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Introducing Functional Thinking in Year 2: a case study of early algebra teaching

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ABSTRACT Sixty-five Year 2 children with ages ranging from six to seven years participated in a teaching experiment to introduce functional thinking. The results show that young children are capable of generalising, can provide examples of relations and functions, can describe the inverse of such relationships and give valid reasons for how they found the inverse relationships. They also indicate that specific features of instruction assist this process, particularly abstracting underlying mathematical relationships, notably the materials used by the teacher and the children, the types of activities and the questions asked by the teacher. This leads to specific implications for the teaching of arithmetic in the early years.

Research in the past 10 years has focused on formal algebraic education for adolescents (MacGregor & Stacey, 1995; Bell, 1996; Kieran et al, 1996; Warren, 1996; Lamon, 1998). In this research, pedagogical approaches to assist the transition from arithmetic reasoning *to* algebraic reasoning have been defined. These involve generalising the patterns found in functional situations such as number patterns, visual patterns and tables of values; developing an understanding of the variable with concrete materials; and using spreadsheets and computers for introducing the concept of the variable (for example, Kieran, 1990; MacGregor & Stacey, 1995; Filloy & Sutherland, 1996; Redden, 1996; Warren, 1996; Raj & Malone, 1997). Yet, in spite of all these new approaches, difficulties still persist.

Recent research has turned to young children and embedding algebraic reasoning *in* arithmetic reasoning. This is a shift *from* the traditional approach of algebraic reasoning that occurs after the development of arithmetic reasoning *to* algebraic reasoning that occurs in conjunction with arithmetic reasoning (for example, Kaput & Blanton, 2001; Warren 2001, 2003; Carpenter et al, 2003; Dougherty & Zilliox, 2003; Malara & Navarra, 2003). This article investigates instruction that assists young children to generalise and formalise their mathematical thinking, and arrive at an understanding of situations involving functions and relations.

Arithmetic vs. Algebra

Mathematics educators have long believed that arithmetic should precede algebra as it provides the foundations for algebra. This has led to the separation of topics within the mathematics curriculum, with arithmetic being taught many years before algebra. Traditionally, this has meant that the main focus in the primary school has been on operations involving particular numbers (arithmetic) before operations with generalised number, variables and functions (algebra). Traditionally, students are taught to count discrete objects (Dougherty & Zilliox, 2003) and, as they move through different number systems, algorithms and routines are commonly changed. It is conjectured that from this traditional approach to early years numeracy development, students fail to develop a consistent conceptual base that can deal with *all* numbers. There has been a focus in arithmetic on procedural fluency and closure (Biggs, 1991), privileging accuracy of answers at the expense of understanding processes used for reaching answers. This does not serve algebra, let alone arithmetic (Herscovics & Linchevski, 1994).

Past research in the elementary years has evidenced that children possess a narrow and restricted knowledge of arithmetic at an early age (even at the end of Year 2). This is believed to impede the later development of algebraic thinking (for example, Carpenter et al, 2003). An example of this is young children's understanding of the equal sign. Many interpret it as meaning 'here comes the answer' (Warren, 2001; Carpenter et al, 2003). Another example is young children's limited understanding of 'turn arounds'; when Year 3 students were asked if 2+3=3+2 and 2-3=3-2 were both true, 25% responded correctly (Warren, 2001). It seems that, as Malara & Navarra (2003) argued, classroom activities in the early years focus on mathematical products rather than on mathematical processes with resulting limited cognisance and misconceptions. Once these misconceptions exist they are very hard to change (Carpenter et al, 2003), and become even more entrenched as these children progress through schooling (for example, Warren, 2003).

