Intrinsic scales for high-dimensional Lévy-driven models with non-Markovian synchronizing updates

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Abstract

We propose stochastic N-component synchronization models $(x_1(t), \ldots, x_N(t))$, $x_j \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, whose dynamics is described by Lévy processes and synchronizing jumps. We prove that symmetric models reach synchronization in a stochastic sense: differences between components $d_{kj}^{(N)}(t) = x_k(t) - x_j(t)$ have limits in distribution as $t \to \infty$. We give conditions of existence of natural (intrinsic) space scales for large synchronized systems, i.e., we are looking for such sequences $\{b_N\}$ that distribution of $d_{kj}^{(N)}(\infty)/b_N$ converges to some limit as $N \to \infty$. It appears that such sequence exists if the Lévy process enters a domain of attraction of some stable law. For Markovian synchronization models based on α-stable Lévy processes this results holds for any finite N in the precise form with $b_N = (N-1)^{1/\alpha}$. For non-Markovian models similar results hold only in the asymptotic sense. The class of limiting laws includes the Linnik distributions. We also discuss generalizations of these theorems to the case of non-uniform matrix-based intrinsic scales. The central point of our proofs is a representation of characteristic functions of $d_{kj}^{(N)}(t)$ via probability distribution of a superposition of N independent renewal processes.

Keywords: stochastic synchronization systems, non-Markovian models, heavy tails, Lévy processes, stable laws, operator stable laws, Linnik distributions, intrinsic scales, superposition of renewal processes, Laplace transform, generating functions, ME distributions, mean-field models

Contents

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1 Introduction

Time evolution of a multicomponent system with synchronization $x(t) = (x_1(t), \ldots, x_N(t))$, $t \in \mathbb{R}_+$, consists of two parts: a free dynamics and a spontaneous synchronizing interaction between components. $x_j(t) \in \mathbb{R}^d$ denotes the state of the component j at time t. The *synchronizing interaction* is possible only at some random epochs $0 < T_1 < T_2 < \cdots$ and has the form of instantaneous jumps $(x_1, \ldots, x_N) \to (x'_1, \ldots, x'_N)$ where the new configuration $(x'_1, \ldots, x'_N)$ is such that $\{x'_1, \ldots, x'_N\} \subsetneq \{x_1, \ldots, x_N\}$. The most important example is a pairwise synchronizing interaction when for a randomly chosen pair (j_1, j_2) the component j_2 changes its state to the value x_{j_1}:

$$x_{j_2}(T_n + 0) = x_{j_1}(T_n), \qquad x_j(T_n + 0) = x_j(T_n), \quad j \neq j_2. \tag{1}$$

The *free dynamics* means that all components evolve independently between successive epochs of interaction.

The pairwise synchronizing interaction (1) can be interpreted as follows: the component j_1 generates a message containing information about its current state x_{j_1} and sends it to the component j_2; the message reaches the destination instantly; after receiving the message the component j_2 reads it and adjusts its state x_{j_2} to the value x_{j_1} recorded in the message.

In this paper we consider stochastic synchronization systems which are essentially more general than many previously studied mathematical models [46, 47, 49, 52]. For instance, the paper [47] studies a symmetric system of N identical Brownian particles with pairwise synchronization. More precisely, in [47] the free dynamics of a single component is the usual Wiener process with diffusion coefficient $\sigma > 0$ and the sequence

$\{T_n\}$ is a Poisson flow of intensity $\delta > 0$. For breavity we will refer to this system as "$\mathcal{BM}_N(\sigma, \delta)$-model". The Markovian synchronization model of [47] is very interesting because many important questions relevant to its long-time behavior can be answered in an explicit form [47, 52]. The "$\mathcal{BM}_N$-model" appears to be also useful for constructing more sophisticated systems, for example, models of clock synchronization in wireless sensor networks [50]. Nevertheless, the Markovian assumption is not realistic for many modern applications. In the present paper we propose a large class non-Markovian synchronization models. The free dynamics of components will be driven by multi-dimensional Lévy processes. In particular, this assumption permit to consider heavy tail cases. In the current paper the random sequence $\{T_n\}$ is such that, in general, the inter-event intervals $\{T_{n+1} - T_n\}_{n=1}^{\infty}$ are not independent. Hence the sequence $\{T_n\}$ is not even a renewal process. Obviously, in this situation we cannot have any profit from the Markov processes theory. We need to develop new specific methods. Before discussing these methods and describing our main results we would like to say a few words about applications that motivate introducing the synchronizing interaction between components.

Synchronization models have their origins in computer science [4]. The key idea of asynchronous parallel and distributed algorithms is to use many computing units (processors etc.) to do some common job. Most of the time the computing units work independently but sometimes they need to share information. The exchange of information is realized by means of a so called message-passing mechanism [4, 20]. During its work, a computing unit sends *timestamped* messages to other units. After receiving a message the computing unit analyzes the received data and sometimes adjusts its current state to be in agreement with other processors. Such adjustments can be interpreted as synchronizing jumps. Usually in these models the variable x_j denotes a *local time* of the processor j.

Similar problems arise for wireless sensor networks (WSNs) [73, 74]. In such networks the nodes (sensors) are almost autonomous. Each sensor is equipped with a non-perfect noisy clock. To work with data collected by different nodes the network needs a common notion of time. There exist many *clock synchronization protocols* [74] designed for wireless sensor networks. Most of them are based on the message-passing mechanism.

The first mathematical paper on stochastic synchronization models was [56]. Mitra and Mitrani studied a two-dimensional system which corresponds to parameters $N = 2$, $d = 1$, $T_k = k$ in terms of the above general description. Multi-dimensional models of distributed computations were proposed by many authors. Unlike [56] some of their papers [1,19,38,39,59,77,78] were focused on numerical simulations and had only auxiliary mathematical sections. Another papers [14,32,61,72] were devoted to very specific parallel algorithms. The first rigorous treatment of a multi-dimensional mathematical model with time stamp synchronization was done in [43]. In [41, 43, 46, 47, 49, 52] different N-component synchronization systems were considered as stochastic particle systems with special interaction. Such interpretation is useful for invoking physical intuition. It should be noted however that the synchronizing interaction was never studied before in the framework of traditional interacting particle systems [34]. Of course, we may also describe the place of the stochastic synchronization models in purely probabilistic terms as special perturbations of multi-dimensional random walks.

Stochastic synchronization models with *large* number of components are of special interests. The goal is to analyze their behavior as both the number of components N and the time t go to infinity. Before formulating this general problem in precise terms it is necessary to understand what kind of a long time behavior we can expect from a stochastic synchronization system. The word "synchronization" can be used in two senses. In a local sense we speak about *synchronization* (or equating) of some components as the results of a single synchronization jump. In a global sense we may ask whenever the total N-component system will *synchronize* as $t \to \infty$ and what is the meaning of this synchronization. Of course, this question should be considered only for "irreducible" multi-component systems that cannot be divided into two noninteacting subsystems. It is clear that due to the random nature of dynamics the so called *perfect synchronization* ($x_1 = \cdots = x_N$) is not possible. Moreover, as it was explained in [47] for the "$\mathcal{BM}_N$-model", the stochastic process $x(t)$ does not have even a *limit in law* as $t \to \infty$. Nevertheless, according to [47] the long time stabilization in law is expected for $x(t)$ considered in a moving coordinate system related, for example, to a tagged particle or to the center of mass. Note that differences $d_{ij}(x) := x_i - x_j$ are the same in both the absolute and the moving coordinate systems. Hence all $x_i(t) - x_j(t)$ are expected to have limits in law as $t \to \infty$. In [52] this statement was proved for the symmetric "$\mathcal{BM}_N$-model" in dimension $d = 1$. Moreover, it was also proved that $(x_i(\infty) - x_j(\infty)) / \sqrt{(N-1)N}$ has a symmetric Laplace distribution which parameter does not depend on N. This means that if t is large then components of $x(t)$ form a "collective" which typical space size is of order N. In this sense, one says that N is the typical *space scale* for the synchronized system. Note that coordinates of the center of mass are not stochastically bounded as $t \to \infty$. It is worth pointing out that a joint distribution of $(x_i(\infty) - x_j(\infty),\ 1 \le i < j \le N)$ cannot be found explicitly and the study of its properties for large N is a challenging problem. Another interesting problem concerning synchronization models with *large* number of components is related to a "prestationary" evolution of $x(t)$. The problem is to find different *time scales* ($t = t_N \to \infty$ as $N \to \infty$) on which the synchronization system $x(t_N)$ demonstrates completely different qualitative behaviour. The complete description of times scales was obtained for several models [40, 46–48, 51]. For example, in [47] it was shown that the "$\mathcal{BM}_N$-model" passes three different phases before it reaches the final synchronization. The model of clock synchronization in WSNs (see [51]) has 5 different consecutive phases of qualitative behaviour. As it was explained in [47] and [51, Sect. 5], each phase in evolution of a stochastic synchronization system is a cumulative result of competition between two opposite tendencies: with the course of time the free dynamics increases the "desynchronization" in the system while the interaction tries to decrease it.

In the present paper we study multi-component models $x(t) = (x_1(t), \ldots, x_N(t))$ with pairwise synchronizing interaction. These models generalize the "$\mathcal{BM}_N$-model" of [47, 52] in several directions. It is assumed that the free dynamics of components are general Lévy processes with values in $\mathbb{R}^d$. This assumption makes our models very flexible. Lévy processes have independent and stationary increments. Probability distributions of these increments may have heavy tails. Note that many modern stochastic models in finance [62, 71], insurance [63], data networks [7, 21, 55], physics [69] etc. use heavy-tailed Lévy processes. The theory of such processes is well developped and we will take advantage of it. We will also see that the stable Lévy processes and domains of attractions of stable laws play an important role in asymptotic analysis of synchronized system with large number of components N. Assumptions about the sequence $\boldsymbol{T} = \{T_n\}$ of synchronization epochs are very natural in the context of multi-component systems. It is assumed that each component j generates messages at epochs of some renewal process $\boldsymbol{\tau}^{(j)} = \left\{\tau_m^{(j)}\right\}$ independently of other components. Hence the point process $\{T_n\}$ is the superposition of N renewal processes: $\boldsymbol{T} = \cup_j \boldsymbol{\tau}^{(j)}$. In general, the superposition of renewal processes no longer forms a renewal process and therefore an analysis of $\boldsymbol{T} = \{T_n\}$ is a difficult task. There exists a huge number of studies in this field [10,11,16,18,75,80], most of them are devoted to limit theorems. Unfortunately, none of them is applicable to our situation.

The paper is organized as follows. In § 2.2–2.4 we introduce a general synchronization model. The precise definition of the pairwise interaction is given in § 2.2 in terms of parameters $F_k(s)$, $k = \overline{1,N}$, and $R = (r_{kj})_{k,j=1}^N$ where $F_k(s) := \mathsf{P}\left(\tau_{q+1}^{(k)} - \tau_q^{(k)} \le s\right)$, $s \in \mathbb{R}$, is the c.d.f. of inter-event intervals in the flow $\boldsymbol{\tau}^{(k)}$ and R is a routing matrix used for choosing message destinations. To introduce free dynamics we recall some classic results from the Lévy processes theory. The free dynamics is determined by a set of Lévy exponents $\{\boldsymbol{\eta}_j^\circ(\lambda),\ j = 1,\dots,N\}$, $\lambda \in \mathbb{R}^d$, (see § 2.3). We show that such approach includes, as examples of free dynamics, Brownian motions, random walks in $\mathbb{R}^d$ and, in particular, random walks with heavy-tailed jumps. The general N-component synchronization model with the above parameters will be denoted by $\mathcal{GG}_N\left(\{\boldsymbol{\eta}_j^\circ\}_{j=1}^N; \{F_j\}_{j=1}^N, R\right)$. While the free dynamics, the flows $\boldsymbol{\tau}^{(k)}$, the random routing and the initial configuration $x(0)$ are assumed to be independent the stochastic process $(x(t),\ t \in \mathbb{R}_+)$ is very complicated and, in general, *non-Markovian*. The only exception is the situation when all c.d.f. $F_j(s)$ correspond to exponential distributions: $F_j(s) = (1 - \exp(-s/m_j))_+$, $m_k > 0$. Under such assumption the point process $\boldsymbol{T} = \{T_n\}$ is a Poisson flow and the process $(x(t),\ t \in \mathbb{R}_+)$ is Markovian. In this case we will use notation $\mathcal{GM}_N\left(\{\boldsymbol{\eta}_j^\circ\}_{j=1}^N; \{F_j\}_{j=1}^N, R\right)$.

The paper is focused on symmetric synchronization models whose definition is given in § 3.1. The symmetry assumption means that evolutions of all components follows the same probabilistic rules with the same parameters ($\boldsymbol{\eta}_j^\circ = \boldsymbol{\eta}^\circ$, $F_j = F\ \forall j$) and the routing R is uniform. We will use short notation $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$ for the general symmetric model and $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$ for the Markovian symmetric model with c.d.f. $F(s) = (1 - \exp(-s/m))_+$. For the general model $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$ we assume that distribution of the inter-arrival intervals in the flows $\boldsymbol{\tau}^{(j)}$ has a *rational Laplace transform* (see RPFN class in § 3.3). This class of distribution was discussed in 1955 by Cox [8]. It is large enough to cover a variety of applications in queueing theory. These distributions are very convenient for analytical treatment and numerical simulations. Moreover, any probability distribution on $\mathbb{R}_+$ can be approximated arbitrarily close (in terms of weak convergence) by distributions with rational Laplace transforms. As it was shown in [3] this class of probability laws coincides with the *ME* (matrix-exponential) *distributions*. It contains as proper subsets the phase-type distributions [58, 60], the Coxian distributions [29], the general Erlangian distributions [9, 35] etc. We believe that most of results of our paper remain true for more general class of distributions but such generalization would make some of our proofs much longer.

In § 3.4 we assume that N is fixed and $t \to +\infty$. Under general assumptions on the free dynamics of the symmetric model $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$ in Theorem 1 we prove existence of the limit in law for the differences $x_k(t) - x_j(t)$,

$$d_{kj}^{(N)}(t) := x_k(t) - x_j(t) \xrightarrow{d} d_{kj}^{(N)}(\infty).$$

Next step is to study the distribution of $d_{kj}^{(N)}(\infty)$ for large values of N. This problem has different answers for Markovian and non-Markovian cases. Theorem 2 is devoted to characteristic function $\chi_N(\infty; \lambda)$ of the limiting law. Under general assumptions on $\{T_n\}$

it gives the following asymptotic representation of $\chi_N(\infty;\lambda)$ for large N:

$$\chi_N(\infty;\lambda) := \mathsf{E}\,\exp\, i\left\langle \lambda, d_{kj}^{(N)}(\infty)\right\rangle = \frac{1}{1+\theta_{1,N}\boldsymbol{\eta}(\lambda)} + \theta_{2,N}(\lambda).$$

Here $\boldsymbol{\eta}(\lambda) = -2\,\mathrm{Re}\,\boldsymbol{\eta}^\circ(\lambda)$, the real sequence $\{\theta_{1,N}\}$ is such that $\theta_{1,N} \sim \frac{1}{2}mN$ as $N\to\infty$,

$$m := \mathsf{E}\,\left(\tau_{q+1}^{(j)} - \tau_q^{(j)}\right) = \int_0^\infty y\, dF_j(y),$$

and the sequence of functions $\{\theta_{2,N}(\lambda)\}$ vanishes uniformly in $\lambda\in\mathbb{R}^d$. This representation is of great importance for subsequent sections.

It appears (Theorem 4 in § 3.8) that for the Markovian symmetrical model $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ;m)$ we have $\theta_{1,N} = \frac{1}{2}(N-1)m$ and $\theta_{2,N}(\lambda)\equiv 0$. This implies (Theorem 5 in § 3.8) that if the free dynamics of the model is driven by an α-stable Lévy process then the probability distribution of $d_{jk}^{(N)}(\infty)/\,(N-1)^{1/\alpha}$ has the characteristic function

$$\frac{1}{1+\frac{1}{2}m\boldsymbol{\eta}(\lambda)}$$

and hence it does not depend on N. In this case we may say that the sychronized system possesses an intrinsic space scale $(N-1)^{1/\alpha} \sim N^{1/\alpha}$. Indeed, since typical distances between components of the synchronized system are of order $N^{1/\alpha}$ it is natural to consider this system on a new space scale with a new unit which is equal to $N^{1/\alpha}$ old units.

For non-Markovian models $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ;F)$ the function $\theta_{2,N}(\lambda)$ is necessarily nonzero (see § 4.7). Therefore we cannot expect such nice result on the existence of the intrinsic scale for any fixed N as in the Markovian case. Nevertheless similar results hold in the asymptotic sense (when $N\to\infty$) if we make additional assumptions about the free dynamics. For asymptotic results it is not strictly necessary to assume that the free dynamics is a stable Lévy process. It is sufficiently to take the free dynamics from the domain of attraction of some stable law in $\mathbb{R}^d$ (§ 3.5). The theory of attraction to stable laws is classical and well developped [15,54,64]. Theorem 3 states that if the free dynamics belongs to the domain of attraction of some stable law and $\{b_n\}$ is a corresponding normalizing sequence then the distribution of $d_{jk}^{(N)}(\infty)/b_N$ weakly converges as $N\to\infty$ to some distribution $Q_{\infty,\infty}(dx)$ on $\mathbb{R}^d$. A situation when the attracting stable law has the index of stability α and the normalizing sequence is $b_n = n^{1/\alpha}$ is known as the normal attraction [15]. Hence the distribution of $d_{jk}^{(N)}(\infty)/b_N$ is asymptotically not depending on N. So we may say that b_N is the *instrinsic space scale* of a *large* synchronized N-component system. It is interesting to note that the limit distribution $Q_{\infty,\infty}(dx)$ belongs to the class of symmetric geometric stable distributions [57] (see Remark 3 in § 3.6). In particular, this class contains the Laplace distribution and the famous Linnik distribution [36].

In § 3.7 we generalize Theorem 3 to the case of matrix-based scales when intrinsic space transformations have the form of linear operators $d_{jk}^{(N)}(\infty)\mapsto B_N d_{jk}^{(N)}(\infty)$ for some special $d\times d$ matrices B_N. We show that existence of such intrinsic matrix scales is related to the problem of attraction to *operator stable* laws in $\mathbb{R}^d$ [31,54]. In the case $B_N = N^{-B}$ these *non-uniform* scales can be described in terms of Jurek coordinates [24,54]. Hence in dimensions $d>1$ the class of N-component synchronization systems discussed in § 3.7 is much wider than the class of models of § 3.5 with "scalar" intrinsic scales.

Section 4 contains proofs of all theorems. These proof use the representation of the characteristic function $\chi_N(t;\lambda) := \mathsf{E} \exp i \left\langle \lambda, d_{kj}^{(N)}(t) \right\rangle$ in terms of the Lévy exponent $\boldsymbol{\eta}(\lambda)$ and generation functions related to the superposition $\boldsymbol{T} = \cup_j \boldsymbol{\tau}^{(j)}$ of the renewal processes $\boldsymbol{\tau}^{(j)}$ (Lemmas 7 and 8 in § 4.3). To get this representation we need a chain of auxiliary results on the free dynamics and the interaction (Lemmas 4–6 in § 4.1–4.2). These lemmas are similar to their analogues proved for Markovian models in [46, 47]. Nevertheless, the proof of Lemma 6 meets additional difficulties related to the involved nature of the sequence of synchronization epochs $\boldsymbol{T}$. The symmetry assumption is very essential for the proof of Lemma 6. Note that Lemma 4 can be generalized for symmetric synchronizing "multi-particle" interactions (see [46] and § 2.1) which are more general than the pairwise interactions. This possibility opens the way to an obvious generalization of the present paper.

The representation for the characteristic function $\chi_N(t;\lambda)$ provided by Lemmas 7 and 8 gives an explicit formula for Markovian models $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$ (§ 4.4). Therefore Theorems 4 and 5 (including the convergence in Theorem 1) easily follow from that explicit formula.

The non-Markovian case $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$ is more complicated. Even the existence of the limit $\lim_{t\to\infty} \chi_N(t;\lambda)$ in Theorem 1 is not evident. At first look a special adaptation of the classic Key Renewal Theorem (KRT) might be helpful for calculating such limits. But as it is explained in § 4.5 it is very unlikely that the classical sufficient conditions for the KRT could be effectively checked in our concrete problem. So we restrict ourself to renewal processes $\boldsymbol{\tau}^{(j)}$ with the ME distribution of inter-event intervals. Keeping in mind this assumption, in § 4.6 we develop some simple rules for manipulating expressions arising in Lemmas 7 and 8. These rules permit us to get a short proof of Theorem 1 in an "algebraic manner". Theorem 1 follows from Lemmas 9 and 10 which proofs are given in § 4.6. Lemma 10 also provides an integral representation for the limiting characteristic function $\chi_N(\infty;\lambda)$. This representation will be useful for proving Theorems 2 and 3 in 4.7. The method of these proofs is based on using the Laplace transform for generating functions. It reduces to an analysis of singularities of rational complex functions. Such approach is standard in the context of the classical renewal theory [9, 16]. But it is necessary to pay attention to coefficients in decomposions (Lemma 12) because they depend on N. The problem is to find singularities giving the principal asymptotics (Lemma 13) and to obtain precise bounds for the coefficients. § 4.7.2 completes proofs of Theorems 2 and 3.

