Chapter 1
Introduction to Accelerator Physics

The development of charged particle accelerators and it's underlying principles has its basis on the theoretical and experimental progress in fundamental physical phenomena. While active particle accelerator experimentation started seriously only in the twentieth century, it depended on the basic physical understanding of electromagnetic phenomena as investigated both theoretically and experimentally mainly during the nineteenth and beginning twentieth century. In this introduction we will recall briefly the history leading to particle accelerator development, applications and introduce basic definitions and formulas governing particle beam dynamics.

1.1 Short Historical Overview

The history and development of particle accelerators is intimately connected to the discoveries and understanding of electrical phenomena and the realization that the electrical charge comes in lumps carried as a specific property by individual particles. It is reported that the Greek philosopher and mathematician Thales of Milet, who was born in 625 BC first observed electrostatic forces on amber. The Greek word for amber is electron or $\eta\lambda\epsilon\kappa\tau\rho o\nu$ and has become the origin for all designations of electrical phenomena and related sciences. For more than 2000 years this observation was hardly more than a curiosity. In the eighteenth century, however, electrostatic phenomena became quite popular in scientific circles and since have been developed into a technology which by now completely embraces and dominates modern civilization as we know it.

It took another 100 years before the carriers of electric charges could be isolated. Many systematic experiments were conducted and theories developed to formulate the observed electrical phenomena mathematically. It was Coulomb, who in 1785 first succeeded to quantify the forces between electrical charges which we now call

H. Wiedemann, *Particle Accelerator Physics*, Graduate Text in Physics,
https://doi.org/10.1007/978-3-319-18317-6_1

Coulomb forces. As more powerful sources for electrical charges became available, glow discharge phenomena were observed and initiated an intensive effort on experimental observations during most of the second half of the nineteenth century. It was the observations of these electrical glow discharge phenomena that led the scientific community to the discovery of elementary particles and electromagnetic radiation which are both basic ingredients for particle acceleration.

Research leading to the discovery of elementary particles and to ideas for the acceleration of such particles is dotted with particularly important milestones which from time to time set the directions for further experimental and theoretical research. It is obviously somewhat subjective to choose which discoveries might have been the most influential. Major historical discoveries leading to present day particle accelerator physics started to happen more than a 150 years ago:

1815 The physician and chemist W. Proust postulates, initially anonymous, that all atoms are composed of hydrogen atoms and that therefore all atomic weights come in multiples of the weight of a hydrogen atom.

1839 M. Faraday [1] publishes his experimental investigations of electricity and described various phenomena of glow discharge.

1858 J. Plücker [2] reports on the observation of cathode rays and their deflection by magnetic fields. He found the light to become deflected in the same spiraling direction as Ampere's current flows in the electromagnet and therefore postulated that the electric light, as he calls it, under the circumstances of the experiment must be magnetic.

1867 L. Lorenz working in parallel with J.C. Maxwell on the theory of electromagnetic fields formulates the concept of retarded potentials although not yet for moving point charges.

1869 J.W. Hittorf [3], a student of Plücker, started his thesis paper with the statement (translated from german): *"The undisputed darkest part of recent theory of electricity is the process by which in gaseous volumes the propagation of electrical current is effected"*. Obviously observations with glow discharge tubes displaying an abundance of beautiful colors and complicated reactions to magnetic fields kept a number of researchers fascinated. Hittorf conducted systematic experiments on the deflection of the light in glow discharges by magnetic fields and corrected some erroneous interpretations by Plücker.

1871 C.F. Varley postulates that cathode rays are particle rays.

1874 H. von Helmholtz postulates atomistic structure of electricity.

1883 J.C. Maxwell publishes his *Treatise on Electricity and Magnetism*.

1883 T.A. Edison discovers thermionic emission.

1886 E. Goldstein [4] observed positively charged rays which he was able to isolate from a glow discharge tube through channels in the cathode. He therefore calls these rays Kanalstrahlen.

1887 H. Hertz discoveries transmission of electromagnetic waves and photoelectric effect.

1891 G.J. Stoney introduces the name electron.

1895 H.A. Lorentz formulates electron theory, the Lorentz force equation and Lorentz contraction.

1894 P. Lenard builts a discharge tube that allows cathode rays to exit to atmospheric air.

1895 W. Röntgen discovers x-rays.

1895 E. Wiedemann [5] reports on a new kind of radiation studying electrical sparks.

1897 J.J. Thomson measures the e/m-ratio for kanal and cathode rays with electromagnetic spectrometer and found the e/m ratio for cathode rays to be larger by a factor of 1,700 compared to the e/m ratio for kanal rays. He concluded that cathode rays consist of free electricity giving evidence to free electrons.

1897 J. Larmor formulates concept of Larmor precession.

1898 A. Liènard calculates the electric and magnetic field in the vicinity of a moving point charge and evaluated the energy loss due to electromagnetic radiation from a charged particle travelling on a circular orbit.

1900 E. Wiechert derives expression for retarded potentials of moving point charges.

1901 W. Kaufmann, first alone, and in 1907 together with A.H. Bucherer measure increase of electron mass with energy. First experiment in support of theory of special relativity.

1905 A. Einstein publishes theory of special relativity.

1906 J.J. Thomson [6] explains the emission of this radiation as being caused by acceleration occurring during the collision of charged particles with other atoms and calculated the energy emitted per unit time to be $(2e^2f^2)/(3V)$, where e is the charge of the emitting particle, f the acceleration and V the velocity of light.

1907 G.A. Schott [7, 8] formulated the first theory of synchrotron radiation in an attempt to explain atomic spectra.

1909 R.A. Millikan starts measuring electric charge of electron.

1913 First experiment by J. Franck and G. Hertz to excite atoms by accelerated electrons.

1914 E. Marsden produces first proton beam irradiating paraffin with alpha particles.

1920 H. Greinacher [9] builts first cascade generator.

1922 R. Wideroe as a graduate student sketches ray transformer (betatron).

1924 G. Ising [10] invents as a student the electron linac with drift tubes and spark gap excitation.

1928 R. Wideroe [11] reports first operation of linear accelerator with potassium and sodium ions. Discusses operation of betatron and failure to get beam for lack of focusing.

1928 P.A.M. Dirac predicts existence of positrons.

1931 R.J. Van de Graaff [12] builts first high voltage generator.

1932 Lawrence and Livingston [13] accelerate first proton beam from 1.2 MeV cyclotron employing weak focusing.

1932 J.D. Cockcroft and E.T.S. Walton [14] use technically improved cascade generator to accelerate protons and initiate first artificial atomic reaction: Li + $p \rightarrow$ 2 He.
1932 in the same year, C.D. Andersen discovers positrons, neutrons were discovered by J. Chadwick, and H.C. Urey discoveries deuterons.
1939 W.W. Hansen, R. Varian and his brother S. Varian invent klystron microwave tube at Stanford.
1941 D.W. Kerst and R. Serber [15] complete first functioning betatron.
1941 B. Touschek and R. Wideroe formulate storage ring principle.
1944 D. Ivanenko and I.Ya. Pomeranchuk [16] and J. Schwinger [17] point out independently an energy limit in circular electron accelerators due to synchrotron radiation losses.
1945 V.I. Veksler [18] and E.M. McMillan [19] independently discover the principle of phase focusing.
1945 J.P. Blewett [20] experimentally discovers synchrotron radiation by measuring the energy loss of electrons.
1947 L.W. Alvarez [21] designs first proton linear accelerator at Berkeley.
1948 E.L. Ginzton et al. [22] accelerate electrons to 6 MeV with Mark I at Stanford.
1949 McMillan et al. commissioned 320 MeV electron synchrotron.
1950 N. Christofilos [23] formulates concept of strong focusing.
1952 M.S. Livingston et al. [24] describe design for 2.2 GeV *Cosmotron* in Brookhaven.
1951 H. Motz [25] builds first wiggler magnet to produce quasi monochromatic synchrotron radiation.
1952 E. Courant et al. [26] publish first paper on strong focusing.
1954 R.R. Wilson et al. operate first AG electron synchrotron in Cornell at 1.1 GeV.
1954 Lofgren et al. accelerate protons to 5.7 GeV in *Bevatron.*
1955 M. Chodorow et al. [27] complete 600 MeV *MARK III* electron linac.
1955 M. Sands [28] define limits of phase focusing due to quantum excitation.
1959 E. Courant and Snyder [29] publish their paper on the *Theory of the Alternating-Gradient Synchrotron.*

Research and development in accelerator physics blossomed significantly during the 1950s supported by the development of high power radio frequency sources and the increased availability of government funding for accelerator projects. Parallel with the progress in accelerator technology, we also observe advances in theoretical understanding, documented in an increasing number of publications. It is beyond the scope of this text to only try to give proper credit to all major advances in the past 60 years and refer the interested reader to more detailed references.

1.2 Particle Accelerator Systems

Particle accelerators come in many forms applying a variety of technical principles. All are based on the interaction of the electric charge with static and dynamic electromagnetic fields and it is the technical realization of these interactions that leads to the different types of particle accelerators. Electromagnetic fields are used over most of the available frequency range from static electric fieldsto ac magnetic fields in betatrons oscillating at 50 or 60 Hz, to radio frequency fields in the MHz to GHz range and ideas are being explored to use laser beams to generate high field particle acceleration.

In this text, we will not discuss the different technical realization of particle acceleration but rather concentrate on basic principles which are designed to help the reader to develop technical solutions for specific applications meeting basic beam stability requirements. For particular technical solutions we refer to the literature. Further down we will discuss briefly basic accelerator types and their theoretical back ground. Furthermore, to discuss basic principles of particle acceleration and beam dynamics it is desirable to stay in contact with technical reality and reference practical and working solutions. We will therefore repeatedly refer to certain types of accelerators and apply theoretical beam dynamics solutions to exhibit the salient features and importance of the theoretical ideas under discussion. In these references we use mostly such types of accelerators which are commonly used and are extensively publicized.

1.2.1 Main Components of Accelerator Facilities

In the following paragraphs we describe components of particle accelerators in a rather cursory way to introduce the terminology and overall features. Particle accelerators consist of two basic units, the particle source or injector and the main accelerator. The particle source comprises all components to generate a beam of desired particles.

Generally glow discharge columns are used to produce proton or ion beams, which then are first accelerated in electrostatic accelerators like a *Van de Graaff* or *Cockcroft-Walton* accelerator and then in an *Alvarez-type* linear accelerator. To increase the energy of heavy ion beams the initially singly charged ions are, after some acceleration, guided through a thin metal foil to strip more electrons off the ions. More than one stripping stage may be used at different energies to reach the maximum ionization for most efficient acceleration.

Much more elaborate measures must be used to produce antiprotons. Generally a high energy proton beam is aimed at a heavy metal target, where, through hadronic interactions with the target material, among other particles antiprotons are generated. Emerging from the target, these antiprotons are collected by strong focusing devices and further accelerated.

Electrons are commonly generated from a heated cathode, also called a thermionic gun, which is covered on the surface by specific alkali oxides or any other substance with a low work function to emit electrons at technically practical temperatures. Another method to create a large number of electrons within a short pulse uses a strong laser pulse directed at the surface of a photo cathode. Systems where the cathode is inserted directly into an accelerating rf field are called rf guns. Positrons are created the same way as antiprotons by aiming high energy electrons on a heavy metal target where, through an electromagnetic shower and pair production, positrons are generated. These positrons are again collected by strong magnetic fields and further accelerated.

