Introduction to Coherent States and Quantum Information Theory

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Abstract

The purpose of this paper is to introduce several basic theorems of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1), and to give some applications of them to quantum information theory for graduate students and/or non–experts who are interested in both Geometry and Quantum Information Theory.

In the first half we make a general review of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1) from the geometric point of view and, in particular, prove that each resolution of unity can be obtained by the curvature form of some bundle on the parameter space.

We also make a short review of Holonomic Quantum Computation (Computer) and show a geometric construction of the well–known Bell states by making use of generalized coherent states.

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In the latter half we apply a method of generalized coherent states to some important topics in Quantum Information Theory, in particular, swap of coherent states and cloning of coherent ones.

We construct the swap operator of coherent states by making use of a generalized coherent operator based on su(2) and show an “imperfect cloning” of coherent states, and moreover present some related problems.

We also present a problem on a possibility of calculation or approximation of coherent state path integrals on Holonomic Quantum Computer.

In the Appendix some related advanced topics are discussed.

In conclusion we state our dream, namely, a construction of **Geometric Quantum Information Theory**.

1 Introduction

This paper is the pair to the preceding one [18] and the aim is to introduce geometric aspects of coherent states and generalized coherent ones based on Lie algebras $su(1,1)$ and $su(2)$ and to apply them to quantum information theory for graduate students and/or non–experts (in this field) who are interested in both Geometry and Quantum Information Theory.

Coherent states or generalized coherent states play a crucial role in quantum physics, in particular, quantum optics, see [1] and its references or [2], [3]. They also play an important one in mathematical physics, see [4] or [5]. For example, they are very useful in performing stationary phase approximations to path integral, see [8], [9] and [10].

In the theory of coherent states or generalized coherent ones the resolution of unity is just a key concept, see [1]. Is it possible to understand this fact from the geometric point of view ? For a set of coherent or generalized coherent states we can define a projector from the manifold consisting of parameters of them to infinite–dimensional Grassmann manifold (called classifying spaces in K–Theory). Making use of this projector we can calculate several geometric quantities such as Chern characters, see for example [13]. In particular, we prove that each resolution of unity can be obtained by the curvature form of some bundle on the parameter space.

Let us turn to Quantum Information Theory (QIT). The main subjects in QIT are

(i) Quantum Computation

(ii) Quantum Cryptgraphy

(iii) Quantum Teleportation

As for general introduction to QIT see [15], [16] and [17], [18]. The aim of this paper is to apply geometric methods to QIT, or more directly

A Geometric Construction of Quantum Information Theory.

We are developing the theory of geometric quantum computation called Holonomic Quantum Computation, see [24], [25], [26] and [19]–[23], and we are also studying geometric construction of the Bell states or the generalized Bell ones, see [48], [49]. We are interested in geometric method of Homodyne Tomography [31], [32] or geometric one of Quantum Cryptgraphy [33], [34].

On the other hand, the method of path integral plays a very important role in Quantum Mechanics or Quantum Field Theory. However it is not easy to calculate complicated path integrals with classical computers. We are interested in it from the quantum information theory's point of view. That is, can we calculate or approximate some path integral in polynomial times with Quantum Computers (Holonomic Quantum Computer especially) ? Unfortunately we cannot answer this question, however we believe this problem becomes crucial for Quantum Computers.

By the way it seems to the author that our calculations suggest some profound relation to recent non–commutative differential geometry or non–commutative field theory, see for example [35] or [36], [37]. This topic is very interesting, but beyond the scope of this paper. We show the relation diagramatically

$$
\begin{array}{ccc}
\textbf{Classical Information Theory} & \Longleftrightarrow & \textbf{Classical Geometry} \\
\Downarrow & \textbf{21 Century} & \Downarrow \\
\textbf{Quantum Information Theory} & \Longleftrightarrow & \textbf{Quantum Geometry}
\end{array}
$$

We expect that some readers would develop this subject.

In the latter half of this paper we treat special topics in Quantum Information Theory, namely, swap of coherent states and cloning of coherent states. It is not difficult to construct a universal swap operator (see Appendix), however for coherent states we can construct a special and better one by making use of a generalized coherent operator based on $su(2)$. On the other hand, to construct a cloning operator is of course not easy by the no cloning theorem [51]. However for coherent states we can make an approximate cloning (“imperfect cloning” in our terminology) by making use of the same coherent operator based on $su(2)$. This and some method in [45] may develop a better approximate cloning method. We also present some related problems on these topics.

We have so many problems to be solved in the near future. The author expects strongly that young mathematical physicists and/or information theorists will take part in this fruitful field.

The contents of this paper are as follows :

F Representation from $SU(2)$ to $SO(3)$

2 Coherent States

We make a review of some basic properties of displacement (coherent) operators within our necessity. For the proofs see [4] or [1].

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger \ , \ [N, a] = -a \ , \ [a^\dagger, a] = -\mathbf{1} \ . \tag{1}$$

Let $\mathcal{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle |\ n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , N|n\rangle = n|n\rangle \tag{2}$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \ . \tag{3}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn} \ , \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1} \ . \tag{4}$$

Let us state coherent states. For the normalized state $|z\rangle \in \mathcal{H}$ for $z \in \mathbf{C}$ the following three conditions are equivalent :

$$\text{(i)} \quad a|z\rangle = z|z\rangle \quad \text{and} \quad \langle z|z\rangle = 1 \tag{5}$$

$$\text{(ii)} \quad |z\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle = \mathrm{e}^{-|z|^2/2} e^{z a^\dagger}|0\rangle \tag{6}$$

$$\text{(iii)} \quad |z\rangle = \mathrm{e}^{z a^\dagger - \bar{z} a}|0\rangle. \tag{7}$$

In the process from (6) to (7) we use the famous elementary Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{A+B} = \mathrm{e}^{-\frac{1}{2}[A,B]}\mathrm{e}^{A}\mathrm{e}^{B} \tag{8}$$

whenever $[A,[A,B]] = [B,[A,B]] = 0$, see [1] or [4]. This is the key formula.

Definition The state $|z\rangle$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following resolution (partition) of unity.

$$\int_{\mathbf{C}} \frac{[d^2z]}{\pi}|z\rangle\langle z| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbf{1}\ , \tag{9}$$

where we have put $[d^2z] = d(\mathrm{Re}z)d(\mathrm{Im}z)$ for simplicity. We note that

$$\langle z|w\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2+\bar{z}w} \Longrightarrow |\langle z|w\rangle| = \mathrm{e}^{-\frac{1}{2}|z-w|^2},\ \langle w|z\rangle = \overline{\langle z|w\rangle}, \tag{10}$$

so $|\langle z|w\rangle| < 1$ if $z \neq w$ and $|\langle z|w\rangle| \ll 1$ if z and w are separated enough. We will use this fact in the following.

