Introduction to Classical Integrable Systems

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Contents

1
Introduction

The aim of this book is to introduce the reader to classical integrable systems. Because the subject has been developed by several schools having different perspectives, it may appear fragmented at first sight. We develop here the thesis that it has a profound unity and that the various approaches are simply changes of point of view on the same underlying reality. The more one understands each approach, the more one sees their unity. At the end one gets a very small set of interconnected methods.

This fundamental fact sets the tone of the book. We hope in this way to convey to the reader the extraordinary beauty of the structures emerging in this field, which have illuminated many other branches of theoretical physics.

The field of integrable systems is born together with Classical Mechanics, with a quest for exact solutions to Newton's equations of motion. It turned out that apart from the Kepler problem which was solved by Newton himself, after two centuries of hard investigations, only a handful of other cases were found. In the nineteenth century, Liouville finally provided a general framework characterizing the cases where the equations of motion are "solvable by quadratures". All examples previously found indeed pertained to this setting. The subject stayed dormant until the second half of the twentieth century when Gardner, Greene, Kruskal and Miura invented the Classical Inverse Scattering Method for the Korteweg–de Vries equation, which had been introduced in fluid mechanics. Soon afterwards, the Lax formulation was discovered, and the connection with integrability was unveiled by Faddeev, Zakharov and Gardner. This was the signal for a revival of the domain leading to an enormous amount of results, and truly general structures emerged which organized the subject. More recently, the extension of these results to Quantum Mechanics already led to remarkable results and is still a very active field of research.

Let us give a general overview of the ideas we present in this book. They all find their roots in the notion of Lax pairs. It consists of presenting the equations of motion of the system in the form $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$, where the matrices $L(\lambda)$ and $M(\lambda)$ depend on the dynamical variables and on a parameter λ called the spectral parameter, and $[\ ,\ ]$ denotes the commutator of matrices. The importance of Lax pairs stems from the following simple remark: the Lax equation is an isospectral evolution equation for the Lax matrix $L(\lambda)$. It follows that the curve defined by the equation $\det(L(\lambda) - \mu I) = 0$ is time-independent. This curve, called the spectral curve, can be seen as a Riemann surface. Its moduli contain the conserved quantities. This immediately introduces the two main structures into the theory: groups enter through the Lie algebra involved in the commutator $[M, L]$, while complex analysis enters through the spectral curve.

As integrable systems are rather rare, one naturally expects strong constraints on the matrices $L(\lambda)$ and $M(\lambda)$. Constructing consistent Lax matrices may be achieved by appealing to factorization problems in appropriate groups. Taking into account the spectral parameter promotes this group to a loop group. The factorization problem may then be viewed as a Riemann–Hilbert problem, a central tool of this subject.

In the group theoretical setting, solving the equations of motion amounts to solving the factorization problem. In the analytical setting, solutions are obtained by considering the eigenvectors of the Lax matrix. At any point of the spectral curve there exists an eigenvector of $L(\lambda)$ with eigenvalue μ. This defines an analytic line bundle $\mathcal{L}$ on the spectral curve with prescribed Chern class. The time evolution is described as follows: if $\mathcal{L}(t)$ is the line bundle at time t then $\mathcal{L}(t)\mathcal{L}^{-1}(0)$ is of Chern class 0, i.e. is a point on the Jacobian of the spectral curve. It is a beautiful result that this point evolves linearly on the Jacobian. As a consequence, one can express the dynamical variables in terms of theta-functions defined on the Jacobian of the spectral curve. The two methods are related as follows: the factorization problem in the loop group defines transition functions for the line bundle $\mathcal{L}$.

The framework can be generalized by replacing the Lax matrix by the first order differential equation $(\partial_\lambda - M_\lambda(\lambda))\Psi = 0$, where $M_\lambda(\lambda)$ depends rationally on λ. The solution Ψ acquires non-trivial monodromy when λ describes a loop around a pole of M_λ. The isomonodromy problem consists of finding all M_λ with prescribed monodromy data. The solutions depend, in general, on a number of continuous parameters. The deformation equations with respect to these parameters form an integrable system. The theta-functions of the isospectral approach are then promoted to more general objects called the tau-functions.