2 Model. Definitions. Assumptions. Notation

In § 2.1 for explanatory purposes only we describe a general approach to constructing a large class of stochastic synchronization models. We try to show that different existing synchronization models may be considered within the unified framework of special perturbations of simple stochastic evolutions. A definition of our model and precise assumptions are given in § 2.2–2.4.

2.1 Perturbation of independent dynamics by synchronization

Imagine there is some system consisting of N components which are labeled by the set $\mathcal{N}_N = \{1, \ldots, N\}$. First we introduce independent dynamics of the components.

Let $(x_1^\circ(t),\, t\in\mathbb{R}_+)\,,\dots,(x_N^\circ(t),\, t\in\mathbb{R}_+)$ be independent stochastic processes taking their values in $\mathbb{R}^d$. Assume that each process $\left(x_j^\circ(t),\, t\in\mathbb{R}_+\right)$ has independent increments. We interpret the variable $x_j^\circ\in\mathbb{R}^d$ as a state of the component j and the set of processes $x^\circ(t)=(x_1^\circ(t),\dots,x_N^\circ(t))$ as a free dynamics of the system.

Next we add a perturbation to the system. We modify the evolution x° by introducing a special interaction between components. This interaction happens at random times and consists in a partial synchronization of component states.

For any map $M:\mathcal{N}_N\to\mathcal{N}_N$ define $\nu_M=\operatorname{card} M\mathcal{N}_N$ which is the number of different elements in the image $M\mathcal{N}_N=\{M(j): j\in\mathcal{N}_N\}$. Consider also a set of fixed points $U_M=\{j: M(j)=j\}$. The map M is called a *synchronization map* if $\nu_M<N$ and $\operatorname{card} U_M=\nu_M$. Denote by $\mathcal{M}_N$ a set of all synchronization maps of the set $\mathcal{N}_N$.

Let $\{T_n,\, n\in\mathbb{Z}_+\}$ be a random sequence

$$0\equiv T_0<T_1<\dots<T_n<\cdots$$

and $\{M_n,\, n\in\mathbb{N}\}$ be a sequence of $\mathcal{M}_N$-valued random variables. We do not assume that $\{T_n\}$ and $\{M_n\}$ are independent. Consider a new stochastic process $x(t)=(x_1(t),\dots,x_N(t))\in\left(\mathbb{R}^d\right)^N$ which paths are determined by the following relations

$$x(t)-x(0)=x^\circ(t)-x^\circ(0),\quad t\in[0,T_1],\qquad x(T_n+0)=(x\circ M_n)\,(T_n)\,,\quad n\geq 1,\quad (2)$$

$$x(t)=x(T_n+0)+(x^\circ(t)-x^\circ(T_n+0))\qquad t\in(T_n,T_{n+1}],\qquad (3)$$

where $y=(x\circ M)$ is the vector $y=(y_1,\dots,y_N)\in\left(\mathbb{R}^d\right)^N$ with coordinates $y_j=x_{M(j)}$, $j\in\mathcal{N}_N$. The correspondence $x\mapsto y=(x\circ M)$ between points of the configuration space $\left(\mathbb{R}^d\right)^N$ will be called a *synchronization jump*. In some sense the process $x(t)$ is the special perturbation of the free dynamics $x^\circ(t)$. We will call the process $x(t)$ a *stochastic synchronization system*.

We always assume that *initial configuration* $x(0)$ is *independent* of $x^\circ(\cdot)$, $\{T_n\}$ and $\{M_n\}$.

Sometimes another terminology is useful. We can speak about interacting particle systems (instead of multi-component systems) and consider $x_j(t)$ as a coordinate of j-th particle. In [46] we studied a system of N identical particles moving as independent random walks (free dynamics $x^\circ(t)$) and interacting by means of special m-particle synchronizations happened at epochs $\{T_n\}$ of some Poisson flow. In that case all synchronizing maps M_n satisfy the condition $\nu_M=N-m+l$ for some $l\leq m/2$. Multiparticle synchronizations ($m>2$) will not be considered further in this paper. Starting from § 2.2 we consider only pairwise interactions.

Hence the N-component stochastic synchronization system $x(t)=(x_1(t),\dots,x_N(t))$ is determined by specifying the following ingredients:

(F) the free dynamics $x^\circ(t)=(x_1^\circ(t),\dots,x_N^\circ(t))$

(T)+(M) the random flow of synchronization epochs $\{T_n\}$ and the sequence of synchronization maps $\{M_n\}$

(I) the initial distribution of $x(0)=(x_1(0),\dots,x_N(0))$

The above assumptions on (F), (T)+(M) and (I) need to be precised when defining a concrete model. In some models it is convenient to consider a marked point process [6,12]

$$(T_1,\kappa_1)\,,\dots,(T_n,\kappa_n)\,,\dots$$

with a finite set of marks K and a marked sequence of synchronization maps $\left\{M_n^{(\kappa_n)}\right\}$. The interaction (T)+(M) is build by a two-stage construction: first, the generation of the sequence $\{(T_n,\kappa_n)\}_{n=1}^{\infty}$, and then the generation of conditionally independent maps $\left\{M_n^{(\kappa_n)}\right\}$. For nonsymmetric models probability distributions of $M_n^{(\kappa)}$ may be different for different marks κ. Such situation will be considered in the current paper, see § 2.2 for details.

The above ingredients (F) and (T)+(M) may be correlated. For example, papers [43] and [42] were devoted to particular models in which the probability distribution of M_n depends on $x(T_n)$.

In models studied in the present paper the free dynamics $x^{\circ}(t)$ and the couple $(\{T_n\},\{M_n\})$ are independent.

2.2 Assumptions on synchronization epochs and synchronization maps

In this paper we consider a pairwise synchronization which is based on the well known message-passing mechanism [4,20]. This means that components of the system can share the data with other components by sending and receiving messages containing information about a current state of the sender. Below we will use terminology of particle systems and speak about particles instead of components.

Each particle k has its own sequence of times

$$0<\tau_1^{(k)}<\tau_2^{(k)}<\cdots$$

when it sends messages to other particles. For convenience we put $\tau_0^{(k)}\equiv 0$. The choice of recipients will be discussed below. Denote $\Delta_n^{(k)}=\tau_n^{(k)}-\tau_{n-1}^{(k)}$.

Let the random variables $\left(\Delta_n^{(k)},\, n\in\mathbb{N}\right)$ be independent and identically distributed. This means that $\Pi_t^{(k)}=\max\left\{n:\,\tau_n^{(k)}\le t\right\}$, $t\ge 0$, is a simple renewal process. Assume that for any k a c.d.f. $F_k(s)=\mathsf{P}\left\{\Delta_n^{(k)}\le s\right\}$ is continuous. We assume also that the renewal processes $\left(\Pi_t^{(k)},\, t\ge 0\right)$, $k=1,\ldots,N$, are independent. Consider events $C_{k_1,k_2}=\left\{\exists n,m:\,\tau_n^{(k_1)}=\tau_m^{(k_2)}\right\}$. It follows that

$$\mathsf{P}\left(\bigcup_{1\le k_1<k_2\le N} C_{k_1,k_2}\right)=0.$$

Consider a point process

$$0=T_0<T_1<T_2<\cdots$$

generated by the superposition of the renewal processes $\Pi_t^{(k)}$, $k=1,\ldots,N$. In general, inter-arrival times T_q-T_{q-1} are not independent. Denote $\Pi_t^S=\sum_{j=1}^N\Pi_t^{(j)}$. In other words, $\Pi_t^S=\max\{m:\ T_m\le t\}$.

Fix some $N\times N$ matrix $R=(r_{ij})_{i,j=1}^N$, $r_{ii}=0$, $r_{ij}\ge 0$, $\sum_{j=1}^N r_{ij}=1$. We define the interaction between particles of $x(t)$ by means of synchronization jumps which occur at times of the point process $\{T_q\}$. Namely, for any point T_q there exists a unique (random) pair (j_1,n), $j_1\in\{1,\ldots,N\}$, $n\in\mathbb{N}$, such that $T_q=\tau_n^{(j_1)}$. It means that at time T_q the particle j_1 sends a message to some another particle j_2 which is chosen independently with probability $r_{j_1 j_2}$. The message contains information on the current value of x_{j_1}. Messages reach their destinations instantly. After receiving the message from j_1 the particle j_2 ajusts its coordinate to the value x_{j_1}: $x_{j_2}(T_q+0) = x_{j_1}(T_q)$. This is the only jump in the system at the time T_q: $x_j(T_q+0) = x_j(T_q)$ for all $j \neq j_2$. Define a map $S_{j_1 j_2} : \mathcal{N}_N \to \mathcal{N}_N$ as follows

$$S_{j_1 j_2}(j) = \begin{cases} j, & j \neq j_2, \\ j_1, & j = j_2 \,. \end{cases}$$

We see that if $T_q = \tau_n^{(j_1)}$ then the random synchronization map M_q is such that

$$\mathsf{P}\left\{M_q = S_{j_1 j_2}\right\} = r_{j_1 j_2}$$

for all $j_2 \neq j_1$. In particular, $\nu_{M_q} = N - 1$.

Hence the synchronization is determined by the following parameters: $F_k(s)$, $k = 1, \ldots, N$, and the matrix $R = (r_{ij})_{i,j=1}^N$.

As it was mentioned in Subsection 2.1 between receiving of subsequent messages the particles evolve according to the free dynamics.

Note that the above defined random sequences $\{T_q\}$ and $\{M_q\}$ correspond to the formal scheme of § 2.1. Namely, $\{T_q\}$ can be obtained from a marked point process $\{(T_q, \kappa_q)\}$ where the set of marks K is $\{1, \ldots, N\}$ and κ_q is such that $T_q = \tau_n^{(\kappa_q)}$ for some n. The probability distribution of $M_q^{(\kappa_q)}$ depends on the mark κ_q because the values $S_{\kappa_q j}$, $j = \overline{1, N}$, are taken with probabilities $r_{\kappa_q j}$.

2.3 Free evolution

Assume that $x_1^\circ(t), \ldots, x_N^\circ(t)$ are independent Lévy processes. This means that here we make an assumption stronger than the independence of increments condition (see Subsection 2.1). The Lévy processes theory is well developped (see, for example, [2, 68]) and we want to make use of it. We recall basic definitions and introduce some notation.

Definition 1 *A stochastic process* $\left(x_j^\circ(t),\, t \in \mathbb{R}_+\right)$ *is called a Lévy process if*

- *it starts from the origin:* $x_j^\circ(0) = 0 \in \mathbb{R}^d$
- *it has independent and stationary increments*
- *it is stochastically continuous.*

Let y_1 and y_2 be two vectors in $\mathbb{R}^d$, $y_m = (y_m^1, \ldots, y_m^d)$, $m = 1, 2$. Denote by $\langle y_1, y_2 \rangle$ their scalar product, i.e., $\langle y_1, y_2 \rangle = \sum_{l=1}^d y_1^l y_2^l$. If Y is a random vector in $\mathbb{R}^d$ then $\psi_Y(\lambda)$ denotes its characteristic function:

$$\psi_Y(\lambda) = \exp\left(i \left\langle \lambda, Y \right\rangle\right), \quad \lambda \in \mathbb{R}^d.$$

The random vector Y is said to be *infinitely divisible* if for all $n \in \mathbb{N}$ there exist i.i.d. random vectors $Z_1^{(1)}, \ldots, Z_n^{(n)}$ such that

$$Y \stackrel{d}{=} Z_1^{(1)} + \cdots + Z_n^{(n)}.$$

As usual the notation $V_1 \stackrel{d}{=} V_2$ means that random vectors V_1 and V_2 have the same distribution. The fundamental result established by Lévy and Khinchine states that

$\psi_Y(\lambda) = \exp \rho_Y(\lambda)$ where the function $\rho_Y : \mathbb{R}^d \to \mathbb{C}$ can be represented in a special form known as the Lévy-Khinchine formula [2, 15, 68]. We will not use here this formula explicitely. When we need to say that Y has an infinitely divisible distribution with the Lévy exponent $\rho_Y(\lambda)$ we will simply write $Y \sim \mathcal{ID}(\rho_Y(\lambda))$.

It it clear that increments of a Lévy process are infinitely divisible. In the sequel we will use the following classical result [2]. Let $\phi^j(t-s;\lambda)$ the characteristic function of the increment $x_j^\circ(t) - x_j^\circ(s)$:

$$\phi^j(t-s;\lambda) = \mathsf{E}\,\exp\left(i\left\langle \lambda, x_j^\circ(t) - x_j^\circ(s)\right\rangle\right), \qquad 0 \le s \le t.$$

Then $\phi^j(t;\lambda) = e^{t\boldsymbol{\eta}_j^\circ(\lambda)}$, $t \ge 0$, with some function $\boldsymbol{\eta}_j^\circ : \mathbb{R}^d \to \mathbb{C}$ having the Lévy-Khinchine form. For such Lévy process $\left(x_j^\circ(t),\, t \ge 0\right)$ we will use a short notation $x_j^\circ \sim \mathcal{LP}\left(\boldsymbol{\eta}_j^\circ\right)$.

We see that the set of Lévy exponents $\left\{\boldsymbol{\eta}_j^\circ(\lambda),\ j = 1, \ldots, N\right\}$ completely determines free dynamics of our model.

Examples of the free dynamics driven by Lévy processes.

- Each component $x_j^\circ(t)$ is a d-dimensional *Brownian motion* with a constant drift:

$$d\,x_j^\circ(t) = \sigma_j dB_j(t) + b_j dt,$$

  where σ_j is a real $d \times d$ matrix, $b_j \in \mathbb{R}^d$, $B_j(t) = (B_j^1(t), \ldots, B_j^d(t))$ and $B_j^1, \ldots, B_j^d$ are independent standard Wiener processes with values in $\mathbb{R}^1$. This case corresponds to the function

$$\boldsymbol{\eta}_j^\circ(\lambda) = i\left\langle b_j, \lambda\right\rangle - \frac{1}{2}\left\langle \sigma_j \sigma_j^T \lambda, \lambda\right\rangle . \tag{4}$$

- *Random walks in* $\mathbb{R}^d$. The component $x_j^\circ(t)$ is a continuous time jump Markov process with generator

$$\left(L_j f\right)(y) = \beta_j \int_{\mathbb{R}^d} \left(f(y+q) - f(y)\right)\mu_j(dq), \qquad f \in C_b(\mathbb{R}^d, \mathbb{R}), \tag{5}$$

  where $C_b(\mathbb{R}^d, \mathbb{R})$ is the Banach space of bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$, $\beta_j > 0$ is the intensity of jumps and a probability measure μ_j is the distribution of jumps. It is easy to see that in this case

$$\boldsymbol{\eta}_j^\circ(\lambda) = \beta_j \int_{\mathbb{R}^d} \left(e^{i\langle \lambda, q\rangle} - 1\right)\mu_j(dq)\,.$$

- *Random walks in* $\mathbb{Z}^d$. This is a subcase of (5) with the measures $\mu_j(dq)$ supported in $\mathbb{Z}^d$:

$$S_j = \operatorname{supp} \mu_j \subset \mathbb{Z}^d.$$

  Then

$$\left(L_j f\right)(y) = \beta_j \sum_{q \in S_j} \left(f(y+q) - f(y)\right)\mu_j(\{q\}), \qquad f \in C_b(\mathbb{Z}^d, \mathbb{R}),$$

  and

$$\boldsymbol{\eta}_j^\circ(\lambda) = \beta_j \sum_{q \in S_j} \left(e^{i\langle \lambda, q\rangle} - 1\right)\mu_j(\{q\})\,.$$

- Consider the following particular subcase of (5)

$$\mu_j(dq) = \frac{C_{\boldsymbol{a}} \mathbf{1}_{\{|q|\geq 1\}}\, dq}{|q|^{d+\boldsymbol{a}}}, \qquad q \in \mathbb{R}^d, \tag{6}$$

  where the papameter $\boldsymbol{a}$ is positive and $C_{\boldsymbol{a}}$ is a normalizing factor. Evidently, the distribution (6) has a finite expectation iff $\boldsymbol{a} > 1$. Moreover, it has a finite variance iff $\boldsymbol{a} > 2$.

2.4 General synchronization models

Assume that $x_1^\circ(t), \ldots, x_N^\circ(t)$ satisfy to assumptions of § 2.3 and $(\{T_n\},\{M_n\})$ satisfy to assumptions of § 2.2. Assume also that the free dynamics x°, the pair $(\{T_n\},\{M_n\})$ and an initial configuration $x(0)$ are *independent*. A stochastic process $x(t) = (x_1(t), \ldots, x_N(t)) \in \left(\mathbb{R}^d\right)^N$ defined by (2)–(3) will be called an *N-component synchronization system*. To specify parameters of the model we will use notation $\mathcal{GG}_N\left(\{\boldsymbol{\eta}_j^\circ\}_{j=1}^N ; \{F_j\}_{j=1}^N, R\right)$.

We list some simple properties of the *general* model $\mathcal{GG}_N\left(\{\boldsymbol{\eta}_j^\circ\}_{j=1}^N ; \{F_j\}_{j=1}^N, R\right)$:

- Under assumptions of § 2.2–2.3 the process $x(t)$ is stochastically continuous.
- $x(t)$ is not a process with independent increments.
- The process $x(t)$ is neither Markovian nor semi-Markovian.

The lack of Markovian property is explained by the complicated structure of the sequence $\{T_n\}$. However there is an important exclusion.

Remark 1 *If all $F_k(s)$ correspond to exponential distributions,*

$$F_k(s) = (1 - \exp(-s/m_k))_+, \quad s \in \mathbb{R}, \quad m_k > 0,$$

then $x(t)$ is a Markov process. Indeed, in this case the point process $\{T_n\}$ is a Poissonian flow as the superposition of independent Poissonian flows $\left\{\tau_n^{(j)}\right\}$, $j = \overline{1,N}$.

Sometimes we will denote the Markovian model by $\mathcal{GM}_N\left(\{\boldsymbol{\eta}_j^\circ\}_{j=1}^N ; \{F_j\}_{j=1}^N, R\right)$.

3 Symmetric models: main results

In this paper we mainly study a *symmetric* synchronization model which will be introduced in Subsection 3.1

3.1 Symmetry assumptions

The general synchronization system was introduced in Subsections 2.2–2.4. Here we add more assumptions to define symmetric model.

Free dynamics. We assume that all functions $\boldsymbol{\eta}_j^\circ$, $j = \overline{1,N}$, defining the independent Lévy processes $x_j^\circ(t)$ are equal: $\boldsymbol{\eta}_j^\circ(\lambda) \equiv \boldsymbol{\eta}^\circ(\lambda)$.

Synchronization epochs. $F_j(y) = F(y)$ for all $j = \overline{1,N}$.

Routing matrix. Senders choose destinations for their messages uniformly: $r_{jk} = 1/(N-1)$ for all $k \neq j$, $r_{jj} = 0$.

In other words, the symmetric model means that all components are identical. Their evolutions follow the same probabilistic rules with the same parameters.

For any random vector $z = (z_1, \ldots, z_N) \in \left(\mathbb{R}^d\right)^N$ with components $z_j \in \mathbb{R}^d$ we denote by $\mathcal{P}_z$ the distribution law of z. Hence $\mathcal{P}_z$ is some probability measure on $\left(\mathbb{R}^d\right)^N$. Let $\pi : (1, \ldots, N) \to (i_1, \ldots, i_N)$ be an arbitrary permutation. The permutation π generates a map on $\left(\mathbb{R}^d\right)^N$:

$$\pi \star (z_1, \ldots, z_N) = (z_{i_1}, \ldots, z_{i_N}) .$$

Initial distribution. Assume that the initial distribution $\mathcal{P}_{x(0)}$ is invariant with respect to permutations of indices, i.e.,

$$\mathcal{P}_{\pi \star x(0)} = \mathcal{P}_{x(0)} \tag{7}$$

for all π. Note that the denenerated case when all components start from the origin, i.e.,

$$x_i(0) = 0 \quad \text{for all} \quad i = 1, \ldots, N,$$

is a particular example of the assumption (7).

