ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 4, 2014

# INTRODUCTION TO A GRAM-SCHMIDT-TYPE BIORTHOGONALIZATION METHOD

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ABSTRACT. The aim of this expository/pedagogical paper is to describe a Gram-Schmidt biorthogonalization method in such a way that it can be used as an introduction to the subject for undergraduate presentation. The task of biorthogonalization naturally arises when the scalar product of vectors formed are linear combinations of two sets of linearly independent vectors, as the case may be. If one wants the scalar product to have the usual form, the two sets of basis vectors should be biorthogonal. If they are not, the question of biorthogonalization arises. New is the detailed description of the biorthogonalzation method for teaching purposes as well as the comparison of this method with Schmidt's orthogonalization method in the case when the sets of linearly independent vectors are identical.

## 1. Introduction.

**1.1. Aim of this article.** The aim of this expository/pedagogical paper is to describe a Gram-Schmidt biorthogonalization method such that it is suitable for undergraduate classroom teaching and/or research as well as for undergraduate textbooks. In its intention, it is similar to the author's papers [**10, 11, 12, 14**].

To motivate with simple words the need to have a biorthogonalization method available, the following can be said. The task of biorthogonalization naturally arises when the scalar product of vectors are formed that are linear combinations of two sets of linearly independent vectors, as the case may be. If one wants the scalar product to have the usual form, the two sets of basis vectors should be biorthogonal. If they are not, the question of biorthogonalization arises.

*Keywords and phrases.* Gram-Schmidt orthogonalization method, Gram-Schmidt-type biorthogonalization method.

Received by the editors on December 13, 2011, and in revised form on May 16, 2012.

DOI:10.1216/RMJ-2014-44-4-1265

New is:

- a somewhat different proof that the version of the biorthogonalization method given in [**15,** Section 2] does not stop abnormally before the end of the procedure,
- the detailed description of the biorthogonalzation method for teaching purposes, as well as
- the comparison of this method with Schmidt's orthogonalization method in the case when the sets of linearly independent vectors are identical.

The following information is for the advanced reader. The author came across the problem of biorthogonalization in the context of linear functionals in finite-dimensional spaces, see [**20,** page 37, formula (1.61-3)]. There, the question arises as how to biorthogonalize a set of linearly independent functionals and a set of linearly independent vectors. Since, in a finite-dimensional space $X$ with scalar product $(\cdot,\cdot)$, every functional $f$ may be represented in the form $f(u)=(u,v)$, $u\in X$, with an element $v\in X$ that is uniquely determined by the functional $f$ (see [**8,** Chapter III, Section 3, (4)]), this leads to the question of biorthogonalization of two sets of linearly independent vectors. This last problem is more elementary and thus more appropriate for undergraduate teaching than the original one.

### 1.2. Remarks on orthogonalization methods.

(i) *On the history of orthogonalization methods.* The Schmidt orthogonalization method of linearly independent vectors, as it is used nowadays, is first described in a paper on integral equations in 1907 (cf. [**16,** pages 442–443, Section 3]). In [**16,** page 442], Schmidt himself refers to an article of Gram [**5**] published in 1883. Therefore, Schmidt's orthogonalization method is also called the Gram-Schmidt orthogonalization method in the literature, especially in English literature. We note, however, that in [**5**], Gram describes the construction of an orthogonal system of functions from linearly independent functions in a very different form (see [**5,** pages 42–47, Chapter I]), and notably [**5,** page 44, formula (8)]). Further, in [**5,** page 45, footnote], Gram himself refers to an article of Chebyshev in Liouville's Journal, Sér. II, T. III. (1858), page 320. Moreover, in [**24**], on the Gram-Schmidt orthogonalizazion method, it is mentioned that this method already appears in the works of Laplace and Cauchy, but no references for this are given. The fundamental articles of Schmidt [**16**] and Gram [**5**] are referred to, by the way, in [**8,** Reference 96, Reference 176] as well as in [**9,** Reference 96, Reference 176], the article of Schmidt [**16**] is also cited in [**24**]. The orthogonalization (unitarization) of matrices by Schmidt's method is discussed in [**22**].

(ii) *Schmidt's orthogonalization method in textbooks.* Schmidt's orthogonalization method can be found in a large number of textbooks on matrices, on linear algebra, and on numerical analysis, cf., e.g., [**1,** subsection 2.5.2], [**2,** subsection 9.6], [**4,** subsection 5.2.7], [**6,** Section 65], [**13,** subsection 5.3], [**18,** subsection 7.1] and [**19,** subsection 7.1].