Resulting from this research has been a call for a new level of coherence, depth and power in elementary mathematical experiences (Kaput & Blanton, 2001). It is believed that the most pressing factor for algebraic reform is the ability of elementary teachers to 'algebrafy' arithmetic (Kaput & Blanton, 2001), that is, to develop in their students the arithmetic underpinnings of algebra (Warren & Cooper, 2001) and extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). As was argued by Carpenter & Levi (2000), the artificial separation of arithmetic and algebra 'deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years' (p. 1). In this research in the early years we are following the distinction delineated by Malara & Navarra (2003), that is, algebraic thinking is about processes and arithmetic thinking is about products. One of the major components of algebraic thinking is functional thinking.

Functional Thinking

Mathematics has been categorised by Scandura (1971) as having only three foci: things (that is, numbers, shapes and variables); relations (that is, relationships) between things; and transformations (that is, changes) of things. The power of mathematics lies in relations and transformations which give rise to patterns and generalisations, not in the things. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning in the research literature (Sfard, 1991; Johnassen et al, 1993). Thus, the focus of mathematics teaching should be on fostering fundamental skills in generalising, and expressing and systematically justifying mathematical generalisation (Kaput & Blanton, 2001). Such experiences give rise to understandings that are independent of the numbers or things being operated on (for example, a + b = b + a regardless of whether a and b are whole numbers, decimals or variables). Ohlsson (1993) calls such understanding *abstract schema* and argues they are more likely to promote transfer to other mathematical notions than a schema based on particular numbers or content.

Traditionally, primary schools give little emphasis to relations and transformations as objects of study. Arithmetical experiences in the primary schools tend to focus on the development of concepts about things and the use of particular relationships for computation. Unlike algebra, strings of numbers and operations in arithmetic are not considered as mathematical objects but as procedures for arriving at answers (Kieran, 1990). Fundamental to relation and transformation is the concept of the function, a schema about how the value of certain quantities relates to the value of other quantities (Chazan, 1996) or how values are changed or mapped to other quantities. Thus, a function is defined as a variable quantity regarded in its relation to another variable in terms of which it may be expressed, for example one's height in relation to age or numbers in relation to decreasing each by five.

The National Council of Teachers of Mathematics (2000) suggests that the study of mathematical change is fundamental to understanding functions, and the higher levels of mathematics that are based on it (for example, calculus). A study of change not only serves higher levels of mathematics but also assists in a better understanding of the processes of arithmetic. Early experiences changing attributes (for example, colour, shape) are natural and interesting for young children. We are suggesting that these experiences go beyond simply finding and describing patterns of attribute change but also encompass ideas such as qualitative change (for example, *I grew taller*), quantitative change (for example, *I grew 2 cm taller*), relationships between these changes (for example, *If everyone grew taller by the same amount and John's height changed from 143 cm to 145 cm, how much did the class grow by?*) and using these relationships to solve problems (for example, *If Alison's height is now 133 cm, what height was she before she grew?*). Addition itself can also be represented as change: for example, *If I have 3 and increase it by 2 how many do I have*? This is the beginning of functional thinking.

Functional thinking also assists in developing an understanding of the relationships between the operations, particularly the inverse relationship: for example, *If my number is increased by 2 and is now 8, what was my original number?* We conjecture that these ideas support thinking about functions at later stages, help children explore arithmetic as change, make connections between the various operations (that is, addition and subtraction are inverses of each other) and provide opportunities for conjectures and justification at an early age. Research indicates that function is not an area that students typically understand (Chazan, 1996), and that the difficulties students have with the notion of function are due to the abrupt and abstract way they are commonly introduced. Thus, the exploration of functional thinking should be gradual and occur over a long period of time.

The Focus of This Article

This article reports on the initial stages of a four-year longitudinal study investigating young children's development of algebraic reasoning. The study focuses on three components: (1) equivalence and equations (the development of the arithmetic underpinnings of algebraic expressions and equations – equality and inequality, balance and compensation, comparison and order relations [Warren & Cooper, 2003]); (2) generalisation of arithmetic (for example, arithmetic properties [Boulton-Lewis et al, 1998]); and (3) the development of function (pattern and operational change [Warren & Cooper, 2001]).

