

Introduction to  
Analytic and Probabilistic  
Number Theory

Gérald Tenenbaum  
*Professor at Université Henri Poincaré–Nancy I*



# Contents

Preface .....	xiii
Notation .....	xv
<b>Part I Elementary methods</b> .....	1
<b>Chapter I.0 Some tools from real analysis</b> .....	3
§ 0.1 Abel summation .....	3
§ 0.2 The Euler–Maclaurin summation formula .....	5
Exercises .....	7
<b>Chapter I.1 Prime numbers</b> .....	9
§ 1.1 Introduction .....	9
§ 1.2 Chebyshev’s estimates .....	10
§ 1.3 $p$ -adic valuation of $n!$ .....	13
§ 1.4 Mertens’ first theorem .....	14
§ 1.5 Two new asymptotic formulae .....	15
§ 1.6 Mertens’ formula .....	17
§ 1.7 Another theorem of Chebyshev .....	19
Notes .....	20
Exercises .....	20
<b>Chapter I.2 Arithmetic functions</b> .....	23
§ 2.1 Definitions .....	23
§ 2.2 Examples .....	23
§ 2.3 Formal Dirichlet series .....	25
§ 2.4 The ring of arithmetic functions .....	26
§ 2.5 The Möbius inversion formulae .....	28
§ 2.6 Von Mangoldt’s function .....	30
§ 2.7 Euler’s totient function .....	32
Notes .....	33
Exercises .....	34
<b>Chapter I.3 Average orders</b> .....	36
§ 3.1 Introduction .....	36
§ 3.2 Dirichlet’s problem and the hyperbola method .....	36
§ 3.3 The sum of divisors function .....	39
§ 3.4 Euler’s totient function .....	39
§ 3.5 The functions $\omega$ and $\Omega$ .....	41
§ 3.6 Mean value of the Möbius function and the summatory functions of Chebyshev .....	42
§ 3.7 Squarefree integers .....	46

§ 3.8	Mean value of a multiplicative function with values in $[0, 1]$	48
	Notes .....	50
	Exercises .....	53
<b>Chapter I.4</b>	<b>Sieve methods</b> .....	56
§ 4.1	The sieve of Eratosthenes .....	56
§ 4.2	Brun's combinatorial sieve .....	57
§ 4.3	Application to prime twins .....	60
§ 4.4	The large sieve – analytic form .....	62
§ 4.5	The large sieve – arithmetic form .....	68
§ 4.6	Applications .....	71
	Notes .....	74
	Exercises .....	76
<b>Chapter I.5</b>	<b>Extremal orders</b> .....	80
§ 5.1	Introduction and definitions .....	80
§ 5.2	The function $\tau(n)$ .....	81
§ 5.3	The functions $\omega(n)$ and $\Omega(n)$ .....	83
§ 5.4	Euler's function $\varphi(n)$ .....	84
§ 5.5	The functions $\sigma_\kappa(n)$ , $\kappa > 0$ .....	85
	Notes .....	87
	Exercises .....	87
<b>Chapter I.6</b>	<b>The method of van der Corput</b> .....	90
§ 6.1	Introduction .....	90
§ 6.2	Trigonometric integrals .....	91
§ 6.3	Trigonometric sums .....	92
§ 6.4	Application to the theorem of Voronoï .....	96
	Notes .....	99
	Exercises .....	100
<b>Part II</b>	<b>Methods of complex analysis</b> .....	103
<b>Chapter II.1</b>	<b>Generating functions: Dirichlet series</b> .....	105
§ 1.1	Convergent Dirichlet series .....	105
§ 1.2	Dirichlet series of multiplicative functions .....	106
§ 1.3	Fundamental analytic properties of Dirichlet series .....	107
§ 1.4	Abscissa of convergence and mean value .....	114
§ 1.5	An arithmetic application: the kernel of an integer .....	116
§ 1.6	Order of magnitude in vertical strips .....	118
	Notes .....	122
	Exercises .....	127