(iii) *Other numerical orthogonalization methods.* Nowadays, a large number of other orthogonalization methods exist. First of all, we mention the modified Gram-Schmidt method in [**4,** subsection 5.28], the method of Householder reflections (transformations) and the method of Givens rotations, cf., [**4,** subsection 5.1] and [**17,** subsection 7.2]. However, of all the orthogonalization methods, the Gram-Schmidt orthogonalization method is the simplest and thus especially appropriate for undergraduate teaching and/or research.

### 1.3. Remarks on biorthogonalization methods.

(i) *On the history of biorthogonalization methods.* The earliest article on biorthogonalization the author has found is that of Unger [**23**] published in 1953, where the biorthogonalization of the principal vectors of matrices $A$ and $A^*$ is discussed. This paper is cited in [**25,** subsection 9.1, page 102].

The earliest publication on a biorthogonalization method for two general sets of linearly independent vectors the author has found is that given by Hestenes [**7**] in 1958. There, a biorthogonalization method is described for two set of linearly independent vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ such that the vectors $u_1, \ldots, u_n$ remain unchanged. Hestenes uses his method to compute the inverse of a matrix, and thus this method cannot reproduce the results by Schmidt's method in the case

$u_i = v_i$, $i = 1, \ldots, n$. Therefore, it cannot be called a Schmidt-type biorthogonalization method.

The second article we want to mention is that by Parlett, Taylor and Liu [**15**] in 1985 where, in Section 2, a Gram-Schmidt biorthogonalization method is stated in a very concise form and where another new Lanczos-type method for computing a biorthogonal set of vectors is described.

Many other biorthogonalization methods exist in the literature. We do not try to give a survey on all biorthogonalization methods here.

(ii) *Biorthogonalization methods in textbooks.* The appearance of biorthogonalization methods in textbooks is rare. No biorthogonalization method is found in the above textbooks on matrices, linear algebra, or numerical analysis containing orthogonalization methods, except in [**21,** Chapter 39], a reference to [**23**] in [**25,** subsection 9.2] and the Lanczos-type method in the newer textbook [**1,** subsection 4.7] that is downloadable from the Internet.

**1.4. Outline of the contents.** Now, we outline the contents of the present paper.

In Section 2, we review the classical Schmidt orthogonalization method. In Section 3, Schmidt's method is generalized to the biorthogonalization of two sets of linearly independent vectors that avoids breakdown where the proof is not based on the Hahn-Banach theorem. In Section 4, the special case when both sets are identical is discussed. It will turn out that we then get essentially the same result as that obtained by Schmidt's orthogonalization method; more precisely, there is a close relationship between the results of both methods in this special case. In Section 5, numerical examples follow illustrating the Gram-Schmidt-type biorthogonalization method and comparing it with Gram-Schmidt's orthogonalization method in the case of two identical sets of linearly independent vectors. In Section 6, conclusions are drawn.

## 2. Schmidt's orthogonalization method.

In this section, we review Schmidt's orthogonalization method since we want to use it as a model for a biorthogonalization method.

Let $X$ be a vector space over the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ with scalar product $(\cdot, \cdot)$. Let $u_1, \ldots, u_n \in X$ be *linearly independent vectors.* Further, let $\| \cdot \|$ be the norm on $X$ induced by the scalar product. Then, *Schmidt's orthogonalization method* is given by

$$\boxed{\begin{array}{lll} v_1 := u_1 & ; \quad w_1 := \dfrac{v_1}{\|v_1\|} & \\[2ex] v_j := u_j - \displaystyle\sum_{k=1}^{j-1} (u_j, w_k) w_k & ; \quad w_j := \dfrac{v_j}{\|v_j\|}, & j = 2, \ldots, n \end{array}}$$

(see [**18,** subsection 7.1]), [**6,** Section 65], and [**14**]).

One has

$$M_1 := [u_1] = [v_1] = [w_1]$$

$$M_j := [u_1, \ldots, u_j] = [v_1, \ldots, v_j] = [w_1, \ldots, w_j], \quad j = 2, \ldots, n$$

and

$$(v_j, v_k) = 0, \quad j \neq k, \ j, k = 1, \ldots, n$$

as well as

$$(w_j, w_k) = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

**Remark 1.** If $u_i$, $i = 1, \ldots, n$, is already an orthonormal system, then we get $w_j = v_j = u_j$, $j = 1, \ldots, n$.