As it was already mentioned in Subsection 2.1 we always assume that initial configuration $x(0)$ is *independent* of the free dynamics and synchronizations.

If all above assumptions hold then for all $t > 0$ the distribution of the N-component system $x(t)$ remains invariant with respect to permutations of indices. In such case we will simply call the process $x(t)$ a symmetric synchronization model.

Note that, in general, the symmetric model $x(t)$ is not Markovian nor semi-Markovian stochastic process. The only exception is the situation discussed in Remark 1.

The *general* (non-Markovian) *symmetric* model will be denoted by $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$. For *Markovian symmetric* model we use notation $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$ where $m > 0$ is the mean of the exponential distribution with c.d.f. $F(s) = (1 - \exp(-s/m))_+$.

3.2 Desynchronization between components

Since the free dynamics of different components are independent our stochastic system will never reach the *perfect synchronization regime* when states of all components $x_1(t)$, ..., $x_N(t)$ become equal after some (possibly random) time t_0. Such phenomenon is impossible due to the stochastic nature of the dynamics. What we can expect is a long time stabilization of synchronization errors in the distributional sense. To get some control over magnitudes of the synchronization errors we will consider differences $d_{jk}^{(N)}(t) = x_j(t) - x_k(t)$ between states of any pair (j,k) at time t.

Let a probability measure $P_{N,t}(dx)$ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ be the distribution of $d_{jk}^{(N)}(t) = x_j(t) - x_k(t)$. By the symmetry assumptions it is the same for all $j \neq k$. Similarly, the characteristic function of $d_{jk}^{(N)}(t)$,

$$\chi_{N,j,k}(t; \lambda) = \mathsf{E} \exp\left(i \left\langle \lambda, x_j(t) - x_k(t) \right\rangle\right), \qquad \lambda \in \mathbb{R}^d, \tag{8}$$

does not depend on j and k for the symmetric model. So we will omit indices j, k and use notation $\chi_N(t; \lambda)$,

$$\chi_N(t; \lambda) = \int_{\mathbb{R}^d} e^{i\langle \lambda, x\rangle} P_{N,t}(dx) .$$

Our aim is to study the characteristic function $\chi_N(t; \lambda)$ for large t and N. Main results will be presented in Subsections 3.4 and 3.5.

3.3 Assumptions on inter-event interval distribution

The independent renewal processes $\left(\Pi_t^{(k)},\, t\geq 0\right)$, $k=1,\ldots,N$, defined in Subsection 2.2 are identically distributed in the symmetric model. Up to the end of this paper we will assume that the below conditions holds.

Assumption P1. The probability distribution function F is absolutely continuous:

$$\begin{aligned} F(s) &= \mathsf{P}\left\{\Delta_n^{(k)}\leq s\right\} = \int_0^s p(s')\,ds', \qquad s\geq 0,\\ F(s) &= 0, \qquad p(s)\,=\,0, \qquad s<0. \end{aligned}$$

Note that this assumption concerns only inter-event intervals in each $\Pi_t^{(k)}$. The point process $\{T_n\}$ which is the superposition of $\Pi_t^{(k)}$, $k=1,\ldots,N$, is very complicated.

Given a function $q=q(s)$ such that $q(s)=0$ for $s<0$, we denote by $q^*(z)$ its Laplace tranform [9],

$$q^*(z)=\int_0^{+\infty} e^{-zs}q(s)\,ds, \qquad z\in\mathbb{C}.$$

If $q(s)$ is a probability density function then $q^*(z)$ is well defined at least in the complex half-plane $\{z:\ \operatorname{Re} z\geq 0\}$.

Before introducing the next assumption we discuss a special class of complex functions $f=f(z)$, $f:\mathbb{C}\to\mathbb{C}$. We say that $f(z)$ is a *RPF-function* if it can be represented as a proper fraction $f(z)=\frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are some polynomials such that $\deg P<\deg Q$. Note that summation and mutiplication of RPF-functions again give a RPF-function. Evidently, any RPF-function has finite number of poles and is vanishing as $z\to\infty$. Such functions can be written as

$$f(z)=\sum_{j=1}^{v}\sum_{k=1}^{n_j}\left(z-z_j\right)^{-k}\,c_{j,k}\,, \tag{9}$$

where $n_j\geq 1$ are natural numbers, $z_j\in\mathbb{C}$ are poles of f and $c_{j,k}\in\mathbb{C}$. The representation (9) is just the sum of principal parts of Laurent expansions about poles, the number n_j is the order of the pole z_j.

If all poles z_i have *strictly negative* real parts ($\operatorname{Re} z_i<0$) we say that the function f belongs to *the class RPFN*.

Assumption P2. The probability density function $p(s)$ is such that its Laplace transform $p^*(z)$ is a RPFN-function.

As it was already mentioned in Introduction the probability distributions satisfying to the Assumption P2 are exactly the ME distributions [3]. An important role of distributions with rational Laplace tranform for the queueing theory was discovered by Cox in [8].

In particular, Assumption P2 implies the existence of an exponential moment

$$\mathsf{E}\,\exp\left(\delta\Delta_n^{(k)}\right)=\int_0^{\infty}\exp(\delta u)\,p(u)\,du<\infty$$

for some $\delta>0$ and hence the existence of all moments

$$m_r=\mathsf{E}\,\left(\Delta_n^{(k)}\right)^r=\int_0^{+\infty}s^r p(s)\,ds\,, \qquad r\in\mathbb{N}. \tag{10}$$

For shortness we will use also notation m for the mean: $m = m_1 = \int s\, p(s)\, ds$.

The function $p(s)$ is a probability density hence $p^*(0) = 1$. If $p(s)$ satisfies Assumption P2 then the equation

$$1 - p^*(z) = 0 \tag{11}$$

has a finite number of roots. Let $\{r_0, r_1, \ldots, r_q\}$, $r_0 = 0$, be the set of different roots of the equation (11).

Lemma 1 *All numbers $r_1, \ldots, r_q$ belong to the subplane $\mathrm{Re}\, z < 0$.*

Proof. Since $(p^*)'(0) = -m < 0$ the root $r_0 = 0$ is simple. Note that $|p^*(v)| < 1$ for any $v \in \{z \in \mathbb{C}: \mathrm{Re}\, z \geq 0\} \setminus \{0\}$. Indeed,

$$|p^*(a + ib)| \leq p^*(a) \qquad a, b \in \mathbb{R}.$$

Evidently, $p^*(a) < 1$ for $a > 0$ so $|p^*(z)| < 1$ if $\mathrm{Re}\, z > 0$. Moreover, $|p^*(ib)| < 1$, $b \neq 0$, as a characteristic function of a non-lattice distribution [70]. □

Assumption P3. The roots $r_1, \ldots, r_q$ are **simple** that is $(p^*)'(r_j) \neq 0$.

Assumption P3 is not necessary for the main results but it makes some proofs shorter. Obviously, Assumption P3 corresponds to the general case situation.

3.4 Limiting distributions

We consider the symmetric synchronization model of Subsection 3.1 under Assumptions P1 and P2. Recall that $P_{N,t}(dx)$ and $\chi_N(t; \lambda)$ denote the distribution law and the characteristic function of $d_{jk}^{(N)}(t) = x_j(t) - x_k(t)$.

Theorem 1 *For any fixed N the distribution of $d_{jk}^{(N)}(t) = x_j(t) - x_k(t)$ has a (weak) limit as $t \to \infty$:*

$$P_{N,t} \xrightarrow{w} P_{N,\infty}\,.$$

This theorem follows from the Lévy continuity theorem (Theorem 3.6.2 in [37]) and the next lemma.

Lemma 2 *For any fixed N the family of characteristic functions $\{\chi_N(t; \lambda),\ t \geq 0\}$ convergences to some function $\chi_N(+\infty; \lambda)$ as $t \to +\infty$ and, moreover, this convergence is uniform in $\lambda \in \mathbb{R}^d$.*

It is well known [37, Th. 3.6.2] that the function $\chi_N(+\infty; \lambda)$ is the characteristic function of the limiting distribution $P_{N,\infty}$.

In the next theorem we need additional Assumptions P3.

Theorem 2 *Let Assumptions P1-P3 hold. The characteristic function $\chi_N(+\infty; \lambda)$ admits the following representation*

$$\chi_N(+\infty; \lambda) = \frac{1}{1 + \theta_{1,N}\boldsymbol{\eta}(\lambda)} + \theta_{2,N}(\lambda).$$

Here $\boldsymbol{\eta}(\lambda) = -2\, \mathrm{Re}\, \boldsymbol{\eta}^\circ(\lambda)$, the real sequence $\{\theta_{1,N}\}$ is such that $\theta_{1,N} \sim \frac{1}{2} mN$ as $N \to \infty$ and the sequence of functions $\{\theta_{2,N}(\lambda)\}$ vanishes uniformly in λ:

$$\sup_{\lambda \in \mathbb{R}^d} |\theta_{2,N}(\lambda)| \to 0 \qquad (N \to \infty)\,. \tag{12}$$

Theorem 2 is proved in Subsection 4.7. We will see from Subsection 4.7.2 that for the sequence of functions $\{\theta_{2,N}(\lambda)\}$ a result stronger than (12) holds. Namely, there exists a real sequence $\{\theta_N\}$ such that $\theta_N \to 0$ as $N \to \infty$,

$$\sup_{\lambda\in\mathbb{R}^d} |\theta_{2,N}(\lambda)| \le \theta_N, \tag{13}$$

and $\{\theta_N\}$ is the same for any function $\boldsymbol{\eta} = \boldsymbol{\eta}(\lambda) \ge 0$.

3.5 Intrinsic scales for synchronized N-component systems

Distributions of the differences $d_{jk}^{(N)} = x_j - x_k$ are important from practical and theoretical viewpoints because many reasonable synchronization error estimates are functions of $d_{jk}^{(N)}$. When we consider the symmetric N-component system for large N we may ask about a proper space scale which depends on N and corresponds to typical values of the synchronization errors. It appears that probabilistic properties of the free dynamics have an important impact on the typical scale of the synchronized system.

Stable random vectors

We need to remind some classical facts about stable distributions [67, 76].

Definition 2 *A random vector $U \in \mathbb{R}^d$ has a stable distribution if there exist an $\alpha \in (0,2]$ and a sequence $\{D_n\}$ of nonrandom vectors in $\mathbb{R}^d$ such that for any $n \in \mathbb{N}$*

$$U_1 + \cdots + U_n \overset{d}{=} n^{1/\alpha} U + D_n \tag{14}$$

where $U_1, \ldots, U_n$ are independent copies of U.

Definition 3 *The vector $U \in \mathbb{R}^d$ is called strictly stable if (14) holds with $D_n = 0$.*

Recall that the probability distribution of a random vector V is called symmetric if $V \overset{d}{=} -V$. A symmetric stable vector is strictly stable.

The stable laws are infinitely divisible [2, 68]. Hence the characteristic function $\boldsymbol{\psi}_U(\lambda) = \mathsf{E}\exp(i\langle\lambda, U\rangle)$ of a stable vector U has the form $\boldsymbol{\psi}_U(\lambda) = \exp\boldsymbol{\zeta}_U(\lambda)$. Therefore the distribution of the stable vector U is completely determined by the function $\boldsymbol{\zeta}_U(\lambda)$. We will denote the stable distribution defined in (14) by $\mathcal{S}(\alpha, \boldsymbol{\zeta}_U(\lambda))$ and write $U \sim \mathcal{S}(\alpha, \boldsymbol{\zeta}_U(\lambda))$. Note that the parameter α is also determined by $\boldsymbol{\zeta}_U(\lambda)$. The presence of α in $\mathcal{S}(\alpha, \boldsymbol{\zeta}_U(\lambda))$ is not necessary but it makes the notation more informative. The number α is called the index of stability. It is evident that $(-U) \sim \mathcal{S}\left(\alpha, \overline{\boldsymbol{\zeta}_U(\lambda)}\right)$ where $\overline{z}$ denotes the complex conjugation of z.

The general form of the function $\boldsymbol{\zeta}_U(\lambda)$ is known [67, 76] but we will not use it. We simply note that (14) can be rewritten as

$$\exp(n\boldsymbol{\zeta}_U(\lambda)) = \exp\left(i(\lambda, D_n) + \boldsymbol{\zeta}_U(n^{1/\alpha}\lambda)\right), \qquad \lambda \in \mathbb{R}^d. \tag{15}$$

In the case $\alpha = 2$ the stable laws are exaclty the d-dimensional Gaussian distributions.

Stable laws are the only possible limiting distributions of scalar-normalized sums of i.i.d. random vectors. The following definition is equivalent to Definition 2 (see [67]).

Definition 4 *A random vector $U \in \mathbb{R}^d$ is stable if it has a domain of attraction, i.e., if there is a random vector V and sequences of positive numbers $\{b_n\}$ and nonrandom vectors $\{C_n\}$, $C_n \in \mathbb{R}^d$, such that*

$$\frac{V_1 + \cdots + V_n}{b_n} + C_n \xrightarrow{d} U \tag{16}$$

where $V_1, \ldots V_n, \ldots$ are independent copies of V and the notation $\xrightarrow{d}$ denotes convergence in distribution.

In the situation of Definition 4 the random vector V is said to be in the *domain of attraction* of the stable vector U. Following the book [54] we will write $V \in \mathbf{DOA}(U)$. In the case when the normalizing sequence $\{b_n\}$ has the form $b_n = n^{1/\alpha}$ we say that V belongs to the *domain of normal attraction* of U and write $V \in \mathbf{DONA}(U)$. Sometimes we will put in these notation distributions instead of random vectors. Evidently, $\mathbf{DOA}(U) \supset \mathbf{DONA}(U) \ni U$.

The exhaustive study of domains of attraction for one-dimensional stable laws were presented in [15]. In dimensions $d \geq 2$ the first results about domains of attraction belong to Rvacheva [64], the disciple of B.V. Gnedenko.

We will need the next simple facts following directly from (16) and Definition 4.

Lemma 3 *Let V' be an independent copy of some random vector V. Let a random vector U be stable with the index α.*

(i) *If $V \in \mathbf{DOA}(U)$ then $V - V' \in \mathbf{DOA}(\underline{U})$ where $\underline{U} \sim \mathcal{S}(\alpha, 2\,\mathrm{Re}\,\boldsymbol{\zeta}_U(\lambda))$. Moreover, the normalyzing sequence $\{b_n\}$ in (16) is the same for V and $V - V'$.*

(ii) *The statement (i) remains true if we replace* **DOA** *by* **DONA**.

(iii) *Assume aditionally that V is infinitely divisible: $V \in \mathcal{ID}(\rho(\lambda))$. Then $V - V'$ is infinitely divisible too: $V - V' \in \mathcal{ID}(2\,\mathrm{Re}\,\rho(\lambda))$*

Infinite divisible laws in the domains of attraction

Let $\boldsymbol{y}(t) \in \mathbb{R}^d$, $t \geq 0$, be a Lévy process with the characteristic function of increments $\phi^{\boldsymbol{y}}(t;\lambda) = e^{t\boldsymbol{\rho}(\lambda)}$, $\lambda \in \mathbb{R}^d$, i.e., $\boldsymbol{y} \sim \mathcal{LP}(\boldsymbol{\rho}(\lambda))$ in notation of Subsection 2.3. Let $\mathcal{S}(\alpha, \boldsymbol{\rho}_{\mathrm{st}}(\lambda))$ be some stable distribution with the index of stability α, $0 < \alpha \leq 2$.

Definition 5 *We say that the Lévy process $\boldsymbol{y} = (\boldsymbol{y}(t),\, t \geq 0)$* ***belongs to the domain of attraction*** *of the stable law $\mathcal{S}(\alpha, \boldsymbol{\rho}_{st}(\lambda))$ if*

$$\boldsymbol{y}(1) \in \mathbf{DOA}(\mathcal{S}(\alpha, \boldsymbol{\rho}_{st}(\lambda))).$$

We say that $\boldsymbol{y} = (\boldsymbol{y}(t),\, t \geq 0)$ belongs to the domain of ***normal attraction*** *of the stable law $\mathcal{S}(\alpha, \boldsymbol{\rho}_{st}(\lambda))$ if $\boldsymbol{y}(1) \in \mathbf{DONA}(\mathcal{S}(\alpha, \boldsymbol{\rho}_{st}(\lambda)))$.*

Remark 2 *Recall that a Lévy process $\boldsymbol{y} = (\boldsymbol{y}(t),\, t \geq 0)$ is called* stable *if each $\boldsymbol{y}(t)$ is stable. In this case, evidently, the process $\boldsymbol{y}$ belongs to the domain of* normal *attraction of $\boldsymbol{y}(1)$.*

According to assumptions of Subsections 2.3 and 3.1 $x_j^\circ \sim \mathcal{LP}(\boldsymbol{\eta}^\circ(\lambda))$, i.e., the free dynamics of any component of $x(t)$ is the Lévy process with the common Lévy exponent $\boldsymbol{\eta}^\circ : \mathbb{R}^d \to \mathbb{C}$.

Assumption D. There exist a stable law $\mathcal{S}(\alpha, \boldsymbol{\zeta}^\circ(\lambda))$ in $\mathbb{R}^d$ such that any component $x_j^\circ(t)$ of the free dynamics $x^\circ(t) = (x_1^\circ(t), \dots, x_N^\circ(t))$ belongs to the *domain of attraction* of $\mathcal{S}(\alpha, \boldsymbol{\zeta}^\circ(\lambda))$.

According to Definition 4 under Assumption D there exist sequences $\{b_n\}$ and $\{C_n\}$ such that for all $\lambda \in \mathbb{R}^d$

$$\exp\left(n\boldsymbol{\eta}^\circ(\lambda/b_n) + i\left\langle C_n, \lambda\right\rangle\right) \to \exp \boldsymbol{\zeta}^\circ(\lambda) \quad \text{as} \quad n \to \infty\,. \tag{17}$$

Assumption DN. There exist a stable law $\mathcal{S}(\alpha, \boldsymbol{\zeta}^\circ(\lambda))$ in $\mathbb{R}^d$ such that any component $x_j^\circ(t)$ of the free dynamics $x^\circ(t) = (x_1^\circ(t), \dots, x_N^\circ(t))$ belongs to the domain of *normal attraction* of $\mathcal{S}(\alpha, \boldsymbol{\zeta}^\circ(\lambda))$.

Define a stochastic process $d_{jk}^{\circ,N}(t) = x_j^\circ(t) - x_k^\circ(t)$, $t \ge 0$. According to Lemma 3(iii) $d_{jk}^{\circ,N} \sim \mathcal{LP}\left(2\operatorname{Re}\boldsymbol{\eta}^\circ(\lambda)\right)$, i.e., all $d_{jk}^{\circ,N}(t)$ are Lévy processes in $\mathbb{R}^d$ with the common characteristic function

$$\left|\phi\left(t;\lambda\right)\right|^2 = e^{-t\boldsymbol{\eta}(\lambda)} \tag{18}$$

where

$$\boldsymbol{\eta}(\lambda) := -(\boldsymbol{\eta}^\circ(\lambda) + \boldsymbol{\eta}^\circ(-\lambda)) = -2\operatorname{Re}\boldsymbol{\eta}^\circ(\lambda).$$

The function (18) is real and, moreover, $\boldsymbol{\eta}(\lambda) \ge 0$ for all $\lambda \in \mathbb{R}^d$ since $e^{-t\boldsymbol{\eta}(\lambda)}$ is a characteristic function of some probability distribution. Hence distributions of the increments of $\left(d_{jk}^{\circ,N}(t),\, t \ge 0\right)$ are symmetric.

It follows from Lemma 3(i) that if Assumption D holds then the process $d_{jk}^{\circ,N}(t)$ belongs to the *domain of attraction* of the *symmetric* stable law $\mathcal{S}(\alpha, -\boldsymbol{\zeta}(\lambda))$ with the characteristic function $e^{-\boldsymbol{\zeta}(\lambda)}$ where

$$\boldsymbol{\zeta}(\lambda) := -(\boldsymbol{\zeta}^\circ(\lambda) + \boldsymbol{\zeta}^\circ(-\lambda)) = -2\operatorname{Re}\boldsymbol{\zeta}^\circ(\lambda) \tag{19}$$

and $\boldsymbol{\zeta}^\circ$ is the same as in Assumption D. It is evident that $\boldsymbol{\zeta}(\lambda) \ge 0$ for all $\lambda \in \mathbb{R}^d$. According to (17) the conclusion that $d_{jk}^{\circ,N}(1) \in \mathbf{DOA}\left(\mathcal{S}(\alpha, -\boldsymbol{\zeta}(\lambda))\right)$ implies that

$$\exp\left(-n\boldsymbol{\eta}(\lambda/b_n)\right) \to \exp(-\boldsymbol{\zeta}(\lambda)) \qquad (n \to \infty)$$

for the same sequence $\{b_n\}$ as in (17). Applying the logarithmic function to the above convergence we get that for any $\lambda \in \mathbb{R}^d$

$$n\boldsymbol{\eta}(\lambda/b_n) \to \boldsymbol{\zeta}(\lambda) \qquad (n \to \infty)\,. \tag{20}$$

Similarly, using the item (ii) of Lemma 3 we get that under Assumption DN the process $d_{jk}^{\circ,N}(t)$ belongs to the domain of *normal attraction* of the same stable law $\mathcal{S}(\alpha, -\boldsymbol{\zeta}(\lambda))$. Of course, under Assumption DN the condition (20) takes the following form

$$n\boldsymbol{\eta}(\lambda/n^{1/\alpha}) \to \boldsymbol{\zeta}(\lambda) \qquad (n \to \infty)\,.$$

The space scaling

Consider the model $\mathcal{GGS}_N(\boldsymbol{\eta}^\circ; F)$, i.e., the N-component synchronization system $x(t) = (x_1(t), \dots, x_N(t))$ which satisfies the symmetry assumptions of Subsection 3.1.

