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Intrinsic Fluctuation and Its Critical Scaling in a Class of Populations of Oscillators with Distributed Frequencies

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A theory is developed of the fluctuation of an order parameter in a class of large populations of oscillators with distributed natural frequencies, which reveals in particular a unique scaling behavior of the fluctuation at the onset of mutual entrainment for which numerical evidence is given.

Large populations of coupled dissipative oscillators exhibit macroscopic mutual entrainment when coupling is strong enough to compensate for the disordering effect due to the distribution of natural frequencies of oscillators. This phenomenon has much relevance to a variety of fields in science (e.g., see Ref. 1)), and quite a few investigations have been devoted to elucidating it. Of these, studies with emphasis on the aspect of a phase transition may be particularly interesting and important since they open a new area in the field of critical phenomena to clarify how the onset of mutual entrainment in oscillator assemblies compares to conventional phase transitions in diverse equilibrium systems.[2)] In this paper we attempt a study along such a line using the following model:[3)]

$$d\theta_j/dt=\Omega_j+\varepsilon(2\pi N)^{-1}\sum_{i=1}^{N}\sin 2\pi(\theta_i-\theta_j)\,, \tag{1}$$

where θ_j is the phase of the jth oscillator ($j=1, 2, \cdots, N$. Hereafter we omit the range of j since it is always the same.) and Ω_j is the natural frequency which is assumed to obey a distribution $f(\Omega)$ over the population. Coupling strength is controled by the parameter ε. In the limit $N\to\infty$ the onset of mutual entrainment beyond a threshold ($=\varepsilon_c$) is well established for the model[2),3)] where the role of an order parameter is played by $\hat{Z}=\lim_{t\to\infty}Z(t)$ in which $Z(t)=N^{-1}\sum_{j=1}^{N}\exp\{2\pi i'(\theta_j-\tilde{\Omega}t)\}$, where $\tilde{\Omega}$ is the frequency of entrainment and $i'=\sqrt{-1}$. Namely $\hat{Z}=0$ for $\varepsilon<\varepsilon_c$ while $|\hat{Z}|$ grows like $(\varepsilon-\varepsilon_c)^\beta$ for $\varepsilon>\varepsilon_c$ where $\beta=1/2$ for typical $f(\Omega)$.[2),3)] On the other hand, for finite-size systems, it was found numerically that Z exhibits persistent fluctuation whose magnitude σ, defined by $\sigma=\lim_{N\to\infty}\sqrt{N}\langle|Z-\langle Z\rangle|^2\rangle^{1/2}$, behaves as

$$\sigma(\varepsilon)\propto|\varepsilon-\varepsilon_c|^{-\gamma'} \tag{2}$$

near the threshold ε_c.[4),5),*)] (In the above and hereafter the brackets ⟨ ⟩ stand for a long time average.) The purpose of this paper is to develop a theory of the fluctuation and especially derive the above scaling behavior of σ. A finite-size scaling analysis, part of which was first attempted in Ref. 5), will also be performed to support the theory.

*) While numerical simulations in Refs. 4) and 5) are for a discrete-time model, their results may be viewed as virtually for the model (1) owing to the small width of $f(\Omega)$ adopted.

Let us first introduce $w=Z-\hat{Z}$ which may be regarded as small after an initial transient dies out, provided N is large. In fact our previous simulations suggest $w=O(N^{-1/2})$,[5] which will be proved later. Correspondingly we may divide every oscillator phase into two parts as

$$\theta_j-\tilde{\Omega}t=\psi_j+\phi_j\,. \tag{3}$$

The first term, ψ_j, is the dominant part obeying

$$d\psi_j/dt=\Delta_j+(\varepsilon/2\pi)\mathrm{Im}(\hat{Z}e^{-2\pi i'\psi_j}) \tag{4}$$

with $\psi_j(0)=\theta_j(0)$, where $\Delta_j=\Omega_j-\tilde{\Omega}$, and Im denotes an imaginary part. The second term in (3), ϕ_j, is a deviation from the dominant phase motion induced by w. Putting Eqs. (3) and (4) into Eq. (1), and retaining only the terms of $O(w)$, we find