**3. The Gram-Schmidt-type biorthogonalization method.** In this section, we describe in detail a biorthogonalization method. This method is inspired by Schmidt's orthogonalization method and is stated in a very concise form in [**15,** Section 2]. The version presented is well defined, that is, it does not stop before the procedure normally ends since no division by zero occurs. Thus, breakdown (as it is termed in [**15,** Section 2]) is avoided. The proof for this is somewhat different from that in the above-mentioned paper.

Let $X$ again be a vector space over the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, and let $(\cdot, \cdot)$ be a scalar product on $X$. Let $a_1, \ldots, a_n \in X$ and $e_1, \ldots, e_n \in X$ be two sets of linearly independent vectors that span the space $X$, i.e., with $[a_1, \ldots, a_n] = X$ and $[e_1, \ldots, e_n] = X$.

We shall show that vectors $c_1, \ldots, c_n \in X$ and $g_1, \ldots, g_n \in X$ can be constructed with the properties

(P1) $\quad (c_i, g_j) = \delta_{ij}, \quad i, j = 1, \ldots, n$

and

(P2) $\quad [c_1, \ldots, c_n] = [a_1, \ldots, a_n], [g_1, \ldots, g_n] = [e_1, \ldots, e_n].$

More generally, we obtain the following:

*Algorithm for the biorthogonalization method*:

*First step.* Set

$$\boxed{b_1 := a_1}$$

and $f_1 \in \{e_1, \ldots, e_n\}$ exists such that

$$(b_1, f_1) \neq 0,$$

since otherwise $(b_1, f) = 0$, $f \in X$. Then, the choice of $f = b_1$ would entail $b_1 = 0$, which is not possible because $b_1 = a_1 \neq 0$.

Define

$$c_1 := \frac{1}{(b_1, f_1)}\, b_1.$$

This entails

$$(c_1, f_1) = 1.$$

Define

$$\boxed{g_1 := f_1.}$$

From this,

$$(c_1, g_1) = 1.$$

Further,

$$\begin{aligned} [c_1] &= [b_1] = [a_1]\,, \\ [g_1] &= [f_1]\,. \end{aligned}$$

*Step from $k$ to $k+1$ with $1 \leq k \leq n-1$.* Let $c_1, \ldots, c_k$ and $g_1, \ldots, g_k$ be constructed such that $(c_i, g_j) = \delta_{i,j}$, $i, j = 1, \ldots, k$ with

$$\begin{aligned} [c_1, \ldots, c_k] &= [b_1, \ldots, b_k] = [a_1, \ldots, a_k]\,, \\ [g_1, \ldots, g_k] &= [f_1, \ldots, f_k]\,. \end{aligned}$$

Define

$$\boxed{b_{k+1} := a_{k+1} - \sum_{i=1}^{k} (a_{k+1}, g_i)\, c_i.}$$

There exists an $f_{k+1} \in \{e_1, \ldots, e_n\}$ such that

$$(b_{k+1}, f_{k+1}) \neq 0,$$

for the same reason as above for $(b_1, f_1) \neq 0$, from which we conclude

$$f_{k+1} \neq f_i, \quad i = 1, \ldots, k$$

since $(b_{k+1}, g_l) = 0$, $l = 1, \ldots, k$, and thus $(b_{k+1}, f_l) = 0$, $l = 1, \ldots, k$. Set

$$\boxed{c_{k+1} := \frac{1}{(b_{k+1}, f_{k+1})}\, b_{k+1}.}$$

This leads to

$$(c_{k+1}, g_i) = 0, \quad i = 1, \ldots, k$$

and

$$(c_{k+1}, f_{k+1}) = 1.$$

Define

$$\boxed{g_{k+1} := f_{k+1} - \sum_{i=1}^{k} (f_{k+1}, c_i)\, g_i = f_{k+1} - \sum_{i=1}^{k} \overline{(c_i, f_{k+1})}\, g_i.}$$

This implies

$$(c_i, g_{k+1}) = \delta_{i,k+1}, \quad i = k+1, k, \ldots, 2, 1.$$

Further,

$$\begin{aligned} [c_1, \ldots, c_{k+1}] &= [b_1, \ldots, b_{k+1}] = [a_1, \ldots, a_{k+1}], \\ [g_1, \ldots, g_{k+1}] &= [f_1, \ldots, f_{k+1}], \end{aligned}$$