<b>Chapter II.2 Summation formulae</b>	130
§ 2.1 Perron formulae	130
§ 2.2 Application : a convergence theorem	134
§ 2.3 The mean value formula	136
Notes	137
Exercises	138
<b>Chapter II.3 The Riemann zeta function</b>	139
§ 3.1 Introduction	139
§ 3.2 Analytic continuation	139
§ 3.3 Functional equation	142
§ 3.4 Approximations and bounds in the critical strip	143
§ 3.5 Initial localisation of zeros	147
§ 3.6 Lemmas from complex analysis	149
§ 3.7 Global distribution of zeros	151
§ 3.8 Expansion as a Hadamard product	155
§ 3.9 Zero-free regions	157
§ 3.10 Bounds for $\zeta'/\zeta$ , $1/\zeta$ and $\log \zeta$	158
Notes	160
Exercises	162
<b>Chapter II.4 The prime number theorem and the Riemann hypothesis</b>	167
§ 4.1 The prime number theorem	167
§ 4.2 Minimal hypotheses	168
§ 4.3 The Riemann hypothesis	170
Notes	174
Exercises	177
<b>Chapter II.5 The Selberg–Delange method</b>	180
§ 5.1 Complex powers of $\zeta(s)$	180
§ 5.2 Hankel's formula	183
§ 5.3 The main result	184
§ 5.4 Proof of Theorem 3	187
§ 5.5 A variant of the main theorem	191
Notes	195
Exercises	197
<b>Chapter II.6 Two arithmetic applications</b>	200
§ 6.1 Integers having $k$ prime factors	200
§ 6.2 The average distribution of divisors: the arcsine law	207
Notes	212
Exercises	214

<b>Chapter II.7 Tauberian theorems</b> .....	217
§ 7.1 Introduction: Abelian/Tauberian theorems duality .....	217
§ 7.2 Tauber's theorem .....	220
§ 7.3 The theorems of Hardy–Littlewood and Karamata .....	222
§ 7.4 The remainder term in Karamata's theorem .....	227
§ 7.5 Ikehara's theorem .....	234
§ 7.6 The Berry–Esseen inequality .....	240
Notes .....	242
Exercises .....	244
<b>Chapter II.8 Prime numbers in arithmetic progressions</b> .....	248
§ 8.1 Introduction: Dirichlet characters .....	248
§ 8.2 $L$ -series. The prime number theorem for arithmetic progressions .....	252
§ 8.3 Lower bounds for $ L(s, \chi) $ when $\sigma \geq 1$ . Proof of Theorem 4 .....	256
Notes .....	262
Exercises .....	264
<b>Part III Probabilistic methods</b> .....	267
<b>Chapter III.1 Densities</b> .....	269
§ 1.1 Definitions. Natural density .....	269
§ 1.2 Logarithmic density .....	272
§ 1.3 Analytic density .....	273
§ 1.4 Probabilistic number theory.....	275
Notes .....	275
Exercises .....	276
<b>Chapter III.2 Limiting distribution of arithmetic functions</b> .	281
§ 2.1 Definition – distribution functions .....	281
§ 2.2 Characteristic functions .....	285
Notes .....	288
Exercises .....	295
<b>Chapter III.3 Normal order</b> .....	299
§ 3.1 Definition .....	299
§ 3.2 The Turán–Kubilius inequality.....	300
§ 3.3 Dual form of the Turán–Kubilius inequality .....	304
§ 3.4 The Hardy–Ramanujan theorem and other applications .	305
§ 3.5 Effective mean value estimates for multiplicative functions	308
§ 3.6 Normal structure of the set of prime factors of an integer	311
Notes .....	313
Exercises .....	319

<b>Chapter III.4 Distribution of additive functions and mean values of multiplicative functions .....</b>	<b>325</b>
§ 4.1 The Erdős–Wintner theorem .....	325
§ 4.2 Delange's theorem .....	331
§ 4.3 Halász' theorem .....	335
§ 4.4 The Erdős–Kac theorem .....	347
Notes .....	350
Exercises .....	353
<b>Chapter III.5 Integers free of large prime factors.</b>	
<b>The saddle-point method .....</b>	<b>358</b>
§ 5.1 Introduction. Rankin's method .....	358
§ 5.2 The geometric method .....	363
§ 5.3 Functional equations .....	365
§ 5.4 Dickman's function .....	370
§ 5.5 Approximations to $\Psi(x, y)$ by the saddle-point method ..	377
Notes .....	387
Exercises .....	391
<b>Chapter III.6 Integers free of small prime factors .....</b>	<b>395</b>
§ 6.1 Introduction .....	395
§ 6.2 Functional equations .....	398
§ 6.3 Buchstab's function .....	403
§ 6.4 Approximations to $\Phi(x, y)$ by the saddle-point method ..	408
Notes .....	418
Exercises .....	420
<b>Bibliography .....</b>	<b>424</b>
<b>Index .....</b>	<b>443</b>