Whatever the method of generating particles may be, in general they do not have the time structure desired for further acceleration or special application. Efficient acceleration by rf fields occurs only during a very short period per oscillation cycle and most particles would be lost without proper preparation. For high beam densities it is desirable to compress the continuous stream of particles from a thermionic gun or a glow discharge column into a shorter pulse with the help of a chopper device and/or a prebuncher. The chopper may be a mechanical device or a deflecting magnetic or rf field moving the continuous beam across the opening of a slit. At the exit of the chopper we observe a series of beam pulses, called bunches, to be further processed by the prebuncher. Here early particles within a bunch are decelerated and late particles accelerated. After a well defined drift space, the bunch length becomes reduced due to the energy dependence of the particle velocity. Obviously this compression works only as long as the particles are not relativistic while the particle velocity can be modulated by acceleration or deceleration.

No such compression is required for antiparticles, since they are produced by high energetic particles having the appropriate time structure. Antiparticle beams emerging from a target have, however, a large beam size and beam divergence. To make them suitable for further acceleration they are generally stored for some time in a cooling or damping ring. Such cooling rings are circular "accelerators" where particles are not accelerated but spend just some time circulating. Positrons circulating in such storage rings quickly lose their transverse momenta and large beam divergence through the emission of synchrotron radiation. In the case of antiprotons, external fields are applied to damp the transverse beam size or they circulate against a strong counterrotating electron beam loosing transverse momentum through scattering.

Antiparticles are not always generated in large quantities. On the other hand, the accelerator ahead of the conversion target can often be pulsed at a much higher rate than the main accelerator can accept injection. In such cases, the antiparticles are collected from the rapid cycling injector in an accumulator ring and then transferred to the main accelerator when required.

Particle beams prepared in such a manner may now be further accelerated in linear or circular accelerators. A linear accelerator consists of a linear sequence of many accelerating units where accelerating fields are generated and timed such that particles absorb and accumulate energy from each acceleration unit. Most commonly used linear accelerators consist of a series of cavities excited by radio frequency sources to high accelerating fields. In the induction accelerator , each accelerating unit consists of a transformer which generates from an external electrical pulse a field on the transformer secondary which is formed such as to allow the particle beam to be accelerated. Such induction accelerators can be optimized to accelerate very high beam currents to medium beam energies.

For very high beam energies linear accelerators become very long and costly. Such practical problems can be avoided in circular accelerators where the beam is held on a circular path by magnetic fields in bending magnets and passing repeatedly every turn through accelerating sections, similar to those in a linear accelerator. This way, the particles gain energy from the accelerating cavities at each turn and reach higher energies while the fields in the bending magnets are raised.

The basic principles to accelerate particles of different kind are similar and we do not need to distinguish between protons, ions, and electrons. Technically, individual accelerator components differ more or less to adjust to the particular beam parameters which have mostly to do with the particle velocities. For highly relativistic particles the differences in beam dynamics vanish. Protons and ions are more likely to be nonrelativistic and therefore vary the velocity as the kinetic energy is increased, thus generating problems of synchronism with the oscillating accelerating fields which must be solved by technical means.

After acceleration in a linear or circular accelerator the beam can be directed onto a target, mostly a target of liquid hydrogen, to study high energy interactions with the target protons. Such fixed target experimentation dominated nuclear and high energy particle physics from the first applications of artificially accelerated particle beams far into the 1970s and is still a valuable means of basic research. Obviously, it is also the method in conjunction with a heavy metal target to produce secondary particles like antiparticles for use in colliding beam facilities and mesons for basic research.

To increase the center-of-mass energy for basic research, particle beams are aimed not at fixed targets but to collide head on with another beam. This is one main goal for the construction of colliding beam facilities or storage rings. In such a ring, particle and antiparticle beams are injected in opposing directions and made to collide in specifically designed interaction regions. Because the interactions between counter orbiting particles is very rare, storage rings are designed to allow the beams to circulate for many turns with beam life times of several hours to give the particles ample opportunity to collide with other counter rotating particles. Of course, beams can counter rotate in the same magnetic fields only if one beam is made of the antiparticles of the other beam while two intersecting storage rings must be employed to allow the collision of unequal particles.

The circulating beam in an electron storage ring emits synchrotron radiation due to the transverse acceleration during deflection in the bending magnets. This radiation is highly collimated in the forward direction, of high brightness and therefore of great interest for basic and applied research, technology, and medicine.

Basically the design of a storage ring is the same as that for a synchrotron allowing some adjustment in the technical realization to optimize the desired features of acceleration and long beam lifetime, respectively. Beam intensities are generally very different in a synchrotron from that in a storage ring. In a synchrotron, the particle intensity is determined by the injector and this intensity is much smaller than desired in a storage ring. The injection system into a storage ring is therefore designed such that many beam pulses from a linear accelerator, an accumulator ring or a synchrotron can be accumulated. A synchrotron serving to accelerate beam from a low energy preinjector to the injection energy of the main facility, which may be a larger synchrotron or a storage ring, is also called a booster synchrotron or short a booster.

Although a storage ring is not used for particle acceleration it often occurs that a storage ring is constructed long after and for a higher beam energy than the injector system. In this case, the beam is accumulated at the maximum available injection energy. After accumulation the beam energy is slowly raised in the storage ring to the design energy by merely increasing the strength of the bending and focusing magnets.

Electron positron storage rings have played a great role in basic high-energy research. For still higher collision energies, however, the energy loss due to synchrotron radiation has become a practical and economic limitation. To avoid this limit, beams from two opposing linear accelerators are brought into head on collision at energies much higher than is possible to produce in circular accelerators. To match the research capabilities in colliding beam storage rings, such linear colliders must employ sophisticated beam dynamics controls, focusing arrangements and technologies similar to X-ray laser systems now operating.

1.2.2 Applications of Particle Accelerators

Particle accelerators are mainly known for their application as research tools in nuclear and high energy particle physics requiring the biggest and most energetic facilities. Smaller accelerators, however, have found broad applications in a wide variety of basic research and technology, as well as medicine. In this text, we will not discuss the details of all these applications but try to concentrate only on the basic principles of particle accelerators and the theoretical treatment of particle beam dynamics and instabilities. An arbitrary and incomplete listing of applications for charged particle beams and their accelerators is given for reference to the interested reader:

Nuclear physics
Electron/proton accelerators
Ion accelerators/colliders
Continuous beam facility
High-energy physics
Fixed target accelerator
Colliding beam storage rings
Linear colliders

Power generation
Inertial fusion
Reactor fuel breeding
Industry
Radiography by x-rays
Ion implantation
Isotope production/separation
Materials testing/modification

Food sterilization
X-ray lithography
Synchrotron radiation
Basic atomic and molecular physics
Condensed matter physics
Earth sciences
Material sciences
Chemistry
Molecular and cell biology
Surface/interface physics
Coherent radiation
Free electron lasers, X-FEL
Microprobe
Holography
Medicine
Radiotherapy
Health physics
Microsurgery with tunable FEL
Sterilization

This list is by no means exhaustive and additions must be made at an impressive pace as the quality and characteristics of particle beams become more and more sophisticated, predictable and controllable. Improvements in any parameter of particle beams create opportunities for new experiments and applications which were not possible before. More detailed information on specific uses of particle accelerators as well as an extensive catalogue of references has been compiled by Scharf [30].

1.3 Definitions and Formulas

Particle beam dynamics can be formulated in a variety units and it is therefore prudent to define the units used in this text to avoid confusion. In addition, we recall fundamental relations of electromagnetic fields and forces as well as some laws of special relativity to the extend that will be required in the course of discussions.

1.3.1 Units and Dimensions

A set of special physical units, selected primarily for convenience, are most commonly used to quantify physical constants in accelerator physics. The use of many such units is often determined more by historical developments than based on the choice of a consistent set of quantities useful for accelerator physics.

Generally, accelerator physics theory is formulated in the metric mks-system of units or SI-units which we follow also in this text. For readers used to cgs units, we include here conversion tables for convenience. To measure the energy of charged particles the unit Joule is actually used very rarely. The basic unit of energy in particle accelerator physics is the electron Volt (eV), which is the kinetic energy a particle with one basic unit of electrical charge e would gain while being accelerated between two conducting plates at a potential difference of 1 V. Therefore, 1 eV is equivalent to $1.60217733 \times 10^{-19}$ J. Specifically, we will often use derivatives of the basic units to express actual particle energies in a convenient form:

$$1\,\text{keV} = 1000\ \text{eV};\ 1\,\text{MeV} = 10^6\ \text{eV};\ 1\,\text{GeV} = 10^9\ \text{eV};\ 1\,\text{TeV} = 10^{12}\ \text{eV}$$

To describe particle dynamics we find it necessary to sometimes use the particle's momentum and sometimes the particle's energy. The effect of the Lorentz force from electric or magnetic fields is inversely proportional to the momentum of the particle. Acceleration in rf fields, on the other hand, is most conveniently measured by the increase in kinetic or total energy.

In an effort to simplify the technical jargon used in accelerator physics the term energy is used for all three quantities although mathematically the momentum is then multiplied by the velocity of light for dimensional consistency. There are still numerical differences which must be considered for all but very highly relativistic particles. Where we need to mention the pure particle momentum and quote a numerical value, we generally use the total energy divided by the velocity of light with the unit eV/c. With this definition a particle of energy $cp = 1$ eV would have a momentum of $p = 1$ eV/c.

An additional complication arises in the case of composite particles like heavy ions, consisting of protons and neutrons. In this case, the particle energy is not quoted for the whole ion but in terms of the energy per nucleon.

The particle beam current is measured generally in Amperes, no matter what general system of units is used but also occasionally in terms of the total charge or number of particles. The current is then the total charge Q passing a point during the time t. Depending on the time duration one gets an instantaneous current or some average current. Therefore a quotation of the particle current requires also the definition of the time structure of the beam. In circular accelerators, for example, the average beam current I relates directly to the beam intensity or the number of circulating particles N. If βc is the velocity of the particle and Z the charge multiplicity, we get for the relation of beam current and beam intensity

$$I = eZf_{\text{rev}}N\,, \tag{1.1}$$

where the revolution frequency $f_{\text{rev}} = \beta c/C$ and C is the circumference of the circular accelerator. This is the average circulating current to be distinguished from the bunch current or peak bunch current, which is the charge per bunch q divided by the duration of the bunch.

For a linear accelerator or beam transport line where particles come by only once, the definition of the beam current is more subtle. We still have a simple case if the particles come by in a continuous stream in which case the beam current is proportional to the particle flux $\dot{N}$ or $I = eZ\dot{N}$. This case, however, occurs very rarely since particle beams are generally accelerated by rf fields. As a consequence there is no continuous flux of particles reflecting the time varying acceleration of the rf field. The particle flux therefore is better described by a series of equidistant particle bunches separated by an integral number of wavelengths of the accelerating

Table 1.1 Numerical conversion factors

Quantity	Replace cgs parameter by practical units	
Potential	1 esu	300 V
Electrical field	1 esu	$3\ 10^4$ V/m
Current	1 esu	$0.1 \cdot c$ A
Charge	1 esu	$0.3333\ 10^{-9}$ C
Force	1 dyn	10^{-5} N
Energy	1 eV	$1.602\ 10^{-19}$ J
	1 eV	$1.602\ 10^{-12}$ erg

rf field. Furthermore, the acceleration often occurs only in bursts or pulses producing either a single bunch of particles or a string of many bunches. In these cases we distinguish between different current definitions. The peak current is the peak instantaneous beam current for a single bunch, while the average current is defined as the particle flux averaged over the duration of the beam pulse or any other given time period, e.g. 1 s.

Magnetic fields are quoted either in Tesla or Gauss.[1] Similarly, field gradients and higher derivatives are expressed in Tesla per meter or Gauss per centimeter. Frequently we find the need to perform numerical calculations with parameters given in different units. Some helpful numerical conversions from cgs to mks-units are compiled in Table 1.1.