$k = 1, \ldots, n-1$, as well as

$$[g_1, \ldots, g_n] = [f_1, \ldots, f_n] = [e_1, \ldots, e_n]. \qquad \square$$

**Remark 2.** For $n = 3$, one obtains

| | | |
|---|---|---|
| $\boxed{1}$ $(c_1, g_1) = 1$ | $\boxed{4}$ $(c_1, g_2) = 0$ | $\boxed{9}$ $(c_1, g_3) = 0$ |
| $\boxed{2}$ $(c_2, g_1) = 0$ | $\boxed{3}$ $(c_2, g_2) = 1$ | $\boxed{8}$ $(c_2, g_3) = 0$ |
| $\boxed{5}$ $(c_3, g_1) = 0$ | $\boxed{6}$ $(c_3, g_2) = 0$ | $\boxed{7}$ $(c_3, g_3) = 1$ |

Here, $\boxed{i}$ indicates the $i$th step of the procedure. Further,

$$[c_1] = [b_1] = [a_1]\,; \quad [g_1] = [f_1]$$
$$[c_1, c_2] = [b_1, b_2] = [a_1, a_2]\,; \quad [g_1, g_2] = [f_1, f_2]$$
$$[c_1, c_2, c_3] = [b_1, b_2, b_3] = [a_1, a_2, a_3]\,;$$
$$[g_1, g_2, g_3] = [f_1, f_2, f_3] = [e_1, e_2, e_3]\,. \quad \square$$

**Remark 3.** If the systems $\{a_1, \dots, a_n\}$, $\{e_1, \dots, e_n\}$ are already biorthonormal, that is, $(a_i, e_j) = \delta_{ij}$, then we obtain $c_j = b_j = a_j$, $g_j = f_j = e_j$, $j = 1, \dots, n$. Another special case will be discussed in the next section.

**Remark 4.** For the advanced reader, we mention that the proof above that an element $f_1 \in \{e_1, \dots, e_n\}$ exists such that $(b_1, f_1) \neq 0$, respectively an element $f_{k+1} \in \{e_1, \dots, e_n\}$, such that $(b_{k+1}, f_{k+1}) \neq 0$ could have also been based on the Hahn-Banach theorem, cf., [**8,** Chapter IV, Section 2, Theorem 2 and Corollary 1] for $\mathbf{F} = \mathbf{R}$ as well as [**8,** Chapter IV, Section 2, Theorem 6] for $\mathbf{F} = \mathbf{C}$.

**4. Special case: Biorthogonalization of identical systems.** In this section, we discuss an important special case, namely, the biorthogonalization of two identical systems; it leads essentially to the same result as Schmidt's orthogonalization method.

As earlier, let $X$ be a vector space over the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ with scalar product $(\cdot, \cdot)$ and consider the case of identical systems, that is,

$$\boxed{e_i = a_i, \quad i = 1, \dots, n.}$$

We shall see that, in this case, we obtain essentially Schmidt's orthogonalization method. Of course, this will be no surprise since the presented biorthogonalization method is inspired by Schmidt's procedure.

Let us *demonstrate the biorthogonalization method in this special case* for $n = 3$. Here,

$$f_i = e_i = a_i, \quad i = 1, 2, 3.$$

We have

$$c_1 = b_1 = \frac{a_1}{\|a_1\|^2}.$$

Further, with $d_1 := \dfrac{a_1}{\|a_1\|}$,

$$\begin{aligned} b_2 &= a_2 - (a_2, f_1)\, c_1 = a_2 - (a_2, a_1)\, \frac{a_1}{\|a_1\|^2} \\ &= a_2 - \left(a_2, \frac{a_1}{\|a_1\|}\right) \frac{a_1}{\|a_1\|} = a_2 - (a_2, d_1)\, d_1 \perp d_1. \end{aligned}$$

Moreover,

$$\begin{aligned} c_2 &= \frac{1}{(b_2, f_2)}\, b_2 = \frac{1}{(b_2, a_2)}\, b_2, \\ g_2 &= f_2 - (f_2, c_1)\, g_1 = a_2 - (f_2, c_1)\, g_1 = a_2 - (a_2, a_1) \frac{a_1}{\|a_1\|^2} \\ &= a_2 - \left(a_2, \frac{a_1}{\|a_1\|}\right) \frac{a_1}{\|a_1\|} = a_2 - (a_2, d_1)\, d_1 = b_2 \end{aligned}$$

so that

$$g_2 = b_2, \qquad c_2 = \frac{1}{(b_2, a_2)}\, b_2.$$

This entails

$$(c_2, g_2) = 1.$$

Further, since $g_1 = f_1 = a_1$ and $g_2 = b_2$,

$$b_3 = a_3 - (a_3, g_1)\, c_1 - (a_3, g_2)\, c_2 = a_3 - (a_3, a_1)\, \frac{a_1}{\|a_1\|^2} - (a_3, b_2)\, \frac{b_2}{(b_2, a_2)}.$$