$$d\phi_j/dt=(\varepsilon/2\pi)\mathrm{Im}\{(-2\pi i'\hat{Z}\phi_j+w)e^{-2\pi i'\psi_j}\} \tag{5}$$

with $\phi_j(0)=0$. Then, by self-consistency, we can derive

$$w(t)=-\hat{Z}+N^{-1}\sum_{j=1}^{N}e^{2\pi i'\psi_j(t)}$$
$$+\lambda\int_0^t dt'\{w(t')A_-(t,t')-w^*(t')A_+(t,t')\}\,, \tag{6}$$

again by keeping $O(w)$ terms alone, where $\lambda=\varepsilon/2$, * is a complex conjugate, and

$$A_\pm(t,t')=\lim_{N\to\infty}N^{-1}\sum_{j=1}^{N}\exp\Big\{2\pi i'(\psi_j(t)\pm\psi_j(t'))$$
$$+(\lambda/\pi)\int_{t'}^{t}d\tau\mathrm{Im}(-2\pi i'\,\hat{Z}e^{-2\pi i'\psi_j(\tau)})\Big\}\,. \tag{7}$$

Contribution from higher-order terms may be neglected in the limit $N\to\infty$.

Let us first focus upon the subcritical regime where $\hat{Z}=0$, so that

$$A_-(t,t')=\int_{-\infty}^{\infty}d\Delta\tilde{f}(\Delta)e^{2\pi i'\Delta(t-t')}\equiv A(t-t')\,, \tag{8}$$

where $\tilde{f}(\Delta)\equiv f(\tilde{\Omega}+\Delta)$. Our interest is in the behavior of $w(t)$ for $t\to\infty$ on which the term in Eq. (6) including A_+ has no influence since $A_+(t,t')$ vanishes fast as $t,t'\to\infty$ (e.g., $A_+(t,t')\sim e^{-2\pi\gamma(t+t')}$ for $\tilde{f}(\Delta)=(\gamma/\pi)(\Delta^2+\gamma^2)^{-1}$). Noting this fact, we obtain

$$w(t)=N^{-1}\sum_{j=1}^{N}\{1-\lambda\tilde{A}(2\pi\Delta_j)\}^{-1}e^{2\pi i'(\Delta_j t+\theta_j(0))} \tag{9}$$

for $t\gg1$, where $\tilde{A}(\omega)=\int_0^\infty d\tau A(\tau)e^{-i'\omega\tau}$. This leads to

$$\langle|Z-\langle Z\rangle|^2\rangle=N^{-1}\int_{-\infty}^{\infty}d\Delta\tilde{f}(\Delta)|1-\lambda\tilde{A}(2\pi\Delta)|^{-2}\,, \tag{10}$$

indicating $w=O(N^{-1/2})$ in accordance with the observation in Ref. 5). We may thus find

$$\sigma(\varepsilon)=\Big(\int_{-\infty}^{\infty}d\Delta\tilde{f}(\Delta)|1-\lambda\tilde{A}(2\pi\Delta)|^{-2}\Big)^{1/2}. \tag{11}$$

For simplicity suppose hereafter that $f(\Omega)$ satisfies the following as typical distributions do: It is maximum at a certain value of $\Omega(=\hat{\Omega})$, being symmetric with respect to $\hat{\Omega}$. Then it is known that $\tilde{\Omega}=\hat{\Omega}$ and $\varepsilon_c=4/\tilde{f}(0)$.[3] As $\varepsilon\to\varepsilon_c$ from below, we find from Eq. (11)

$$\sigma(\varepsilon)\cong\tilde{f}(0)\sqrt{\varepsilon_c/|b|}(\varepsilon_c-\varepsilon)^{-1/2}, \tag{12}$$

where $b=(2/\pi^2)\int_0^\infty dy\tilde{f}'(y)/y$ with $\tilde{f}'\equiv d\tilde{f}/dy$. Therefore $\gamma'=1/2$ for $\varepsilon<\varepsilon_c$. (Remark: for $\tilde{f}(\Delta)=(\gamma/\pi)(\Delta^2+\gamma^2)^{-1}$, $\sigma(\varepsilon)=\sqrt{\varepsilon_c}(\varepsilon_c-\varepsilon)^{-1/2}$ exactly for $^\forall\varepsilon<\varepsilon_c=4\pi\gamma$ by Eq. (11).)