One can study the behaviour around each singularity of the differential operator quite independently. In the group theoretical version, the above extension of the framework corresponds to centrally extending the loop groups. Around a singularity the most general extended group is the group $GL(\infty)$ which corresponds to the KP hierarchy. It can be represented in a fermionic Fock space. Fermionic monomials acting on the vacuum yield decomposed vectors, which describe an infinite Grassmannian introduced by Sato. In this setting, the time flows are induced by the action of commuting one-parameter subgroups, and the tau-function is defined on the Grassmannian, i.e. the orbit of the vacuum, and characterizes it. Finally the Plücker equations of the Grassmannian are identified with the equations of motion, written in the bilinear Hirota form.

We have tried, as much as possible, to make the book self-contained, and to achieve that each chapter can be studied quite independently. Generally, we first explain methods and then show how they can be applied to particular examples, even though this does not correspond to the historical development of the subject.

In Chapter 2 we introduce the classical definition of integrable systems through the Liouville theorem. We present the Lax pair formulation, and describe the symplectic structure which is encoded into the so-called r-matrix form. In Chapter 3 we explain how to construct Lax pairs with spectral parameter, for finite and infinite-dimensional systems. The Lax matrix may be viewed as an element of a coadjoint orbit of a loop group. This introduces immediately a natural symplectic structure and a factorization problem in the loop group. We also introduce, at this early stage, the notion of tau-functions. In Chapter 4 we discuss the abstract group theoretical formulation of the theory. We then describe the analytical aspects of the theory in Chapter 5. In this setting, the action variables are g moduli of the spectral curve, a Riemann surface of genus g, and the angle variables are g points on it. We illustrate the general constructions by the examples of the closed Toda chain in Chapter 6, and the Calogero model in Chapter 7.

The following two Chapters, 8 and 9, describe respectively the isomonodromic deformation problem and the infinite Grassmannian. Soliton solutions are obtained using vertex operators. Chapters 10 and 11 are devoted to the classical study of the KP and KdV hierarchies. We develop and use the formalism of pseudo-differential operators which allows us to give simple proofs of the main formal properties. Finite-zone solutions of KdV allow us to make contact with integrable systems of finite dimensionality and soliton solutions.

In the next Chapter, 12, we study the class of Toda and sine-Gordon field theories. We use this opportunity to exhibit the relations between their conformal and integrable properties. The sine-Gordon model is presented in the framework of the Classical Inverse Scattering Method in Chapter 13. This very ingenious method is exploited to solve the sine-Gordon equation.

The last three chapters may be viewed as mathematical appendices, provided to help the reader. First we present the basic facts of symplectic geometry, which is the natural language to speak about Classical Mechanics and integrable systems. Since mathematical tools from Riemann surfaces and Lie groups are used almost everywhere, we have written two chapters presenting them in a concise way. We hope that they will be useful at least as an introduction and to fix notations.

Let us say briefly how we have limited our discussion. First we choose to remain consistently at a relatively elementary mathematical level, and have been obliged to exclude some important developments which require more advanced mathematics. We put the emphasis on methods and we have not tried to make an exhaustive list of integrable systems. Another aspect of the theory we have touched only very briefly, through the Whitham equations, is the study of perturbations of integrable systems. All these subjects are very interesting by themselves, but the present book is big enough!

A most active field of recent research is concerned with quantum integrable systems or the closely related field of exactly soluble models in statistical mechanics. When writing this book we always had the quantum theory present in mind, and have introduced all classical objects which have a well-known quantum counterpart, or are semi-classical limits of quantum objects. This explains our emphasis on Hamiltonians methods, Poisson brackets, classical r-matrices, Lie–Poisson properties of dressing transformations and the method of separation of variables. Although there is nothing quantum in this book, a large part of the apparatus necessary to understand the literature on quantum integrable systems is in fact present.

The bibliography for integrable systems would fill a book by itself. We have made no attempt to provide one. Instead, we give, at the end of each chapter, a short list of references, which complements and enhances the material presented in the chapter, and we highly encourage the reader to consult them. Of course these references are far from complete, and we apologize to the numerous authors having contributed to the domain, and whose due credit is not acknowledged. Finally we want to thank our many colleagues from whom we learned so much and with whom we have discussed many parts of this book.