Since the operator

$$D(z) = \mathrm{e}^{za^\dagger-\bar{z}a} \quad \text{for} \quad z \in \mathbf{C} \tag{11}$$

is unitary, we call this a displacement (coherent) operator. For these operators the following properties are crucial. For $z,\ w \in \mathbf{C}$

$$D(z)D(w) = \mathrm{e}^{z\bar{w}-\bar{z}w}\ D(w)D(z), \tag{12}$$

$$D(z+w) = \mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\ D(z)D(w). \tag{13}$$

Here we list some basic properties of this operator.

(a) Matrix Elements The matrix elements of $D(z)$ are

$$\text{(i)} \quad n \le m \quad \langle n|D(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{n!}{m!}}(-\bar{z})^{m-n}L_n{}^{(m-n)}(|z|^2), \tag{14}$$

$$\text{(ii)} \quad n \ge m \quad \langle n|D(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{m!}{n!}}z^{n-m}L_m{}^{(n-m)}(|z|^2), \tag{15}$$

where ${L_n}^{(\alpha)}$ is the associated Laguerre's polynomial defined by

$$ {L_n}^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!} \ . \tag{16} $$

In particular ${L_n}^{(0)} \equiv L_n$ is the usual Laguerre's polynomial and these are related to diagonal elements of $D(z)$. Here let us list the generating function and orthogonality condition of associated Laguerre's polynomials :

$$ \frac{\mathrm{e}^{-xt/(1-t)}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} {L_n}^{(\alpha)}(x) t^n \quad \text{for} \quad |t| < 1, \tag{17} $$

$$ \int_0^{\infty} \mathrm{e}^{-x} x^{\alpha} {L_n}^{(\alpha)}(x) {L_m}^{(\alpha)}(x) dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{nm} \quad \text{for} \quad \mathrm{Re}(\alpha) > -1. \tag{18} $$

As interesting applications of this formula see the recent [57] or [58]. See also Appendix A.

(b) Trace Formula We have

$$ \mathrm{Tr} D(z) = \pi \delta^2(z) \equiv \pi \delta(x) \delta(y) \quad \text{if } z = x + iy. \tag{19} $$

This is just a fundamental property.

(c) Glauber Formula Let A be any observable. Then we have

$$ A = \int_{\mathbf{C}} \frac{[d^2 z]}{\pi} \mathrm{Tr}[A D^{\dagger}(z)] D(z) \tag{20} $$

This formula plays an important role in the field of homodyne tomography, [31] and [32].

(d) Projection on Coherent State The projection on coherent state $|z\rangle$ is given by $|z\rangle\langle z|$. But this projection has an interesting expression :

$$ |z\rangle\langle z| = : \mathrm{e}^{-(a-z)^{\dagger}(a-z)} : \tag{21} $$

where the notation : : means normal ordering.

This formula has been used in the field of quantum cryptgraphy, [33] and [34]. We note that

$$ |z\rangle\langle w| \neq : \mathrm{e}^{-(a-z)^{\dagger}(a-w)} : $$

for $z,\ w \in \mathbf{C}$ with $z \neq w$.

A comment is in order. Several properties of displacement operator discussed in this section can be generalized to the operator

$$D(z,t) = \mathrm{e}^{za^\dagger - \bar{z}a + itN} \quad \text{for} \quad z \in \mathbf{C}, \quad t \in \mathbf{R},$$

where N is the number operator, see [39].

3 Generalized Coherent States Based on su(1,1)

In this section we introduce some basic properties of generalized coherent operators based on $su(1,1)$, see [8] or [4]. As for Lie groups or Lie algebras in the following refer to [6].

3.1 General Theory

We consider a spin K (> 0) representation of $su(1,1) \subset sl(2,\mathbf{C})$ and set its generators $\{K_+, K_-, K_3\}$ $((K_+)^\dagger = K_-)$,

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \tag{22}$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K,n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and whose actions are

$$\begin{aligned} K_+|K,n\rangle &= \sqrt{(n+1)(2K+n)}|K,n+1\rangle, \\ K_-|K,n\rangle &= \sqrt{n(2K+n-1)}|K,n-1\rangle, \\ K_3|K,n\rangle &= (K+n)|K,n\rangle, \end{aligned} \tag{23}$$

where $|K,0\rangle$ is a normalized vacuum ($K_-|K,0\rangle = 0$ and $\langle K,0|K,0\rangle = 1$). We have written $|K,0\rangle$ instead of $|0\rangle$ to emphasize the spin K representation, see [8]. We also denote by $\mathbf{1}_K$ the unit operator on $\mathcal{H}_K$. ¿From (23), states $|K,n\rangle$ are given by

$$|K,n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}}|K,0\rangle, \tag{24}$$

where $(a)_n$ is the Pochammer's notation $(a)_n \equiv a(a+1)\cdots(a+n-1)$. These states satisfy the orthogonality and completeness conditions

$$\langle K,m|K,n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty}|K,n\rangle\langle K,n| \ = \mathbf{1}_K. \tag{25}$$

Now let us consider a generalized version of coherent states :

Definition We call a state

$$|w\rangle \equiv \mathrm{e}^{wK_+ - \bar{w}K_-}|K,0\rangle \quad \text{for} \quad w \in \mathbf{C}. \tag{26}$$

the generalized coherent state based on $su(1,1)$, [40].

We note that this is the extension of (7) not (5), see [4]. For this the following disentangling formula is well–known :

$$\begin{aligned}\mathrm{e}^{wK_+ - \bar{w}K_-} &= \mathrm{e}^{\zeta K_+}\mathrm{e}^{\log(1-|\zeta|^2)K_3}\mathrm{e}^{-\bar{\zeta}K_-} \quad \text{or} \\ &= \mathrm{e}^{-\bar{\zeta}K_-}\mathrm{e}^{-\log(1-|\zeta|^2)K_3}\mathrm{e}^{\zeta K_+}.\end{aligned} \tag{27}$$

where

$$\zeta = \zeta(w) \equiv \frac{w\tanh(|w|)}{|w|} \quad (\Longrightarrow |\zeta| < 1). \tag{28}$$

This is the key formula for generalized coherent operators. Therefore from (23)

$$|w\rangle = (1-|\zeta|^2)^K \mathrm{e}^{\zeta K_+}|K,0\rangle \equiv |\zeta\rangle. \tag{29}$$

This corresponds to the right hand side of (6). Moreover since

$$\mathrm{e}^{\zeta K_+}|K,0\rangle = \sum_{n=0}^{\infty}\frac{\zeta^n}{n!}K_+^n|K,0\rangle = \sum_{n=0}^{\infty}\sqrt{\frac{(2K)_n}{n!}}\frac{\zeta^n K_+^n}{\sqrt{(2K)_n n!}}|K,0\rangle = \sum_{n=0}^{\infty}\sqrt{\frac{(2K)_n}{n!}}\zeta^n|K,n\rangle$$

we have

$$|w\rangle = (1-|\zeta|^2)^K \sum_{n=0}^{\infty}\sqrt{\frac{(2K)_n}{n!}}\zeta^n|K,n\rangle \equiv |\zeta\rangle. \tag{30}$$

This corresponds to the left hand side of (6). Then the resolution of unity corresponding to (9) is

$$\begin{aligned}&\int_{\mathbf{C}}\frac{2K-1}{\pi}\frac{\tanh(|w|)[d^2w]}{\left(1-\tanh^2(|w|)\right)|w|}|w\rangle\langle w| = \int_{\mathbf{C}}\frac{2K-1}{\pi}\frac{\sinh(2|w|)[d^2w]}{2|w|}|w\rangle\langle w| \\ =&\int_{\mathrm{D}}\frac{2K-1}{\pi}\frac{[d^2\zeta]}{(1-|\zeta|^2)^2}|\zeta\rangle\langle\zeta| = \sum_{n=0}^{\infty}|K,n\rangle\langle K,n| \ = \mathbf{1}_K,\end{aligned} \tag{31}$$

where $\mathbf{C} \to \mathrm{D} : w \mapsto \zeta = \zeta(w)$ and D is the Poincare disk in $\mathbf{C}$, see [38].