The article investigates instruction that helps young children generalise and formalise their mathematical thinking, and come to some understanding of situations involving functions and relations. A lesson designed to develop the underpinnings of functional thinking, that is, change and reversing change (backtracking), was prepared and given to three Year 2 classrooms. The specific aims of the investigation were to: (1) document the implementation of the lesson; (2) identify examples of children's algebraic and functional thinking; and (3) determine teacher actions, children's material use and classroom activities that facilitate functional thinking.

Method

The methodology adopted was that of Teaching Experiment, the conjecture-driven approach of Confrey & Lachance (2000). This form takes account of a well-informed theoretical base (conjecture) in order to create and investigate new instructional strategies in natural classroom settings, examining simultaneously children's, teachers' and researchers' perspectives. The conjecture consists of two dimensions, mathematical content and pedagogy linked to the content. The design aims to produce both theoretical analyses and instructional innovations (Cobb, 2000). The process is ever evolving and adjustments are made to mathematical content and pedagogy as a result of an ongoing analysis of the three perspectives.

The researchers taught the same lessons to Year 2 children from three classes from an upper middle class state primary school in an inner city suburb of a major city. The sample, therefore, comprised 65 children from the three classes, their three teachers (Alice, Barbara and Cynthia) and the two researchers. One of the classes, Barbara's class, was a composite Years 1 and 2 class, but only the Year 2 students participated in the study. The three classes had shown differing levels of ability in earlier experiments. This decision was based on (1) previous teaching experiments conducted in these classes (Warren & Cooper, 2002, 2003), (2) the beliefs of the three classroom teachers and (3) students' records of achievement. Alice's class was of average ability, Barbara's Year 2 students were of high ability and Cynthia's class was of average to low ability. The children had been in Year 2 for six months and their ages ranged from six to seven years. The researchers worked collaboratively with the teachers to trial teaching ideas and document student learning in relation to teaching actions. The lesson trialled in this article was an initial activity designed to demonstrate functional thinking. The lesson was taught by one of the researchers.

The task chosen for this teaching experiment was a task that did not require an understanding of number but did exhibit the fundamental characteristic of a function, that is, consistently mapping one set of variables onto another set. The selection of the task was based on the conjecture that from the traditional approach to early years numeracy development, students fail to develop a consistent conceptual base that can deal with *all* numbers and *all* mathematical situations (Dougherty & Zilliox, 2003). The advantage of unmeasured quantities (that is, situations without number) is that young children can investigate and conjecture about the 'big' ideas of mathematics, focus on processes rather than products and develop relevant language and sign systems before they even formally begin number (Davydov, 1975). In this instance the task involved mapping colours of the rainbow onto other colours in the rainbow, according to a consistent rule (for example, the next colour in the rainbow). We also chose a rule where backtracking involved reversing the relationship (for example, find the previous colour in the rainbow) rather than simply reversing the colour order (for example, red goes to blue so blue goes to red), as it is reversing the relationship that is the core understanding required to identify the inverse function (Chazan, 1996).

Function Lesson

The aim of this lesson was to introduce young children to (1) change using a functional approach, and (2) backtracking as an inverse process (that is, the process that 'undoes' the function). The teaching experiment reflected a Vygotskiian approach (Vygotskii, 1978) where young children's potential for learning (zone of proximal development) is inferred from intervention in active learning situations. The format incorporated concrete, real-life, relevant materials and consisted of an interactive discussion between the researcher and the young participants. Because of the age of the children, it was decided to use materials with one attribute, colour, and investigate change with respect to this one attribute, for example changing the red stick to a green stick. To do this, children were provided with a set of large coloured wooden tongue depressors (described by the teacher as 'large paddle pop sticks') and one box (to act as a function machine). The colours were the six colours of the rainbow: red, orange, yellow, green, blue and purple. One of the researchers took the role of teacher. The children were seated on the floor around the researcher and the function machine throughout the lesson.