Theorem 3 *Let Assumption D hold with some $\boldsymbol{\zeta}^{\circ}(\lambda)$. Let $\{b_n\}$ be the normalizing sequence in (17). Rescale the system $x(t) = (x_1(t), \dots, x_N(t))$ as follows*

$$y^{(N)}(t) = \frac{x(t)}{b_N}, \qquad y^{(N)}(t) = \left(y_1^{(N)}(t), \dots, y_N^{(N)}(t)\right).$$

Let $Q_{N,t}$ be the probability law of the rescaled differences $y_j^{(N)}(t) - y_k^{(N)}(t)$. Then for any fixed $N \geq 2$ the weak limit of $Q_{N,t}$ exists,

$$Q_{N,t} \xrightarrow{w} Q_{N,\infty} \quad as \quad t \to +\infty,$$

and the characteristic function of the limiting distribution has asymptotically explicit form as $N \to \infty$

$$\int_{\mathbb{R}^d} \exp\left(i \langle \lambda, y \rangle\right) Q_{N,\infty}(dy) \;\to\; \frac{1}{1 + \frac{1}{2} m \boldsymbol{\zeta}(\lambda)}. \tag{21}$$

Here the real function $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\lambda)$, $\lambda \in \mathbb{R}^d$, is the same as in (19) and m is defined by (10).

We have an immediate corollary of this theorem under the stronger condition that the synchronized system $x(t)$ satisfied to Assumption DN with respect to some stable law $\mathcal{S}\left(\alpha, \boldsymbol{\zeta}^{\circ}(\lambda)\right)$. In this case $b_n = n^{1/\alpha}$ and the statement of Theorem 3 is true for the rescaled synchronization system

$$y^{(N)}(t) = \frac{x(t)}{N^{1/\alpha}}, \qquad y^{(N)}(t) = \left(y_1^{(N)}(t), \dots, y_N^{(N)}(t)\right).$$

This result can be interpreted as follows: distances between components in the synchronized system are of order $N^{1/\alpha}$ provided the free dynamics belongs to the domain of normal attraction of an α-stable law in the sense of [15].

Note also that the Lévy continuity theorem and (21) imply the weak convergence of $Q_{N,\infty}$ to some probability law $Q_{\infty,\infty}$ in $\mathbb{R}^d$ having the characteristic function $\left(1 + \frac{1}{2} m \boldsymbol{\zeta}(\lambda)\right)^{-1}$.

3.6 Free dynamics attracting to stable laws. Linnik distributions

Symmetric stable laws

It is very useful to illustrate the result of Theorem 3 by different concrete examples of free dynamics. Before doing this we need to recall some classical results about representation of stable laws. It is known [2, 68] that the characteristic function of a d-dimensional *symmetric* α-stable law has the following form

- for $0 < \alpha < 2$ (the *heavy tail* case):

$$e^{-t\boldsymbol{\zeta}(\lambda)} = \exp\left(-t \int_{S^{d-1}} \left|\langle \lambda, \xi \rangle\right|^{\alpha} \nu(d\xi)\right) \tag{22}$$

  where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and ν is some finite measure on S^{d-1},

- for $\alpha = 2$ (the *Gaussian* case):

$$e^{-t\boldsymbol{\zeta}(\lambda)} = \exp\left(-t \langle A\lambda, \lambda \rangle / 2\right) \tag{23}$$

  where A is a positive definite symmetric $d \times d$ matrix.

Corresponding formula for rotationally invariant α-stable laws, $0 < \alpha \leq 2$, is simpler:

$$e^{-t\boldsymbol{\zeta}(\lambda)} = \exp\left(-tc^\alpha\, |\lambda|^\alpha\,\right), \qquad c > 0, \quad \lambda \in \mathbb{R}^d, \quad |\lambda| = \sqrt{\langle \lambda, \lambda \rangle}\,.$$

It is clear from Theorem 3 that any of functions

$$\int_{S^{d-1}} |\langle \lambda, \xi \rangle|^\alpha\, \nu(d\xi), \quad \langle A\lambda, \lambda \rangle /2, \quad c^\alpha\, |\lambda|^\alpha \tag{24}$$

can participate as $\boldsymbol{\zeta}(\lambda)$ in the limit (21). Indeed, to see this one should consider the free dynamics $x^\circ(t)$ driven by symmetric stable Lévy processes with the Lévy exponent $\boldsymbol{\eta}^\circ(\lambda) = -\frac{1}{2}\boldsymbol{\zeta}(\lambda)$ where $\boldsymbol{\zeta}(\lambda)$ is taken from the list (24).

Remark 3 *Note that the limiting characteristic function in (21) has the form*

$$\frac{1}{1 - \log \phi(\lambda)}\,, \qquad \lambda \in \mathbb{R}^d,$$

where $\phi(\lambda)$ is a characteristic function of some symmetric α-stable distribution. As it follows from [57, Prop. 1] the class of limiting laws in (21) are exactly the symmetric ***geometric stable distributions*** *(GSDs). The GSDs are obtained as limiting laws of appropriately normalized* random *sums of i.i.d. random vectors in $\mathbb{R}^d$ where the* number *of summands is* geometrically distributed *and independent of the summands. There is a large bibliography devoted to this topic, see, for example, [17, 23, 25, 28, 30, 31, 57].*

Free dynamics of the Gaussian type. The Laplace distribution

Let $x_j^\circ(t)$, $j = \overline{1, N}$, be the same as in example (4),

$$d\, x_j^\circ(t) = \sigma dB_j(t) + b\, dt,$$

where σ is a real $d \times d$ matrix, $b \in \mathbb{R}^d$ and $B_j(t) = (B_j^1(t), \ldots, B_j^d(t)) \in \mathbb{R}^d$ are independent standard d-dimensional *Brownian motions* . We know from (4) that any $x_j^\circ(t)$ is a Lévy process $\mathcal{LP}(\boldsymbol{\eta}^\circ)$ determined by the Lévy exponent $\boldsymbol{\eta}^\circ(\lambda) = i\,\langle b, \lambda \rangle - \frac{1}{2}\left\langle \sigma\sigma^T\lambda, \lambda \right\rangle$. Using (15) it is easy to check that $\mathcal{LP}(\boldsymbol{\eta}^\circ)$ is stable with $\alpha = 2$. Hence

$$\boldsymbol{\zeta}(\lambda) = \boldsymbol{\eta}(\lambda) = -2\,\mathrm{Re}\,\boldsymbol{\eta}^\circ(\lambda) = \left\langle \sigma\sigma^T\lambda, \lambda \right\rangle.$$

By Theorem 3 the proper scaling for differences $d_{jk}^{(N)}(t) = x_j(t) - x_k(t)$, is $N^{-1/2}$. Namely, $d_{jk}^{(N)}(t)/\sqrt{N}$ weakly converges to some law $Q_{N,\infty}$ as $t \to \infty$. Letting $N \to \infty$ we get from (21) that $Q_{N,\infty}$ weakly converges to the distribution with characteristic function

$$\frac{1}{1 + \frac{1}{2}m\,\langle \sigma\sigma^T\lambda, \lambda \rangle}\,, \qquad \lambda \in \mathbb{R}^d\,.$$

In the case $d = 1$ this characteristic function takes the form $(1 + \frac{1}{2}m\sigma^2\lambda^2)^{-1}$ and corresponds to the Laplace distribution with density

$$p_L(y) = \frac{1}{2c_0}\, e^{-|y|/c_0}, \qquad -\infty < y < +\infty, \quad c_0 = \sigma\sqrt{\frac{m}{2}}\,. \tag{25}$$

This result generalizes the result obtained in [52] and cited in Introduction of the current paper. Indeed, putting $d = 1$ and $b = 0$ we have equivalence of the following models

$$\mathcal{GMS}_N(-\frac{1}{2}\sigma^2\lambda^2; m) = \mathcal{BM}_N(\sigma, N/m).$$

Using self-similarity of the Wiener process one can derive that the intrinsic scale of $\mathcal{BM}_N(\sigma, N/m)$ is $N^{1/2}$ times smaller than the intrinsic scale of the model $\mathcal{BM}_N(\sigma, m^{-1})$ studied in [52].

One-dimensional random walks. Linnik distribution

Let $d = 1$ and the free dynamics of each component $x_j^\circ(t)$ be a continuous time symmetric *random walk* with the Markov generator

$$(Lf)(y) = \beta \int_{\mathbb{R}} (f(y+q) - f(y))\, \mu(dq), \qquad f \in C_b(\mathbb{R}, \mathbb{R}). \tag{26}$$

Here $\beta > 0$ is the intensity of jumps and $\mu(dq) = \frac{1}{2}\boldsymbol{a}\, |q|^{-1-\boldsymbol{a}}\, 1_{\{|q|\geq 1\}} dq$ is the distribution of an individual jump $x \mapsto x + q$. This is a one-dimensional subcase of the example (6). Please, note that the sequence $\{T_n\}$ is considered here under general assumptions of Subsections 2.2 and 3.3. Hence the synchronization system $x(t)$ is not Markovian while the free dynamics $x^\circ(t)$ is a Markov process.

The jump distribution $\mu(dq)$ has the "Pareto tails" and, as it will be seen below, the conditions of Theorem 3 can be easily checked. Let ξ be a random variable with distribution $\mu(dq)$. If $\boldsymbol{a} > 2$ then ξ has a finite variation $\boldsymbol{D}_0 = \mathrm{Var}\,(\xi) = a/(a-2)$. It follows from [15, § 35, Th. 4] that $\xi \in \mathbf{DONA}\,(\mathcal{N}(0, \boldsymbol{D}_0))$ where $\mathcal{N}(0, \boldsymbol{D}_0)$ is the Gaussian law with zero mean and variance $\boldsymbol{D}_0$. It follows from (26) that $x_j^\circ(t)$ is a compound Poisson process, i.e.,

$$x_j^\circ(t) \sim \sum_{r=1}^{N_\beta(t)} \xi_r$$

where $(N_\beta(t),\, t \geq 0)$ is the Poisson process with intencity β and $\xi_1, \ldots, \xi_r, \ldots$ are independent copies of ξ. It is well known that $N_\beta(t) \sim \beta t$ as $t \to \infty$. Arguments similar to [54, § 4.4] show that $x_j^\circ(1) \in \mathbf{DONA}\,(\mathcal{N}(0, \boldsymbol{D}))$ where $\boldsymbol{D} = \beta a/(a-2) > 0$. Hence Assumption DN holds with $\boldsymbol{\zeta}^\circ(\lambda) = -\frac{1}{2}\boldsymbol{D}\lambda^2$, $\alpha = 2$, and we can apply Theorem 3. It is readily seen that $\boldsymbol{\zeta}(\lambda) = \boldsymbol{D}\lambda^2$. The distribution of rescaled differences $d_{jk}^{(N)}(t)/\sqrt{N}$ converges to some law $Q_{N,\infty}$ as $t \to \infty$. The sequence of laws $Q_{N,\infty}$ converges as $N \to \infty$ to the Laplace distribution (25) where σ is replaced by $\sqrt{D}$.

If $0 < \boldsymbol{a} < 2$ (the *heavy tail* case) then by [15, § 35, Th. 5] the random variable ξ with the distribution $\mu(dq)$ belongs to the domain of *normal* attraction of a symmetric $\boldsymbol{a}$-stable law. As in the above paragraph we conclude that $x_j^\circ(1)$ also belongs to the domain of normal attraction of some symmetric $\boldsymbol{a}$-stable law. So Assumption DN holds. Again Theorem 3 implies that rescaled differences $d_{jk}^{(N)}(t)/N^{1/\boldsymbol{a}}$ converge in law as $t \to \infty$ to some distribution $Q_{N,\infty}$. The sequence $Q_{N,\infty}$ converges as $N \to \infty$ to a symmetric law which characteristic function is

$$\frac{1}{1 + c^{\boldsymbol{a}}\, |\lambda|^{\boldsymbol{a}}}, \qquad \lambda \in \mathbb{R}, \tag{27}$$

for some $c = c(\boldsymbol{a}, \beta, m) > 0$. This is characteristic function of the famous symmetric Linnik distribution [36] usually denoted as $\mathcal{L}_{\boldsymbol{a},c}$. It is known [30,37] that this distribution is unimodal, absolutely continuous, geometric stable (see Remark 3) and infinitely divisible. If $0 < \boldsymbol{a} < 2$ then the Linnik distribution has heavy tails [22,30]: $q^{\boldsymbol{a}} \mathsf{P}\,(\mathcal{L}_{\boldsymbol{a},c} > q) \sim \mathrm{const}$ as $q \to +\infty$. For $\boldsymbol{a} = 2$ the law (27) is the Laplace distribution.

It is not hard to modify this example to obtain domains of non-normal attraction. Let $0 < \boldsymbol{a} < 2$. Using notation of [27, Th. 4] we introduce a probability measure $\mu(dq)$ on $\mathbb{R}^1$ such that

$$\begin{aligned} \mu\,((-\infty, -q]\,) &= q^{-\boldsymbol{a}}\,(c_1 + h_1(q))\; L(q), \\ \mu\,([q, +\infty)\,) &= q^{-\boldsymbol{a}}\,(c_2 + h_2(q))\; L(q), \end{aligned}$$

where $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$, $L(q)$ is a slowly varying function and $h_i(q) \to 0$, $i = 1, 2$, as $q \to \infty$. Let each component $x_j^\circ(t)$ be a continuous time *random walk* with the Markov generator (26). Again denote by ξ a random variable with distribution $\mu(dq)$. The classical results [15] states that there exists a stable law $U_{\boldsymbol{a},c_1,c_2}$ in $\mathbb{R}^1$ with index of stability $\boldsymbol{a}$ such that $\xi \in \mathbf{DOA}(U_{\boldsymbol{a},c_1,c_2})$. This statement implies that $x_j^\circ(1)$ belongs to the $\mathbf{DOA}(U_{\boldsymbol{a},\beta c_1,\beta c_2})$. We don't need an explicit definition of $U_{\boldsymbol{a},c_1,c_2}$ here. The law $U_{\boldsymbol{a},c_1,c_2}$ is symmetric iff $c_1 = c_2$. It is important to note [27, 67] that the choice of normalizing sequence $\{b_N\}$ arising in Theorem 3 depends on the function $L(q)$. If there exists a limit

$$L(q) \to L(\infty), \qquad (q \to \infty), \qquad 0 < L(\infty) < \infty,$$

then Assumption DN holds and $b_N = N^{1/\alpha}$. In the other case $x_j^\circ(1) \in \mathbf{DOA}\backslash\mathbf{DONA}$. In general, $b_N = N^{1/\alpha}\ell(N)$ where $\ell = \ell(N)$ is a slowly varying function at infinity. See [27, 67] for details.

In any case the limiting characteristic function in (21) is from the class of Linnik distributions (27).

Multi-dimensional random walks with heavy-tailed jumps

We consider a special subclass of *random walks* $x_j^\circ(t)$ in $\mathbb{R}^d$, $d \geq 2$, introduced in (5). According to the symmetry assumption we put

$$\beta_j = \beta, \qquad \mu_j(dq) = \mu(dq) \qquad \forall j = \overline{1, N}.$$

As in previous examples we restrict ourself to consideration of power law jumps. Let $\xi \in \mathbb{R}^d$ denotes a random vector with distribution $\mu(dq)$, $q \in \mathbb{R}^d$: $\mathsf{P}\,(\xi \in G) = \mu(G)$ for any Borel set $G \in \mathcal{B}\left(\mathbb{R}^d\right)$. Following [54, § 6.4] we represent it as $\xi = W\Theta$ where W is a scalar random variable and $\Theta \in S^{d-1}$ is a random vector taking values on the unit sphere in $\mathbb{R}^d$. Assume that W and Θ are independent, and

$$\mathsf{P}\,(W > R) = CR^{-\alpha}, \qquad R \geq R_0 > 0, \qquad \mathsf{P}\,(W \geq 0) = 1,$$
$$\mathsf{P}\,(\Theta \in B) = M(B), \qquad B \in \mathcal{B}(S^{d-1}),$$

where $C > 0$ is some constant, $C \leq R_0^\alpha$, and $M(\cdot)$ is a probability measure on S^{d-1}. Consider only the *heavy tail* case $\alpha \in (0, 2)\backslash\{1\}$ excluding $\alpha = 1$ for breavity of formulae. By Theorem 6.17 in [54] $\xi \in \mathbf{DONA}(\mathcal{S}\,(\alpha, \rho_\alpha^\circ(\lambda)))$ where

$$\rho_\alpha^\circ(\lambda) = -C\,K_\alpha \int_{S^{d-1}} |\langle \lambda, \theta \rangle|^\alpha \left(1 - i\,\mathrm{sgn}\,\langle \lambda, \theta \rangle \tan\frac{\pi\alpha}{2}\right) M(d\theta), \tag{28}$$

for some $K_\alpha > 0$. The same arguments as for the one-dimensional random walks imply that $x_j^\circ(1) \in \mathbf{DONA}\left(\mathcal{S}\left(\alpha, \beta\rho_\alpha^\circ(\lambda)\right)\right)$. Hence Assumption DN holds with $\boldsymbol{\zeta}^\circ(\lambda) = \beta\rho_\alpha^\circ(\lambda)$. After calculating

$$\boldsymbol{\zeta}(\lambda) = -2\,\mathrm{Re}\,\boldsymbol{\zeta}^\circ(\lambda) = 2C\,K_\alpha \int_{S^{d-1}} |\langle \lambda, \theta \rangle|^\alpha\; M(d\theta)$$

(compare with (22)) we are ready to apply Theorem 3. We conclude that the rescaled differences $d_{jk}^{(N)}(t)/N^{1/\alpha}$ converge in law as $t \to \infty$ to some distribution $Q_{N,\infty}$. If $N \to \infty$ then $Q_{N,\infty}$ is approximated by the distribution with characteristic function

$$\frac{1}{1 + \frac{1}{2}m \int_{S^{d-1}} |\langle \lambda, \theta \rangle|^\alpha\; \nu_\alpha(d\theta)}$$

where $\nu_\alpha(d\theta) = 2C\,K_\alpha M(d\theta)$. In the case $\alpha = 1$ we have essentially the same final conclusion but the intermediate formula (28) is different.

If $\alpha \geq 2$ them the corresponding analysis is based on the multi-dimensional Central Limit Theorem. Here $b_N = N^{1/2}$ and Assumption DN holds for d-dimensional Gaussian law. We omit details.

3.7 Intrinsic scales based on matrix transformations. Jurek coordinates

Theorem 3 justifies the existence of a natural space scale for a large N-components synchronization system. This scale is uniform in any of d coordinate axes in $\mathbb{R}^d$ because the scaling transformation $y = b_N^{-1}x$ is the multiplication by a scalar value b_N^{-1}.

It is also interesting to find conditions when large synchronized systems "are concentrated" in space domains which change non-uniformly in different coordinate directions as $N \to \infty$. Recalling that § 3.5 is related with attraction to stable laws in $\mathbb{R}^d$ it is clear that one can look for generalizations of Theorem 3 by considering the domains of attraction of *operator stable laws* (OSLs) .

Definition 6 *A random vector $U \in \mathbb{R}^d$ is operator stable if it has a **generalized** domain of attraction, i.e., if there is a random vector V and sequences $\{B_n\}$ of linear operators $B_n : \mathbb{R}^d \to \mathbb{R}^d$ and nonrandom vectors $\{C_n\}$, $C_n \in \mathbb{R}^d$, such that*

$$B_n\left(V_1 + \cdots + V_n\right) + C_n \xrightarrow{d} U \tag{29}$$

where $V_1, \ldots V_n, \ldots$ are independent copies of V. The random vector V is said to be in the generalized domain of attraction of the stable vector U, the short notation for it is $V \in \mathbf{GDOA}(U)$.