Here let us construct an example of spin K–representations.

If we set

$$K_+ \equiv \frac{1}{2}\left(a^\dagger\right)^2 \ , \ K_- \equiv \frac{1}{2}a^2 \ , \ K_3 \equiv \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \ , \tag{32}$$

then we have

$$[K_3, K_+] = K_+ \ , \ [K_3, K_-] = -K_- \ , \ [K_+, K_-] = -2K_3 \ . \tag{33}$$

That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1,1)$ with spin $K = 1/4$ and $3/4$. Now we also call an operator

$$S(w) = \mathrm{e}^{\frac{1}{2}\{w(a^\dagger)^2 - \bar{w}a^2\}} \quad \text{for} \quad w \in \mathbf{C} \tag{34}$$

the squeezed operator.

3.2 Some Formulas

We make some preliminaries for the following section. For that we list some useful formulas on generalized coherent states based ob $su(1,1)$. Since the proofs are not so difficult, we leave them to the readers.

Formulas For w_1, w_2 we have

$$\text{(i)} \quad \langle w_1|w_2\rangle = \left\{\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{(1-\bar{\zeta}_1\zeta_2)^2}\right\}^K , \tag{35}$$

$$\text{(ii)} \quad \langle w_1|K_+|w_2\rangle = \langle w_1|w_2\rangle\frac{2K\bar{\zeta}_1}{1-\bar{\zeta}_1\zeta_2} \ , \tag{36}$$

$$\text{(iii)} \quad \langle w_1|K_-|w_2\rangle = \langle w_1|w_2\rangle\frac{2K\zeta_2}{1-\bar{\zeta}_1\zeta_2} \ , \tag{37}$$

$$\text{(iv)} \quad \langle w_1|K_-K_+|w_2\rangle = \langle w_1|w_2\rangle\frac{2K + 4K^2\bar{\zeta}_1\zeta_2}{(1-\bar{\zeta}_1\zeta_2)^2} \ . \tag{38}$$

where

$$\zeta_j = \frac{w_j \tanh(|w_j|)}{|w_j|} \quad \text{for} \quad j = 1,\ 2. \tag{39}$$

When $w_1 = w_2 \equiv w$, then $\langle w|w\rangle = 1$, so we have

$$\langle w|K_+|w\rangle = \frac{2K\bar{\zeta}}{1-|\zeta|^2}\ , \quad \langle w|K_-|w\rangle \frac{2K\zeta}{1-|\zeta|^2}\ , \tag{40}$$

$$\langle w|K_-K_+|w\rangle = \frac{2K+4K^2|\zeta|^2}{(1-|\zeta|^2)^2}\ . \tag{41}$$

Here let us make a comment. From (35)

$$|\langle w_1|w_2\rangle|^2 = \left\{\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{|1-\bar{\zeta}_1\zeta_2|^2}\right\}^{2K}\ ,$$

so we want to know the property of

$$\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{|1-\bar{\zeta}_1\zeta_2|^2}.$$

It is easy to see that

$$1-\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{|1-\bar{\zeta}_1\zeta_2|^2} = \frac{|\zeta_1-\zeta_2|^2}{|1-\bar{\zeta}_1\zeta_2|^2} \geq 0 \tag{42}$$

and (42) $= 0$ if and only if (iff) $\zeta_1 = \zeta_2$. Therefore

$$|\langle w_1|w_2\rangle|^2 = \left\{\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{|1-\bar{\zeta}_1\zeta_2|^2}\right\}^{2K} \leq 1 \tag{43}$$

because $2K > 1$ ($2K-1>0$ from (31)). Of course

$$|\langle w_1|w_2\rangle| = 1 \quad \text{iff} \quad \zeta_1 = \zeta_2 \quad \text{iff} \quad w_1 = w_2. \tag{44}$$

by (39).

3.3 A Supplement

Before ending this section let us make a brief comment on generalized coherent states (26). The coherent states $|z\rangle$ has been defined by (5) : $a|z\rangle = z|z\rangle$. Why do we define the generalized coherent states $|w\rangle$ as $K_-|w\rangle = w|w\rangle$ because K_- is an annihilation operator corresponding to a ? First let us try to calculate $K_-|w\rangle$ making use of (29).

$$K_-|w\rangle = (1-|\zeta|^2)^K K_- \mathrm{e}^{\zeta K_+}|K,0\rangle = (1-|\zeta|^2)^K \mathrm{e}^{\zeta K_+}\mathrm{e}^{-\zeta K_+}K_-\mathrm{e}^{\zeta K_+}|K,0\rangle.$$

Here it is easy to see

$$
\begin{aligned}
\mathrm{e}^{-\zeta K_+}K_-\mathrm{e}^{\zeta K_+} &= \sum_{n=0}^{\infty}\frac{1}{n!}[-\zeta K_+,[-\zeta K_+,[\ ,\cdots,[-\zeta K_+,K_-]\cdots]]] \\
&= K_- + 2\zeta K_3 + \zeta^2 K_+\ ,
\end{aligned}
$$

from the relations (22), so that

$$
\begin{aligned}
K_-|w\rangle &= (1-|\zeta|^2)^K \mathrm{e}^{\zeta K_+}(K_- + 2\zeta K_3 + \zeta^2 K_+)|K,0\rangle \\
&= 2\zeta K(1-|\zeta|^2)^K \mathrm{e}^{\zeta K_+}|K,0\rangle + \zeta^2 K_+(1-|\zeta|^2)^K \mathrm{e}^{\zeta K_+}|K,0\rangle \\
&= (2K\zeta + \zeta^2 K_+)|w\rangle \qquad (45)
\end{aligned}
$$

because $K_-|K,0\rangle = \mathbf{0}$. Namely we have the equation

$$
(K_- - \zeta^2 K_+)|w\rangle = 2K\zeta|w\rangle, \quad \text{where} \quad \zeta = \frac{w\tanh(|w|)}{|w|}, \qquad (46)
$$

or more symmetrically

$$
(\zeta^{-1}K_- - \zeta K_+)|w\rangle = 2K|w\rangle, \quad \text{where} \quad \zeta = \frac{w\tanh(|w|)}{|w|}. \qquad (47)
$$

This equation is completely different from (5).

