRLP* approach can help decision makers of various preferences identify desired schemes while demands of conservative ones can be satisfied. The applicable scope of the *RTS* method is just a particular case of the *IRLP* approach. Moreover, all existing improved *ILP* methods cannot provide the largest decision space which does not include solutions that may cause constraint violation. Part of decision alternatives are neglected due to ineffectiveness of these methods. This limitation is overcome by the *IRLP* approach, which is helpful for enhancing reliability of the decision support for *REM* problems under interval uncertainties.

7.3. Comparison with Discrete ILP Methods

In addition to aforementioned methods (e.g., *MILP*, *ThS* and *RTS*), multiple efforts are made to solving of *ILP* models. For instance, Tong (1994), Levin (1994) and Chi-nneck and Ramadan (2000) proposed methods that could obtain the most optimistic decision scheme (denoted as X_u) of *ILP* model (1) from a best-case sub-model as well as the most conservative decision scheme (denoted as X_l) from a worst-case sub-model. Let these methods be named as discrete interval linear programming and be abbreviated as *DILP* in this study. These methods only focus on two extreme cases corresponding to particular combinations of inexact coefficients in *ILP* model (1). If at least one optimal solution exists for the best-case sub-model, it can be deduced from (Tong, 1994) and (Huang et al., 1992) that there would be infinite possibly optimal solutions corresponding to infinite combinations of coefficients. Either X_u or X_l is only a particular case out of infinite ones. Almost all of decision alternatives including the possibly most desired decision scheme that may be located between the two extreme cases are neglected due to discreteness of the decision space in *DILP* methods. Furthermore, constraints may be violated for *DILP* methods under the assumption of continuity of decision space. Let

the value of objective function (1-1) corresponding to X_u and X_l be denoted as F_{uopt} and F_{lopt} , respectively. The decision space (denoted as *DILPDS*) constrained by the two most extreme schemes (i.e., X_u or X_l) may involve solutions that are infeasible for all combinations of inexact coefficients. Let the set of these solutions be denoted as *DILPInF*. Any element in *DILPInF*, especially ones around X_u or X_l , may lead to extremely high penalties for infeasibility, and needs to be removed from *DILPDS*. In comparison, these challenges are effectively mitigated in the proposed *IRLP* approach.

7.4. Comparison with Robust Linear Programming

The *IRLP* approach is developed for solving *ILP* models and guiding *ILP* problems. Management schemes can also be provided by robust linear programming (*RLP*) methods. The *ILP* models are equivalent with *RLP* problems in particular cases. An *RLP* model corresponding to *ILP* model (1) can be formulated as $\text{Max } \{CX \mid AX \leq b; X \geq 0\}$ where X is a $n \times 1$ vector of non-negative decision variables x_j ($j = 1, 2, \dots, n$), $0 = \{0\}^{n \times 1}$; $C = \{c_j\}^{1 \times n}$, $A = \{a_{ij}\}^{m \times n}$, and $b = \{b_i\}^{m \times 1}$; positive integers n and m are numbers of decision variables $\{x_j\}$ and constraints, respectively; any vector or matrix of coefficients (i.e., C , A , or b) belongs to an uncertainty set U where $U = \{C = \{c_j\}^{1 \times n} \mid c_j^- \leq c_j \leq c_j^+, j = 1, 2, \dots, n\} \cup \{A = \{a_{ij}\}^{m \times n} \mid a_{ij}^- \leq a_{ij} \leq a_{ij}^+, i = 1, 2, \dots, m; j = 1, 2, \dots, n\} \cup \{b = \{b_i\}^{m \times 1} \mid b_i^- \leq b_i \leq b_i^+, i = 1, 2, \dots, m\}$. The appeal of robust linear programming is that its solutions and performances remain relatively unchanged when exposed to uncertain conditions (Beyer and Sendhoff, 2007). A robust solution of *ILP* model (1) can be obtained from a robust counterpart of the *RLP* model (abbreviated as *RCLP*). The model can be expressed as $\text{Max } \{f \mid F \leq CX; AX \leq b; X \geq 0; \forall A \in U; \forall C \in U; \forall b \in U\}$. As stated in (Beyer and Sendhoff, 2007), however, the robust solution obtained from the *RCLP* model is the most conservative decision scheme. Higher system profits are sacrificed for pursuing higher robustness of solutions, which cannot be acceptable for all decision makers of diverse preferences in real-world programming problems, at least for optimistic ones who are willing to chase high profits while taking risks of solution infeasibility. In contrast, the *IRLP* approach for *ILP* models is of higher adaptability and enables provision of a continuous decision space for decision making under interval uncertainties.