The lesson began with the researcher discussing with the children what they thought 'function' was and using the children's answers or lack of answers to introduce the notion of change. The researcher then introduced the box as representing a 'function machine' that would change the colour of sticks put into it. The lesson then proceeded through the following three phases.

Phase 1. The children were given the red and green paddle pop sticks. The researcher wrote a change rule, 'Red goes to green and green goes to red', on a piece of paper and placed it in the function box without telling the class what it was. Children in turn gave the researcher one of their paddle pop sticks. It was 'moved' through the machine, changed in colour and the resultant stick handed back. The process was repeated a number of times and the children were asked to identify the rule the researcher was following in making the changes (called the 'change rule'). This 'What's my rule?' activity was repeated as often as needed.

Once they had correctly identified the change rule and the rule was written on the whiteboard, the researcher moved to discussing how to reverse the process (backtrack), that is, how to work out what was put in the machine if you knew what was taken out. To do this, the researcher modelled the reverse process with materials by handing a child a red stick as she asked 'If I gave you a red paddle pop stick, what colour did you put in the box?'

Phase 2. Two more colours were added to the task, blue and yellow, and a second symmetric change rule placed in the function box (for example, 'Blue goes to yellow and yellow goes to blue'). Children now had four colours from which to choose and to take into account in determining the change rule as the proceeding processes were repeated. Because of the concern that children might simply see backtracking as reversing the colour order rather than reversing the relationship, other change rules where there was no symmetry in the change were also experienced (for example, blue goes to red, red goes to yellow, yellow goes to blue). This was also seen as a prerequisite task to the rainbow task. The researcher asked the children to reverse the process after each change was identified.

Phase 3. All six colours of the rainbow were included in the task. The researcher repeated the processes developed in phase 1 and phase 2, but in this instance the colour change involved changing the colour to the next colour in the rainbow. Again, the change was reversed after it had been identified.

As the researchers expected that these tasks (particularly those at the end of phase 2 and in phase 3) would be difficult for Year 2 children, it was decided to record the changes on an IN/OUT table to assist students to find the pattern. A visual cue in the form of a picture of a rainbow was also prepared for phase 3.

Data-gathering Techniques and Procedure

The lessons occurred sequentially, starting with classroom 1, Alice's class, and finishing with classroom 3, Cynthia's class. As the lesson was taught to each class by one of the researchers, the class was videotaped and the other researcher and the classroom teacher acted as participant observers. The video focused on the class as a whole, recording teaching actions and major interactions between teacher and children. The other researcher and classroom teacher recorded field notes of significant events that indicated evidence of learning and algebraic thinking.

The basis of rigour in participant observation is 'the careful and conscious linking of the social process of engagement in the field with the technical aspects of data collection and decisions which that linking involves' (Ball, 1997, p. 311). Thus, both observers acknowledged the interplay between them as classroom participants and their role in the research process. At the completion of teaching in each class, the researcher and teacher briefly reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and to come to some common perspective of the instruction that occurred, and the thinking exhibited by the children participating in the classroom discussions. As a result of these discussions, instruction in the following lesson was modified, particularly in relation to use of materials and form of questioning. The three phases remained unchanged in general.

Results

The videotapes were viewed by both researchers and what was seen on these was combined with the field notes to produce rich descriptions of the teaching and learning in each of the lessons. These descriptions were first analysed in terms of teaching actions (materials, activities and questions) and student activity that indicated functional thinking. Then the three lessons were compared in terms of differences and similarities between these teaching actions and student responses. Finally, conclusions were drawn with respect to the relative effectiveness of the teaching and the form and nature of any development of functional thinking. The three lessons are described in this section and the conclusions drawn are discussed.

Classroom 1

Alice's class was seated on the floor and the materials distributed. Most of the children had little understanding of 'function'. One response was:

> *Researcher*: Does anyone know what a function is?
> *Tim*: It's like a factory machine?
> *Researcher*: What is a factory machine.
> *Tim*: My dad works at a factory it changes things.