We see that operator stable laws arise as limiting distributions of matrix-normalized sums of i.i.d. random vectors. The study of OSLs was originated by G.N. Sakovich, the disciple of B.V. Gnedenko, and M. Sharpe. Here we cannot go too deeply in details of this vast theory and refer to [24, 54, 65, 66]. Below we list a limited number of facts on OSLs which are necessary to state our result. We will consider only *full* OSLs . The probability distribution of a random vector U on $\mathbb{R}^d$ is full if $\langle\lambda, U\rangle$ is nondegenerate for every $\lambda \in \mathbb{R}^d \setminus \{0\}$.

The simplest examples of OSLs are laws U in $\mathbb{R}^d$ with marginal stable distributions which possess a stability property similar to (14),

$$U_1 + \cdots + U_n \stackrel{d}{=} n^E U + D_n, \tag{30}$$

where E is a diagonal matrix $E = \mathrm{diag}\left(\alpha_1^{-1}, \ldots, \alpha_d^{-1}\right)$, $\alpha_i \in (0, 2]$. In this case, evidently, $B_n = n^{-E}$. To have OSLs with dependent coordinated one should replace in (30) the diagonal matrix n^E by the multiplier n^B where B is a real $d \times d$-matrix with whose eigenvalues all have real part in $[\frac{1}{2}, +\infty)$, see [54]. The matrix n^B is defined by using the matrix exponent as $n^B = \exp(B \log n)$. Any full operator stable U is infinitely divisible [24, § 4.2], hence its characteristic function has the form

$$\boldsymbol{\psi}_U(\lambda) = \mathsf{E}\,\exp\left(i\,\langle\lambda, U\rangle\right) = \exp \boldsymbol{\zeta}_U(\lambda).$$

We will write $U \sim \mathcal{OS}(\boldsymbol{\zeta}_U(\lambda))$ to have a short notation for this situation.

We are ready to state a ***result* generalizing Theorem 3**. Consider the symmetric synchronization model $x(t) \in \mathbb{R}^{dN}$ with the free dynamics $x_j^\circ \sim \mathcal{LP}(\boldsymbol{\eta}^\circ(\lambda))$, $j = \overline{1, N}$. *Assume* that there exist an operator stable law $\mathcal{OS}(\boldsymbol{\zeta}^\circ(\lambda))$ in $\mathbb{R}^d$ such that any component $x_j^\circ(1) \in \mathbf{GDOA}\left(\mathcal{OS}(\boldsymbol{\zeta}^\circ(\lambda))\right)$. According to Definition 6 this assumption means that there exist sequences $\{B_n\}$ and $\{C_n\}$ such that for all $\lambda \in \mathbb{R}^d$

$$\exp\left(n\boldsymbol{\eta}^\circ(B_n^T\lambda) + i\left\langle C_n, \lambda\right\rangle\right) \to \exp\boldsymbol{\zeta}^\circ(\lambda) \quad \text{as} \quad n \to \infty\,. \tag{31}$$

Define a transformed system

$$y^{(N)}(t) = \left(B_N x_1(t)\,, \ldots,\, B_N x_N(t)\right). \tag{32}$$

Then the differences $d_{jk}^{(N)}(t) = y_j^{(N)}(t) - y_k^{(N)}(t) = B_N\left(x_j(t) - x_k(t)\right)$ converge in law as $t \to \infty$ to some distribution $Q_{N,\infty}$. If $N \to \infty$ then $Q_{N,\infty}$ is approximated by some distribution with characteristic function given in the explicit form:

$$\int_{\mathbb{R}^d} \exp\left(i\left\langle\lambda, y\right\rangle\right)\, Q_{N,\infty}(dy) \;\to\; \frac{1}{1 + \frac{1}{2}m\boldsymbol{\zeta}(\lambda)}\,, \qquad \lambda \in \mathbb{R}^d, \tag{33}$$

where $\boldsymbol{\zeta}(\lambda) = -2\,\mathrm{Re}\,\boldsymbol{\zeta}^\circ(\lambda)$.

The proof of this generalization is very similar to the proof of Theorem 3 and is based on the representation for $\chi_N(\infty;\lambda)$ of Theorem 2. So we omit it.

We end this subsection by two remarks. The characterization of GDOA in the operator stable case and the description of all possible functions $\boldsymbol{\zeta}(\lambda)$ in (33) are not easy. They demand many additional constructions and are out of scope of this paper. We refer interested readers to [24, 54].

In the case when $B_n = n^{-B}$ the transformation (32) is deeply connected with so called *Jurek coordinates.* The Jurek coordinates in $\mathbb{R}^d$ is a pair (r, Θ) such that $y = r^B\Theta$, where $y \in \mathbb{R}^d$, $r \geq 0$ and $\Theta \in S^{d-1}$. Details can be found in [24, 54].

3.8 The Markovian case

Here we consider $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$, the symmetric N-component synchronization model in any dimension d with the special choice of inter-event distribution:

$$F(s) \;=\; \mathsf{P}\left\{\Delta_n^{(k)} \leq s\right\} = 1 - \exp\left(-s/m\right), \qquad s \geq 0, \quad m > 0. \tag{34}$$

This is the exponential distribution with the mean m. In this case the sequence $\{T_n\}$ is the Poissonian flow of intensity N/m and $x(t)$ is a Markov process.

In the Markovian case it is possible to precise main results of Subsections 3.4 and 3.5. Theorem 2 is replaced by the following one.

Theorem 4 *For the Markovian symmetric synchronization model* $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$

$$\chi_N(+\infty;\lambda) = \frac{1}{1 + \frac{1}{2}(N-1)m\boldsymbol{\eta}(\lambda)}$$

where the function $\boldsymbol{\eta}(\lambda)$ *is the same as in Theorem 2.*

The proof of this theorem is given at the end of Subsection 4.3.

The next theorem holds for finite N. It immediately follows from Theorem 4.

Theorem 5 *Let a Markovian N-component symmetric synchronization model $\mathcal{GMS}_N(\boldsymbol{\eta}^\circ; m)$ be such that its free dynamics $x^\circ(t)$ is an α-stable Lévy process, $0 < \alpha \le 2$. Then for any fixed N the distribution of rescaled differences $d_{jk}^{(N)}(\infty)/\,(N-1)^{1/\alpha}$ does not depend on N.*

The Markov assumption is essential for Theorem 5. For the non-Markov case the statement (21) of Theorem 3 is asymptotic and does not hold for finite N.

Theorem 5 generalizes results of the paper [52] where the role of the α-stable free dynamics was played by Brownian motions ($\alpha = 2$).

For the Markovian symmetric model the function $\chi_N(t;\lambda)$ satisfies to the following differential equation

$$\frac{d}{dt}\,\chi_N(t;\lambda) = -q_N(\lambda)\,\chi_N(t;\lambda) + w_N, \tag{35}$$

where

$$w_N = \frac{2}{(N-1)m}, \qquad q_N = \boldsymbol{\eta}(\lambda) + w_N.$$

This equation directly follows from the representation for $\chi_N(t;\lambda)$ which will be obtained in Subsection 4.3. In particular, the statement of Theorem 4 easily follows from this equation.

It is important to note that for non-Markovian models the function $\chi_N(t;\lambda)$ don't satisfy to any differential equation of such type.

3.9 Some generalizations

According to Subsection 3.1 and Assumption P1 the general (non-Markovian) symmetric synchronization model $x(t)$ is determining by the quadruple $\big(N,\, \boldsymbol{\eta}^\circ(\lambda),\, p(s), \mathcal{P}_{x(0)}\big)$. Here we briefly discuss a possibility to extend our asymptotic results to the case

$$\big(N,\, \boldsymbol{\eta}_N^\circ(\lambda),\, p_N(s), \mathcal{P}_{x(0)}\big)$$

when $\boldsymbol{\eta}^\circ(\lambda)$ and $p(s)$, the functions defining the dynamics, depend on N. The main task is *to generalize* Theorem 2. Note that this problem is interesting only for *non-Markovian models*. Indeed, in the Markovian case Theorem 4 already gives the exact and explicit answer to the question.

We will restrict ourself to the special situation when

$$p_N(s) = \beta_N p(\beta_N s) \tag{36}$$

for some sequence $\{\beta_N\}$, $\beta_N > 0$. This situation corresponds to the rescaling of the time t and is quite simple. Obviously,

$$m_{N,1} = m/\beta_N \tag{37}$$

where

$$m_{N,1} = \int_0^\infty s p_N(s)\, ds, \qquad m_1 = \int_0^\infty s p(s)\, ds\,.$$

The main idea is to compare models with different quadruples. Indeed, in distributional sense

$$x(t)\,\Big|_{\left(N,\boldsymbol{\eta}_N^\circ(\lambda),\,p_N(s),\mathcal{P}_{x(0)}\right)} = x(\beta_N t)\,\Big|_{\left(N,\boldsymbol{\eta}_N^\circ(\lambda)/\beta_N,\,p(s),\mathcal{P}_{x(0)}\right)}\,.$$

Hence

$$\chi_N(t;\lambda)\,\Big|_{\left(N,\boldsymbol{\eta}_N^\circ(\lambda),\,p_N(s),\mathcal{P}_{x(0)}\right)} = \chi_N(\beta_N t;\lambda)\,\Big|_{\left(N,\boldsymbol{\eta}_N^\circ(\lambda)/\beta_N,\,p(s),\mathcal{P}_{x(0)}\right)}\,.$$

Let the probability density function $p(s)$ satisfies to Assumptions P1–P3. Then by Lemma 2

$$\chi_N(+\infty;\lambda)\,\Big|_{\left(N,\,\boldsymbol{\eta}_N^\circ(\lambda),\,p_N(s),\mathcal{P}_{x(0)}\right)} = \chi_N(+\infty;\lambda)\,\Big|_{\left(N,\,\boldsymbol{\eta}_N^\circ(\lambda)/\beta_N,\,p(s),\mathcal{P}_{x(0)}\right)}.$$

Note that these limiting characteristic functions do not depend on the initial distribution $\mathcal{P}_{x(0)}$ so we can omit it in the notation. From Theorem 2 and remark (13) we get the following representation

$$\chi_N(+\infty;\lambda)\,\Big|_{\left(N,\,\boldsymbol{\eta}_N^\circ(\lambda),\,p_N(s)\right)} = \frac{1}{1+\theta_{1,N}\boldsymbol{\eta}_N(\lambda)/\beta_N} + \rho_{2,N}(\lambda).$$

Here $\boldsymbol{\eta}_N(\lambda) = -2\operatorname{Re}\boldsymbol{\eta}_N^\circ(\lambda)$, the real sequence $\{\theta_{1,N}\}$ is such that $\theta_{1,N}\sim\frac{1}{2}mN$ as $N\to\infty$ and the sequence of functions $\{\rho_{2,N}(\lambda)\}$ vanishes uniformly in λ. Taking into account (37) we can rewrite this representation as follows

$$\chi_N(+\infty;\lambda)\,\Big|_{\left(N,\,\boldsymbol{\eta}_N^\circ(\lambda),\,p_N(s)\right)} = \frac{1}{1+\rho_{1,N}\boldsymbol{\eta}_N(\lambda)} + \rho_{2,N}(\lambda) \tag{38}$$

where the real sequence $\{\rho_{1,N}\}$ is such that $\rho_{1,N}\sim\frac{1}{2}m_{N,1}N$ as $N\to\infty$. Using this result one can study intrinsic scales of the corresponding synchronization models with large number of components similarly to Theorem 3.

It would be interesting to know if the decomposition (38) holds for other sequences $\{p_N(s)\}$ different from (36).

4 Proofs

4.1 Lemmas of dynamics

As in paper [52] we start from introducing useful functions. Fix some even function $g=g(a)$ on $\mathbb{R}^d$:

$$g:\ \mathbb{R}^d\to\mathbb{C},\qquad g(a)=g(-a).$$

Consider also $g_0(a)=g(a)-g(0)$. Now define the following functions on the configuration space $\mathbb{R}^{Nd}$

$$V(x) := \frac{2}{(N-1)N}\sum_{j_1<j_2} g\left(x_{j_1}-x_{j_2}\right),\qquad V_0(x) := \frac{2}{(N-1)N}\sum_{j_1<j_2} g_0\left(x_{j_1}-x_{j_2}\right)$$

where $x=(x_1,\ldots,x_N)$, $x_j\in\mathbb{R}^d$. Evidently, $V(x)=V_0(x)+g(0)$. Note that

$$x_1=\cdots=x_N\quad\Rightarrow\quad V_0(x)=0.$$

Keeping in mind notation of Subsections 2.1 and 2.2 we introduce a map $S_{(i,j)}$: $\mathbb{R}^N\to\mathbb{R}^N$, as follows $S_{(i,j)}x := x\circ S_{i,j}$. In other words,

$$S_{(i,j)}:\quad (x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N)\mapsto(x_1,\ldots,x_i,\ldots,\underset{j}{x_i},\ldots,x_N). \tag{39}$$

Define a map-valued random variable $\mathcal{S}$ such that

$$\mathsf{P}\left\{\mathcal{S}=S_{(i,j)}\right\} = \frac{1}{(N-1)N},\qquad i\neq j. \tag{40}$$

Lemma 4 *There exists $\varkappa > 0$ such that for any $x \in \mathbb{R}^N$*

$$\mathsf{E}\, V_0(\mathcal{S}x) = k_N V_0(x),$$

where $k_N = 1 - \varkappa / ((N-1)N)$.

Lemma 4 was proved in [46] for much more general synchronization jumps. For the pairwise synchronization interaction considered in the current paper in the framework of the symmetric model the value of $\varkappa$ is known: $\varkappa = 2$.

From this point we take the following concrete even function $g(y) := \cos\langle y, \lambda\rangle$. Its dependence on the variable $\lambda \in \mathbb{R}^d$ will usually be omitted. Consider the function

$$V(x) := \frac{2}{(N-1)N} \sum_{j_1<j_2} \cos\langle \lambda, x_{j_1} - x_{j_2}\rangle \tag{41}$$

corresponding to this choice of g. It follows form Lemma 4 that

$$\mathsf{E}\, V(\mathcal{S}x) = k_N V(x) + l_N, \tag{42}$$

where

$$k_N = 1 - \frac{\varkappa}{(N-1)N}, \qquad l_N := 1 - k_N = \frac{\varkappa}{(N-1)N}. \tag{43}$$

Lemma 5 *For $s > 0$, $x \in \mathbb{R}^{Nd}$*

$$\mathsf{E}\, V(x + x^\circ(s)) = V(x)\, e^{-s\boldsymbol{\eta}(\lambda)} \tag{44}$$

where $\boldsymbol{\eta}(\lambda) = -2\,\mathrm{Re}\,\boldsymbol{\eta}^\circ(\lambda)$ and V is defined in (41).

Proof of Lemma 5.

$$\cos\langle\lambda, y\rangle = \frac{\exp(i\langle\lambda, y\rangle) + \exp(-i\langle\lambda, y\rangle)}{2},$$

$$\begin{aligned}
\mathsf{E}\exp\left(i\left\langle\lambda, x_{j_1} + x^\circ_{j_1}(s) - x_{j_2} - x^\circ_{j_2}(s)\right\rangle\right) &= \exp(i\langle\lambda, x_{j_1} - x_{j_2}\rangle)\ \mathsf{E}\exp\left(i\left\langle\lambda, x^\circ_{j_1}(s) - x^\circ_{j_2}(s)\right\rangle\right)\\
&= \exp(i\langle\lambda, x_{j_1} - x_{j_2}\rangle)\ \phi^{j_1}(s;\lambda)\phi^{j_2}(s;-\lambda)\\
&= \exp(i\langle\lambda, x_{j_1} - x_{j_2}\rangle)\ |\phi(s;\lambda)|^2\\
\mathsf{E}\exp\left(-i\left\langle\lambda, x_{j_1} + x^\circ_{j_1}(s) - x_{j_2} - x^\circ_{j_2}(s)\right\rangle\right) &= \exp(-i\langle\lambda, x_{j_1} - x_{j_2}\rangle)\ |\phi(s;-\lambda)|^2
\end{aligned}$$

Note that $|\phi(s;\lambda)|^2$ is the real symmetric characteristic function and

$$|\phi(s;\lambda)|^2 = |\phi(s;-\lambda)|^2 = |\exp(s\boldsymbol{\eta}^\circ(\lambda))|^2 = \exp(2\,\mathrm{Re}\,\boldsymbol{\eta}^\circ(\lambda)s).$$

So

$$\mathsf{E}\cos\left\langle\lambda, x_{j_1} + x^\circ_{j_1}(s) - x_{j_2} - x^\circ_{j_2}(s)\right\rangle = \cos\langle\lambda, x_{j_1} - x_{j_2}\rangle\, e^{-s\boldsymbol{\eta}(\lambda)}.$$

Summing over $j_1 < j_2$ as in (41) we get (44). □

The function V defined by (41) is very important because

$$\mathsf{E}\, V(x(t)) = \chi_N(t;\lambda) \tag{45}$$

where $\chi_N(t;\lambda)$ is the characteristic function of $d^{(N)}_{j_1 j_2}(t) = x_{j_1}(t) - x_{j_2}(t)$ for the symmetric synchronization model $x(t)$ of Subsection 3.1. Indeed, in symmetric model random variables $d^{(N)}_{jk}(t)$ are symmetrically distributed hence $\chi_N(t;\lambda)$ is real and

$$\chi_N(t;\lambda) = \mathsf{E}\exp(i\langle\lambda, x_{j_1}(t) - x_{j_2}(t)\rangle) = \mathsf{E}\cos\langle\lambda, x_{j_1}(t) - x_{j_2}(t)\rangle.$$

Now (45) easily follows from (41).

4.2 Recurrent equations

Recall that the symmetric N-component synchronization model $x(t)$, $t \geq 0$, is the stochastic process with values in $\mathbb{R}^{Nd}$. Let $f = f(x)$ be some function on the configuration space $\mathbb{R}^{Nd}$. Put

$$f^{(n)} = \mathsf{E}\left(f(x(T_n+0)) \mid \{T_q\}_{q=1}^{\infty}\right), \qquad n = 1, 2, \dots . \tag{46}$$

Hence $f^{(n)}$ is a random variable functionally depending on the sequence $\underline{T} := \{T_q\}_{q=1}^{\infty}$. In particular, we may consider $\{V^{(n)}\}$ where V is defined in (41). Main result of this subsection will be given in Lemma 6 below.

Remark 4 *Note that conditional expectations*

$$\mathsf{E}\left(\cdot \mid \left\{\tau_l^{(j)}\right\}_{l=1}^{\infty}, \, j = \overline{1,N}\right) \quad \text{and} \quad \mathsf{E}\left(\cdot \mid \{T_q\}_{q=1}^{\infty}\right)$$

are different. The first one carries the total information about senders at epochs T_q but in the second conditional expectation such information is unavailable.