3.4 Barut–Girardello Coherent States

Now let us make a brief comment on Barut–Girardello coherent states, [41].

The states $||w\rangle\rangle$ ($w \in \mathbf{C}$) defined by

$$
K_-||w\rangle\rangle = w||w\rangle\rangle \qquad (48)
$$

are called the Barut–Girardello coherent states. This definition is a natural genelization of (5) because K_- is an annihilation operator. In the preceding section we denoted by a capital letter K a level of the representation of $su(1,1)$. But to avoid some confusion in this subsection we use a small letter k instead of K.

The solution is easy to find and given by

$$
||w\rangle\rangle = \sum_{n=0}^{\infty}\frac{w^n}{\sqrt{n!(2k)_n}}|k,n\rangle \qquad (49)
$$

up to the normalization factor. Compare this with (30). Let us determine the inner product.

$$\langle\langle w||w'\rangle\rangle = \sum_{n=0}^{\infty}\frac{(\bar{w}w')^n}{n!(2k)_n} = \sum_{n=0}^{\infty}\frac{(\sqrt{\bar{w}w'})^{2n}}{n!(2k)_n}$$

Noting that

$$(2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)} \quad\Longrightarrow\quad \frac{1}{(2k)_n} = \frac{\Gamma(2k)}{\Gamma(2k+n)}$$

we have

$$\langle\langle w||w'\rangle\rangle = \Gamma(2k)(\sqrt{\bar{w}w'})^{-2k+1}I_{2k-1}(2\sqrt{\bar{w}w'}),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind :

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}.$$

Therefore

$$||w|| = \langle\langle w||w'\rangle\rangle^{1/2} = \left\{\Gamma(2k)|w|^{-2k+1}I_{2k-1}(2|w|)\right\}^{1/2}. \tag{50}$$

This gives the normalization factor of (49). Therefore the normalized solution of (48) corresponding to (6) is given by

$$||w\rangle\rangle = \left\{\Gamma(2k)|w|^{-2k+1}I_{2k-1}(2|w|)\right\}^{-1/2}\sum_{n=0}^{\infty}\frac{w^n}{\sqrt{n!(2k)_n}}|k,n\rangle. \tag{51}$$

Next we show the resolution of unity.

$$\int_{\mathbf{C}}d\mu(\bar{w},w)||w\rangle\rangle\langle\langle w|| \equiv \int_{\mathbf{C}}\frac{2K_{2k-1}(2|w|)}{\pi\Gamma(2k)}[d^2w]||w\rangle\rangle\langle\langle w|| = \mathbf{1}_k, \tag{52}$$

where $K_\nu(z)$ is the modified Bessel function whose integral representation is given by

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{z}{2}\right)^\nu\int_1^\infty dy\mathrm{e}^{-zy}(y^2-1)^{\nu-1/2}, \quad \nu > -\frac{1}{2}.$$

The proof of (52) is not so easy, so we give it.

$$\begin{aligned}\int_{\mathbf{C}}d\mu(\bar{w},w)||w\rangle\rangle\langle\langle w|| &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{\mathbf{C}}d\mu(\bar{w},w)\frac{\bar{w}^n w^m}{\sqrt{n!(2k)_n m!(2k)_m}}|k,n\rangle\langle k,m| \\ &= \sum_{n=0}^{\infty}\int_0^\infty d\mu(r)\frac{r^{2n}}{n!(2k)_n}|k,n\rangle\langle k,n| \\ &= \sum_{n=0}^{\infty}\frac{1}{n!(2k)_n}\left\{\int_0^\infty d\mu(r)r^{2n}\right\}|k,n\rangle\langle k,n|\end{aligned} \tag{53}$$

where we have integrated on θ making use of $w = re^{i\theta}$. Here

$$\begin{aligned}
\int_0^\infty d\mu(r) r^{2n} &= \frac{4}{\Gamma(2k)} \int_0^\infty dr r^{2k+2n} K_{2k-1}(2r) \\
&= \frac{4}{\Gamma(2k)} \frac{1}{4} \Gamma(\frac{2k+2n+1+2k-1}{2}) \Gamma(\frac{2k+2n+1-2k+1}{2}) \\
&= \frac{\Gamma(2k+n)}{\Gamma(2k)} \Gamma(n+1) = (2k)_n n!\ ,
\end{aligned}$$

where we have used the famous formula

$$\int_0^\infty dx x^{\mu-1} K_\nu(ax) = \frac{1}{4}\left(\frac{2}{a}\right)\Gamma(\frac{\mu+\nu}{2})\Gamma(\frac{\mu-\nu}{2}) \qquad a > 0, \quad \mathrm{Re}\mu > |\mathrm{Re}\nu|\ .$$

For the proof see [42]; Appendix B. Therefore we have

$$\int_{\mathbf{C}} d\mu(\bar{w}, w) ||w\rangle\rangle\langle\langle w|| = \sum_{n=0}^{\infty} \frac{1}{n!(2k)_n} (2k)_n n! |k, n\rangle\langle k, n| = \sum_{n=0}^{\infty} |k, n\rangle\langle k, n| = \mathbf{1}_k.$$

Their states have several interesting structures, but we don't consider them in this paper. See [42], [43] and [44] as for further developments and applications.

A comment is in order. Here let us compare two types of coherent states based on Lie algebra $su(1,1)$ $\cdots$ Perelomov type (section 3.1) and Barut–Girardello one (section 3.4). The measures satisfying resolution of unity must be positive, so we have

(1) Perelomov type $\quad K > \frac{1}{2}$ $(\Longleftarrow (31))$

(2) Barut–Girardello type $\quad K > 0$ $(\Longleftarrow (52))$

4 Generalized Coherent States Based on su(2)

In this section we introduce some basic properties of generalized coherent operators based on $su(2)$, see [8] or [4].

4.1 General Theory

We consider a spin J (>0) representation of $su(2) \subset sl(2,\mathbf{C})$ and set its generators $\{J_+, J_-, J_3\}$ ($(J_+)^\dagger = J_-$),

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{54}$$

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which $\{J_+, J_-, J_3\}$ act is $\mathcal{H}_J \equiv \{|J,m\rangle | 0 \le m \le 2J\}$ and whose actions are

$$\begin{aligned} J_+|J,m\rangle &= \sqrt{(m+1)(2J-m)}|J,m+1\rangle, \\ J_-|J,m\rangle &= \sqrt{m(2J-m+1)}|J,m-1\rangle, \\ J_3|J,m\rangle &= (-J+m)|J,m\rangle, \end{aligned} \tag{55}$$

where $|J,0\rangle$ is a normalized vacuum ($J_-|J,0\rangle = 0$ and $\langle J,0|J,0\rangle = 1$). We have written $|J,0\rangle$ instead of $|0\rangle$ to emphasize the spin J representation, see [8]. We also denote by $\mathbf{1}_J$ the unit operator on $\mathcal{H}_J$. ¿From (23), states $|J,m\rangle$ are given by

$$|J,m\rangle = \frac{(J_+)^m}{\sqrt{m!\,{}_{2J}\mathrm{P}_m}}|J,0\rangle, \tag{56}$$

where ${}_{2J}\mathrm{P}_m = (2J)(2J-1)\cdots(2J-m+1)$. These states satisfy the orthogonality and completeness conditions

$$\langle J,m|J,n\rangle = \delta_{mn}, \quad \sum_{m=0}^{2J} |J,m\rangle\langle J,m| = \mathbf{1}_J. \tag{57}$$

Now let us consider a generalized version of coherent states :

Definition We call a state

$$|v\rangle \equiv \mathrm{e}^{vJ_+ - \bar{v}J_-}|J,0\rangle \quad \text{for} \quad v \in \mathbf{C}. \tag{58}$$

the generalized coherent state based on $su(2)$, [40].