Now, since $b_2 \perp d_1$,

$$\frac{b_2}{(b_2, a_2)} = \frac{b_2}{\|b_2\|^2}$$

so that

$$b_3 = a_3 - \left(a_3, \frac{a_1}{\|a_1\|}\right) \frac{a_1}{\|a_1\|} - \left(a_3, \frac{b_2}{\|b_2\|}\right) \frac{b_2}{\|b_2\|}$$

and thus, with $d_2 := b_2/\|b_2\|$,

$$b_3 = a_3 - (a_3, d_1)\, d_1 - (a_3, d_2)\, d_2.$$

Consequently, since $f_3 = a_3$ and $b_3 \perp d_1$, $d_2$,

$$c_3 = \frac{b_3}{(b_3, f_3)} = \frac{b_3}{\|b_3\|^2}.$$

Moreover,

$$\begin{aligned} g_3 &= f_3 - (f_3, c_1)\, g_1 - (f_3, c_2)\, g_2 = a_3 - (a_3, c_1)\, g_1 - (a_3, c_2)\, g_2 \\ &= a_3 - \left(a_3, \frac{a_1}{\|a_1\|^2}\right) a_1 - \left(a_3, \frac{b_2}{\|b_2\|^2}\right) b_2 \\ &= a_3 - (a_3, d_1)\, d_1 - (a_3, d_2)\, d_2 = b_3. \end{aligned}$$

Since $b_3 \perp d_1$, $d_2$, one has

$$c_3 = \frac{b_3}{(b_3, f_3)} = \frac{b_3}{(b_3, a_3)} = \frac{b_3}{\|b_3\|^2}.$$

This entails

$$(c_3, g_3) = (c_3, b_3) = \left(\frac{b_3}{\|b_3\|^2}, b_3\right) = \frac{(b_3, b_3)}{\|b_3\|^2} = 1.$$

Define

$$d_3 := \frac{b_3}{\|b_3\|}.$$

Summarizing, we obtain

$$\boxed{\begin{array}{lcl} b_1 = a_1 & ; & d_1 := \frac{b_1}{\|b_1\|} \\ b_2 = a_2 - (a_2, d_1)\, d_1 & ; & d_2 := \frac{b_2}{\|b_2\|} \\ b_3 = a_3 - (a_3, d_1)\, d_1 - (a_3, d_2)\, d_2 & ; & d_3 := \frac{b_3}{\|b_3\|} \end{array}}$$

This is Schmidt's orthogonalization method for $a_1$, $a_2$, $a_3$, which we obtain from the biorthogonalization method by using the normalized vectors $d_i := b_i/\|b_i\|$ (with $\|d_i\| = 1$) instead of the normalized vectors $c_i = b_i/\|b_i\|^2$ (with $(c_i, b_i) = 1$). $\square$

**5. Numerical examples.** In this section, we present numerical examples for the biorthogonalization process and for Schmidt's orthogonalization method. We have written three MATLAB programs, invoked as follows:

1 $[C, B, G, F] = \text{biorth}\,(A, E)$
2 $[D] = \text{biorthsm}\,(A)$
3 $[W, V] = \text{schmidt}\,(A)$

The programs may be solicited from the author. The input and output arguments are made up of $n \times n$ matrices, whose columns are the vectors of interest. For instance, in the program *biorth*, $A = [a_1, \ldots, a_n]$ and $E = [e_1, \ldots, e_n]$ are the vectors to be biorthogonalized. In a similar way, the matrices $C$, $B$, $G$, $F$, $D$, $W$, $V$ have to be interpreted. The program *biorth* generates biorthogonal vectors contained in the columns of the matrices $C$, $G$. The program *biorthosm* delivers orthogonal vectors like Schmidt's orthogonalization method. However, the procedure is based on the biorthogonalization method; it treats the special case $E = A$ and delivers numerically the same result $D$ as the output $W$ of the program *schmidt*, which is based on Schmidt's orthogonalization method.

Now, some numerical examples follow.