Similar conversion factors can be derived for electromagnetic quantities in formulas by comparisons of similar equations in the MKS and cgs-system. Table 1.2 includes some of the most frequently used conversions. The absolute dielectric constant is

$$\varepsilon_0 = \frac{10^7}{4\pi c^2}\frac{\mathrm{C}}{\mathrm{Vm}} = 8.854 \times 10^{-12}\frac{\mathrm{C}}{\mathrm{Vm}} \tag{1.2}$$

and the absolute is

$$\mu_0 = 4\pi 10^{-7}\frac{\mathrm{Vs}}{\mathrm{Am}} = 1.2566 \times 10^{-6}\frac{\mathrm{Vs}}{\mathrm{Am}} \tag{1.3}$$

Both constants are related by $c^2\varepsilon_0\mu_0 = 1$. Using these conversion factors it is possible to convert formulas in cgs units into the equivalent form for mks-units.

1.3.2 Maxwell's Equations

Predictable control of charged particles is effected only by electric and magnetic fields and beam dynamics is the result of such interaction. We try to design and

[1]Because of its wide use, we use in rare cases the unit Gauss even though it is not a SI unit (1 Gauss = 0.0001 Tesla = 0.1 mT).

Table 1.2 Conversion factors for equations

Quantity	Replace cgs-parameter by mks-parameter	
Potential	V_{cgs}	$\sqrt{4\pi\varepsilon_0}V_{\mathrm{mks}}$
Electric field	E_{cgs}	$\sqrt{4\pi\varepsilon_0}E_{\mathrm{mks}}$
Current	I_{cgs}	$\frac{1}{\sqrt{4\pi\varepsilon_0}}I_{\mathrm{mks}}$
Current density	j_{cgs}	$\frac{1}{\sqrt{4\pi\varepsilon_0}}j_{\mathrm{mks}}$
Charge	q_{cgs}	$\frac{1}{\sqrt{4\pi\varepsilon_0}}q_{\mathrm{mks}}$
Charge density	ρ_{cgs}	$\frac{1}{\sqrt{4\pi\varepsilon_0}}\rho_{\mathrm{mks}}$
Conductivity	σ_{cgs}	$\frac{1}{\sqrt{4\pi\varepsilon_0}}\sigma_{\mathrm{mks}}$
Inductance	L_{cgs}	$4\pi\varepsilon_0 L_{\mathrm{mks}}$
Capacitance	C_{cgs}	$\frac{1}{4\pi\varepsilon_0}C_{\mathrm{mks}}$
Magnetic field	H_{cgs}	$\sqrt{4\pi\mu_0}H_{\mathrm{mks}}$
Magnetic induction	B_{cgs}	$\frac{4\pi}{\mu_0}B_{\mathrm{mks}}$

formulate electromagnetic fields in a way that can be used to accurately predict the behavior of charged particles. To describe the general interaction of fields based on electric currents in specific devices and free charged particles which we want to preserve, guide and focus, we use as a starting point Maxwell's equations:

$$\begin{aligned}
&\nabla\,(\epsilon\boldsymbol{E}) = \tfrac{\rho}{\epsilon_0}, && \text{Coulomb's law,}\\
&\nabla\boldsymbol{B} = 0, && \\
&\nabla\times\boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{B}, && \text{Faraday's law,}\\
&\nabla\times\left(\tfrac{1}{\mu}\boldsymbol{B}\right) = \mu_0\boldsymbol{j} + \tfrac{1}{c^2}\tfrac{\partial}{\partial t}(\epsilon\boldsymbol{E})\ . && \text{Ampère's law,}
\end{aligned} \tag{1.4}$$

consistent with the SI-system of units by inclusion of the unit scale factors ϵ_0 and μ_0. The quantities ϵ and μ are the relative dielectric constant and magnetic permeability of the surrounding materials, respectively. Integration of one or the other of Maxwell's equations results, for example, in the fields from singly charged particles or those of an assembly of particles travelling along a common path and forming a beam. Applying Maxwell's equations, we will make generous use of algebraic relations which have been collected in Appendix A.

1.4 Primer in Special Relativity

In accelerator physics the dynamics of particle motion is formulated for a large variety of energies from nonrelativistic to highly relativistic values and the equations of motion obviously must reflect this. Relativistic mechanics is therefore a fundamental ingredient of accelerator physics and we will recall a few basic relations of relativistic particle mechanics from a variety of more detailed derivations in generally available textbooks. Beam dynamics is expressed in a laboratory by a fixed system of coordinates but some specific problems are better discussed in the moving coordinate system of particles. Transformation between the two systems is effected through a Lorentz transformation.

1.4.1 Lorentz Transformation

Physical phenomena can appear different for observers in different systems of reference. Yet, the laws of nature must be independent of the reference system. In classical mechanics, we transform physical laws from one to another system of reference by way of the Galileo transformation $z^* = z - vt$ assuming that one system moves with velocity v along the z-axis of the other system.

As the velocities of bodies under study became faster, it became necessary to reconsider this simple transformation leading to Einstein's special theory of relativity. Maxwell's equations result in electromagnetic waves expanding at a finite velocity and do not contain any reference to a specific system of reference. Any attempt to find a variation of the "velocity of light" with respect to moving reference systems failed, most notably in Michelson's experiments. The expansion velocity of electromagnetic waves is therefore independent of the reference system and is finite.

Any new transformation laws must include the observation that no element of energy can travel faster than the speed of light. The new transformation formulae combine space and time and are for a reference system $\mathcal{L}^*$ moving with velocity $v_z = c\beta_z$ along the z-axis with respect to the stationary system $\mathcal{L}$.

$$\begin{aligned} x &= x^*, \\ y &= y^*, \\ z &= \gamma\,(z^* + \beta_z\,ct^*)\ , \\ ct &= \gamma\ (\beta_z\,z^* + ct^*)\ , \end{aligned} \tag{1.5}$$

where the relativistic factor is

$$\gamma = \frac{1}{\sqrt{1-\beta_z^2}} \tag{1.6}$$

with

$$\beta_z = v_z/c \tag{1.7}$$

and where all quantities designated with * are defined in the moving system $\mathcal{L}^*$. Of course, either system is moving relative to the other and we will use this relativity in various circumstances depending on whether quantities are known in the laboratory or moving system. The Lorentz transformations can be expressed in matrix formulation by

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & +\beta\gamma \\ 0 & 0 & +\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix} = \mathcal{M}_\mathrm{L} \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix} \tag{1.8}$$

and the inverse transformation is the same except that the velocity or β changes sign $(v \rightarrow -v)$.

Lorentz Transformation of Fields

Without proof, electromagnetic fields transform between reference systems in relative motion like

$$\begin{pmatrix} E_x \\ E_y \\ E_z \\ cB_x \\ cB_y \\ cB_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & 0 & +\gamma\beta_z & 0 \\ 0 & \gamma & 0 & -\gamma\beta_z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\gamma\beta_z & 0 & \gamma & 0 & 0 \\ +\gamma\beta_z & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x^* \\ E_y^* \\ E_z^* \\ cB_x^* \\ cB_y^* \\ cB_z^* \end{pmatrix}. \tag{1.9}$$

Again, for the inverse transformation only the sign of the relative velocity must be changed, $\beta_z \rightarrow -\beta_z$. According to this transformation of fields, a pure static magnetic field in the laboratory system $\mathcal{L}$, for example, becomes an electromagnetic field in the moving system $\mathcal{L}^*$. An undulator field, therefore, looks to an electron like a virtual photon with an electromagnetic field like a laser field and both interactions can be described by Compton scattering.

Lorentz Contraction

Characteristic for relativistic mechanics is the Lorentz contraction and time dilatation, both of which become significant in the description of particle dynamics. To describe the Lorentz contraction, we consider a rod at rest in the stationary system $\mathcal{L}$ along the z-coordinate with a length $\ell = z_2 - z_1$. In the system $\mathcal{L}^*$, which is moving with the velocity v_z in the positive z-direction with respect to $\mathcal{L}$, the rod appears to have the length $\ell^* = z_2^* - z_1^*$. By a Lorentz transformation we can relate that to the length in the $\mathcal{L}$-system. Observing both ends of the rod at the same time the lengths of the rod as observed from both systems relate like $\ell = z_2 - z_1 = \gamma(z_2^* + v_z t_2^*) - \gamma(z_1^* + v_z t_1^*) = \gamma\ell^*$ or

$$\ell = \gamma\ell^*. \tag{1.10}$$

A rod at rest in system $\mathcal{L}$ appears shorter in the moving particle system $\mathcal{L}^*$ by a factor γ and is always longest in it's own rest system. For example, the periodicity of an undulator λ_p becomes Lorentz contracted to λ_p/γ as seen by relativistic electrons. Because of the Lorentz contraction, the volume of a body at rest in the system $\mathcal{L}$ appears also reduced in the moving system $\mathcal{L}^*$ and we have for the volume of a body in three dimensional space

$$V = \gamma V^* . \tag{1.11}$$

Only one dimension of this body is Lorentz contracted and therefore the volume scales only linearly with γ. As a consequence, the charge density ρ of a particle bunch with the volume V is lower in the laboratory system $\mathcal{L}$ compared to the density in the system moving with this bunch and becomes

$$\rho = \frac{\rho *}{\gamma} . \tag{1.12}$$

Time Dilatation

Similarly, we may derive the time dilatation or the elapsed time between two events occurring at the same point in both coordinate systems. Applying the Lorentz transformations we get from (1.5) with $z_2^* = z_1^*$

$$\Delta t = t_2 - t_1 = \gamma \left(t_2^* + \frac{\beta_z z_2^*}{c} \right) - \gamma \left(t_1^* + \frac{\beta_z z_1^*}{c} \right) \tag{1.13}$$

or

$$\Delta t = \gamma \Delta t^* . \tag{1.14}$$

For a particle at rest in the moving system $\mathcal{L}^*$ the time t^* varies slower than the time in the laboratory system. This is the mathematical expression for the famous twin paradox where one of the brothers moving in a space capsule at relativistic speed would age slower than his twin brother staying back. This phenomenon gains practical importance for unstable particles. For example, high-energy pions, observed in the laboratory system, have a longer lifetime by the factor γ compared to low-energy pions with $\gamma = 1$. As a consequence, high energy unstable particles, like pions and muons, live longer and can travel farther as measured in the laboratory system, because the particle decay time is a particle property and is therefore measured in its own moving system. This is important. For example, in medical applications when a beam of pions has to be transported from the highly radioactive target area to a radiation free environment for the patient for cancer treatment.

1.4.2 Lorentz Invariance

Briefly, we have to introduce 4-vectors, because they will make later discussions much easier and illuminate fundamental properties of synchrotron radiation which is emitted in the particle system, but observed in the laboratory system as we will see later in this section. Four-vectors have a special significance in physics. As their name implies, four physical quantities can form a 4-vector which has convenient properties when viewed in different reference systems. The components of space-time, for example, form a 4-vector $\tilde{\boldsymbol{s}} = (x, y, z, \mathrm{i}ct)$. To identify 4-vectors, we add a tilde $\tilde{\boldsymbol{s}}$ to the symbols. All true 4-vectors transform like the space-time coordinates through Lorentz transformations.

$$\tilde{\boldsymbol{a}} = \mathcal{M}_{\mathrm{L}}\,\tilde{\boldsymbol{a}}^{*}. \tag{1.15}$$

Invariance to Lorentz Transformations

The length of 4-vectors is the same in all reference systems and is therefore open to measurements and comparisons independent of the location of the experimenter. In fact, it can be shown (exercise) that even the product of two arbitrary 4-vectors is Lorentz invariant. Take two 4-vectors in an arbitrary frame of reference $\tilde{\boldsymbol{a}}^{*} = \left(a_1^{*}, a_2^{*}, a_3^{*}, \mathrm{i}a_4^{*}\right)$ and $\tilde{\boldsymbol{b}}^{*} = \left(b_1^{*}, b_2^{*}, b_3^{*}, \mathrm{i}b_4^{*}\right)$ and form the product $\tilde{\boldsymbol{a}}^{*}\tilde{\boldsymbol{b}}^{*}$ in component form. A Lorentz transformation on both 4-vectors gives $\tilde{\boldsymbol{a}}^{*}\tilde{\boldsymbol{b}}^{*} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}$, which is the same in any reference system and is therefore Lorentz invariant. Specifically, the length of any 4-vector is Lorentz invariant.