The correlation function of Z may also be obtained from Eq. (9) as

$$C_Z(\tau)=N^{-1}\int_{-\infty}^{\infty}d\Delta\tilde{f}(\Delta)|1-\lambda\tilde{A}(2\pi\Delta)|^{-2}e^{2\pi i'\Delta\tau}, \tag{13}$$

where $C_Z(\tau)\equiv\langle\{Z(t+\tau)-\langle Z\rangle\}\{Z(t)-\langle Z\rangle\}^*\rangle$, which is shown to decay exponentially near ε_c with the correlation time τ_c diverging at ε_c as

$$\tau_c\cong\tilde{f}(0)^{-1}(\varepsilon_c|b|/2)(\varepsilon_c-\varepsilon)^{-1}. \tag{14}$$

(Remark: for $\tilde{f}(\Omega)$ in the above remark Eq. (13) gives exactly $C_Z(\tau)=N^{-1}\varepsilon_c(\varepsilon_c-\varepsilon)^{-1}e^{-(\varepsilon_c-\varepsilon)\tau/2}$ with $\varepsilon_c=4\pi\gamma$.)

Let us now go on to the supercritical regime where it seems difficult to obtain exact expressions such as Eq. (11) in the subcritical regime mainly because of the nontrivial behavior of $\psi_j(t)$. Therefore Eq. (6) has been solved only approximately for this regime as will be outlined below (details will be published elsewhere). The problem is how to deal with the kernels $A_\pm$ in Eq. (6). It is easy to evaluate contribution to them from entrained oscillators (i. e., those satisfying $|\Delta_j|\leq\varepsilon|\hat{Z}|/(2\pi)$). As to nonentrained oscillators, noting that the bracket { } of A_- in Eq. (7) is rewritten as

$$2\pi i'(t-t')\Delta_j-2\lambda\hat{Z}^*\int_{t'}^{t}d\tau e^{2\pi i'\psi_j(\tau)},$$

we replace it by

$$2\pi i'(t-t')\Delta_j-2\lambda\hat{Z}^*\langle e^{2\pi i'\psi_j}\rangle(t-t')\equiv\Phi_j(t-t'). \tag{15}$$

This approximation should be reasonable for large t which is sufficient for our purpose. Moreover we replace exp{ } of A_+ in Eq. (7) by $\langle e^{4\pi i'\psi_j}\rangle e^{\Phi_j(t-t')}$. Then we find

$$\langle|Z-\langle Z\rangle|^2\rangle=N^{-1}\int_{|\Delta|>\lambda|\hat{Z}|/\pi}d\Delta\tilde{f}(\Delta)\sum_{m=-\infty}^{\infty}|d(\Delta,m)P(m\Delta')-d^*(\Delta,-m)R(m\Delta')|^2, \tag{16}$$

where $\Delta'=2\pi\{\Delta^2-(\lambda|\hat{Z}|/\pi)^2\}^{1/2}$, $d(\Delta_j,m)=T_j^{-1}\int_0^{T_j}dte^{2\pi i'\psi_j-i'm\Delta_j't}$ with $T_j=2\pi/\Delta_j'$, and $P(\omega)=\{1-\lambda C(\omega)\}/S(\omega)$, $R(\omega)=\lambda D(\omega)/S(\omega)$ with $S(\omega)=\{1-\lambda C(\omega)\}^2-\lambda^2D(\omega)D^*(-\omega)$. C and D are the Fourier transforms of A_- and A_+ (defined as $\tilde{A}(\omega)$), respectively. A detailed analysis of (16) shows