We note that this is the extension of (7) not (5), see [4]. For this the following disentangling formula is well–known :

$$\begin{aligned} \mathrm{e}^{vJ_+ - \bar{v}J_-} &= \mathrm{e}^{\eta J_+}\mathrm{e}^{\log(1+|\eta|^2)J_3}\mathrm{e}^{-\bar{\eta}J_-} \quad \text{or} \\ &= \mathrm{e}^{-\bar{\eta}J_-}\mathrm{e}^{-\log(1+|\eta|^2)J_3}\mathrm{e}^{\eta J_+}. \end{aligned} \tag{59}$$

where

$$\eta = \eta(v) \equiv \frac{v\tan(|v|)}{|v|} \ . \tag{60}$$

This is the key formula for generalized coherent operators. Therefore from (55)

$$|v\rangle = \frac{1}{(1+|\eta|^2)^J}\mathrm{e}^{\eta J_+}|J,0\rangle \equiv |\eta\rangle. \tag{61}$$

This corresponds to the right hand side of (6). Moreover since

$$\begin{aligned}\mathrm{e}^{\eta J_+}|J,0\rangle &= \sum_{m=0}^{\infty}\frac{\eta^m}{m!}J_+^m|J,0\rangle = \sum_{m=0}^{\infty}\sqrt{\frac{{}_{2J}\mathrm{P}_m}{m!}}\frac{\eta^m J_+^m}{\sqrt{{}_{2J}\mathrm{P}_m m!}}|J,0\rangle = \sum_{m=0}^{2J}\sqrt{\frac{{}_{2J}\mathrm{P}_m}{m!}}\eta^m|J,m\rangle \\ &= \sum_{m=0}^{2J}\sqrt{{}_{2J}\mathrm{C}_m}\eta^m|J,m\rangle\end{aligned} \tag{62}$$

we have

$$|v\rangle = \frac{1}{(1+|\eta|^2)^J}\sum_{m=0}^{2J}\sqrt{{}_{2J}\mathrm{C}_m}\eta^m|J,m\rangle \equiv |\eta\rangle. \tag{63}$$

This corresponds to the left hand side of (6). Then the resolution of unity corresponding to (9) is

$$\begin{aligned}&\int_{\mathbf{C}}\frac{2J+1}{\pi}\frac{\tan(|v|)[d^2v]}{(1+\tan^2(|v|))\,|v|}|v\rangle\langle v| = \int_{\mathbf{C}}\frac{2J+1}{\pi}\frac{\sin(2|v|)[d^2v]}{2|v|}|v\rangle\langle v| \\ =&\int_{\mathbf{C}}\frac{2J+1}{\pi}\frac{[d^2\eta]}{(1+|\eta|^2)^2}|\eta\rangle\langle\eta| = \sum_{m=0}^{2J}|J,m\rangle\langle J,m| \ = \mathbf{1}_J,\end{aligned} \tag{64}$$

where $\mathbf{C} \rightarrow \mathbf{C} \subset \mathbf{C}\mathrm{P}^1 : v \mapsto \eta = \eta(v)$, see [38].

4.2 Some Formulas

We make some preliminaries for the following section. For that we list some useful formulas on generalized coherent states based on $su(2)$. Since the proofs are not so difficult, we leave them to the readers.

Formulas For v_1, v_2 we have

$$\text{(i)} \quad \langle v_1|v_2\rangle = \left\{\frac{(1+\bar{\eta}_1\eta_2)^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)}\right\}^J, \tag{65}$$

$$\text{(ii)} \quad \langle v_1|J_+|v_2\rangle = \langle v_1|v_2\rangle\frac{2J\bar{\eta}_1}{1+\bar{\eta}_1\eta_2} \ , \tag{66}$$

$$\text{(iii)} \quad \langle v_1|J_-|v_2\rangle = \langle v_1|v_2\rangle \frac{2J\eta_2}{1+\bar{\eta}_1\eta_2} \ , \tag{67}$$

$$\text{(iv)} \quad \langle v_1|J_-J_+|v_2\rangle = \langle v_1|v_2\rangle \frac{2J+4J^2\bar{\eta}_1\eta_2}{(1+\bar{\eta}_1\eta_2)^2} \ . \tag{68}$$

where

$$\eta_j = \frac{v_j \tan(|v_j|)}{|v_j|} \quad \text{for} \quad j = 1,\ 2. \tag{69}$$

When $v_1 = v_2 \equiv v$, then $\langle v|v\rangle = 1$, so we have

$$\langle v|J_+|v\rangle = \frac{2J\bar{\eta}}{1+|\eta|^2} \ , \quad \langle v|J_-|v\rangle \frac{2J\eta}{1+|\eta|^2} \ , \tag{70}$$

$$\langle v|J_-J_+|v\rangle = \frac{2J+4J^2|\eta|^2}{(1+|\eta|^2)^2} \ . \tag{71}$$

Here let us make a comment. From (65)

$$|\langle v_1|v_2\rangle|^2 = \left\{ \frac{|1+\bar{\eta}_1\eta_2|^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)} \right\}^{2J} ,$$

so we want to know the property of

$$\frac{|1+\bar{\eta}_1\eta_2|^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)}.$$

It is easy to see that

$$1 - \frac{|1+\bar{\eta}_1\eta_2|^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)} = \frac{|\eta_1-\eta_2|^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)} \geq 0 \tag{72}$$

and (72) $= 0$ if and only if (iff) $\eta_1 = \eta_2$. Therefore

$$|\langle v_1|v_2\rangle|^2 = \left\{ \frac{|1+\bar{\eta}_1\eta_2|^2}{(1+|\eta_1|^2)(1+|\eta_2|^2)} \right\}^{2J} \leq 1 \tag{73}$$

because $2J > 1$ (from (64)). Of course

$$|\langle v_1|v_2\rangle| = 1 \quad \text{iff} \quad \eta_1 = \eta_2 \quad \text{iff} \quad v_1 = v_2. \tag{74}$$

by (69).