The children completed the first two phases of the lesson reasonably successfully, but they did exhibit some difficulties with the reversing process. The following excerpt exemplifies the discussion that ensued at the completion of phase 2:

> *Researcher*: What do you think the box does?
> *Alison*: It's changing.
> *Researcher*: How is it changing?
> *Simon*: Red green red green red green [a sequential pattern].
> *Researcher*: Is that what it is doing? Red green red green red green.
> *Kelly*: No. It is changing the colours of the paddle pop sticks?
> *Researcher*: How is it changing them?
> *Ben*: It's swapping it over with another colour – red to green and green to red [a relational pattern].

With regard to the backtracking component, from an analysis of the videos it appeared that they needed to have the correct coloured stick in their hand in order to ascertain the colour that they had placed in the box. That is, in a task where the change rule was 'Green goes to red', for example, children with a green stick in their hand quickly answered the question 'If I gave you a red stick what colour did you give me?', while children without the correct stick in their hand were more hesitant. There was a concern as to whether the children were actually reversing the processes or simply responding from knowing that green and red go together. This phase was changed for classroom 2.

With the rainbow phase of the lesson, the changes were recorded on the whiteboard using a table in Figure 1.

IN	OUT
Red	Orange
Blue	Purple
Yellow	Green
Orange	Yellow
Purple	Red
Green	Blue

Figure 1. Recording of response to the rainbow function.

Children were asked to identify the pattern. Some responses were 'They are changing!' and 'They are trading!' All children experienced difficulties in recognising the pattern. It was conjectured by the researchers that children's difficulties with this task were for three reasons: the children found the task too complex; they were not helped to see the pattern by the random recording of responses in the table; and they required a visual representation of the problem to help them see the pattern. Thus, the visual picture of a rainbow prepared beforehand was introduced to the class. While this appeared to help some children, many still experienced difficulties. One child responded 'They all change' and another 'They all change position, they go around'. Finally, Susan came out to the rainbow and demonstrated the pattern by running her finger down the rainbow (that is, red to orange to yellow to green and so on), saying as she did so 'Red goes to orange, orange goes to yellow, yellow goes to green…'. Other volunteers were then asked to demonstrate the pattern Susan had found and most were able to do so.

The majority of children in the class appeared to be able to backtrack when the lesson moved to reversing (backtracking). However, it seemed that their focus was on the table rather than on the rainbow: they appeared to be reading the corresponding IN colour from the table.

Classroom 2

Barbara's class was a composite class comprising Year 1 and Year 2 children. The researchers worked with the Year 2 component, consisting of 14 children who previous experience had shown to be high achievers, an observation that was confirmed by Barbara, the classroom teacher.

In response to some of the difficulties that the preceding class experienced with reversing, it was decided to introduce the lesson in Barbara's class by requiring children to initially follow rules rather than guess rules. One child acted as the function machine. The first rule was as before, 'Red gives green and green gives red', only this time all the children knew the rule, as the child holding the function machine shared it with the whole class. Using red and green paddle pop sticks, children were asked to enact the change.

All the children appeared to successfully follow the simple change rule given in phase 1, as was acknowledged in the many children's responses of 'This is too easy', and 'We know this, make it harder'. Therefore, before introducing the 'guess my rule' activity in phase 1, the teacher gave the children another rule to follow ('Green gives two reds and red gives green'). All of the children appeared to also be successful in following this more complicated rule.

In phase 2, more complicated rules were also considered in activities where rules had to be followed, guessed and reversed (for example, 'Red gives green, green gives blue and blue gives red'). The third colour was introduced to ascertain if they could reverse the process without simply using the relationship between two colours (red and green). It was also believed that this step was a necessary intermediate step before introducing the rainbow.

In phase 3 of this lesson, in response to the conjecture that the random recording of responses in the table did not assist in 'seeing' the pattern, the researcher ensured that the recording of the pattern on the whiteboard was in the rainbow sequence. Figure 2 illustrates the recording used for this phase.