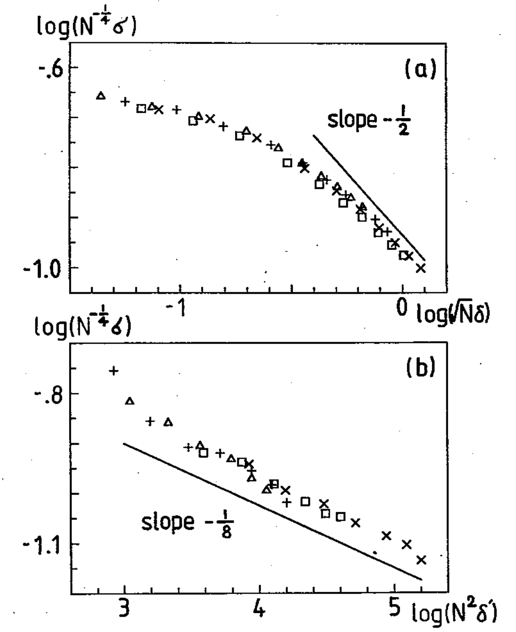

Fig. 1. Finite-size scaling plots of σ where $\delta=\varepsilon_c-\varepsilon$, $\delta'=\varepsilon-\varepsilon_c$ and $\log=\log_{10}$. (a) the subcritical regime: $N=6000(\varDelta)$, $10000(+)$, $14000(\square)$, $20000(\times)$. (b) the supercritical regime: $N=1600(\varDelta)$, $1900(+)$, $3000(\square)$, $6000(\times)$. The straight lines show the theoretical slopes. σ was computed through averaging over nearly $130000(\varepsilon<\varepsilon_c)$ and $30000\sim520000$ $(\varepsilon>\varepsilon_c)$ iterations of the Euler difference equation for Eq. (1).

$$\sigma(\varepsilon)\propto(\varepsilon-\varepsilon_c)^{-1/8} \tag{17}$$

near ε_c. Namely, $\gamma'=1/8$ for $\varepsilon>\varepsilon_c$.

Remarkably the critical exponent γ' is not the same for below and above ε_c. When compared to previous numerical results,[5] the theory is consistent with $\gamma'=0.123\pm0.003$ for $\varepsilon>\varepsilon_c$ while it disagrees with $\gamma'=0.126\pm0.009$ for $\varepsilon<\varepsilon_c$. A finite-size scaling analysis has been performed with N larger than in Ref. 5) to get conclusive evidence for σ which is expected to fit

$$\sigma=N^{1/4}\Psi(|\varepsilon-\varepsilon_c|N^s), \tag{18}$$

where $s=2$, $\Psi(x)\sim x^{-1/8}(x\gg1)$ for $\varepsilon>\varepsilon_c$ (as first proposed and verified in Ref. 5)), and $s=1/2$, $\Psi(x)\sim x^{-1/2}(x\gg1)$ for $\varepsilon<\varepsilon_c$. Evidence is presented in Fig. 1 which was obtained for $\tilde{f}(\varDelta)=(\gamma/\pi)(\varDelta^2+\gamma^2)^{-1}$ with $\gamma=10^{-3}$, and for fairly random initial conditions. The reason for the discrepancy of the result in Ref. 5) with the theory in the subcritical regime is now clear: As is evident in Fig. 1(a), the finite-size effect is so strong there that $N=1600$ chosen in Ref. 5) was still too small. It has also been found that $\tau_c\propto(\varepsilon-\varepsilon_c)^{-1/4}$ for $\varepsilon\to\varepsilon_c{}^+$, which was verified numerically together with Eq. (14) as will be reported in detail elsewhere.

To conclude our theory developed above provides us with a quantitative understanding of the order parameter fluctuation for populations of oscillators modeled by Eq. (1). In particular, it reveals that the fluctuation plays really an important role in characterizing the onset of mutual entrainment since (a) it exhibits critical divergence and more importantly (b) the critical exponent takes different values below and above the threshold in remarkable contrast with those in conventional phase transitions. The true order parameter $\hat{Z}$ only shows a stereotyped behavior, so that it is far from sufficient to capture the nature of the phase transition. The origin of the feature (b) is now being studied.

The framework of the theory and the results for the subcritical regime were first reported in an earlier version of this article,[6] where it was also pointed out that our theory can be used to study the linear stability of macroscopic states specified by $\hat{Z}$ in infinite-size systems as will be elaborated elsewhere. Very recently a plan of an approach different from ours has been announced.[7] The author is grateful to Professor Y. Kuramoto for some comments on the basic framework of the theory.

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Note added in proop: The approximation for A_+ has been found not to be consistent with the one for A_-, so that it has to be modified in a way. The result (17), however, remains unchanged.