Below we will use the telescopic property of the conditional expectation

$$\mathsf{E}\left(\mathsf{E}\left(\cdot \mid \xi, \underline{T}\right) \mid \underline{T}\right) = \mathsf{E}\left(\cdot \mid \underline{T}\right)$$

where ξ is some random variable. Let V be as in (41). Then

$$V^{(n)} = \mathsf{E}\left(V(x(T_n+0)) \mid \underline{T}\right) = \mathsf{E}\left(\mathsf{E}\left(V(x(T_n+0)) \mid x(T_n), \underline{T}\right) \mid \underline{T}\right). \tag{47}$$

Consider now $\mathsf{E}\left(V(x(T_n+0)) \mid x(T_n), \underline{T}\right)$. What is the difference between configurations $x(T_n)$ and $x(T_n+0)$? This difference is produced by a single message (j_1, j_2) sent from some component j_1 to another component j_2. Obviously, the index j_1 of the sender is random. What is the distribution of j_1? For the symmetric model the answer is simple: since the dynamics of the stochastic process $x(t)$ is invariant with respect to permutations of indices the distribution of j_1 is uniform:

$$\mathsf{P}\{j_1 = k\} = \frac{1}{N}, \qquad k = \overline{1,N}.$$

In symmetric model the recipient of the message is chosen with probability $\frac{1}{N-1}$ among the components different from the sender. So in the symmetric model all messages (j_1, j_2) have the same probability $\frac{1}{(N-1)N}$ to be sent at epoch T_n. This means that

$$\mathsf{E}\left(V(x(T_n+0)) \mid x(T_n), \underline{T}\right) = \mathsf{E}_S V(\mathcal{S}x(T_n))$$

where averaging E_S is taken over distribution of the map-valued random variable $\mathcal{S}$ introduced in (40). Hence by (42) we get

$$\mathsf{E}\left(V(x(T_n+0)) \mid x(T_n), \underline{T}\right) = k_N V(x(T_n)) + l_N. \tag{48}$$

Consider now

$$\mathsf{E}\left(V(x(T_n)) \mid \underline{T}\right) = \mathsf{E}\left(\mathsf{E}\left(V(x(T_n)) \mid x(T_{n-1}+0), \underline{T}\right) \mid \underline{T}\right).$$

There are no synchronization jumps inside the time interval (T_{n-1}, T_n) hence by Lemma 5

$$\mathsf{E}\,\left(V(x(T_n))\,|\,x(T_{n-1}+0),\underline{T}\right) = V(x(T_{n-1}+0))\,\exp\left(-(T_n - T_{n-1})\boldsymbol{\eta}(\lambda)\right).$$

Applying the conditional averaging $\mathsf{E}\,(\cdot\,|\,\underline{T})$ we get

$$\mathsf{E}\,\left(V(x(T_n))\,|\,\underline{T}\right) = V^{(n-1)}\exp\left(-(T_n - T_{n-1})\boldsymbol{\eta}(\lambda)\right). \tag{49}$$

Collecting (47), (48) and (49) together we obtain

Lemma 6

$$V^{(n)} = k_N V^{(n-1)}\exp\left(-(T_n - T_{n-1})\boldsymbol{\eta}(\lambda)\right) + l_N\,. \tag{50}$$

On the time interval $(T_{\Pi_t^S}, t)$ there are no synchronization jumps, so similar arguments give

$$\mathsf{E}\,\left(V(t)\,|\,\underline{T}\right) = V^{(\Pi_t^S)}\exp\Big(-(t - T_{\Pi_t^S})\boldsymbol{\eta}(\lambda)\Big). \tag{51}$$

4.3 Representations for the characteristic function

Recall notation: the point process $\{T_q\}$ is a superposition of the renewal processes $\left\{\tau_l^{(j)}\right\}$, $j = 1, \ldots, N$,

$$\Pi_t^S = \sum_{j=1}^N \Pi_t^{(j)} = \max\{q \ge 0:\ T_q \le t\}\,.$$

Let k_N and l_N be the same as in (43).

Lemma 7 *For any $t > 0$ and $\lambda \in \mathbb{R}^d$*

$$\chi_N(t;\lambda) = \chi_N(0;\lambda)\exp(-t\boldsymbol{\eta}(\lambda))\mathsf{E}\,k_N^{\Pi_t^S} + l_N\mathsf{E}\sum_{q=1}^{\Pi_t^S}\exp\left(-(t - T_q)\boldsymbol{\eta}(\lambda)\right)k_N^{\Pi_t^S - q}. \tag{52}$$

Similar decompositions were used in [52] and [51].

Proof of Lemma 7. Denote $\underline{\Delta}_q = T_q - T_{q-1}$. Iterating (50) we get

$$V^{(n)} = k_N^2 V^{(n-2)}\exp\big(-(\underline{\Delta}_{n-1} + \underline{\Delta}_n)\boldsymbol{\eta}(\lambda)\big) + k_N\exp\left(-\underline{\Delta}_n\boldsymbol{\eta}(\lambda)\right)l_N + l_N\,,$$

$$\begin{aligned} V^{(n)} &= k_N^n V^{(0)}\exp\left(-(\underline{\Delta}_1 + \cdots + \underline{\Delta}_n)\boldsymbol{\eta}(\lambda)\right) + k_N^{n-1}\exp\left(-(\underline{\Delta}_2 + \cdots + \underline{\Delta}_n)\boldsymbol{\eta}(\lambda)\right)l_N + \cdots + \\ &\quad + k_N\exp\left(-\underline{\Delta}_n\boldsymbol{\eta}(\lambda)\right)l_N + l_N\,. \end{aligned}$$

Taking into account (51) and using identity $\sum_{i=q}^n \underline{\Delta}_i = T_n - T_{q-1}$ we come to the following representation

$$\begin{aligned} \mathsf{E}\,\left(V(x(t))\,|\,\underline{T}\right) &= V^{(\Pi_t^S)}\exp\Big(-(t - T_{\Pi_t^S})\boldsymbol{\eta}(\lambda)\Big) = \\ &= k_N^{\Pi_t^S}V^{(0)}\exp(-t\boldsymbol{\eta}(\lambda)) + k_N^{\Pi_t^S - 1}\exp\left(-(t - T_1)\boldsymbol{\eta}(\lambda)\right)l_N + \cdots \\ &\quad + k_N\exp\Big(-(t - T_{\Pi_t^S - 1})\boldsymbol{\eta}(\lambda)\Big)l_N + \exp\Big(-(t - T_{\Pi_t^S})\boldsymbol{\eta}(\lambda)\Big)l_N\,. \end{aligned}$$

The statement of Lemma 7 will now follow from (45) if we apply the unconditional expectation E to the both sides of this representation. □

We introduce some notation. Since for the symmetric model all renewal processes $\left\{\tau_m^{(k)}\right\}$, $k = 1, \ldots, N$, are equally distributed they have the common renewal function

$$H(t) = \mathsf{E}\,\Pi_t^{(k)} . \tag{53}$$

Similarly, $\left(\Pi_t^{(k)},\, t \geq 0\right)$ have the same moment generating function

$$\varphi(u, v) = \mathsf{E}\,\left(v^{\Pi_u^{(k)}}\right), \quad u \geq 0, \quad v \in \mathbb{R}. \tag{54}$$

Denote also $\overline{F}(s) := 1 - F(s)$ where $F(s)$ is the common inter-event probability distribution function of the renewal processes $\left\{\tau_m^{(k)}\right\}$ (see Subsection 3.3). If $f_1 = f_1(t)$ and $f_2 = f_2(t)$ are two functions vanishing for $t < 0$ then their convolution $(f_1 * f_2)\,(t)$ is the function defined as

$$(f_1 * f_2)\,(t) = \int_0^t f_1(s) f_2(t - s)\, ds$$

for $t \geq 0$ and $(f_1 * f_2)\,(t) = 0$ for $t < 0$.

Define the following functions

$$\begin{aligned} \varphi_{1,s}(u, v) \;&=\; \int_0^s dH(y)\, \overline{F}(s + u - y) + \overline{F}(s + u) + \\ &\quad + v\, \left(g_s * \varphi(\cdot, v)\right)(u) \end{aligned} \tag{55}$$

$$g_s(w) = \int_0^s dH(y)\, p(s + w - y) + p(s + w)\, .$$

Here $s, u, w \geq 0$, $v \in \mathbb{R}$. By definition $\varphi_{1,s}(u, v) = 0$ for $u < 0$ and $g_s(w) = 0$ for $w < 0$.

The function $\varphi_{1,s}(u, v)$ has a very clear meaning: it is the moment generating function for the number renewals in $\left\{\tau_m^{(k)}\right\}$ happened on the interval $[s, s+u]$. Note that the first two summands in (55) is the probability that the flow $\left\{\tau_m^{(k)}\right\}$ has no renewal on $[s, s+u]$. The probability that the first renewal in $\left\{\tau_m^{(k)}\right\}$ fits to a small interval $[s+w,\, s+w+dw]$ is equal to $g_s(w)\, dw + o(dw)$.

Lemma 8 *The expectation in the second term of (52) is*

$$I_N(t, \lambda) := N \int_0^t dH(s)\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \left(\varphi_{1,s}(t - s, k_N)\right)^{N-1} \varphi(t - s, k_N)\, .$$

Proof of Lemma 8. For any $A \subset \mathbb{R}_+$ denote by $\#^{(j)}A$ a random variable "the number of epochs of the point process $\left\{\tau_l^{(j)}\right\}_{l=0}^{\infty}$ belonging to A":

$$\#^{(j)}A := \sum_{l=1}^{\infty} \mathbf{1}_{\left\{\tau_l^{(j)} \in A\right\}}.$$

Denote also $\#^S A := \sum_{j=1}^{N} \#^{(j)} A$. In particular, $\Pi_t^{(j)} = \#^{(j)}[0,t]$, $\Pi_t^S = \#^S[0,t]$. Define a function

$$a(s) = \begin{cases} \exp(-s\boldsymbol{\eta}(\lambda)), & s \geq 0 \\ 0, & s < 0. \end{cases} \tag{56}$$

Then

$$I_N(t,\lambda) := \mathsf{E} \sum_{q=1}^{\Pi_t^S} \exp\left(-(t-T_q)\boldsymbol{\eta}(\lambda)\right) k_N^{\Pi_t^S - q} = \mathsf{E} \sum_{q=1}^{\infty} a(t-T_q) k_N^{\Pi_t^S - q}$$

Consider a single summand in these sums

$$\begin{aligned} \exp\left(-(t-T_q)\boldsymbol{\eta}(\lambda)\right) k_N^{\Pi_t^S - q} &= a(t-T_q) k_N^{\#^S(T_q,t)} \\ &= a(t-T_q) \prod_{j=1}^{N} k_N^{\#^{(j)}(T_q,t)} \\ &= \sum_{r=1}^{N} \mathbf{1}_{\{T_q \in \underline{\tau}^{(r)}\}} a(t-T_q)\, k_N^{\#^{(r)}(T_q,t)} \prod_{j \neq r}^{N} k_N^{\#^{(j)}(T_q,t)} \end{aligned}$$

So

$$I_N(t,\lambda) = \sum_{r=1}^{N} \mathsf{E} \sum_{q=1}^{\infty} \mathbf{1}_{\{T_q \in \underline{\tau}^{(r)}\}} a(t-T_q)\, k_N^{\#^{(r)}(T_q,t)} \prod_{j \neq r}^{N} k_N^{\#^{(j)}(T_q,t)}.$$

The point process $\underline{T}$ is the superposition of the point processes $\underline{\tau}^{(j)}$, $j = \overline{1,N}$. Hence the sum $\sum_{q=1}^{\infty} \mathbf{1}_{\{T_q \in \underline{\tau}^{(r)}\}}$ is the summation over all point of $\underline{\tau}^{(r)} = \left\{\tau_n^{(r)}\right\}_{n=1}^{\infty}$. Therefore

$$\begin{aligned} I_N(t,\lambda) &= \sum_{r=1}^{N} \mathsf{E} \sum_{n=1}^{\infty} a(t-\tau_n^{(r)})\, k_N^{\#^{(r)}(\tau_n^{(r)},t)} \prod_{j \neq r}^{N} k_N^{\#^{(j)}(\tau_n^{(r)},t)} \\ &= N\mathsf{E} \sum_{n=1}^{\infty} a(t-\tau_n^{(1)})\, k_N^{\#^{(1)}(\tau_n^{(1)},t)} \prod_{j=2}^{N} k_N^{\#^{(j)}(\tau_n^{(1)},t)} \end{aligned}$$

since in the symmetric model all renewal processes $\underline{\tau}^{(j)}$, $j = \overline{1,N}$, are independent and identically distributed. Note also that the random variables $\#^{(1)}(\tau_n^{(1)},t)$ and $\#^{(j)}(\tau_n^{(1)},t)$, $j = \overline{2,N}$, are conditionally independent when the value of $\tau_n^{(1)}$ is known. So we can proceed with our calculation as follows

$$\begin{aligned} I_N(t,\lambda) &= N\mathsf{E} \sum_{n=1}^{\infty} \mathsf{E} \left(a(t-\tau_n^{(1)})\, k_N^{\#^{(1)}(\tau_n^{(1)},t)} \prod_{j=2}^{N} k_N^{\#^{(j)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)} \right) \\ &= N\mathsf{E} \sum_{n=1}^{\infty} a(t-\tau_n^{(1)}) \mathsf{E} \left(k_N^{\#^{(1)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)} \right) \prod_{j=2}^{N} \mathsf{E} \left(k_N^{\#^{(j)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)} \right). \end{aligned}$$

Note that

$$\mathsf{E} \left(k_N^{\#^{(1)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)} \right) = \varphi(t-\tau_n^{(1)}, k_N)$$

where φ is the generating function (54). Here we have used the fact that there is a renewal at point $\tau_n^{(1)}$. If $j \geq 2$ then $\mathsf{E} \left(k_N^{\#^{(j)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)} \right)$ differs from the generating function φ

because the point process $\underline{\tau}^{(j)}$ has a memory and $\tau_n^{(1)}$ is not a renewal point for $\underline{\tau}^{(j)}$. If we denote

$$\varphi_{1,s}(u,v) := \mathsf{E}\left(v^{\#^{(j)}(s,s+u)}\right). \tag{57}$$

then $\mathsf{E}\left(k_N^{\#^{(j)}(\tau_n^{(1)},t)} \mid \tau_n^{(1)}\right) = \varphi_{1,\tau_n^{(1)}}(t-\tau_n^{(1)}, k_N)$. Hence the following representation

$$I_N(t,\lambda) = N\mathsf{E}\sum_{n=1}^{\infty} f(\tau_n^{(1)})$$

holds with the function

$$f(s) := \, a(t-s)\,\varphi(t-s,k_N)\,(\varphi_{1,s}(t-s,k_N))^{N-1}.$$

It follows from the renewal theory that

$$\mathsf{E}\sum_{n=1}^{\infty} f(\tau_n^{(1)}) = \int_0^{\infty} f(s)\,dH(s)$$

where $H(s)$ is the renewal function of the point process $\underline{\tau}^{(1)}$ (see (53)). Recalling notation (56) we conclude that the proof of Lemma 8 is almost done. The only thing remains to be proved is the formula (55) for the function defined as (57). This is a standard exercise form the renewal theory so we leave it to readers. □

The representation (52) will be very useful for Subsection 4.5. At this point we discuss the next two immediate corollaries of Lemma 8.

Recall that $\boldsymbol{\eta}(\lambda) \geq 0$. The first summand in (52) tends to 0 as $t \to +\infty$ uniformly in $\lambda \in \mathbb{R}^d$. Indeed, for any fixed $v \in (0,1)$ the generating function

$$\phi_S(t,v) = \mathsf{E}\left(v^{\Pi_t^S}\right)$$

tend to 0 as $t \to +\infty$ since $\Pi_t^S \to \infty$ (a.s.) [9]. Hence we come to the following result.

Corollary. *For any fixed N*

$$\lim_{t\to\infty} \chi_N(t;\lambda) = l_N \lim_{t\to\infty} I_N(t,\lambda). \tag{58}$$

The existence of these limits will be proved in Subsection 4.5.

The *second corollary* of Lemma 8 is a short proof of Theorems 4 and 5 for the Markovian model. In the Markovian situation the representation (52) turns in a simple explicit formula. Details are given in a separate subsection.

4.4 The Markovian case: proofs of Theorems 4 and 5

Assume that (34) holds. The inter-event distribution is exponential and has the "lack of memory" property. Hence the generating functions $\varphi(u,v)$ and $\varphi_2(u,v)$ are equal and, moreover,

$$\varphi(u,v) = \varphi_2(u,v) = e^{t(v-1)/m},$$

the probability generating function of the Poisson law with the mean t/m. Since the renewal processes $\Pi_t^{(j)}$ are Poissonian we have $H(t) = t/m$ (see [9]). Then

$$\begin{aligned} I_N(t,\lambda) &= \frac{N}{m}\int_0^t ds\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \left(\varphi(t-s,k_N)\right)^N = \\ &= \frac{N}{m}\int_0^t ds\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \exp\left((t-s)(k_N-1)N/m\right) = \\ &= \frac{N}{m}\int_0^t ds\, \exp\left(-A\cdot(t-s)\right), \end{aligned} \tag{59}$$

where $A = \boldsymbol{\eta}(\lambda) + l_N N/m$. By (58) we have

$$\begin{aligned} \chi_N(\infty;\lambda) &= l_N I_N(\infty,\lambda) = \frac{l_N N}{m}\int_0^\infty du\, \exp\left(-Au\right) = \\ &= \frac{l_N N}{m}\,\frac{1}{\boldsymbol{\eta}(\lambda)+l_N N/m} = \frac{1}{\boldsymbol{\eta}(\lambda)\cdot m/(l_N N)+1} = \\ &= \frac{1}{\boldsymbol{\eta}(\lambda)\cdot (N-1)m/\varkappa+1}\,. \end{aligned}$$

In these calcucations the notation (43) were used. Theorem 4 is proved.

Theorem 5 immediately follows from Theorem 4 and the definition of a stable law.

Using (59) we derive an explicit formula for the characteristic function $\chi_N(t;\lambda)$:

$$\chi_N(t;\lambda) = \exp\left(-At\right)\chi_N(0;\lambda) + \frac{l_N N}{m}\int_0^t ds\, \exp\left(-A\cdot(t-s)\right).$$

It is straightforward to check that $\chi_N(t;\lambda)$ satisfies to the differential equation (35).

4.5 The general case: around the Key Renewal Theorem

We go back to the general non-Markovian synchronization model. In this subsection N is fixed. Define functions

$$\varphi_2(u,v) = \frac{1}{m}\int_0^\infty \overline{F}(w+u)\,dw + v\cdot\left(\frac{1}{m}\overline{F}*\varphi(\cdot,v)\right)(u), \quad u\ge 0, \quad v\in\mathbb{R}, \tag{60}$$

and

$$J_N(t,\lambda) = N\int_0^t dH(s)\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \left(\varphi_2(t-s,k_N)\right)^{N-1}\varphi(t-s,k_N)\,. \tag{61}$$

Roughly speaking, the function $J_N(t,\lambda)$ differs from the function $I_N(t,\lambda)$ by the formal replacing $\varphi_{1,s}(u,v)$ by $\varphi_2(u,v)$.

We are going to give a probabilistic interpretation to (60). Following [9, p. 62] we denote by V_t the forward recurrence-time in $\Pi_t^{(j)}$, defined as the time measured from t to the next renewal. It is well known [9] that the law of V_t converges to some absolutely continuous distribution as $t\to\infty$. Moreover, the probability density function of the limiting law is

$$p_{V_\infty}(w) = \frac{1}{m}\overline{F}(w), \qquad w\ge 0. \tag{62}$$

We see that $\varphi_2(u,v)$ is the generating function for the number of renewals on $[0,u]$ in the *modified* renewal process for which the distribution of the first interval Δ_1 is (62),

$$\mathsf{P}\left(\Delta_1 \le s\right) = \int_0^s p_{V_\infty}(w)\, dw\,, \tag{63}$$

but the intervals $\Delta_2, \Delta_3, \ldots$ have the same distribution as before $\mathsf{P}\left(\Delta_n \le s\right) = F(s)$, $n \ge 2$. Note that the modified renewal process with the first interval Δ_1 distributed as (62)–(63) is a *stationary* renewal process. We recall also that $\varphi(u,v) = \mathsf{E}\, v^{\Pi_u}$ is the corresponding generating function for the *ordinary* renewal process.

As it was explained in (58) the main task is to study the limit of the function $I_N(t,\lambda)$. Our idea is to reduce this problem to the analysis of the function $J_N(t,\lambda)$.

Lemma 9 *Let N be fixed and $t \to \infty$. Then*

$$\sup_{\lambda\in\mathbb{R}^d} |I_N(t,\lambda) - J_N(t,\lambda)| \to 0 \qquad (t\to\infty).$$

Lemma 10 *Let N be fixed and $t \to \infty$. The family of functions $\{J_N(t,\lambda),\, t\ge 0\}$ converges to some limit $J_N(\infty,\lambda)$ as $t \to \infty$. This limit is uniform in $\lambda \in \mathbb{R}^d$,*

$$\sup_{\lambda\in\mathbb{R}^d} |J_N(t,\lambda) - J_N(\infty,\lambda)| \to 0\,,$$

and the limiting function is

$$J_N(\infty,\lambda) = \frac{N}{m}\int_0^\infty du\, e^{-u\boldsymbol{\eta}(\lambda)} \left(\varphi_2(u,k_N)\right)^{N-1} \varphi(u,k_N)\,. \tag{64}$$

Hence using (58), Lemma 9 and Lemma 10 we get the following representation for the limiting characteristic function:

$$\lim_{t\to\infty} \chi_N(t;\lambda) = l_N J_N(\infty,\lambda).$$

Lemmas 9 and 10 will be proven in Subsection 4.6. Before proceeding with the proofs we want to discuss some connection of the above results with the classical renewal processes theory.

The Smith theorem states that if a function $f:\mathbb{R}_+\to\mathbb{R}$ satisfies to certain *sufficient conditions* then

$$\int_0^t dH(s)\, f(t-s) \to \frac{1}{m}\int_0^\infty f(t)\, dt \quad (t\to+\infty) \tag{65}$$

Here $H(s)$ is the renewal function of the ordinary renewal process and m is the expectation of the inter-event interval. This statement is known also as the Key Renewal Theorem (KRT).