IN	OUT
Red	Orange
Orange	Yellow
Yellow	Green
Green	Blue
Blue	Purple
Purple	Red

Figure 2. Recording used for the three colours.

Recording the responses in sequence appeared to assist children to identify the function in the rainbow problem and to backtrack. One child stated that the function was 'Red gives orange, green gives blue, and blue gives red. They go round!' With the exception of Sam, all children could successfully backtrack the function. However, further time spent following or acting out the process involved in change assisted Sam in seeing the relationship. At this point, when Sam 'pretended' to put a stick in the function machine, a blue paddle pop stick was given back to him and he was asked 'What colour did you put in? How did you know?', he explained his correct answer by referring to the rule on the whiteboard and then picking up a green paddle pop stick and putting it in the machine.

The rainbow problem was conducted in this lesson initially without the visual model being used. Recording the responses on a table in the correct sequence (see Figure 2) appeared to obviate the need for visual support. However, the picture of the rainbow did provide a more integrated understanding of the change. The following protocol provides evidence of the role of both table and picture (the first protocols are from a time before the table was completed):

Researcher: What colour would green become?

Susan: Blue.

Researcher: How do you know?

Susan: Because yellow and blue make green. We are mixing colours.

Researcher: What does mixing colours mean?

Susan: It is the colour in the middle, the next colour in the colour wheel.

Researcher: What colour would yellow become?

Phoebe: Green.

Researcher: Why did you say green?

Phoebe: Green is not up there.

Researcher: What does the machine do?

Mark: It is a colour change.
Ben: They are in the middle. Those two go together to make orange [pointing at red and yellow in the table].

Further probing ascertained that Ben believed it was the next colour rather than a colour mix. 'We are going up and down the colours. Red goes to orange and orange goes to yellow.'

The picture of the rainbow was then introduced to the class and the question was posed:

Researcher: What is happening?
Amy: You start like this [points to the top of the rainbow and runs her finger towards the centre].

It is conjectured that the rainbow picture presented them with an easier way of expressing their thinking, that is, clarifying the difference between mixing colours and the next colour. Running your finger up and down the rainbow seemed easier than saying red gives orange, orange gives yellow and so on. It appeared to assist in overcoming the difficulties some children experienced in expressing the relationship in their own language, as it became clear that *mixing colours* was referring to an imagined movement up and down the rainbow or around the colour wheel. The success of the children in this class in following rules, finding rules and backtracking was obvious. The question is, what was it due to? From the observations, it appears that the teaching actions of the researcher were as important as the innate intelligence of the 14 Year 2s.

Classroom 3

As Cynthia's class had shown itself to be the lowest ability of the three in earlier teaching experiments, it was expected that it would have more difficulty with the function tasks than the other classes. Because of the difficulties Alice's class and Sam in Barbara's class had experienced in imagining the backtracking process, the researchers decided to formalise the operation of the function machine in this lesson. This was done by introducing IN and OUT cards and organising children volunteers, labelled with these cards, to put sticks presented by children in the class in the box and to give the resultant colour-changed or unchanged stick back to the children.

Even with this addition to the teaching framework, the children in Cynthia's class did not achieve as much as the children in the preceding two classes. The lesson did not even reach phase 3, the rainbow task. The children had so much difficulty with phase 2, where there were more than two colours, that the researcher/teacher decided not to proceed to phase 3.

Interestingly, the incorporation of IN and OUT appeared to add to the task's complexity when modelling the more complicated functions such as 'Red goes to green, green goes to blue and red'. The number of children involved in acting out the function and the backtracking may have been the cause. For example, the actions involved in 'Red goes to green, green goes to blue, and blue goes to red' when Ned was IN, Bonnie was OUT and Frank was providing the green stick were:

Frank gives green stick to Ned;
Ned puts green stick in box;
Researcher changes the green stick to a red stick and gives this to Bonnie;
Bonnie gives the red to Frank; and
Teacher records the change on the IN/OUT table.