Space-Time

Imagine a light flash to originate at the origin of the coordinate system $\mathcal{L}(x, y, z)$. At the time t, the edge of this expanding light flash has expanded with the velocity of light to

$$x^2 + y^2 + z^2 = c^2 t^2. \tag{1.16}$$

Observing the same light flash from a moving system, we apply a Lorentz transformation from the laboratory system $\mathcal{L}$ to the moving system $\mathcal{L}^{*}$ and get

$$x^{*2} + y^{*2} + z^{*2} = c^2 t^{*2} \tag{1.17}$$

demonstrating the invariance of the velocity of light c as has been experimentally verified by Michelson and Morley in 1887. The velocity of light is the same in all reference systems and its value is

$$c = 299{,}792{,}458 \text{ m/s}. \tag{1.18}$$

The components of the space-time 4-vector are

$$\tilde{\boldsymbol{s}} = (x_1, x_2, x_3, x_4) = (x, y, z, \mathrm{i}ct) \;, \tag{1.19}$$

where the time component has been multiplied by c to give all components the same dimension. From the Lorentz invariant world time τ, defined as

$$c\tau = \sqrt{-\tilde{\boldsymbol{s}}^2} \tag{1.20}$$

we get

$$\begin{aligned} c\mathrm{d}\tau &= \sqrt{c^2\,(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2} = \sqrt{c^2 - \left(v_x^2 + v_y^2 + v_z^2\right)}\mathrm{d}t \\ &= \sqrt{c^2 - v^2}\mathrm{d}t = \sqrt{1-\beta^2}c\mathrm{d}t, \end{aligned} \tag{1.21}$$

a relation, we know from the Lorentz transformation as time dilatation $\mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t$.

Other 4-vectors can be formulated and often become relevant in accelerator physics as, for example, those listed below. More 4-vectors are listed in Appendix B.

Four-Velocity

A velocity 4-vector can be derived from the space-time 4-vector by simple differentiation

$$\tilde{\boldsymbol{v}} = \frac{\mathrm{d}\tilde{\boldsymbol{s}}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}\tilde{\boldsymbol{s}}}{\mathrm{d}t} = \gamma\,(\dot{x}, \dot{y}, \dot{z}, \mathrm{i}c) \;. \tag{1.22}$$

Evaluating the square of the velocity 4-vector we find $\tilde{\boldsymbol{v}}^2 = \gamma\boldsymbol{v}^2 - \gamma c^2 = -c^2$ in the rest frame and in any other reference frame. The velocity of light is the same in any reference system as experimentally verified by Michelson and Morley.

Four-Acceleration

From the velocity 4-vector, we derive the 4-acceleration

$$\tilde{\boldsymbol{a}} = \frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\frac{\mathrm{d}\tilde{\boldsymbol{s}}}{\mathrm{d}t}\right) = \gamma^2\frac{\mathrm{d}^2\tilde{\boldsymbol{s}}}{\mathrm{d}t^2} + \gamma\tilde{\boldsymbol{v}}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma^2\frac{\mathrm{d}^2\tilde{\boldsymbol{s}}}{\mathrm{d}t^2} + \tilde{\boldsymbol{v}}\frac{\gamma^3}{c^2}\,(\boldsymbol{v}\boldsymbol{a}) \tag{1.23}$$

or in component form $\tilde{\boldsymbol{a}} = \left(\tilde{a}_x, \tilde{a}_y, \tilde{a}_z, \mathrm{i}\,\tilde{a}_t\right)$, we get $\tilde{a}_x = \gamma^2 a_x + \gamma^4\beta_x\,(\boldsymbol{\beta}\,\boldsymbol{a})\,, .., \tilde{a}_t = \gamma^4\,(\boldsymbol{\beta}\,\boldsymbol{a})$ where $\boldsymbol{a} = (\ddot{x}, \ddot{y}, \ddot{z})$ is the ordinary acceleration. The Lorentz invariance of $\tilde{\boldsymbol{a}}^2$ becomes important to describe the emission of synchrotron radiation from a relativistic charged particle and observation in a laboratory reference frame. Conversely, experimental verification of the theory of synchrotron radiation validates the invariance of $\tilde{\boldsymbol{a}}^2$.

Momentum-Energy 4-Vector

An important 4-vector is the 4-momentum or momentum-energy 4-vector defined by the canonical momentum $c\boldsymbol{p}$ and total energy E

$$c\tilde{\boldsymbol{p}} = (cp_x, cp_y, cp_z, \mathrm{i}E). \tag{1.24}$$

The length of the energy-momentum 4-vector $c\tilde{\boldsymbol{p}} = (cp_x, cp_y, cp_z, \mathrm{i}E)$ can be determined by going into the rest frame where the momentum is zero and we get

$$c^2\tilde{\boldsymbol{p}}^2 = c^2p_x^2 + c^2p_y^2 + c^2p_z^2 - E^2 = -A^2m^2c^4\,, \tag{1.25}$$

where we have set $E_0 = Amc^2$ for a particle with atomic mass A. From this the total energy is

$$E^2 = c^2\boldsymbol{p}^2 + A^2m^2c^4, \tag{1.26}$$

demonstrating the experimentally verifiable fact that the particle mass is Lorentz invariant.

We look now for an expression of (1.26) without the use of velocities and derive from the product of the velocity and momentum-energy 4-vectors

$$(\gamma\boldsymbol{v},\mathrm{i}\gamma c)\,(c\boldsymbol{p},\mathrm{i}E) = \gamma\boldsymbol{v}\,c\boldsymbol{p} - c\gamma E = -cAmc^2 \tag{1.27}$$

an expression for the momentum $cp = \frac{\gamma E - Amc^2}{\gamma\beta}$ since $\boldsymbol{p}\|\boldsymbol{\beta}$. Inserting this into (1.26), we get $E^2 = \left(\frac{\gamma E - Amc^2}{\gamma\beta}\right)^2 + A^2m^2c^4$,and with $\beta^2\gamma^2 = \gamma^2 - 1$

$$\gamma = \frac{E}{Amc^2} \tag{1.28}$$

defining the relativistic factor γ in terms of energies. Sometimes, authors attach this relativistic factor to the mass and assume thereby an increasing moving mass. Einstein's point of view is expressed in the following quote: "It is not good to introduce the concept of the mass of a moving body $M = \gamma m_0$ for which no clear definition can be given. It is better to introduce no mass concept other than the 'rest mass' m_0. Instead of introducing M it is better to mention the expression for the momentum and energy of a body in motion." In this book, we take the rest mass m_0 as an invariant.

The total energy of a particle is given by

$$E = \gamma E_0 = \gamma A m c^2, \tag{1.29}$$

where $E_0 = Amc^2$ is the rest energy of the particle and A the atomic mass. For electrons we assume that $A = 1$ and $m = m_e$. Since in this text we concentrate mainly on electrons and protons, we assume $A = 1$. The kinetic energy is defined as the total energy minus the rest energy

$$E_{\text{kin}} = E - E_0 = (\gamma - 1) m c^2. \tag{1.30}$$

The change in kinetic energy during acceleration is equal to the product of the accelerating force and the path length over which the force acts on the particle. Since the force may vary along the path we use the integral

$$\Delta E_{\text{kin}} = \int_{L_{\text{acc}}} \mathbf{F} \mathrm{d}\mathbf{s} \tag{1.31}$$

to define the energy increase. The length L_{acc} is the path length through the accelerating field. In discussions of energy gain through acceleration, we consider only energy differences and need therefore not to distinguish between total and kinetic energy. The particle momentum finally is defined by

$$c^2 p^2 = E^2 - E_0^2 \tag{1.32}$$

or

$$cp = \sqrt{E^2 - E_0^2} = mc^2 \sqrt{\gamma^2 - 1} = \gamma \beta m c^2 = \beta E\,, \tag{1.33}$$

where $\beta = v/c$. The simultaneous use of the terms energy and momentum might seem sometimes to be misleading as we discussed earlier. In this text, however, we will always use physically correct quantities in mathematical formulations even though we sometimes use the term energy for the quantity cp. In electron accelerators the numerical distinction between energy and momentum is insignificant since we consider in most cases highly relativistic particles. For proton accelerators and even more so for heavy ion accelerators the difference in both quantities becomes, however, significant.

Often we need differential expressions or expressions for relative variations of a quantity in terms of variations of another quantity. Such relations can be derived from the definitions in this section. By variation of (1.33), for example, we get

$$\mathrm{d}cp = \frac{mc^2}{\beta} \mathrm{d}\gamma = \frac{\mathrm{d}E}{\beta} = \frac{\mathrm{d}E_{\text{kin}}}{\beta} \tag{1.34}$$

and

$$\frac{\mathrm{d}cp}{cp} = \frac{1}{\beta^2}\frac{\mathrm{d}\gamma}{\gamma}\,.$$

Varying (1.32) and replacing $\mathrm{d}\gamma$ from (1.33) we get

$$\mathrm{d}cp = \gamma^3 mc^2\,\mathrm{d}\beta \tag{1.35}$$

and

$$\frac{\mathrm{d}cp}{cp} = \gamma^2\frac{\mathrm{d}\beta}{\beta}\,.$$

Photon 4-Vector

An analogous 4-vector can be formulated for photons using deBroglie's relations $\boldsymbol{p} = \hbar\boldsymbol{k}$ and $E = \hbar\omega$ for $c\tilde{\boldsymbol{k}} = (ck_x, ck_y, ck_z, \mathrm{i}\omega)$. Since the energy-momentum 4-vector is derived from the canonical momentum, we will have to modify this 4-vector when electromagnetic fields are present.

Force 4-Vector

The force 4-vector is the time derivative of the energy-momentum 4-vector $\left(c\dot{\boldsymbol{p}}, \mathrm{i}\dot{E}\right)$, which is consistent with the observation (so far) that the rest mass does not change with time.

Electro-magnetic 4-Vector

The electromagnetic-potential 4-vector is $(c\boldsymbol{A}, \mathrm{i}\phi)$.

1.4.3 Spatial and Spectral Distribution of Radiation

Of great importance in accelerator and synchrotron radiation physics is the Lorentz invariance of the product of two 4-vectors. Electromagnetic fields emanating from relativistic charges can be described by plane waves $E^* = E_0^*\mathrm{e}^{\mathrm{i}\,\Phi^*}$,where $\Phi^* = \omega^* t^* - k^*\boldsymbol{n}^*\boldsymbol{r}^*$ is the phase of the wave in the particle system and is Lorentz invariant. This invariance stems from the fact that the phase can be formulated as the product of the photon and space-time 4-vectors

$$c\,\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{s}} = [ck\boldsymbol{n},\mathrm{i}\omega]\,[z,\mathrm{i}ct]\;, \tag{1.36}$$

where we have set $\boldsymbol{k} = \boldsymbol{n}k$ with $\boldsymbol{n}$ being the unit vector in the direction of wave propagation. Using $k = \omega/c$ the phase as measured in the laboratory $\mathcal{L}$ is the same as that in the particle frame of reference $\mathcal{L}^*$

$$\omega^* \left[\left(n_x^* x^* + n_y^* y^* + n_z^* z^*\right) - ct^*\right] = \omega \left[\left(n_x x + n_y y + n_z z\right) - ct\right] = \text{invariant}.$$

To derive the relationships between similar quantities in both systems, we use the Lorentz transformation (1.8), noting that the particle reference frame is the frame, where the particle or radiation source is at rest, and replace the coordinates (x^*, y^*, z^*, ct^*) by those in the laboratory system for

$$\begin{aligned} &\omega^* \left[\left(n_x^* x^* + n_y^* y^* + n_z^* z^*\right) - ct^*\right] \\ &= \omega^* \left[n_x^* x + n_y^* y + n_z^* \left(\gamma z - \beta\gamma ct\right) - \left(-\beta\gamma z + \gamma ct\right)\right] \\ &= \omega \left[\left(n_x x + n_y y + n_z z\right) - ct\right], \end{aligned} \tag{1.37}$$

from which one can isolate, for example, a relation between ω^* and ω. Since the space-time coordinates are independent from each other, we may equate their coefficients on either side of the equation separately.