Lemma 10 looks like a formal application of the KRT to the function

$$f(t) = e^{-t\boldsymbol{\eta}(\lambda)} \left(\varphi_2(t,k_N)\right)^{N-1} \varphi(t,k_N). \tag{66}$$

It is well known that the KRT holds if, for example, any of the next *sufficient conditions* SC1 or SC2 is satisfied.

SC1: the function $f(t)$ in (65) is nonnegative, nonincreasing and integrable (the Smith's conditions, [16]);

SC2: the function $f(t)$ is directly integrable (the Feller's condition, [13]).

They should be verified for any fixed λ and N. The function defined by (66) is not convenient to check SC2.

Consider now SC1. The first condition is evidently satisfied. Since $k_N \in (0,1)$ the both generating functions $\varphi(t,k_N)$ and $\varphi_2(t,k_N)$ are nonincreasing in t and the second condition of SC1 is true. To have integrability of $f(t)$ it is sufficient to assume that

$$\int_0^\infty \varphi(t,k_N)\,dt \,<\, +\infty. \tag{67}$$

So we are interested in conditions on the inter-event distribution (on the function $F(s)$ and $p(s)$) that ensure (67). To find such conditions one need to study behavior of $\varphi(t,k_N)$ when $t \to +\infty$. Similar problems arise in the renewal theory [9]. It is natural to attack them by using the classical analytic methods involving the Laplace transform or Tauber theorems. Based on the experience existing in this field we can imagine that it would be rather hard to get exhaustive general description of such distributions in simple and concise terms. In the present paper we would like to avoid too heavy analytical considerations. From the other side there is a hope that for many concrete inter-event distributions the condition (67) could be checked by direct methods. So we chose a "happy medium" and adopt the strategy followed in the classical book [9]. We consider distributions with *rational Laplace transforms* (ME distributions) which are sufficient for most applications and very convenient in the context of the current study. In Subsection 4.6 we construct special classes of functions (we call them the $\mathcal{K}$-classes) and propose a method based on a set of rules for manipulation of these functions. Functions of the form (66) belong to these classes. Moreover, this technique is also very efficient for proving Lemma 9. Proofs can be obtained in a transparent "algebraic" way.

Briefly speeking, we will prove here (65) under assumptions different from SC1 and SC2.

Lemma 11 *Assume that the function $f(t)$, $t \geq 0$, is such that its Laplace transform*

$$f^*(z) = \int_0^\infty f(t)\exp\left(-zt\right)\,dt \tag{68}$$

is a RPFN-function. Then (65) holds.

To conclude this discussion one should mention the paper [75]. It contains an interesting approach based on the derivation of an analog of the KRT for the superposition of renewal processes. Nevertheless, we cannot use results of [75] because they exploite sufficient conditions similar to the direct intergrability (SC2) which is very hard to verify.

4.6 Algebra of functions $\mathcal{K}_t$

We introduce some notation. Let $\mathcal{K}_t$ be a linear space of functions $f = f(t)$, $f : \mathbb{R}_+ \to \mathbb{C}$, having the following form

$$f(t) = \sum_j P_{n_j}(t)e^{\lambda_j t}\,,$$

where the sum is taken over a finite set of indices, $\lambda_j \in \mathbb{C}$ are such that $\operatorname{Re}\lambda_j < 0$, $P_{n_j}(t)$ are polynomials with complex coefficients, n_j is a degree of the polynomial $P_{n_j}(t)$. It is important to note that if $f \in \mathcal{K}_t$ then $f(t) \to 0$ as $t \to +\infty$.

It is easy to see that the Laplace transform (68) maps the set $\mathcal{K}_t$ to a set $\mathcal{K}_z^*$ of complex-valued functions $f^* = f^*(z)$, $z \in \mathbb{C}$, which is exactly the set of RPFN-function. In other words, the Laplace transform provides a one-to-one correspondence between the sets $\mathcal{K}_t$ and $\mathcal{K}_z^*$. Other properties of these sets are listed below.

K**1.** The set $\mathcal{K}_t$ is an algebra over the field $\mathbb{C}$ with the usual operations "+" and "·", the summation and the pointwise multiplication of functions. In particular, if functions $f_1 = f_1(t)$ and $f_2 = f_2(t)$ belong to $\mathcal{K}_t$ then the functions $c_1 f_1(t) + c_2 f_2(t)$ and $f_1(t) f_2(t)$ also belong to $\mathcal{K}_t$ for all $c_1, c_2 \in \mathbb{C}$.

K**2.** Similarly, the set $\mathcal{K}_z^*$ is also an algebra over $\mathbb{C}$ with the operations "+" and "·".

Remark 5 *The Laplace transform is a one-to-one correspondence between the vector spaces $\mathcal{K}_t$ and $\mathcal{K}_z^*$ but it is not an homomorphism of the algebras $\mathcal{K}_t$ and $\mathcal{K}_z^*$.*

K**3.** The set $\mathcal{K}_t$ is closed with respect to the convolution, i.e.,

$$f_1, f_2 \in \mathcal{K}_t \quad \Rightarrow \quad f_1 * f_2 \in \mathcal{K}_t\,,$$

where

$$(f_1 * f_2)(t) = \int_0^t f_1(s) f_2(t-s)\, ds\,.$$

K**4.** If $f \in \mathcal{K}_t$ then

$$\int_0^t dH(s)\, f(t-s) \;=\; \frac{1}{m}\int_0^\infty f(s)\, ds + \gamma_1(t),$$

where γ_1 is some function from $\mathcal{K}_t$. In particular, $\gamma_1(t) \to 0$ as $t \to +\infty$.

Proof of $K4$. Denote $q(t) = \int_0^t dH(s)\, f(t-s)$. Under Assumptions P1 there exists a renewal density function $h(s)$ corresponding to $H(s)$: $dH(s) = h(s)\, ds$. It follows from the classic results [9, p. 54] that

$$q^*(z) = \left(\int_0^t h(s) f(t-s)\, ds\right)^*(z) = \frac{p^*(z)}{1 - p^*(z)}\, f^*(z)\,. \tag{69}$$

By Assumption 2 $p^*(z)$ is a RPFN-function. By Lemma 1 the equation $p^*(z) - 1 = 0$ has a simple root at $z = 0$. Hence the r.h.s of (69) has $z_0 = 0$ as a simple pole. Again by Lemma 1 there is no other singularities in the half-plane $\operatorname{Re} z \ge 0$ and there is a finite number of other poles in $\operatorname{Re} z < 0$. A pole z_j of the order n_j of the Laplace transform $q^*(z)$ corresponds to a summand $P_j(t)\exp(z_j t)$ in the original function $q(t)$ where $P_j(t)$ is some polynomial of degree n_j. Hence

$$q(t) = q_0 + \sum_{j \ne 0} P_j(t) \exp(z_j t)$$

where q_0 is the residual $\operatorname*{res}_{z=0} q^*(z)$. Recall that $p^*(0) = 1$ and $(p^*)'(0) = -m < 0$. Taking the limit of $q(t)$ as $t \to +\infty$ we get

$$\begin{aligned}\lim_{t\to+\infty} \int_0^t dH(s)\, f(t-s) &= q(+\infty) = q_0 \\ &= \operatorname*{res}_{z=0} \frac{p^*(z)}{1-p^*(z)}\, f^*(z) = \frac{1}{m} f^*(0) = \frac{1}{m}\int_0^\infty f(t)\, dt\,.\end{aligned}$$

□

K**5.** If $f \in \mathcal{K}_t$, $n \in \mathbb{Z}_+$, $\beta \in \mathbb{C}$ and $\operatorname{Re}\beta > 0$ then

$$\gamma_2(t) = \int_0^t dH(s)\, s^n e^{-\beta s} f(t-s) \,\in \mathcal{K}_t\,.$$

In particular, $\gamma_2(t) \to 0$ as $t \to +\infty$.

Proof of $K5$. Consider the Laplace transform $\gamma_2^*(z)$:

$$\left(\int_0^t dH(s)\, s^n e^{-\beta s} f(t-s)\right)^*(z) = f^*(z)\,(-1)^m \frac{d^m}{d\beta^m}\left(\frac{p^*(z+\beta)}{1-p^*(z+\beta)}\right).$$

All poles of the function in the r.h.s. belong to the left half-plane $\operatorname{Re} z < 0$. □

Denote by $\mathcal{K}^2_{s,w}$ a class of functions $a = a(s,w)$, $s, w \in \mathbb{R}_+$, of the following form

$$a(s,w) = \sum_{k,l} a_{k,l} e_k(s) f_l(w), \qquad e_k, f_l \in \mathcal{K} \tag{70}$$

where the sum is taken over a finite set of indices, $a_{k,l} \in \mathbb{C}$. In other words, the set $\mathcal{K}^2_{s,w}$ is a tensor product of $\mathcal{K}_s$ and $\mathcal{K}_w$:

$$\mathcal{K}^2_{s,w} = \mathcal{K}_s \otimes \mathcal{K}_w\,.$$

Remark 6 *Note that for any $f \in \mathcal{K}_t$ the function $f(s+w)$ belongs to $\mathcal{K}^2_{s,w}$.*

K**6.** The set of functions $\mathcal{K}^2_{s,w}$ is an algebra, in particular, $a(s,w)b(s,w) \in \mathcal{K}^2_{s,w}$ if $a \in \mathcal{K}^2_{s,w}$ and $b \in \mathcal{K}^2_{s,w}$.

K**7.** If $f = f(t) \in \mathcal{K}_t$ and $a = a(s,w) \in \mathcal{K}^2_{s,w}$ then

a) $a(s,w)f(w) \in \mathcal{K}^2_{s,w}$,

b) $(a(s,\cdot) * f(\cdot))\,(w) = \int_0^w a(s,w-y) f(y)\,dy \in \mathcal{K}^2_{s,w}$,

c) $\int_0^{+\infty} a(s,w)\,ds \in \mathcal{K}_w\,, \quad \int_0^{+\infty} f(s+w)\,ds \in \mathcal{K}_w\,.$

The item $K7$b follows from $K3$ and (70). The next two properties are corollaries of $K4$ and $K5$. Assume that $a(s,w) \in \mathcal{K}^2_{s,w}$ and $H(y)$ is the renewal function (53).

K**8.** $\int_0^s dH(y)\, a(s-y,w) = \frac{1}{m}\int_0^\infty a(s,w)\,ds + \gamma_3(s,w)$
where $\gamma_3(s,w) \in \mathcal{K}^2_{s,w}$.

K**9.** $\int_0^t dH(s)\, a(s,t-s) \,\in \mathcal{K}_t\,.$

K**10.**

a) $\forall\, f(t) \in \mathcal{K}_t\ \exists\, g(t) \in \mathcal{K}_t\ :\ |f(t)| \le g(t)$

b) $\forall\, a(s,w) \in \mathcal{K}^2_{s,w}\ \ \exists\, b(s,w) \in \mathcal{K}^2_{s,w} :\ |a(s,w)| \le b(s,w)\,.$

The item $K10$a can be proved using the following simple bounds:

$$\begin{aligned} \left|t^{2k-1}\right| &\leq 1+t^{2k}, \quad k \in \mathbb{N}, \\ \left|e^{-\mu t}\right| &\leq e^{-t\,\mathrm{Re}\,\mu}. \end{aligned}$$

A proof of the item $K10$b is similar. □

Proof of Lemma 9. The idea is to prove that $I_N(t,\lambda) = J_N(t,\lambda) + \psi_N(t,\lambda)$ where $|\psi_N(t,\lambda)| \leq \psi_{1,N}(t)$ for some function $\psi_{1,N}(t)$ from $\mathcal{K}_t$. This is easy to do by using the above properties $K1$–$K10$. Below we give the chain of conclusions with minor comments.

To analyze $I_N(t,\lambda)$ consider first the function $\varphi_{1,s}(u,v)$ defined by the formula (55).

$\overline{F}(t) \in \mathcal{K}_t$

$\overline{F}(s+u) \in \mathcal{K}^2_{s,u}$ (Remark 6)

Below we use notation $\gamma_4(s,u)$, ..., $\gamma_9(s,u)$ for functions belonging to $\mathcal{K}^2_{s,u}$.

$\int_0^s dH(y)\,\overline{F}(s-y+u) = \frac{1}{m}\int_0^\infty \overline{F}(y+u)\,dy + \gamma_4(s,u), \quad \gamma_4(s,u) \in \mathcal{K}^2_{s,u}$ (see $K8$)

$p(s+w) \in \mathcal{K}^2_{s,u}$(Remark $K9$.)

$g_s(w) = \frac{1}{m}\int_0^\infty p(y+w)\,dy + \gamma_5(s,u), \quad \gamma_5(s,u) \in \mathcal{K}^2_{s,u}$ (see $K8$)

For any fixed $v \in (0,1)$ we have $\varphi^*(z,v) \in \mathcal{K}^*_z$ by the formula (74). Hence $\varphi(u,v) \in \mathcal{K}_u$ for any fixed $v \in (0,1)$. It follows from $K7$ and $K8$

$$(g_s * \varphi(\cdot,v))\,(u) = \frac{1}{m}\int_0^\infty (p(y+\cdot) * \varphi(\cdot,v))\,(u)\,dy + \gamma_6(s,u)\,.$$

Note that the function $\gamma_6(s,u)$ depends on the variable v but in the current lemma its value is fixed ($v = k_N$) so we skip this dependence in the notation $\gamma_6(s,u)$. So we get

$$\begin{aligned} \varphi_{1,s}(u,k_N) &= \frac{1}{m}\int_0^\infty \overline{F}(y+u)\,dy + k_N\,\frac{1}{m}\int_0^\infty (p(y+\cdot) * \varphi(\cdot,k_N))\,(u)\,dy + \gamma_7(s,u) = \\ &= \varphi_2(u,k_N) + \gamma_7(s,u) \end{aligned}$$

where $\varphi_{1,s}$ and φ_2 are defined in (55) and (60).

$\int_0^\infty \overline{F}(y+u)\,dy\,, \int_0^\infty p(y+u)\,dy \in \mathcal{K}_u$ (by $K7$c). Hence $\varphi_2(u,k_N) \in \mathcal{K}_u$ ($K3$).

$(\varphi_{1,s}(u,k_N))^{N-1} = (\varphi_2(u,k_N))^{N-1} + \gamma_8(s,u)$ (by $K7$a)

Since $\varphi(u,k_N) \in \mathcal{K}_u$ by $K7$a we get

$(\varphi_{1,s}(u,k_N))^{N-1}\,\varphi(u,k_N) = (\varphi_2(u,k_N))^{N-1}\,\varphi(u,k_N) + \gamma_9(s,u).$

Hence

$$I_N(t) = J_N(t) + N \int_0^t dH(s)\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \gamma_9(s, t-s)\,. \tag{71}$$

Denoting $\psi_N(t,\lambda) = N \int_0^t dH(s)\, e^{-(t-s)\boldsymbol{\eta}(\lambda)} \gamma_9(s, t-s)$ and recalling that $\boldsymbol{\eta}(\lambda) \geq 0$ we have

$$\begin{aligned} |\psi_N(t,\lambda)| &\leq N \int_0^t dH(s)\; |\gamma_9(s,t-s)| \\ &\overset{K10b}{\leq} N \int_0^t dH(s)\, \gamma_{10}(s,t-s) \,\in \mathcal{K}_t \quad (\text{by } K9)\,. \end{aligned}$$

Hence for any fixed N

$$\sup_{\lambda \in \mathbb{R}^d} |\psi_N(t,\lambda)| \to 0 \qquad (t \to \infty). \qquad \square$$

Proof of Lemma 10. In proving Lemma 9 we obtained inclusions: $\varphi(u,v) \in \mathcal{K}_u$ and $\varphi_2(u,k_N) \in \mathcal{K}_u$. Since $\boldsymbol{\eta}(\lambda) \geq \boldsymbol{\eta}(0) = 0$ we have that for any fixed $\lambda \in \mathbb{R}^d$ and N the function

$$f_{\lambda,N}(u) := e^{-u\boldsymbol{\eta}(\lambda)} \left(\varphi_2(u,k_N)\right)^{N-1} \varphi(u,k_N), \qquad u \geq 0,$$

belongs to the class $\mathcal{K}_u$. Recalling the definition (61) and using $K4$ we conclude that for fixed $\lambda \in \mathbb{R}^d$ and N

$$J_N(t,\lambda) \to J_N(\infty,\lambda) \qquad (t \to \infty) \tag{72}$$

and the limit $J_N(\infty,\lambda)$ is given by the formula (64). Let us show that this convergence in uniform in λ.

We use the following properties of the renewal density function $h(s)$:

$$h(t) \to m^{-1}, \quad (t \to \infty) \qquad M := \sup_{t \geq 0} h(t) \;<\; +\infty.$$

Their proof is similar to the proof of $K4$ (see also [9, § 4.4]). Note that the functions φ and φ_2 are non-negative hence $f_{\lambda,N}(u) \geq 0$. Fix some $A > 0$ and consider $t > A$. Then

$$\begin{aligned} J_N(t,\lambda) &= \int_0^t h(t-s)\, f_{\lambda,N}(s)\, ds = \\ &= \int_0^A h(t-s)\, f_{\lambda,N}(s)\, ds + \int_A^t h(t-s)\, f_{\lambda,N}(s)\, ds. \end{aligned}$$

The second summand can be bounded uniformly in λ as

$$M \int_A^\infty \left(\varphi_2(u,k_N)\right)^{N-1} \varphi(u,k_N)\, du. \tag{73}$$

The intergrand is $f_{0,N}(u) \in \mathcal{K}_u$ hence (73) goes to 0 as $A \to +\infty$. Consider

$$\begin{aligned} \left| \int_0^A h(t-s)\, f_{\lambda,N}(s)\, ds - \frac{1}{m} \int_0^A f_{\lambda,N}(s)\, ds \right| &\leq \int_0^A \left| h(t-s) - m^{-1} \right| \, f_{\lambda,N}(s)\, ds \\ &\leq \int_0^A \left| h(t-s) - m^{-1} \right| \, f_{0,N}(s)\, ds. \end{aligned}$$

By the Lebesque domination theorem the last integral vanishes as $t \to +\infty$. Now it is readily seen that the convergence (72) is uniform in $\lambda \in \mathbb{R}^d$. Lemma 10 is proved. $\square$

4.7 Proofs of Theorems 2 and 3

In this subsection we study asymptotic behavior of the characteristic function

$$\chi_N(\infty;\lambda) = l_N J_N(\infty,\lambda), \qquad \lambda \in \mathbb{R}^d,$$

when N tends to infinity. We will use the representation (64). We start with detailed considerations of the functions φ^* and φ_2^*.

4.7.1 Laplace transforms: decompositions and bounds

Here we obtain decompositions of the functions φ^* and φ_2^*. Let $\varphi^*(z,k_N)$ and $\varphi_2^*(z,k_N)$ be their Laplace transforms:

$$\varphi^*(z,k_N) = \int_0^{+\infty} e^{-zu}\varphi^*(u,k_N)\,du\,, \qquad \varphi_2^*(z,k_N) = \int_0^{+\infty} e^{-zu}\varphi_2(u,k_N)\,du\,.$$

Recall [9, § 3.2] that

$$\varphi^*(z,k_N) = \frac{1-p^*(z)}{z\,(1-k_N p^*(z))} \tag{74}$$

as the Laplace transform of the generating function for the ordinary renewal process. Similarly, $\varphi_2^*(z,k_N)$ is the Laplace transform of the generating function for the modified renewal process (63):

$$\varphi_2^*(z,k_N) = \frac{1-p^*_{V_\infty}(z)}{z\,(1-k_N p^*(z))}\,.$$

It follows from basic properties of the Laplace transform [9, § 1.3] that

$$p^*_{V_\infty}(z) = \frac{1-p^*(z)}{mz}\,.$$

Hence the Laplace transform of the function (60) is

$$\varphi_2^*(z,k_N) = \frac{p^*(z)-1+mz}{mz^2} + k_N\frac{1-p^*(z)}{mz}\,\varphi^*(z,k_N). \tag{75}$$

It can be rewritten as

$$\varphi_2^*(z,k_N) = \varphi^*(z,k_N) + (1-k_N)\,\frac{p^*(z)-1+mz\cdot p^*(z)}{mz^2\cdot(1-k_N p^*(z))}\,.$$

Finally we get

$$\varphi_2^*(z,k_N) = \varphi^*(z,k_N)\,(1+(1-k_N)\,\vartheta(z)) \tag{76}$$

where

$$\vartheta(z) \;=\; \frac{p^*(z)-1+mz\cdot p^*(z)}{mz\cdot(1-p^*(z))}\,.$$

Remark 7 *While the Markovian case (34) was completely discussed in Subsection 4.4 is is interesting to see its exceptionality in the formulae derived for the general situation. If* $p(s) = m^{-1}\exp(-s/m)$, $s \geq 0$, *the density of exponential distribution with the mean* m^{-1}, *then one can easily check that*

$$p^*(z) = \frac{1}{1+mz}\,, \qquad \vartheta(z) = 0\,, \qquad \varphi^*(z,k_N) = \varphi_2^*(z,k_N) = \left(z+\frac{1-k_N}{m}\right)^{-1}.$$

If inter-event intervals have a non-exponential distribution then $\vartheta(z) \neq 0$.