After watching a few examples, the children had to guess or predict the stick that Bonnie would give them. For backtracking the reverse process occurred and the children had to guess/predict the stick they would have given to Ned to get the stick that Bonnie was giving them. The following protocol provides evidence of the difficulties these children experienced with this process:

Researcher: Pick up a colour Bonnie and show the class.
Bonnie picks up a red stick.
Researcher: What colour do you think Ned put in the box?
Michael: A red stick.
Researcher: Would red give red?
Michael: Don't know.

Researcher: Come and we will act it out.
Michael gives the red stick to Ned and Bonnie gives back a green stick to Michael.
Researcher: Is it red?
Michael: No, I don't know I am mixed up.

It appeared that the acting out, in this instance, took the children's focus off the table and thus they were not using the table to help them backtrack the function.

Discussion and Conclusions

The lesson described in this article opposed traditional teaching of early mathematics in that it focused on relations and transformations between things that are related and changed (Scandura, 1971). It introduced functional thinking through a focus on change as recommended in National Council of Teachers of Mathematics (2000). Its focus on reversing change also attempted to build one of the 'powerful schemes of thinking' (Carpenter & Levi, 2000, p. 1) in mathematics, that is, the ability to comprehend and handle the notion of inverse (Krutetskii, 1976). It also focused on following, finding and abstracting patterns upon which structural knowledge can be built as recommended by Johnassen et al (1993), Sfard (1991) and Ohlsson (1993).

The success of the lessons was mixed, with the children of classroom 2 doing much better than those from classrooms 1 and 3. It appears that, similar to research findings on older students' function understanding (Chazan, 1996), young children also find difficulties in comprehending a change or function approach to mathematics.

Findings

Looking at the three lessons and comparing teaching actions and student responses, five findings are evident. First, although there were differences in achievement, it was evident that Year 2 children are capable of early function activities of the type described here. The Vygotskiian approach together with an environment that engendered active learning appeared to both support and challenge individuals' understanding. In addition, children can study change and reverses of change through activities with attributes other than number. Though these children had had some early arithmetic experiences with number, addition and subtraction, none of these experiences appeared to assist in their development of function understanding within the rainbow context. Thus it is conjectured that function understanding can be conducted without number and before operations as simple as addition and subtraction are developed or are familiar. This requires further research.

Second, sequencing of the function activities is important. The modification in classroom 2 so that the lesson started with activities where change was followed (correctly changing sticks after being given the change rule) before it was guessed or identified (determining the change rule from examples of the change) appeared to be productive. It assisted students complete, in particular, the more difficult tasks in phases 2 and 3. This sequencing should be at the pace of the learners. Classrooms 1 and 3 obviously required a slower-paced development than classroom 2 – most possibly more time with two colour changes before moving on to three and four colours.

There were three phases in the lesson but also three types of task within each phase: following a change rule; identifying the change rule; and backtracking. The backtracking was done immediately after the rule was identified so that it was a matter of working backwards; therefore, though more difficult than following a change rule it is still a 'following' not an 'identifying' activity. Focusing on how the children responded to the lesson appears to show the general sequence in Figure 3 for introducing the function tasks.

Although there was no direct evidence, the difficulties experienced by classroom 3 appear to support a view that some classes will benefit from spending time on two-type, single-attribute changes in a variety of attributes (for example, two colours, two lengths, two masses) before progressing to changes with more than two types of the one attribute.

Third, the difficulties experienced in classroom 3 and the success achieved in classroom 2 indicated that the use of pictorial and concrete models 'walks a fine line' between distractor and enabler. For example, the use of tables assisted the children in identifying change rules. This was even more evident when the table was organised so that it was correctly sequenced. As another example, the use of children volunteers to act as IN and OUT with respect to the function machine was a strong distractor in classroom 3.