Spectral Distribution

In so doing, the ct-coefficients define the transformation of the oscillation frequency

$$\omega^* \gamma \left(1 + \beta_z n_z^*\right) = \omega, \tag{1.38}$$

which expresses the relativistic Doppler effect. Looking parallel and opposite to the direction of particle motion $n_z^* = 1$, the observed oscillation frequency is increased by the factor $(1 + \beta_z)\,\gamma \approx 2\gamma$ for highly relativistic particles. The Doppler effect is reduced (red shifted) if the radiation is viewed at some finite angle Θ with respect to the direction of motion of the source. In these cases $n_z^* = \cos\Theta^*$ and the frequency shift can be very large for highly relativistic particles with $\gamma \gg 1$.

Spatial Distribution

Similarly, we obtain the transformation of spatial directions from

$$n_x = \frac{n_x^*}{\gamma\left(1 + \beta_z n_z^*\right)}, \qquad n_y = \frac{n_y^*}{\gamma\left(1 + \beta_z n_z^*\right)}, \qquad n_z = \frac{\beta_z + n_z^*}{\left(1 + \beta_z n_z^*\right)}. \tag{1.39}$$

These transformations define the spatial distribution of radiation in the laboratory system. In case of transverse acceleration the radiation in the particle rest frame is distributed like $\cos^2 \Theta^*$ about the direction of motion. This distribution becomes greatly collimated into the forward direction in the laboratory system. With $n_x^{*2} + n_y^{*2} = \sin^2 \Theta^*$ and $n_x^2 + n_y^2 = \sin^2 \Theta \approx \Theta^2$ and $n_z^* = \cos \Theta^*$, we find

$$\Theta \approx \frac{\sin \Theta^*}{\gamma(1 + \beta \cos \Theta^*)} . \tag{1.40}$$

In other words, radiation from relativistic particles, emitted in the particle system into an angle $-\pi/2 < \Theta^* < \pi/2$ appears in the laboratory system highly collimated in the forward direction within an angle of

$$\Delta\Theta \approx \pm \frac{1}{\gamma} . \tag{1.41}$$

This angle is very small for highly relativistic electrons like those in a storage ring, where γ is of the order of 10^3–10^4.

1.4.4 Particle Collisions at High Energies

The most common use of high-energy particle accelerators has been for basic research in elementary particle physics. Here, accelerated particles are aimed at a target, which incidentally may be just another particle beam, and the researchers try to analyze the reaction of high-energy particles colliding with target particles. The available energy from the collision depends on the kinematic parameters of the colliding particles. We define a center of mass coordinate system which is the system that moves with the center of mass of the colliding particles. In this system the vector sum of all momenta is zero and is preserved through the collision. Similarly, the total energy is conserved and we may define a center of mass energy the same way the rest energy of a single particle is defined by

$$E_{\mathrm{cm}}^2 = \left(\sum\nolimits_i E_i\right)^2 - \left(\sum\nolimits_i cp_i\right)^2 , \tag{1.42}$$

where the summation is taken over all particles forming the center of mass system. The center of mass energy includes all old particle masses but also new masses of new particles which have not been there before. We apply this to two colliding particles with masses m_1 and m_2 and velocities $\mathbf{v_1}$ and $\mathbf{v_2}$, respectively,

$$(m_1, \mathbf{v}_1) \quad \longrightarrow \quad \longleftarrow \quad (m_2, \mathbf{v}_2) .$$

The center of mass energy for this system of two colliding particles is then

$$E_{\mathrm{cm}}^2 = \left[\sum_{i=1}^{2}\left(E_{\mathrm{kin}} + mc^2\right)_i\right]^2 - \left[\sum_{i=1}^{2} cp_i\right]^2 \tag{1.43}$$

$$= (\gamma_1 m_1 + \gamma_2 m_2)^2\, c^4 - (\gamma_1\beta_1 m_1 + \gamma_2\beta_2 m_2)^2\, c^4$$

We apply these kinematic relations to a proton $\left(m_1 = m_{\mathrm{p}}\right)$ of energy γ colliding with a proton at rest in a target. For a target proton at rest with $\gamma_2 = 1$, $m_2 = m_{\mathrm{p}}$, $\beta_2 = 0$ and $\beta\gamma = \sqrt{\gamma^2 - 1}$, the center of mass energy is

$$E_{\mathrm{cm}}^2 = (\gamma + 1)^2 m_{\mathrm{p}}^2 c^4 - (\gamma^2 - 1) m_{\mathrm{p}}^2 c^4$$

or after some manipulations

$$E_{\mathrm{cm}} = \sqrt{2(\gamma + 1)}\, m_{\mathrm{p}} c^2\,. \tag{1.44}$$

The available energy for high-energy reactions after conservation of energy and momentum for the whole particle system is the center of mass energy minus the rest energy of the particles that need to be conserved. If, for example, two protons collide, high-energy physics conservation laws tell us that the hadron number must be conserved and therefore the reaction products must include two units of the hadron number. In the most simple case the reaction will produce just two protons and some other particles with a total energy equal to the available energy

$$E_{\mathrm{avail}} = E_{\mathrm{cm}} - 2m_{\mathrm{p}}c^2 = \left[\sqrt{2(\gamma + 1)} - 2\right] m_{\mathrm{p}} c^2\,. \tag{1.45}$$

The energy available from such reactions increases only like the square root of the energy of the accelerated particle which makes such stationary target physics an increasingly inefficient use of high-energy particles. A significantly more efficient way of using the energy of colliding particles can be obtained by head on collision of two equal particles of equal energy. In this case $\gamma_1 = \gamma_2 = \gamma$, the mass of the colliding particles is $m_1 = m_2 = m_{\mathrm{p}}$, and $\beta_1 = -\beta_2 = \beta$. In this case, the center of mass energy is simply twice the energy of each of the particles

$$E_{\mathrm{cm}} = 2\gamma mc^2 = 2E\,. \tag{1.46}$$

In colliding beam facilities, where particles collide with their antiparticles no particle type conservation laws must be obeyed and therefore the total energy of both particles becomes available for the production of new particles at the collision point. In a similar way we may calculate the available energy for a variety of collision scenarios like the collision of an accelerated electron with a stationary proton, the head on collision of electrons with protons or collisions involving high-energy heavy ions.

1.5 Principles of Particle-Beam Dynamics

Accelerator physics relates primarily to the interaction of charged particles with electromagnetic fields. Detailed knowledge of the functionality of this interaction allows the design of accelerators meeting specific goals and the prediction of charged particle beam behavior in those accelerators. The interplay between particles and fields is called beam dynamics. In this section, we recall briefly some features of electromagnetic fields and fundamental processes of classical and relativistic mechanics as they relate to particle beam dynamics.

1.5.1 Electromagnetic Fields of Charged Particles

Predictable control of charged particles is effected only by electric and magnetic fields and beam dynamics is the result of such interaction. We try to design and formulate electromagnetic fields in a way that can be used to accurately predict the behavior of charged particles. To describe the general interaction of fields based on electric currents in specific devices and free charged particles which we want to preserve, guide and focus, we use as a starting point Maxwell's equations (1.4).

Electric Field of a Point Charge

First, we apply Gauss' theorem to a point charge q at rest. The natural coordinate system is the polar system because the fields of a point charge depend only on the radial distance from the charge. We integrate Coulomb's law (1.4) over a spherical volume containing the charge q at its center. With $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ the integral becomes $\int \nabla \boldsymbol{E} \mathrm{d}V = \int_0^R \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) \mathrm{d}V = 4\pi R^2 E_r(R)$, where R is the radial distance from the charge. On the r.h.s. of Coulomb's law (1.4), an integration over all the charge q gives $\int \frac{\rho}{\epsilon_0 \epsilon} \mathrm{d}V = \frac{q}{\epsilon_0 \epsilon}$ and the electric field of a point charge at distance R is

$$E_r(R) = \frac{1}{4\pi\epsilon_0\epsilon}\frac{q}{R^2}. \tag{1.47}$$

The electric field is proportional to the charge and decays quadratically with distance R.

Fields of a Charged Particle Beam

Many charged particles, travelling along the same path form a beam. This particle beam generates an electric as well as a magnetic field. The proper coordinates are now cylindrical and Coulomb's law is

$$\nabla \boldsymbol{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\underbrace{\frac{\partial E_\varphi}{\partial \varphi}}_{=0} + \underbrace{\frac{\partial E_z}{\partial z}}_{=0} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho_0}{\epsilon_0 \epsilon}, \tag{1.48}$$

where ρ_0 is the charge density in the particle beam. We assume a uniform continuous beam and expect therefore no azimuthal or longitudinal dependence, leaving only the radial dependence. Radial integration over a cylindrical volume of unit length collinear with the beam gives with the volume element $\mathrm{d}V = 2\pi r \mathrm{d}r$, on the l.h.s. $|rE_r|_0^r \, 2\pi$. The r.h.s. is $\frac{\rho_0}{\epsilon_0 \epsilon}\pi r^2$ and the electric field for a uniformly charged particle beam with radius R is

$$E_r(r) = \begin{cases} \frac{\rho_0}{2\epsilon_0\epsilon} r & \text{for } r < R \\ \frac{\rho_0}{2\epsilon_0\epsilon}\frac{R^2}{r} & \text{for } r > R \end{cases}. \tag{1.49}$$

The magnetic field for the same beam can be derived by applying Stoke's theorem on Ampere's law to give after integration

$$B_\varphi(r) = \begin{cases} \frac{1}{2}\mu_0 \mu j_0 r & \text{for } r < R \\ \frac{1}{2}\mu_0 \mu j_0 \frac{R^2}{r} & \text{for } r > R \end{cases}. \tag{1.50}$$

The fields increase linearly within the beam and decay again like $1/r$ outside the beam. Real particle beams do not have a uniform distribution and therefore a form function must be included in the integration. In most cases, the radial particle distribution is bell shaped or Gaussian. Both distributions differ little in the core of the beam and therefore a convenient assumption is that of a Gaussian distribution for which the fields will be derived in Problem 1.3.

1.5.2 Vector and Scalar Potential

By virtue of Maxwell's equation $\nabla \boldsymbol{B} = 0$ one can derive the magnetic field from a vector potential $\boldsymbol{A}$ defined by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.Faraday's law can be used to derive also the electric field from potentials. The equation $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{A})$ can be written like $\nabla \times \left(\boldsymbol{E} + \dot{\boldsymbol{A}}\right) = 0$, and solved by $\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V$, where we added the gradient of a scalar potential function V which does not alter the validity of Maxwell's equations for all fields so defined. To summarize, both, electric and magnetic fields can be derived from a scalar V and vector $\boldsymbol{A}$ potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} - \boldsymbol{\nabla} V, \tag{1.51}$$

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} V. \tag{1.52}$$

These definitions of the magnetic and electric fields from potentials will not alter the validity of Maxwell's equations as can be verified by backinsertion.