In the general case we see from formulae (74)–(76) that $\varphi^*(z,k_N)$ and $\varphi_2^*(z,k_N)$ are RPF-functions. Our goal is to obtain representation (9) for these functions. First of all we will find their poles.

We use the following notation. $\mathcal{P}(g)$ denotes the set of poles of a rational function $g = g(z)$ and $\mathcal{R}(g)$ denotes the set of its roots: $\mathcal{R}(g) = \{z : g(z) = 0\}$. From (75) we see that φ_2^* has the same singularities as φ^*. Hence

$$\mathcal{P}(\varphi_2^*) = \mathcal{P}(\varphi^*). \tag{77}$$

Let all assumptions of Subsection 3.3 hold. Recall that $r_0 = 0$ is a simple root of the equation $1 - p^*(z) = 0$. If

$$\mathcal{R}(1 - p^*(z)) = \{0, r_1, \ldots, r_q\}$$

denotes the set of different roots of the equation $1-p^*(z) = 0$ then by Lemma 1 all numbers $r_1, \ldots, r_q$ belong to the subplane $\operatorname{Re} z < 0$. By Assumption P3 the roots $r_1, \ldots, r_q$ are simple that is $(p^*)'(r_j) \neq 0$.

It is well known that roots of a polynomial depend continuously on its coefficients (see, for example, [26] or [79, Th. 2.7.1]). The coefficients of the equation $1 - k_N p^*(z) = 0$ are analytic in k_N in the vicinity of 1. Hence for sufficiently large N the "perturbed" equation $1 - k_N p^*(z) = 0$ has $q + 1$ different roots

$$\mathcal{R}(1 - k_N p^*(z)) = \left\{\varkappa_N, r_1^{(N)}, \ldots, r_q^{(N)}\right\}. \tag{78}$$

It follows from the general theory [33, Ch. 9, § 2] that the roots (78) are also simple. Any root $r_j^{(N)}$ is close to the root r_j in the following sense

$$r_j^{(N)} \to r_j \quad as \quad N \to \infty. \tag{79}$$

It is straightforward to check that $\varkappa_N$ is real and, moreover,

$$\varkappa_N = -\frac{\gamma_N}{m_1} + \frac{m_2}{2m_1^3}\gamma_N^2 + o(\gamma_N^2) \qquad (N \to \infty) \tag{80}$$

where $\gamma_N = k_N^{-1} - 1$, $m_n = \mathsf{E}\,\Delta^n = \int x^n p(x)\,dx$. In particular, $\varkappa_N < 0$ and $\varkappa_N \to r_0 = 0$. Hence for sufficiently large N all roots listed in (78) belong to the subplane $\operatorname{Re} z < 0$. Moreover, the real parts of $r_1^{(N)}, \ldots, r_q^{(N)}$ are separated from 0. Namely, for sufficiently large N

$$r_1^{(N)}, \ldots, r_q^{(N)} \in \left\{ z : \operatorname{Re} z < \frac{1}{2} \max_{j=1,\ldots,q} \operatorname{Re} r_j < 0 \right\}. \tag{81}$$

The representation (9) for the function φ^* takes the following form

$$\varphi^*(z,k_N) = \frac{c_0^{(N)}}{z - \varkappa_N} + \sum_{j=1}^{q} \frac{c_j^{(N)}}{z - r_j^{(N)}}. \tag{82}$$

Since the function φ_2^* has the same poles as the function φ^* we obtain also

$$\varphi_2^*(z,k_N) = \frac{d_0^{(N)}}{z - \varkappa_N} + \sum_{j=1}^{q} \frac{d_j^{(N)}}{z - r_j^{(N)}}. \tag{83}$$

We need some bounds for the coefficients of these decompositions.

Lemma 12 *There exist $C_1 > 0$ and $C_2 > 0$ such that for sufficiently large N*

$$\left|c_0^{(N)} - 1\right| < C_1\gamma_N, \quad \left|c_j^{(N)}\right| < C_1\gamma_N,$$

$$\left|d_0^{(N)} - 1\right| < C_2\gamma_N^2, \quad \left|d_j^{(N)}\right| < C_2\gamma_N^2, \quad j = 1, \ldots, q\,.$$

Proof of Lemma 12. All we need to prove the lemma is a careful calculation of residuals. Consider (74). We have

$$c_0^{(N)} = \operatorname*{res}_{z=\varkappa_N} \varphi^*(z, k_N) = \frac{1 - p^*(\varkappa_N)}{-k_N\,(p^*)'\,(\varkappa_N)\,\varkappa_N} = \frac{1 - k_N^{-1}}{-k_N\,(p^*)'\,(\varkappa_N)\,\varkappa_N}\,.$$

Using the Taylor's theorem with the Lagrange form of the remainder we have

$$\begin{aligned} p^*(0) - p^*(\varkappa_N) \;&=\; (p^*)'\,(\varkappa_N)\,(-\varkappa_N) + \frac{1}{2!}\,(p^*)''\,(\varkappa_N)\,(-\varkappa_N)^2 + \frac{1}{3!}\,(p^*)^{(3)}\,(\varkappa_N)\,(-\varkappa_N)^3 \\ &\quad + \frac{1}{4!}\,(p^*)^{(4)}\,(\xi_N)\,(-\varkappa_N)^4 \end{aligned} \tag{84}$$

for some $\xi_N \in [\varkappa_N, 0]$. Expanding

$$\begin{aligned} (p^*)''\,(\varkappa_N) \;&=\; (p^*)''\,(0) + (p^*)^{(3)}\,(0)\,\varkappa_N + O\left(\varkappa_N^2\right), \\ (p^*)^{(3)}\,(\varkappa_N) \;&=\; (p^*)^{(3)}\,(0) + O\left(\varkappa_N\right), \qquad N \to \infty, \end{aligned}$$

we get from (84)

$$(p^*)'\,(\varkappa_N)\,\varkappa_N = \left(k_N^{-1} - 1\right) + \frac{1}{2}\,(p^*)''\,(0)\varkappa_N^2 + \left(\frac{1}{2} - \frac{1}{6}\right)(p^*)^{(3)}\,(0)\varkappa_N^3 + O(\varkappa_N^4).$$

Taking into account that $(p^*)^{(n)}\,(0) = (-1)^n m_n$ and $k_N = (1 + \gamma_N)^{-1}$ we obtain

$$c_0^{(N)} = \frac{(1 + \gamma_N)\,\gamma_N}{\gamma_N + \frac{1}{2}m_2\varkappa_N^2 - \frac{1}{3}m_3\varkappa_N^3 + O(\varkappa_N^4)}\,.$$

Using (80) we come to the expansion

$$c_0^{(N)} = 1 + \left(1 - \frac{m_2}{2m_1^2}\right)\gamma_N + \frac{9m_2^2 - 6m_1^2 m_2 - 4m_1 m_3}{12m_1^4}\,\gamma_N^2 + O(\gamma_N^3). \tag{85}$$

It is seen from (76) that

$$d_0^{(N)} = c_0^{(N)}\left(1 + (1 - k_N)\theta(\varkappa_N)\right) \tag{86}$$

It is easy to check that

$$\theta(\varkappa_N) = -1 + \frac{m_2}{2\,{m_1}^2} + \frac{(3\,{m_2}^2 - 2\,m_1\,m_3)\ \varkappa_N}{12\,{m_1}^3} + \frac{(3\,{m_2}^3 - 4\,m_1\,m_2\,m_3)\ \varkappa_N^2}{24\,{m_1}^4} + O(\varkappa_N^3).$$

Taking into account that $(1 - k_N) = \gamma_N - \gamma_N^2 + O(\gamma_N^3)$ and using (80) we get

$$1 + (1 - k_N)\theta(\varkappa_N) = 1 + \left(-1 + \frac{m_2}{2\,{m_1}^2}\right)\gamma_N + \left(1 - \frac{m_2}{2\,{m_1}^2} - \frac{{m_2}^2}{4\,{m_1}^4} + \frac{m_3}{6\,{m_1}^3}\right)\gamma_N^2 + O(\gamma_N^3).$$

Combining the latter decomposition with (86) we see that $d_0^{(N)}$ does not contain a term proportional to the first power of γ_N:

$$d_0^{(N)} = 1 + \frac{(3\,{m_2}^2 - 2\,m_1\,m_3)\ \gamma_N^2}{12\,{m_1}^4} + O(\gamma_N^3). \tag{87}$$

Remark 8 *If $p(x)$ is an exponential p.d.f. then coefficients in front of γ_N and γ_N^2 in (85) and (87) vanish.*

Let us estimate $c_j^{(N)}$ and $d_j^{(N)}$, $j = 1, \ldots, q$. They are residuals of the first order poles. Hence

$$c_j^{(N)} = \operatorname*{res}_{z=r_j^{(N)}} \varphi^*(z, k_N) = \frac{1 - k_N^{-1}}{-k_N \, (p^*)' \, (r_j^{(N)}) \, r_j^{(N)}} = \frac{\gamma_N}{k_N \, (p^*)' \, (r_j^{(N)}) \, r_j^{(N)}} ,$$

$$d_j^{(N)} = \operatorname*{res}_{z=r_j^{(N)}} \varphi_2^*(z, k_N) = k_N \frac{1 - p^*(r_j^{(N)})}{m r_j^{(N)}} \, c_j^{(N)} = -\frac{k_N \gamma_N}{m r_j^{(N)}} \, c_j^{(N)}.$$

By (78)–(79) there exists $N_0 > 0$ such that the numbers $r_j^{(N)}$ and $(p^*)' \, (r_j^{(N)})$, $j = 1, \ldots, q$, are separated from 0 uniformly in $N \geq N_0$. So we come to the conclusion that for some $C_1, C_2 > 0$

$$\left| c_j^{(N)} \right| < C_1 \gamma_N, \quad \left| d_j^{(N)} \right| < C_2 \gamma_N^2, \quad j = 1, \ldots, q \, .$$

Lemma 12 is proved. □

Using simple properties of the Laplace transform and the decompositions (82)–(83) we come to the following representations of the functions $\varphi(t, k_N)$ and $\varphi_2(t, k_N)$:

$$\varphi(t, k_N) = c_0^{(N)} \exp\left(\varkappa_N t\right) + \sum_{j=1}^{q} c_j^{(N)} \exp\left(r_j^{(N)} t \right) ,$$

$$\varphi_2(t, k_N) = d_0^{(N)} \exp\left(\varkappa_N t\right) + \sum_{j=1}^{q} d_j^{(N)} \exp\left(r_j^{(N)} t \right) .$$

Combining the above formulae with (80)–(81) and Lemma 12 we get the next lemma.

Lemma 13 *There exist $N_0 \in \mathbb{N}$, $\delta > 0$ and $C > 0$ such that for all $t \geq 0$*

$$\sup_{N \geq N_0} \left| \varphi(t, k_N) - c_0^{(N)} \exp\left(- \left| \varkappa_N \right| t\right) \right| < C \gamma_N \exp(-\delta t)$$

$$\sup_{N \geq N_0} \left| \varphi_2(t, k_N) - d_0^{(N)} \exp\left(- \left| \varkappa_N \right| t\right) \right| < C \gamma_N^2 \exp(-\delta t)$$

This lemma is very essential for the further proof. Moreover, in order to prove our main results under assumptions weaker than P2 and P3 one should first derive Lemma 13 under that new assumptions.

4.7.2 Asymptotics for large N

From Lemma 10 we know that

$$\chi_N(\infty; \lambda) = l_N J_N(\infty, \lambda) \tag{88}$$

where

$$J_N(\infty, \lambda) = \frac{N}{m} \int_0^\infty du \, e^{-u \boldsymbol{\eta}(\lambda)} \left(\varphi_2(u, k_N) \right)^{N-1} \varphi(u, k_N). \tag{89}$$

Now we will study the limit of $\chi_N(\infty;\lambda)$ as $N\to\infty$. Recall that $l_N\sim c/N^2$. Main idea is to show that

$$\sup_{\lambda\in\mathbb{R}^d}|l_N J_N(\infty,\lambda)-l_N J_N^\circ(\lambda)|\to 0 \quad as \quad N\to\infty \tag{90}$$

where

$$J_N^\circ(\lambda)=\frac{N}{m}\int_0^\infty du\, e^{-u\boldsymbol{\eta}(\lambda)}\left(d_0^{(N)}\right)^{N-1} c_0^{(N)}\exp\left(-N\left|\varkappa_N\right|u\right).$$

In other words $J_N^\circ(\lambda)$ is obtained from $J_N(\infty,\lambda)$ by the formal replacement of the functions $\varphi_2(u,k_N)$ and $\varphi(u,k_N)$ by their principal asymptotics (see Lemma 13). To prove (90) we will use the following bounds

$$|\varphi(t,k_N)|\le(1+C_3\gamma_N)\exp\left(-\left|\varkappa_N\right|t\right)$$

$$|\varphi_2(t,k_N)|\le\left(1+C_3\gamma_N^2\right)\exp\left(-\left|\varkappa_N\right|t\right) \tag{91}$$

where $C_3=C+\max\left(C_1,C_2\right)$ and $N\ge N_0$. Then

$$A_N(\lambda):=\left|J_N(\infty,\lambda)-\frac{N}{m}\int_0^\infty du\, e^{-u\boldsymbol{\eta}(\lambda)}\left(\varphi_2(u,k_N)\right)^{N-1}c_0^{(N)}\exp\left(-\left|\varkappa_N\right|u\right)\right|\le$$

$$\begin{aligned}&\le\ \frac{N}{m}\int_0^\infty du\ \left(\left(1+C_3\gamma_N^2\right)\exp\left(-\left|\varkappa_N\right|u\right)\right)^{N-1}C\gamma_N\exp(-\delta u)\\&\le\ \frac{N}{m}\left(1+C_3\gamma_N^2\right)^{N-1}\left(C\gamma_N\right)\frac{1}{\delta+(N-1)\left|\varkappa_N\right|}.\end{aligned}$$

Recall that $\gamma_N=k_N^{-1}-1\sim\varkappa/N^2$, therefore $\left(1+C_3\gamma_N^2\right)^{N-1}\to 1$ as $N\to\infty$. Hence

$$A_N(\lambda)\ \le\ \frac{N}{m}\delta^{-1},\quad \forall N\ge N_1,$$

for some specially chosen $N_1\ge N_0$. Consider now

$$B_N(\lambda):=\left|\frac{N}{m}\int_0^\infty du\, e^{-u\boldsymbol{\eta}(\lambda)}\left(\varphi_2(u,k_N)\right)^{N-1}c_0^{(N)}\exp\left(-\left|\varkappa_N\right|u\right)-J_N^\circ(\lambda)\right|.$$

Denote $a_N(u)=\varphi_2(u,k_N)$ and $b_N(u)=d_0^{(N)}\exp\left(-\left|\varkappa_N\right|u\right)$. We have

$$\left(a_N(u)\right)^{N-1}-\left(b_N(u)\right)^{N-1}=\left(a_N(u)-b_N(u)\right)\sum_{i=0}^{N-2}\left(a_N(u)\right)^i\left(b_N(u)\right)^{N-1-i}.$$

By (91) and Lemmas 12 and 13 the following bounds hold

$$\max\left(\left|a_N(u)\right|,\left|b_N(u)\right|\right)\le\left(1+C_3\gamma_N^2\right)\exp\left(-\left|\varkappa_N\right|u\right)$$

$$\sup_{N\ge N_0}|a_N(u)-b_N(u)|<C\gamma_N^2\exp(-\delta u).$$

Therefore

$$\left|\left(a_N(u)\right)^{N-1}-\left(b_N(u)\right)^{N-1}\right|\ \le\ C\gamma_N^2e^{-\delta u}\left(N-1\right)\left(1+C_3\gamma_N^2\right)^{N-1}\exp\left(-(N-1)\left|\varkappa_N\right|u\right)$$

So

$$B_N(\lambda) \le \frac{N}{m} \underbrace{C\gamma_N^2 (N-1) \left(1 + C_3\gamma_N^2\right)^{N-1} (1 + C_3\gamma_N)}\int_0^\infty du \exp\left(-\delta u - N\left|\varkappa_N\right| u\right).$$

Since the underbraced expression vanishes as $N \to \infty$ we get $B_N \le \frac{N}{m}\delta^{-1}$ for sufficiently large $N \ge N_2$.

We see that the following estimate

$$\left|l_N J_N(\infty, \lambda) - l_N J_N^\circ(\lambda)\right| \le \left(A_N(\lambda) + B_N(\lambda)\right) l_N \le \frac{2}{m\delta} N l_N \tag{92}$$

holds for $N \ge \max(N_1, N_2)$. Now the statement (90) easily follows because $N l_N \to 0$ as $N \to \infty$.

Remark 9 *It is easy to see that we are able to get a bound even better than (92), namely,* $cl_N/(m\delta N)$.

We just proved that $\chi_N(\infty; \lambda) = l_N J_N^\circ(\lambda) + \theta_{2,N}^\circ(\lambda)$ for some function $\theta_{2,N}^\circ(\lambda)$ such that the bound

$$\left|\theta_{2,N}^\circ(\lambda)\right| \le \frac{2Nl_N}{m\delta}, \qquad N \ge \max(N_1, N_2),$$

holds for any function $\boldsymbol{\eta} = \boldsymbol{\eta}(\lambda) \ge 0$.

Let us calculate $l_N J_N^\circ(\lambda)$. We have

$$\begin{aligned} l_N J_N^\circ(\lambda) &= \left(d_0^{(N)}\right)^{N-1} c_0^{(N)} \frac{l_N N}{m} \int_0^\infty du\, e^{-u\boldsymbol{\eta}(\lambda)} \exp\left(-N\left|\varkappa_N\right| u\right) \\ &= \left(d_0^{(N)}\right)^{N-1} c_0^{(N)} m^{-1} \frac{l_N N}{N\left|\varkappa_N\right| + \boldsymbol{\eta}(\lambda)} \\ &= \frac{1 + \theta_{3,N}}{1 + \theta_{1,N}\boldsymbol{\eta}(\lambda)} \end{aligned}$$

where

$$\theta_{1,N} := \left(N\left|\varkappa_N\right|\right)^{-1}, \qquad \theta_{3,N} := \left(d_0^{(N)}\right)^{N-1} c_0^{(N)} m^{-1} l_N / \left|\varkappa_N\right| - 1.$$

As it is seen from the above estimates (Lemma 12)

$$\left(d_0^{(N)}\right)^{N-1} c_0^{(N)} \to 1 \qquad (N \to \infty).$$

Recall that

$$l_N = \frac{\varkappa}{(N-1)N}, \quad k_N = 1 - l_N, \quad \varkappa_N \sim -\frac{k_N^{-1} - 1}{m} = -\frac{l_N}{k_N m}.$$

We see that if $N \to \infty$ then $\theta_{3,N} \to 0$ and $\theta_{1,N} \sim mN/\varkappa$. Since $\boldsymbol{\eta}(\lambda) \ge 0$ we can write

$$\chi_N(\infty; \lambda) = \frac{1}{1 + \theta_{1,N}\boldsymbol{\eta}(\lambda)} + \theta_{2,N}(\lambda)$$

where the function $\theta_{2,N}(\lambda)$ is bounded by

$$\left|\theta_{2,N}(\lambda)\right| \le \left|\theta_{3,N}\right| + \left|\theta_{2,N}^\circ(\lambda)\right|.$$

It is readily seen that $\theta_{2,N}(\lambda)$ satisfies to the conditions (12) and (13). Theorem 2 is proved. □

Theorem 3 easily follows from Theorem 2 and definitions of domains of attraction to a stable law (§ 3.5).

5 Conclusions

We presented a wide class of stochastic synchronization systems whose dynamics was constructed by means of Lévy processes and superposition of renewal processes. Such systems can be used after minor modification to build non-Markovian mathematically tractable models for various applications in parallel computing, wireless networks etc. For the symmetric N-component models we showed the long time synchronization in the stochastic sense and proved some limit theorems for the synchronized systems as $N \to \infty$. It is interesting to note that the limiting distributions depend on very few parameters (the Lévy exponent $\boldsymbol{\eta}(\lambda)$ and the mean m of an inter-event interval for a single component). This suggests that Theorems 2–3 hold true under more general assumptions.

Future research could be directed at realistic non-Markovian synchronization models generalizing already existing studies of WSNs [51, 53]. Methods of the present paper can also be adapted for studying correlations between components of synchronization systems.

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