Following a change rule → Following a backtracking → Identifying a change rule → Identifying a backtracking

Figure 3. Sequence for introducing change and backtracking.

Some things that are 'common wisdoms' in one mathematics topic do not necessarily always work in another. For example, the 'act out/model the problem' strategy is a powerful problem-solving technique for word problems. However, the acting out of the IN/OUT process in the function machine in classroom 3 was not successful. The use of materials has to focus at the heart of the mathematics of the task. This, for the lesson described in this article, is to identify a relationship or pattern in a table of changes that are presented in no given and sometimes near-random order. For identifying relationships, too much activity around the presentation of the change (for example, IN and OUT volunteers passing sticks) may reduce the effectiveness of the pattern finding. This was also found to be the case when junior secondary students utilised patterns to introduce variable; Warren (1996) found that simple number patterns were more successful than complex geometric patterns when the pattern is presented in a traditional manner.

Fourth, backtracking or reversing appears to be more difficult than changing in a forward direction, particularly for some children, yet it is one of the attributes that determines excellence in mathematics (Krutetskii, 1976). It is also a task that the use of prompts such as IN and OUT cards does not appear to assist (see the description of classroom 3). However, visual aids such as the rainbow picture can be powerful enablers of backtracking (see description of classroom 1). Interestingly, this is the case even if only one stick is in the visual aid, as classroom 1 evidences. Children who held the correct colour in their hand were much more confident and accurate in determining the original colour from the changed colour.

Fifth, although studying change is new to early mathematics, following, identifying and constructing patterns is not – it is a common mathematics activity in the early years. However, there are differences between the two, though both are generalising. Traditionally, patterns are presented in a fixed sequence, whilst change activities are presented in a much more random order. An example of the difference, using numbers, is given in Figure 4.

PATTERN		CHANGE	
1	4	5	16
2	7	22	67
3	10	2	7
4	13	13	40
and so on		and so on	

Figure 4. Comparing representations for patterns and changes.

Thus, to generalise change, a learner must do two things not necessary for traditional patterns: sort instances into sequences that assist generalisation; and directly test hypotheses concerning relationships between IN and OUT numbers. Because of the way they are presented, there is a tendency to analyse PATTERN given in the form in Figure 4 vertically, looking down the right hand side column and stating 'it is increasing by 3', whilst the random nature of CHANGE activities means that they are analysed left to right (or IN to OUT). There is no vertical pattern. We suggest that this forces the children to think relationally rather than to think sequentially (a lower level of understanding). It is the relational thinking that is functional thinking, that is, regarding a variable quantity in its relation to another variable in terms of which it can be expressed (for example, the OUT is three times the IN plus one rather than the OUT is three more than the previous OUT, a relationship with the same variable). Although a similar outcome can be achieved by presenting patterns non-traditionally with gaps in the early parts of the sequence, change activities are important to any early childhood mathematics syllabus because they focus children on generalising between two terms (not generalising along a sequence of terms).

Future Research

Future research areas abound from this lesson. It would be of interest to determine if the development issues in this article also emerged when other attributes (for example, length or mass) are used. It would also extend this research to focus on multi-attribute changes, or attribute changes with more than one step. Other work by the researchers has revealed interesting results in changing from attributes such as mass to number. The tendency in children to want to close-in number situations appears to reduce their ability and motivation to generalise relationships. Is this the case for functions? Our present research in young children's thinking with regard to growing and repeating patterns is also indicating that many children have a sequential understanding of patterns, that is, finding the next term according to the previous term. It is the relational understanding that is functional thinking (that is, relating the growing pattern to its position in the pattern). What teacher actions and classroom activities assist the development of this understanding in the early years? How do we balance continuing patterns (both repeating and growing) with exploring the pattern in relationship to its position? We conjecture that the first supports sequential thinking and the second relational thinking, both necessary for mathematical development, but it is the second that leads to the relationship between two variable quantities, that is, functional thinking, thinking that forms the basis of higher levels of mathematics (Scandura, 1971).

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