1.5.3 Wave Equation

From Ampere's law both the vector and scalar potentials can be derived. Replacing in Ampère's law $\nabla \times \boldsymbol{B} = \mu_0 \mu \boldsymbol{j} + \frac{\epsilon\mu}{c^2}\dot{\boldsymbol{E}}$ the fields with their expressions in terms of potentials, we get $\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \mu \boldsymbol{j} + \frac{\epsilon\mu}{c^2}\left(-\ddot{\boldsymbol{A}} - \boldsymbol{\nabla}\dot{V}\right)$ and with $\boldsymbol{\nabla} \mathbf{x} (\boldsymbol{\nabla} \mathbf{x} \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

$$\nabla^2 \boldsymbol{A} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \mu \boldsymbol{j} + \boldsymbol{\nabla}\underbrace{\left(\nabla \boldsymbol{A} + \frac{\epsilon\mu}{c^2}\dot{V}\right)}_{=0}. \tag{1.53}$$

At this point, we specify the potential function V such that it meets the condition

$$\nabla \boldsymbol{A} + \frac{\epsilon\mu}{c^2}\dot{V} = 0 \tag{1.54}$$

thereby simplifying greatly (1.53) and separating both potentials. This condition is called the Lorenz gauge and the resulting wave equation is

$$\nabla^2 \boldsymbol{A} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \mu \boldsymbol{j}. \tag{1.55}$$

The vector potential is clearly defined by the placement of electrical currents $\boldsymbol{j}$. We will use this property later in the design of, for example, magnets for particle beam guidance. Similarly, the wave equation for the scalar potential is

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0 \epsilon}. \tag{1.56}$$

Knowledge of the placement of electrical charges defines uniquely the scalar potential function. However, because the velocity of electro-magnetic waves is finite, the potentials at the observation point depend on the charges and currents at the retarded time, e.g. the location when the electro-magnetic waves have been emitted. The second order differential equations (1.55), (1.56) can be integrated readily and the potentials are

$$\boldsymbol{A}(P,t) = \frac{\mu_0\mu}{4\pi}\int \left.\frac{\boldsymbol{j}(x,y,z)}{R(x,y,z)}\right|_{t_\mathrm{r}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{1.57}$$

and

$$V(P,t) = \frac{1}{4\pi\epsilon_0\epsilon}\int \left.\frac{\rho(x,y,z)}{R(x,y,z)}\right|_{t_\mathrm{r}} \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.58}$$

Integration over all currents and charges at the retarded distance R_{t_r} from the observation point P results in the definition of the vector and scalar potential at the point P. Both electric and magnetic fields may be derived as discussed in the last section.

The wave equation just derived has special relevance for static fields where the Lorenz gauge reduces to the Coulomb gauge

$$\nabla\boldsymbol{A} = 0 \tag{1.59}$$

and (1.55) and (1.56) reduce in a charge and current free environment to the Laplace equation being equal to zero

$$\begin{aligned}\Delta\boldsymbol{A} &= 0, \\ \Delta V &= 0.\end{aligned} \tag{1.60}$$

Static magnetic and electric fields used in beam dynamics will be derived from theses potentials being solutions of the Laplace equation.

Lienard-Wiechert Potentials

For a point charge e at rest, we can integrate (1.57) readily to get $A(R,t) = 0$ and $V(R,t) = \frac{e}{4\pi\epsilon_0\epsilon R}$. On the other hand, in case of a moving point charge the potentials cannot be obtained by simply integrating over the "volume" of the point charge. The motion of the charge must be taken into account and the result of a proper integration (see Chap. 25) are the Liénard-Wiechert potentials for moving charges [31, 32]

$$\boldsymbol{A}(R,t) = \frac{\mu_0\mu c}{4\pi}\,\frac{q}{R}\left.\frac{\boldsymbol{\beta}}{1+\boldsymbol{n\beta}}\right|_{t_\mathrm{r}} \tag{1.61}$$

and

$$V(R,t) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{R}\left.\frac{1}{1+\boldsymbol{n\beta}}\right|_{t_\mathrm{r}}. \tag{1.62}$$

These potentials describe the radiation fields of synchrotron radiation being emitted from relativistic electrons.

1.5.4 Induction

Applying Stokes' theorem to Faraday's law (1.4), we get on the l.h.s. a line integral along the boundaries of the surface area S, which is equivalent to a voltage. On the r.h.s. the magnetic flux passing through the surface S is integrated and

$$\int_S [\nabla \times \boldsymbol{E}]\,\mathrm{d}\boldsymbol{a} = \oint \boldsymbol{E}\mathrm{d}\boldsymbol{s} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t}\mathrm{d}\boldsymbol{a} = -\frac{\partial \Phi}{\partial t}. \tag{1.63}$$

By virtue of Faraday's law, the time varying magnetic flux Φ through the area S generates an electromotive force along the boundaries of S. In accelerator physics this principle is applied in the design of a betatron. Similarly, from the second term on the right hand side in Ampère's law (1.4), we get a magnetic induction from a time varying electric field. Both phenomena play together to form the principle of induction or, in a particular example, that of a transformer.

1.5.5 Lorentz Force

The trajectories of charged particles can be influenced only by electric and magnetic fields through the Lorentz force

$$\boldsymbol{F}_\mathrm{L} = q\boldsymbol{E} + q\,(\boldsymbol{v} \times \boldsymbol{B})\,. \tag{1.64}$$

Guiding particles through appropriate electric or magnetic fields is called particle beam optics or beam dynamics. Knowledge of the location and amplitudes of electric and magnetic fields allows us to predict the path of charged particles. Closer inspection of (1.64) shows that the same force from electric or magnetic fields can be obtained if $\boldsymbol{E} = v\boldsymbol{B},$ where we have assumed that the particle velocity is orthogonal to the magnetic field, $\boldsymbol{v}\perp\boldsymbol{B}.$ For relativistic particles $v \approx c$ and to get the same force from an electric field as from, say a 1 Tesla magnetic field, one would have to have an unrealistic high field strength of $\boldsymbol{E} \approx 300\,\mathrm{MV/m}$. For this reason, magnetic fields are used to deflect or focus relativistic charged particles. For sub-relativistic particles like ion beams with velocities $v \ll c$, on the other hand, electric fields may be more efficient.

1.5.6 Equation of Motion

Accelerator physics is to a large extend the description of charged particle dynamics in the presence of external electromagnetic fields or of fields generated by other charged particles. We use the Lorentz force to formulate particle dynamics under the influence of electromagnetic fields. Whatever the interaction of charged particles with electromagnetic fields and whatever the reference system may be, we depend in accelerator physics on the invariance of the Lorentz force equation under coordinate transformations. All acceleration and beam guidance in accelerator physics will be derived from the Lorentz force. For simplicity, we use throughout this text particles with one unit of electrical charge e like electrons and protons. In case of multiply charged ions the single charge e must be replaced by eZ where Z is the charge multiplicity of the ion. Both components of the Lorentz force are used in accelerator physics where the force due to the electrical field is mostly used to actually increase the particle energy while magnetic fields are used to guide particle beams along desired beam transport lines. This separation of functions, however, is not exclusive as the example of the betatron accelerator shows where particles are accelerated by time dependent magnetic fields. Similarly electrical fields are used in specific cases to guide or separate particle beams.

Relating the Lorentz force to particle momentum or kinetic energy, we know from definitions in classical mechanics that

$$\left.\begin{aligned}\Delta \boldsymbol{p} &= \int \boldsymbol{F}_{\mathrm{L}} \mathrm{d}t \\ \Delta E_{\mathrm{kin}} &= \int \boldsymbol{F}_{\mathrm{L}} \mathrm{d}\mathbf{s}\end{aligned}\right\} \underset{\mathrm{d}\boldsymbol{s}=\boldsymbol{v}\mathrm{d}t}{\longrightarrow} \quad \boldsymbol{\beta}\,\Delta c\boldsymbol{p} = \Delta E_{\mathrm{kin}}\,, \tag{1.65}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$. The Lorentz force can be expressed in terms of fields and the change of kinetic energy becomes

$$\begin{aligned}\Delta E_{\mathrm{kin}} &= \int \boldsymbol{F}_{\mathrm{L}} \mathrm{d}\boldsymbol{s} = q \int [\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]\,\mathrm{d}\boldsymbol{s} \\ &= q \int \boldsymbol{E}\mathrm{d}\boldsymbol{s} + q \int \underbrace{(\boldsymbol{v} \times \boldsymbol{B})\,\boldsymbol{v}}_{=0}\mathrm{d}t\,,\end{aligned} \tag{1.66}$$

which indicates that an electric field component in the direction of particle motion does increase the particle's kinetic energy, while the magnetic field does not contribute any acceleration. Magnetic fields are used only to deflect a particle's path by changing the direction of its momentum vector.

It becomes obvious that the kinetic energy of a particle changes whenever it travels in an accelerating electric field $\boldsymbol{E}$ and the acceleration occurs in the direction of the electric field. This acceleration is independent of the particle velocity and acts even on a particle at rest $\boldsymbol{v} = 0$. The second component of the Lorentz force in contrast depends on the particle velocity and is directed normal to the direction of propagation and normal to the magnetic field direction. We find therefore from

(1.66) the result that the kinetic energy is not changed by the presence of magnetic fields since the scalar product $(\boldsymbol{v} \times \boldsymbol{B})\,\boldsymbol{v} = 0$ vanishes. The magnetic field causes only a deflection of the particle trajectory.

The Lorentz force (1.64) in conjunction with (1.65) is used to derive the equation of motion for charged particles in the presence of electromagnetic fields

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(Am\gamma\boldsymbol{v}) = eZ\boldsymbol{E} + eZ(\boldsymbol{v} \times \boldsymbol{B})\,, \tag{1.67}$$

where Z is the charge multiplicity of the charged particle and A the atomic mass. For simplicity we drop from here on the factors A and Z since they are different from unity only for ion beams. For ion accelerators we note therefore that the particle charge e must be replaced by eZ and the mass by Am.

Both relations in (1.65) can be used to describe the effect of the Lorentz force on particles. However, ease of mathematics makes us use one or the other. We use the first equation for dynamics in magnetic fields and the second for that in accelerating fields. Since the energy or the particle velocity does not change in a magnetic field it is straightforward to calculate $\Delta\boldsymbol{p}$. On the other hand, accelerating fields do change the particle's velocity which must be included in the time integration to get $\Delta\boldsymbol{p}$. Calculating ΔE_{kin}, we need to know only the spatial extend and magnitude of the accelerating fields to perform the integration.

The particle momentum $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and it's time derivative

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + m\boldsymbol{v}\frac{\mathrm{d}\gamma}{\mathrm{d}t}. \tag{1.68}$$

With

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d},}{\mathrm{d}\beta}\frac{\mathrm{d}\beta}{\mathrm{d}t} = \gamma^3\frac{\beta}{c}\frac{\mathrm{d}v}{\mathrm{d}t} \tag{1.69}$$

we get from (1.68) the equation of motion

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\left(\gamma\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \gamma^3\frac{\beta}{c}\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{v}\right). \tag{1.70}$$

For a force parallel to the particle propagation $\boldsymbol{v}$, we have $\dot{v}\boldsymbol{v} = \dot{\boldsymbol{v}}v$ and (1.70) becomes

$$\frac{\mathrm{d}\boldsymbol{p}_{\parallel}}{\mathrm{d}t} = m\gamma\left(1 + \gamma^2\beta\frac{v}{c}\right)\frac{\mathrm{d}\boldsymbol{v}_{\parallel}}{\mathrm{d}t} = m\gamma^3\frac{\mathrm{d}\boldsymbol{v}_{\parallel}}{\mathrm{d}t}. \tag{1.71}$$

On the other hand, if the force is directed normal to the particle propagation, we have $\mathrm{d}v/\mathrm{d}t = 0$ and (1.70) reduces to

$$\frac{\mathrm{d}\boldsymbol{p}_{\perp}}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\boldsymbol{v}_{\perp}}{\mathrm{d}t}. \tag{1.72}$$

It is obvious from (1.71) and (1.72) how differently the dynamics of particle motion is affected by the direction of the Lorentz force. Specifically the dynamics of highly relativistic particles under the influence of electromagnetic fields depends greatly on the direction of the force with respect to the direction of particle propagation. The difference between parallel and perpendicular acceleration will have a great impact on the design of electron accelerators. As we will see later, the acceleration of electrons is limited due to the emission of synchrotron radiation. This limitation, however, is much more severe for electrons in circular accelerators where the magnetic forces act perpendicularly to the propagation compared to the acceleration in linear accelerators where the accelerating fields are parallel to the particle propagation. This argument is also true for protons or for that matter, any charged particle, but because of the much larger particle mass the amount of synchrotron radiation is generally negligibly small.