4.3 A Supplement

Before ending this section let us make a brief comment on generalized coherent states (58). The coherent states $|z\rangle$ has been defined by (5) : $a|z\rangle = z|z\rangle$. Why do we define the generalized coherent states $|w\rangle$ as $J_-|v\rangle = v|v\rangle$ because J_- is an annihilation operator corresponding to a ? First let us try to calculate $J_-|v\rangle$ making use of (61).

$$J_-|v\rangle = (1+|\eta|^2)^{-J} J_- \mathrm{e}^{\eta J_+}|J,0\rangle = (1+|\eta|^2)^{-J}\mathrm{e}^{\eta J_+}\mathrm{e}^{-\eta J_+}J_-\mathrm{e}^{\eta J_+}|J,0\rangle.$$

Here it is easy to see

$$\begin{aligned}\mathrm{e}^{-\eta J_+}J_-\mathrm{e}^{\eta J_+} &= \sum_{m=0}^{\infty}\frac{1}{m!}[-\eta J_+,[-\eta J_+,[\ ,\cdots,[-\eta J_+,J_-]\cdots]]] \\ &= J_- - 2\eta J_3 - \eta^2 J_+\ ,\end{aligned}$$

from the relations (54), so that

$$\begin{aligned}J_-|v\rangle &= (1+|\zeta|^2)^{-J}\mathrm{e}^{\eta J_+}(J_- - 2\eta J_3 - \eta^2 J_+)|J,0\rangle \\ &= 2\eta J(1+|\eta|^2)^{-J}\mathrm{e}^{\eta J_+}|J,0\rangle - \eta^2 J_+(1+|\zeta|^2)^{-J}\mathrm{e}^{\eta J_+}|J,0\rangle \\ &= (2J\eta - \eta^2 J_+)|v\rangle \qquad (75)\end{aligned}$$

because $J_-|J,0\rangle = \mathbf{0}$ and $J_3|J,0\rangle = -J|J,0\rangle$. Namely we have the equation

$$(J_- + \eta^2 J_+)|v\rangle = 2J\eta|v\rangle, \quad \text{where} \quad \eta = \frac{v\tan(|v|)}{|v|}, \qquad (76)$$

or more symmetrically

$$(\eta^{-1}J_- + \eta J_+)|v\rangle = 2J|v\rangle, \quad \text{where} \quad \eta = \frac{v\tan(|v|)}{|v|}. \qquad (77)$$

This equation is completely different from (5).

5 Schwinger's Boson Method

Here let us construct the spin J and K–representations making use of Schwinger's boson method [7].

We consider the system of two-harmonic oscillators. If we set

$$a_1 = a \otimes 1,\ {a_1}^\dagger = a^\dagger \otimes 1;\ a_2 = 1 \otimes a,\ {a_2}^\dagger = 1 \otimes a^\dagger, \tag{78}$$

then it is easy to see

$$[a_i, a_j] = [{a_i}^\dagger, {a_j}^\dagger] = 0,\ [a_i, {a_j}^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \tag{79}$$

We also denote by $N_i = {a_i}^\dagger a_i$ number operators.

Now we can construct representation of Lie algebras $su(2)$ and $su(1,1)$ by making use of Schwinger's boson method, see for example [8], [9]. Namely if we set

$$su(2): \quad J_+ = {a_1}^\dagger a_2,\ J_- = {a_2}^\dagger a_1,\ J_3 = \frac{1}{2}\left({a_1}^\dagger a_1 - {a_2}^\dagger a_2\right), \tag{80}$$

$$su(1,1): \quad K_+ = {a_1}^\dagger {a_2}^\dagger,\ K_- = a_2 a_1,\ K_3 = \frac{1}{2}\left({a_1}^\dagger a_1 + {a_2}^\dagger a_2 + 1\right), \tag{81}$$

then we have

$$su(2): \quad [J_3, J_+] = J_+,\ [J_3, J_-] = -J_-,\ [J_+, J_-] = 2J_3, \tag{82}$$

$$su(1,1): \quad [K_3, K_+] = K_+,\ [K_3, K_-] = -K_-,\ [K_+, K_-] = -2K_3. \tag{83}$$

In the following we define (unitary) generalized coherent operators based on Lie algebras $su(2)$ and $su(1,1)$.

Definition We set

$$su(2): \quad U_J(v) = e^{vJ_+ - \bar{v}J_-} \quad \text{for} \quad v \in \mathbf{C}, \tag{84}$$

$$su(1,1): \quad U_K(w) = e^{wK_+ - \bar{w}K_-} \quad \text{for} \quad w \in \mathbf{C}. \tag{85}$$

For the latter convenience let us list well-known disentangling formulas once more. We have

$$su(2): \quad U_J(v) = e^{\eta J_+} e^{\log\left(1+|\eta|^2\right)J_3} e^{-\bar{\eta}J_-}, \quad where \quad \eta = \frac{v\tan(|v|)}{|v|}, \tag{86}$$

$$su(1,1): \quad U_K(w) = e^{\zeta K_+} e^{\log\left(1-|\zeta|^2\right)K_3} e^{-\bar{\zeta}K_-}, \quad where \quad \zeta = \frac{w\tanh(|w|)}{|w|}. \tag{87}$$

For the proof see Appendix B. As for a generalization of these formulas see [40].

Now let us make some mathematical preliminaries for the latter sections. We have easily

$$U_J(t)a_1U_J(t)^{-1} = cos(|t|)a_1 - \frac{tsin(|t|)}{|t|}a_2, \tag{88}$$

$$U_J(t)a_2U_J(t)^{-1} = cos(|t|)a_2 + \frac{\bar{t}sin(|t|)}{|t|}a_1, \tag{89}$$

so the map $(a_1, a_2) \longrightarrow (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1})$ is

$$(U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1}) = (a_1, a_2)\begin{pmatrix} cos(|t|) & \frac{\bar{t}sin(|t|)}{|t|} \\ -\frac{tsin(|t|)}{|t|} & cos(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} cos(|t|) & \frac{\bar{t}sin(|t|)}{|t|} \\ -\frac{tsin(|t|)}{|t|} & cos(|t|) \end{pmatrix} \in SU(2).$$

On the other hand we have easily

$$U_K(t)a_1U_K(t)^{-1} = cosh(|t|)a_1 - \frac{tsinh(|t|)}{|t|}a_2^{\dagger}, \tag{90}$$

$$U_K(t)a_2^{\dagger}U_K(t)^{-1} = cosh(|t|)a_2^{\dagger} - \frac{\bar{t}sinh(|t|)}{|t|}a_1, \tag{91}$$

so the map $(a_1, a_2^{\dagger}) \longrightarrow (U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^{\dagger}U_K(t)^{-1})$ is

$$(U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^{\dagger}U_K(t)^{-1}) = (a_1, a_2^{\dagger})\begin{pmatrix} cosh(|t|) & -\frac{\bar{t}sinh(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & cosh(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} cosh(|t|) & -\frac{\bar{t}sinh(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & cosh(|t|) \end{pmatrix} \in SU(1,1).$$

Before ending this section let us ask a question.

What is a relation between (85) and (34) of generalized coherent operators based on $su(1.1)$?