**Example 5.1.** For

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

the commands $[W, V] = \text{schmidt}\,(A)$ and $[D] = \text{biorthsm}\,(A)$ deliver

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example 5.2.** For

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

the commands $[W, V] = \text{schmidt}\,(A)$, $[D] = \text{biorthsm}\,(A)$, and $[C, B, G, F] = \text{biorth}\,(A, A)$ generate

$$W = \begin{bmatrix} 0.57735026918963 & -0.81649658092773 & 0.00000000000000 \\ 0.57735026918963 & 0.40824829046386 & -0.70710678118655 \\ 0.57735026918963 & 0.40824829046386 & 0.70710678118655 \end{bmatrix},$$

$$V=\begin{bmatrix} 1.00000000000000 & -0.66666666666667 & 0.00000000000000 \\ 1.00000000000000 & 0.33333333333333 & -0.50000000000000 \\ 1.00000000000000 & 0.33333333333333 & 0.50000000000000 \end{bmatrix},$$

$$D=\begin{bmatrix} 0.57735026918963 & -0.81649658092773 & 0 \\ 0.57735026918963 & 0.40824829046386 & -0.70710678118655 \\ 0.57735026918963 & 0.40824829046386 & 0.70710678118655 \end{bmatrix},$$

$$C=\begin{bmatrix} 0.33333333333333 & -1.00000000000000 & 0 \\ 0.33333333333333 & 0.50000000000000 & -1.00000000000000 \\ 0.33333333333333 & 0.50000000000000 & 1.00000000000000 \end{bmatrix},$$

$$B=\begin{bmatrix} 1.00000000000000 & -0.66666666666667 & 0 \\ 1.00000000000000 & 0.33333333333333 & -0.50000000000000 \\ 1.00000000000000 & 0.33333333333333 & 0.50000000000000 \end{bmatrix},$$

$$G=\begin{bmatrix} 1.00000000000000 & -0.66666666666667 & 0 \\ 1.00000000000000 & 0.33333333333333 & -0.50000000000000 \\ 1.00000000000000 & 0.33333333333333 & 0.50000000000000 \end{bmatrix},$$

$$F=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

One verifies that numerically $(c_j, g_k) = \delta_{jk}$. For instance, $(c_1, g_1) = 1$, $(c_1, g_2) \doteq 3.699839913606784e - 017 \approx 0$.

**Example 5.3.** For

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad E=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

the command $[C, B, G, F] = \mathrm{biorth}\,(A, E)$ generates

$$C=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$G=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad F=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example 5.4.** For

$$A = \begin{bmatrix} i & 0 & 0 \\ 1 & i & 0 \\ 1 & 1 & i \end{bmatrix}, \qquad E = \begin{bmatrix} i & 1 & 1 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix},$$

the command $[C, B, G, F] = \mathrm{biorth}\,(A, E)$ generates

$$C = \begin{bmatrix} i & 0 & 0 \\ 1 & i & 0 \\ 1 & 1 & i \end{bmatrix}, \qquad B = \begin{bmatrix} i & 0 & 0 \\ 1 & i & 0 \\ 1 & 1 & i \end{bmatrix},$$

$$G = \begin{bmatrix} i & 1 & 1-i \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}, \qquad F = \begin{bmatrix} i & 1 & 1 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}.$$

**Example 5.5.** This example is taken from [**15,** Section 2, Example 2]. For

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

the command $[C, B, G, F] = \mathrm{biorth}\,(A, E)$ generates

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

showing that breakdown is avoided as it is termed in [**15,** Section 2], which means that the algorithm does not end abnormally and thus is well defined.

**6. Conclusion.** In this paper, we first motivated the importance of the question of the biorthogonalization of two sets of linearly independent vectors and put the orthogonalization and biorthogonalization methods in an historical and present time context. Then, Schmidt's classical orthogonalization method is reviewed. This is followed by a detailed description of a Schmidt-type biorthogonalization method for two sets of linearly independent vectors which could serve as a possible template for the teaching of this method and for the presentation in undergraduate textbooks where a proof is given that the method does not end abnormally. The paper can also serve as a subject of undergraduate research, similarly to the earlier papers **[10]**–**[14]**. The relation to the classical Schmidt orthogonalization method is exhibited for the special case of two identical sets of linearly independent vectors showing that the results differ in the normalization of the orthogonalized vectors. The Gram-Schmidt-type biorthogonalization method is illustrated by several numerical examples. Three MATLAB programs may be solicited from the author.

**Acknowledgments.** The author would like to give thanks to Professor Moysey Brio and to anonymous referees for many suggestions that improved the presentation of earlier drafts of the paper.

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