1.5.7 Charged Particles in an Electromagnetic Field

An electromagnetic field exerts a force on a charged particle. A magnetic field or transverse electric field can deflect the beam and we use magnets as guiding and focusing elements for particle beam dynamics. This dynamics guides the particles on a path which is in equilibrium between the Lorentz force and the centrifugal force. A charged particle in a magnetic field follows a path defined by the equilibrium between centrifugal and Lorentz force

$$\frac{\gamma m v^2}{\rho}\boldsymbol{n} + e\left[\boldsymbol{v} \times \boldsymbol{B}\right] = 0, \tag{1.73}$$

where $\boldsymbol{n}$ is the unit vector in the direction of the centrifugal force, $1/\rho$ the local curvature and m the mass of the particle with charge e. For a magnetic field orthogonal to the velocity vector of the particle the vector product is always parallel and opposite to $\boldsymbol{n}$ and (1.73) reduces to

$$\frac{\gamma m v^2}{\rho} = -evB_\perp, \tag{1.74}$$

with the local bending radius

$$\frac{1}{\rho} = \frac{ecB}{\beta E_{\mathrm{tot}}} = \frac{ecB_\perp}{cp}. \tag{1.75}$$

The plane of the particle path is orthogonal to the transverse magnetic field. In a uniform magnetic field the particle follows the path of an arc with radius

$$\frac{1}{\rho}\left[\mathrm{m}^{-1}\right] = 0.2995\frac{B_\perp\,[\mathrm{T}]}{cp\,[\mathrm{GV}]} \approx 0.3\frac{B_\perp\,[\mathrm{T}]}{cp\,[\mathrm{GV}]} \tag{1.76}$$

in more practical units.We have a similar situation with respect to a transverse electrical field. Here, the centrifugal force is now

$$\frac{\gamma m v^2}{\rho} + e\boldsymbol{E}_\perp = 0 \tag{1.77}$$

or

$$\frac{1}{\rho} = -\frac{e\boldsymbol{E}_\perp}{\gamma m c^2 \beta^2} = -\frac{e\boldsymbol{E}_\perp}{\beta^2 E_{\text{tot}}}, \tag{1.78}$$

or in more practical units

$$\frac{1}{\rho}\left[\text{m}^{-1}\right] = -\frac{\boldsymbol{E}_\perp\,[\text{V/m}]}{\beta c p\,[\text{GV}]}. \tag{1.79}$$

Here, some caution is appropriate, because during the deflection the unit vector $\boldsymbol{n}$ is changing direction while the electric field may not change direction as in the case of a field between parallel straight plates. However, if the electrodes are bend along the expected particle path, the direction of the electric field is changing with $\boldsymbol{n}$ or the deflection of the beam.

1.5.8 Linear Equation of Motion

We have now all ingrediences to formulate an equation of motion in linear approximation. Analytical geometry tells us that the curvature is given in cartesian coordinates by

$$\kappa = \frac{-x''}{\sqrt{1 + x'^2}^3}. \tag{1.80}$$

This equation can be simplified if we assume that $x' \approx 0$. We recognize this from light optics as the paraxial approximation where all trajectories or rays are assumed to be close to the optical axis. This approximation suits beam dynamics very well since we try hard to keep all particles within a rather narrow vacuum chamber. Therefore (1.80) reduces with (1.75) to

$$\kappa \approx -x'' = \frac{ecB_y}{cp}. \tag{1.81}$$

The magnetic fields will have two main components, the guiding field for bending and a focusing field. Both fields together can be expressed by $B_y = B_{0y} + gx$, where B_{0y} is the bending field and g the field gradient $g = \partial B_y / \partial x$. The particle beam is not perfectly monochromatic and we account for this by expanding the particle energy to first order $\frac{1}{cp} \approx \frac{1}{cp_0}(1-\delta)$, where $\delta = \Delta p/p_0$.With this we get the equation of motion

$$x" = \frac{ecB_y}{cp} = \frac{ec}{cp_0}(1-\delta)\left(B_{0y} + gx\right)$$

or keeping only linear terms in x and δ

$$x" + kx = -\frac{1}{\rho_0} + \frac{1}{\rho_0}\delta. \tag{1.82}$$

Here we have introduced the quadrupole focusing strength $k = \frac{ec}{cp_0}gx$ and the bending radius is taken to be in the horizontal plane. The solution of this equation of motion will be very complicated due to the arbitrary layout of the beam transport line or $\rho_0(z)$. we are not interested in a mathematical formulation of this layout, but are interested only on the deviation of a particle from the desired transport line layout as defined by the location of magnets. We may transform away the beam line layout by merely dropping the $\frac{1}{\rho_0}$-term from (1.82) to get finally the linear equation of motion for particle dynamics

$$x" + k(z)\,x = +\frac{1}{\rho_0(z)}\delta. \tag{1.83}$$

Later we will introduce this coordinate system rigorously. This looks basically like the differential equation of a harmonic oscillator if it were not for the fact that the magnet strengths are functions of z.However, the solutions will be of oscillatory nature describing the particle motion in the restoring fields of the focusing devices. Actual analytical solutions will be discussed in great detail later in this text.

1.5.9 Energy Conservation

The rate of work done in a charged particle-field environment is defined by the Lorentz force and the particle velocity $\boldsymbol{F}_\mathrm{L}\boldsymbol{v} = e\boldsymbol{E}\boldsymbol{v} + e(\boldsymbol{v}\times\boldsymbol{B})\,\boldsymbol{v}$. Noting that $(\boldsymbol{v}\times\boldsymbol{B})\,\boldsymbol{v} = 0$, we set $e\boldsymbol{E}\boldsymbol{v} = \boldsymbol{j}\boldsymbol{E}$, and the total rate of work done by all particles and fields can be obtained by integrating Ampère's law(1.4) over all currents and fields

$$\int \boldsymbol{j}\boldsymbol{E}\,\mathrm{d}V = \epsilon_0\epsilon\int\left(c^2(\nabla\times\boldsymbol{B}) - \dot{\boldsymbol{E}}\right)\boldsymbol{E}\,\mathrm{d}V. \tag{1.84}$$

With the vector relation $\nabla(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b}\,(\nabla \times \boldsymbol{a}) - \boldsymbol{a}\,(\nabla \times \boldsymbol{b})$

$$\int \boldsymbol{j}\boldsymbol{E}\,\mathrm{d}V = \epsilon_0\epsilon \int \Big[c^2\boldsymbol{B}\,\underbrace{\nabla\times\boldsymbol{E}}_{=-\dot{\boldsymbol{B}}} - c^2\nabla\,(\boldsymbol{E}\times\boldsymbol{B}) - \dot{\boldsymbol{E}}\boldsymbol{E}\Big]\,\mathrm{d}V \tag{1.85}$$

$$= -\int \left[\frac{\mathrm{d}u}{\mathrm{d}t} + \epsilon_0\epsilon c^2\nabla\,(\boldsymbol{E}\times\boldsymbol{B})\right]\mathrm{d}V\,,$$

where an energy density has been defined by

$$u = \tfrac{1}{2}\epsilon_0\left(E^2 + c^2B^2\right). \tag{1.86}$$

Applying Gauss's theorem to the vector product in (1.85), we get an expression for the energy conservation of the complete particle-field system

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int u\mathrm{d}V}_{\substack{\text{change of}\\ \text{field energy}}} + \underbrace{\int \boldsymbol{j}\boldsymbol{E}\mathrm{d}V}_{\substack{\text{particle energy}\\ \text{loss or gain}}} + \underbrace{\oint \boldsymbol{S}\boldsymbol{n}\mathrm{d}a}_{\substack{\text{radiation loss through}\\ \text{closed surface } a}} = 0\,. \tag{1.87}$$

This equation expresses the conservation of energy relating the change in field energy and particle acceleration with a new quantity describing energy loss or gain through radiation.

Poynting Vector

The third integral in (1.87) is performed over a surface enclosing all charges and currents considered. The Poynting vector $\boldsymbol{S}$ is the energy loss/gain through a unit surface element in the direction of the unit vector $\boldsymbol{n}$ normal to the surface defined by

$$\boldsymbol{S} = \frac{1}{\mu_0\mu}\,[\boldsymbol{E}\times\boldsymbol{B}]\,. \tag{1.88}$$

Equation (1.88) exhibits characteristic features of electromagnetic radiation. Both, electric and magnetic radiation fields are orthogonal to each other ($\boldsymbol{E}\perp\boldsymbol{B}$), orthogonal to the direction of propagation ($\boldsymbol{E}\perp\boldsymbol{n},\ \boldsymbol{B}\perp\boldsymbol{n}$), and the vectors $\boldsymbol{E},\boldsymbol{B},\boldsymbol{S}$ form a right handed orthogonal system. For plane waves $\boldsymbol{n}\times\boldsymbol{E} = c\boldsymbol{B}$ and

$$\boldsymbol{S} = \frac{1}{c\mu_0\mu}\boldsymbol{E}^2\boldsymbol{n}\,. \tag{1.89}$$

Knowing the electric fields we may determine the Poynting vector describing electro-magnetic waves or synchrotron radiation.

1.5.10 Stability of a Charged-Particle Beam

Individual particles in an intense beam are under the influence of strong repelling electrostatic forces creating the possibility of severe stability problems. Particle beam transport over long distances could be greatly restricted unless these space-charge forces can be kept under control. First, it is interesting to calculate the magnitude of the problem.

If all particles would be at rest within a small volume, we would clearly expect the particles to quickly diverge from the center of charge under the influence of the repelling space charge forces from the other particles. This situation may be significantly different in a particle beam where all particles propagate in the same direction. We will therefore calculate the fields generated by charged particles in a beam and derive the corresponding Lorentz force due to these fields. Since the Lorentz force equation is invariant with respect to coordinate transformations, we may derive this force either in the laboratory system or in the moving system of the particle bunch.

From (1.49) and (1.50) we determine the Lorentz force due to electro-magnetic fields generated by the beam itself and acting on a particle within that beam. From (1.49) and (1.50) we get

$$F_r = e\left(E_r - vB_\varphi\right) = \frac{e}{2\epsilon_0\epsilon}\frac{\rho_0}{\gamma^2}r. \tag{1.90}$$

Only the radial component of the Lorentz force is finite. The Lorentz force remains repelling but due to a relativistic effect we find that the repelling electrostatic force at higher energies is increasingly compensated by the magnetic field. The total Lorentz force due to space charges therefore vanishes like γ^{-2} for higher energies. Obviously this repelling space charge force is much stronger for proton and especially for ion beams because of the smaller value for γ and, in the case of ions, because of the larger charge multiplicity which increases the space-charge force by a factor of Z.