The answer is given by :

Formula We have

$$U_J(-\frac{\pi}{4})S_1(w)S_2(-w)U_J(-\frac{\pi}{4})^{-1} = U_K(w), \tag{92}$$

where $S_j(w) = (34)$ with a_j instead of a, see [32].

Namely, $U_K(w)$ is given by "rotating" the product $S_1(w)S_2(-w)$ by $U_J(-\frac{\pi}{4})$.

Proof It is easy to see

$$U_J(t)S_1(w)S_2(-w)U_J(t)^{-1} = U_J(t)\mathrm{e}^{\frac{w}{2}\left\{(a_1^\dagger)^2-(a_2^\dagger)^2\right\}-\frac{\bar{w}}{2}\left\{(a_1)^2-(a_2)^2\right\}}U_J(t)^{-1} = \mathrm{e}^{\mathrm{X}} \tag{93}$$

where

$$\begin{aligned}\mathrm{X} = &\frac{w}{2}\left\{(U_J(t)a_1^\dagger U_J(t)^{-1})^2 - (U_J(t)a_2^\dagger U_J(t)^{-1})^2\right\} \\ &- \frac{\bar{w}}{2}\left\{(U_J(t)a_1 U_J(t)^{-1})^2 - (U_J(t)a_2 U_J(t)^{-1})^2\right\}.\end{aligned} \tag{94}$$

¿From (88) and (89) we have

$$\begin{aligned}&\mathrm{X} = \\ &\frac{w}{2}\left\{\left(cos^2(|t|) - \frac{t^2 sin^2(|t|)}{|t|^2}\right)(a_1^\dagger)^2 - \left(cos^2(|t|) - \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\right)(a_2^\dagger)^2 - \frac{(t+\bar{t})sin(2|t|)}{|t|}a_1^\dagger a_2^\dagger\right\} \\ &-\frac{\bar{w}}{2}\left\{\left(cos^2(|t|) - \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\right)a_1^2 - \left(cos^2(|t|) - \frac{t^2 sin^2(|t|)}{|t|^2}\right)a_2^2 - \frac{(t+\bar{t})sin(2|t|)}{|t|}a_1 a_2\right\}.\end{aligned} \tag{95}$$

Here we set $t = \frac{-\pi}{4}$, then

$$\mathrm{X} = \frac{w}{2}(2a_1^\dagger a_2^\dagger) - \frac{\bar{w}}{2}(2a_1 a_2) = w a_1^\dagger a_2^\dagger - \bar{w} a_1 a_2 \implies \mathrm{e}^{X} = U_K(w).$$

Namely, we obtain the formula.

Next let us prove the following

Formula

$$U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = U_J(t)\mathrm{e}^{\left\{\frac{\alpha}{2}(a_1^\dagger)^2-\frac{\bar{\alpha}}{2}(a_1)^2+\frac{\beta}{2}(a_2^\dagger)^2-\frac{\bar{\beta}}{2}(a_2)^2\right\}}U_J(t)^{-1} = \mathrm{e}^{\mathrm{X}} \tag{96}$$

where

$$\begin{aligned}\mathrm{X} = &\frac{\alpha}{2}(U_J(t)a_1^\dagger U_J(t)^{-1})^2 - \frac{\bar{\alpha}}{2}(U_J(t)a_1 U_J(t)^{-1})^2 \\ &+ \frac{\beta}{2}(U_J(t)a_2^\dagger U_J(t)^{-1})^2 - \frac{\bar{\beta}}{2}(U_J(t)a_2 U_J(t)^{-1})^2.\end{aligned}$$

¿From (88) and (89) we have

$$
\begin{aligned}
\mathrm{X} = & \frac{1}{2}\left\{cos^2(|t|)\alpha + \frac{t^2 sin^2(|t|)}{|t|^2}\beta\right\}(a_1^\dagger)^2 - \frac{1}{2}\left\{cos^2(|t|)\bar{\alpha} + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\bar{\beta}\right\}a_1^2 \\
& + \frac{1}{2}\left\{cos^2(|t|)\beta + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\alpha\right\}(a_2^\dagger)^2 - \frac{1}{2}\left\{cos^2(|t|)\bar{\beta} + \frac{t^2 sin^2(|t|)}{|t|^2}\bar{\alpha}\right\}a_2^2 \\
& + (\beta t - \alpha\bar{t})\frac{sin(2|t|)}{2|t|}a_1^\dagger a_2^\dagger - (\bar{\beta}\bar{t} - \bar{\alpha}t)\frac{sin(2|t|)}{2|t|}a_1 a_2. \qquad (97)
\end{aligned}
$$

If we set

$$\beta t - \alpha\bar{t} = 0 \Longleftrightarrow \beta t = \alpha\bar{t}, \qquad (98)$$

then it is easy to check

$$cos^2(|t|)\alpha + \frac{t^2 sin^2(|t|)}{|t|^2}\beta = \alpha, \quad cos^2(|t|)\beta + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\alpha = \beta,$$

so, in this case,

$$X = \frac{1}{2}\alpha(a_1^\dagger)^2 - \frac{1}{2}\bar{\alpha}a_1^2 + \frac{1}{2}\beta(a_2^\dagger)^2 - \frac{1}{2}\bar{\beta}a_2^2\ .$$

Therefore

$$U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = S_1(\alpha)S_2(\beta). \qquad (99)$$

That is, $S_1(\alpha)S_2(\beta)$ commutes with $U_J(t)$ under the condition (98). We use this formula in the following.

6 Universal Bundles and Chern Characters

In this section we introduce some basic properties of pull–backed ones of universal bundles over the infinite–dimensional Grassmann manifolds and Chern characters, see [13].

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$. For $m \in \mathbf{N}$, we set

$$St_m(\mathcal{H}) \equiv \left\{V = (v_1, \cdots, v_m) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V \in GL(m;\mathbf{C})\right\}\ . \qquad (100)$$

This is called a (universal) Stiefel manifold. Note that the unitary group $U(m)$ acts on $St_m(\mathcal{H})$ from the right :

$$St_m(\mathcal{H}) \times U(m) \longrightarrow St_m(\mathcal{H})\ :\ (V, a) \longmapsto Va\ . \qquad (101)$$

Next we define a (universal) Grassmann manifold

$$Gr_m(\mathcal{H}) \equiv \left\{X \in M(\mathcal{H}) \mid X^2 = X, X^\dagger = X \text{ and } \mathrm{tr}X = m\right\} \ , \tag{102}$$

where $M(\mathcal{H})$ denotes a space of all bounded linear operators on $\mathcal{H}$. Then we have a projection

$$\pi : St_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) \ , \quad \pi(V) \equiv V(V^\dagger V)^{-1}V^\dagger \ , \tag{103}$$

compatible with the action (101) ($\pi(Va) = Va\{a^{-1}(V^\dagger V)^{-1}a\}(Va)^\dagger = \pi(V)$).