7.5. Potential Extensions

Although the proposed *IRLP* approach effectively mitigates constraint violation problems in solving *ILP* models, it is of multiple limitations due to complexities of real-world resources and environmental management systems. For instance, contraction ratios are introduced to reflect preferences of decision makers, and they should be determined prior to contraction of the initial decision space. There is still a lack of scientific tool to help decision makers identify the desired values of these ratios. Elaboration of penalties of constraint violation may be helpful for achieving it. In this study, the criterion of constraint violation is assumed to be constraints of sub-model *ILP1* due to relatively high penalties. Effectiveness of the *IRLP* approach

relies on the assumption. As a result, it is still possible that the obtained solutions lead to violation of constraints under other criteria such as constraints of sub-model *ILP2*. In these cases, the *IRLP* approach should be corrected to reflect realities of real-world *ILP* problems. The *IRLP* approach is dependent with assumptions 1 to 6 that hold for most resources and environmental management problems. Nonetheless, it is possible that these assumptions do not hold for some cases. Corresponding revisions are desired for avoiding ineffectiveness of the *IRLP* approach. The suggested solution of the *IRLP* approach is a series of interval sets. It provides a decision space of which any solution may be the desired scheme for decision makers. However, the feasibility and optimality of any solution is connected with whitened values of interval-set coefficients in *ILP* models. To facilitate decision makers identifying the most desired scheme, analysis of the correspondence between them may be demanded from the viewpoint of applicability. In addition, there may be a number of other complexities in *ILP* systems, e.g., multiple objectives, nonlinear relationships, discrete alternatives, parameter interactions, subjective information, random disturbances, and joint uncertainties. Coupling *IRLP* with other optimization methods deserves an attempt to reflect these complexities and provide reliable decision support for engineering practices. Therefore, a lot of efforts will be made in the following studies to overcome these challenges, enlarge adaptability of the *IRLP* approach, enhance reliability of decision support processes, and promote effective management of resources and environmental systems.

8. Conclusions

An *IRLP* approach was proposed in this study. Based on a review of *ILP* models, the *TSA* approach and their significances to *REM* problems, sub-models *ILP1* and *ILP2* in *TSA* were employed to initialize a decision space for *IRLP*. Causes of constraint violation were examined based on identification of a violation criterion. Contraction ratios were defined after revelation of violation ranges of constraints. As a recourse measure to constraint violation problems, two linear programming sub-models (*ILP3* and *ILP4*) were constructed given a series of contraction ratios. A hypercube decision space where infeasible solutions were excluded through sub-models *ILP3* and *ILP4* was obtained for decision makers. Post-optimality analysis was conducted to deal with barriers in applying the *IRLP* approach to real-world *ILP* problems. An *ILP* problem was introduced to demonstrate procedures and effectiveness of the *IRLP* approach. Strengthens and shortcomings of the *IRLP* approach were revealed from comparisons with existing *ILP* methods.

Results indicated that the *IRLP* approach was effective at resolving the constraint-violation problem of the *TSA* approach, reproducing the largest decision space which did not include infeasible solutions and enhancing reliability of decision support for *REM*. A hypercube decision space that covered most alternative schemes could be provided for facilitating *REM* under interval uncertainties. Demands of decision makers of diverse preferences could be satisfied by adjusting values of contraction ratios, e.g., low values for ones who pursued conserva-

tive schemes or a large decision space. The reliability of the optimization process for *ILP* problems was enhanced in comparison with existing *ILP* methods.

Complexities of real-world *REM* problems may require further improvements of the *IRLP* approach in the future studies. For instance, elaboration of penalties of constraint violation is helpful for identifying the desired values of contraction ratios. The *IRLP* approach should be corrected to adapt to various constraint-violation criteria. Corresponding revisions are desired for avoiding ineffectiveness of the *IRLP* approach when related assumptions do not hold. To facilitate decision makers identifying the most desired scheme, analysis of the correspondence between solutions and coefficient combinations may be demanded from the viewpoint of applicability. Coupling *IRLP* with other optimization methods deserves an attempt to reflect diverse complexities (e.g., multiple objectives, nonlinear relationships, discrete alternatives, parameter interactions, subjective information, random disturbances, and joint uncertainties) and provide reliable decision support for *REM*.

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