We find the same result if we derive the Lorentz force in the moving system $\mathcal{L}^*$ of the particle beam and then transform to the laboratory system. In this moving system we have obviously only the repelling electrostatic force since the particles are at rest and the only field component is the radial electrical field which is from (1.49)

$$F_r^* = eE_r^* = \frac{e}{2\epsilon_0\epsilon}\rho_0^* r^*. \tag{1.91}$$

Transforming this equation back into the laboratory system we note that the force is purely radial and therefore acts only on the radial momentum. With $F_r = \mathrm{d}p_r/\mathrm{d}t$ and $p_r = p_r^*$ we find $F^* = \gamma F_r$ since $\mathrm{d}t = \gamma \mathrm{d}t^*$. The charge densities in both systems are related by $\rho^* = \rho/\gamma$, the radii by $r^* = r$, and the Lorentz force in the laboratory system becomes thereby

$$F_r = \frac{e}{2\epsilon_0 \epsilon} \frac{\rho_0}{\gamma^2} r \tag{1.92}$$

in agreement with (1.90).

We obtained the encouraging result that at least relativistic particle beams become stable under the influence of their own fields. For lower particle energies, however, significant diverging forces must be expected and adequate focusing measures must be applied. The physics of such space charge dominated beams is beyond the scope of this book and is treated elsewhere, for example in considerable detail in [33].

Problems

1.1 (S). Use the definition for β, the momentum, the total and kinetic energy and derive expressions $p(\beta, E_{\text{kin}})$, $p(E_{\text{kin}})$ and $E_{\text{kin}}(\gamma)$. Simplify the expressions for very large energies, $\gamma \gg 1$. Derive from these relativistic expressions the classical nonrelativistic formulas.

1.2 (S). Prove the validity of the field equations $E_r = \frac{1}{2\epsilon_0}\rho_0 r$ and $B_\varphi = \frac{1}{2}\mu_0 \beta \rho_0 r$ for a uniform cylindrical particle beam with constant charge density ρ_0 within a radius $r < R$. Derive the field expressions for $r > R$.

1.3 (S). Derive the electric and magnetic fields of a beam with a radial charge distribution $\rho(r, \varphi, z) = \rho(r)$. Derive the field equations for a Gaussian charge distribution with standard deviation σ given by $\rho(r) = \rho_0 \exp\left[-r^2/\left(2\sigma^2\right)\right]$. What are the fields for $r = 0$ and $r = \sigma$?

1.4 (S). A circular accelerator with a circumference of 300 m contains a uniform distribution of singly charged particles orbiting with the speed of light. If the circulating current is 1 amp, how many particles are orbiting? We instantly turn on an ejection magnet so that all particles leave the accelerator during the time of one revolution. What is the peak current at the ejection point? How long is the current pulse duration? If the accelerator is a synchrotron accelerating particles at a rate of 10 acceleration cycles per second, what is the average ejected particle current?

1.5 (S). A proton with a kinetic energy of 1 eV is emitted parallel to the surface of the earth. What is the bending radius due to gravitational forces? What are the required transverse electrical and magnetic fields to obtain the same bending radius? What is the ratio of electrical to magnetic field? Is this ratio different for a proton energy of say 10 TeV? Why? (gravitational constant $6.67259 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$).

1.6 (S). Consider a highly relativistic electron bunch of $n = 10^{10}$ uniformly distributed electrons. The bunch has the form of a cylindrical slug, $\ell = 1$ mm long and a radius of $R = 0.1\,\mu$m. What is the electrical and magnetic field strength at the surface of the beam. Calculate the peak electrical current of the bunch. If two such beams in a linear collider with an energy of 500 GeV pass by each other at a distance of 10 μm (center to center), what is the deflection angle of each beam due to the field of the other beam?

1.7 (S). Show that for plane waves $\boldsymbol{n} \times \boldsymbol{E} = c\boldsymbol{B}$.

1.8 (S). Show that the product of two 4-vectors is Lorentz invariant.

1.9 (S). Prove that the 4-acceleration is indeed given by (1.23).

1.10 (S). Using 4-vectors, derive the frequency of an outgoing photon from a head-on Compton scattering process of an electron with a photon of frequency ω_{L}.

1.11 (S). Using 4-vectors, derive the frequency of an outgoing photon from a head-on Compton scattering process of an electron with the field of an undulator with period λ_{u}.

1.12. Protons are accelerated to a kinetic energy of 200 MeV at the end of the Fermilab Alvarez linear accelerator. Calculate their total energy, their momentum and their velocity in units of the velocity of light $\left(m_{\mathrm{p}}c^2 = 938.27 \text{ MeV}\right)$.

1.13. Consider electrons to be accelerated in the $L = 3$ km long SLAC linear accelerator with a uniform gradient of 20 MeV/m. The electrons have a velocity $v = \frac{1}{2}c$ at the beginning of the linac. What is the length of the linac in the rest frame of the electron? Assume the particles at the end of the 3 km long linac would enter another 3 km long tube and coast through it. How long would this tube appear to be to the electron?

1.14 (S). A positron beam of energy E accelerated in a linac hits a fixed hydrogen target. What is the available energy from a collision with a target electron assumed to be at rest? Compare this available energy with that obtained in a linear collider where electrons and positrons from two similar linacs collide head on at the same energy.

1.15 (S). The SPEAR colliding beam storage ring has been constructed originally for electron and positron beams to collide head on with an energy of up to 3.5 GeV. At 1.55 GeV per beam a new particle, the ψ/J-particle, was created. In a concurrent fixed target experiment at BNL, such ψ/J-particle have been produced by protons hitting a hydrogen target. What proton energy was required to produce the new particle? Determine the positron energy needed to create ψ/J-particles by collisions with electrons in a fixed target.

1.16. A charged pion meson has a rest energy of 139.568 MeV and a mean life time of $\tau_{0\pi} = 26.029$ ns in its rest frame. What are the life times τ_π, if accelerated to a kinetic energy of 20 MeV? and 100 MeV? A pion beam decays exponentially like

e^{-t/τ_π}. At what distance from the source will the pion beam intensity have fallen to 50 %, if the kinetic energy is 20 MeV? or 100 MeV?

1.17 (S). Assume you want to produce antiprotons by accelerating protons and letting them collide with other protons in a stationary hydrogen target. What is the minimum kinetic energy the accelerated protons must have to produce antiprotons? Use the reaction $p + p \to p + p + p + \bar{p}$.

1.18. Use the results of Problem 1.3 and consider a parallel beam at the beginning of a long magnet free drift space. Follow a particle under the influence of the beam self fields starting at a distance $r_0 = \sigma$ from the axis. Derive the radial particle distance from the axis as a function of z.

1.19. Show that (1.57) is indeed a solution of (1.55).

1.20. Express the equation of motion (1.67) for $Z = 1$ in terms of particle acceleration, velocity and fields only. Verify from this result the validity of (1.71) and (1.72).

1.21. Plot on log-log scale the velocity β, total energy as a function of the kinetic energy for electrons, protons, and gold ions Au^{+14}. Vary the total energy from $0.01mc^2$ to 10^4mc^2. Why does the total energy barely change at low kinetic energies.

1.22. The design for a Relativistic Heavy Ion Collider calls for the acceleration of completely ionized gold atoms in a circular accelerator with a bending radius of $\rho = 242.78$ m and superconducting magnets reaching a maximum field of 34.5 kg. What is the maximum achievable kinetic energy per nucleon for gold ions Au^{+77} compared to protons? Calculate the total energy, momentum, and velocity of the gold atoms (A_{Au}=197).

1.23. Gold ions Au^{+14} are injected into the Brookhaven Alternating Gradient Synchrotron AGS at a kinetic energy per nucleon of 72 MeV/u. What is the velocity of the gold ions? The AGS was designed to accelerate protons to a kinetic energy of 28.1 GeV. What is the corresponding maximum kinetic energy per nucleon for these gold ions that can be achieved in the AGS? The circulating beam is expected to contain $6 \cdot 10^9$ gold ions. Calculate the beam current at injection and at maximum energy assuming there are no losses during acceleration. The circumference of the AGS is $C_{AGS} = 807.1$ m. Why does the beam current increase although the circulating charge stays constant during acceleration?

1.24. Particles undergo elastic collisions with gas atoms. The rms multiple scattering angle is given by $\sigma_\theta \approx Z\frac{20\,(\mathrm{MeV/c})}{\beta p}\sqrt{\frac{s}{\ell_r}}$, where Z is the charge multiplicity of the beam particles, s the distance travelled and ℓ_r the radiation length of the scattering material (for air the radiation length at atmospheric pressure is $\ell_r = 500$ m or 60.2 g/cm^2). Derive an approximate expression for the beam radius as a function of s due to scattering. What is the approximate tolerable gas pressure in a proton storage ring if a particle beam is supposed to orbit for 20 h and the elastic gas scattering shall not increase the beam size by more than a factor of two during that time?

References

1. M. Faraday, Poggendorf Ann. **48**, 430 (1839)
2. J. Pluecker, Ann. Phys. Chem. **13**, 88 (1858)
3. J.W. Hittorf, Ann. Phys. Chem. 5 Reihe **16**, 1 (1869)
4. E. Goldstein, Monatsberichte der Königlich Preussischen Akademie der Wissenschaften, p. 284 (1876)
5. E. Wiedemann, Z. Elektrochem. **8**, 155 (1895)
6. J.J. Thomson, *Conduction of Electricity Through Gases*, vol. 161 (Cambridge University Press, Cambridge, 1906), p. 602
7. G.A. Schott, Ann. Phys. **24**, 635 (1907)
8. G.A. Schott, Phil. Mag. [6] **13**, 194 (1907)
9. H. Greinacher, Z. Phys. **4**, 195 (1921)
10. G. Ising, Ark. Mat. Astron. Fys. **18**, 1 (1924)
11. R. Wideroe, Arch. Elektrotech. **21**, 387 (1928)
12. R.J. Van de Graaff, pr **38**, 1919 (1931)
13. E.O. Lawrence, M.S. Livingston, pr **40**, 19 (1932)
14. J.D. Cockcroft, E.T.S. Walton, Proc. Roy. Soc A **136**, 619 (1932)
15. D.W. Kerst, R. Serber, Phys. Rev. **60**, 47 (1941)
16. D. Ivanenko, A.A. Sokolov, DAN(USSR) **59**, 1551 (1972)
17. J.S. Schwinger, On the classical radiation of accelerated electrons. Phys. Rev. **75**, 1912 (1949)
18. V.I. Veksler, DAN(USSR) **44**, 393 (1944)
19. E.M. McMillan, Phys. Rev. **68**, 143 (1945)
20. J.P. Blewett, Phys. Rev. **69**, 87 (1946)
21. L.W. Alvarez, Phys. Rev. **70**, 799 (1946)
22. E.L. Ginzton, W.W. Hansen, W.R. Kennedy, Rev. Sci. Inst. **19**, 89 (1948)
23. N. Christofilos, US Patent No 2,736,766 (1950)
24. M.S. Livingston, J.P. Blewett, G.K. Green, L.J. Haworth, Rev. Sci. Instrum. **21**, 7 (1950)
25. H. Motz, J. Appl. Phys. **22**, 527 (1951)
26. E.D. Courant, M.S. Livingston, H.S. Snyder, Phys. Rev. **88**, 1190 (1952)
27. M. Chodorow, E.L. Ginzton, W.W. Hansen, R.L. Kyhl, R. Neal, W.H.K. Panofsky, Rev. Sci. Instrum. **26**, 134 (1955)
28. M. Sands, Phys. Rev. **97**, 470 (1955)
29. E.D. Courant, H.S. Snyder, Appl. Phys. **3**, 1 (1959)
30. W. Scharf, *Particle Accelerators and Their Uses* (Harwood Academic, New York, 1985)
31. A. Liènard, L'Eclaire Electrique **16**, 5 (1898)
32. E. Wiechert, Arch. Neerl. **5**, 546 (1900)
33. M. Reiser, *Theory and Design of Charged Particle Beams* (Wiley, New York, 1994)