Now the set

$$\{U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\} \ , \tag{104}$$

is called a (universal) principal $U(m)$ bundle, see [13] and [18]. We set

$$E_m(\mathcal{H}) \equiv \{(X, v) \in Gr_m(\mathcal{H}) \times \mathcal{H} \mid Xv = v\} \ . \tag{105}$$

Then we have also a projection

$$\pi : E_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) \ , \quad \pi((X, v)) \equiv X \ . \tag{106}$$

The set

$$\{\mathbf{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\} \ , \tag{107}$$

is called a (universal) m–th vector bundle. This vector bundle is one associated with the principal $U(m)$ bundle (104).

Next let $\mathcal{M}$ be a finite or infinite dimensional differentiable manifold and the map

$$P : \mathcal{M} \longrightarrow Gr_m(\mathcal{H}) \tag{108}$$

be given (called a projector). Using this P we can make the bundles (104) and (107) pullback over $\mathcal{M}$:

$$\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\right\} \equiv P^* \{U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\} \ , \tag{109}$$

$$\left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, \mathcal{M}\right\} \equiv P^* \{\mathbf{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\} \ , \tag{110}$$

$$
\begin{array}{ccccccc}
U(m) & & U(m) & \qquad & \mathbf{C}^m & & \mathbf{C}^m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{St} & \longrightarrow & St_m(\mathcal{H}) & & \tilde{E} & \longrightarrow & E_m(\mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{M} & \stackrel{P}{\longrightarrow} & Gr_m(\mathcal{H}) & & \mathcal{M} & \stackrel{P}{\longrightarrow} & Gr_m(\mathcal{H})
\end{array}
$$

see [13]. (110) is of course a vector bundle associated with (109).

For this bundle the (global) curvature (2–) form $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = PdP \wedge dP \tag{111}$$

making use of (108), where d is the usual differential form on $\mathbf{\Omega}$. For the bundles Chern characters play an essential role in several geometric properties. In this case Chern characters are given by

$$\mathbf{\Omega},\ \mathbf{\Omega}^2,\ \cdots,\ \mathbf{\Omega}^{m/2};\quad \mathbf{\Omega}^2 = \mathbf{\Omega} \wedge \mathbf{\Omega},\ \text{etc}, \tag{112}$$

where we have assumed that $m = \dim\mathcal{M}$ is even. In this paper we don't take the trace of (112), so it may be better to call them densities for Chern characters.

To calculate these quantities in infinite–dimensional cases is not so easy. In the next section let us calculate these ones in the special cases.

Let us now define our projectors for the latter aim. In the following, for $\mathcal{H}$ we treat $\mathcal{H} = \mathcal{H}$ in section 2, $\mathcal{H} = \mathcal{H}_K$ in section 3 and $\mathcal{H} = \mathcal{H}_J$ in section 4 at the same time. For $u_1, u_2, \cdots, u_m \in \mathbf{C}$ we consider a set of coherent or generalized coherent states $\{|u_1\rangle, |u_2\rangle, \cdots, |u_m\rangle\}$ and set

$$V_m(\mathbf{u}) = (|u_1\rangle, |u_2\rangle, \cdots, |u_m\rangle) \equiv V_m \tag{113}$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_m)$. Since ${V_m}^\dagger V_m = (\langle u_i|u_j\rangle) \in M(m, \mathbf{C})$, we define

$$\mathcal{D}_m \equiv \{\mathbf{u} \in \mathbf{C}^m \mid \det({V_m}^\dagger V_m) \neq 0\}. \tag{114}$$

We note that $\mathcal{D}_m$ is an open set in $\mathbf{C}^m$. For example, for $m = 1$ and $m = 2$

$$V_1{}^\dagger V_1 = 1,$$
$$\det(V_2{}^\dagger V_2) = \begin{vmatrix} 1 & a \\ \bar{a} & 1 \end{vmatrix} = 1 - |a|^2 \geq 0 \ ,$$

where $a = \langle u_1 | u_2 \rangle$. So from (10), (44) and (74) we have

$$\mathcal{D}_1 = \{ u \in \mathbf{C} \mid \text{no conditions} \} = \mathbf{C}, \tag{115}$$

$$\mathcal{D}_2 = \{ (u_1, u_2) \in \mathbf{C}^2 \mid u_1 \neq u_2 \}. \tag{116}$$

For $\mathcal{D}_m$ $(m \geq 3)$ it is not easy for us to give a simple condition like (116).

Problem For the case $m = 3$ make the condition (114) clear like (116).

At any rate $V_m \in St_m(\mathcal{H})$ for $\mathbf{u} \in \mathcal{D}_m$. Now let us define our projector P as follows :

$$P : \mathcal{D}_m \longrightarrow Gr_m(\mathcal{H}) \ , \quad P(\mathbf{u}) = V_m(V_m^\dagger V_m)^{-1} V_m^\dagger \ . \tag{117}$$

In the following we set $V = V_m$ for simplicity. Let us calculate (111). Since

$$dP = V(V^\dagger V)^{-1} dV^\dagger \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} + \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV (V^\dagger V)^{-1} V^\dagger \tag{118}$$

where $d = \sum_{j=1}^m \left(du_j \frac{\partial}{\partial u_j} + d\bar{u}_j \frac{\partial}{\partial \bar{u}_j} \right)$, we have

$$PdP = V(V^\dagger V)^{-1} dV^\dagger \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \}$$

after some calculation. Therefore we obtain

$$PdP \wedge dP = V(V^\dagger V)^{-1} [dV^\dagger \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV] (V^\dagger V)^{-1} V^\dagger \ . \tag{119}$$

Our main calculation is $dV^\dagger \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV$, which is rewritten as

$$dV^\dagger \{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV = \left[\{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV \right]^\dagger \left[\{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV \right] \tag{120}$$

since $Q \equiv \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger$ is also a projector ($Q^2 = Q$ and $Q^\dagger = Q$). Therefore the first step for us is to calculate the term

$$\{ \mathbf{1} - V(V^\dagger V)^{-1} V^\dagger \} dV \ . \tag{121}$$

Let us summarize **our process of calculations** :

1–st step $\{\mathbf{1} - V(V^\dagger V)^{-1}V^\dagger\}dV \;\cdots(121)$,

2–nd step $dV^\dagger\{\mathbf{1} - V(V^\dagger V)^{-1}V^\dagger\}dV \;\cdots(120)$,

3–rd step $V(V^\dagger V)^{-1}[dV^\dagger\{\mathbf{1} - V(V^\dagger V)^{-1}V^\dagger\}dV](V^\dagger V)^{-1}V^\dagger \;\cdots(119)$.

7 Calculations of Curvature Forms

In this section we only calculate the curvature forms ($m = 1$). The calculations even for the case $m = 2$ are complicated enough, see [11] and [12]. For $m \geq 3$ calculations may become miserable.

7.1 Coherent States

In this case $\langle z|z\rangle = 1$, so our projector is very simple to be

$$P(z) = |z\rangle\langle z|. \tag{122}$$

In this case the calculation of curvature is relatively simple. ¿From (119) we have

$$PdP \wedge dP = |z\rangle\{d\langle z|(\mathbf{1} - |z\rangle\langle z|)d|z\rangle\}\langle z| = |z\rangle\langle z|\{d\langle z|(\mathbf{1} - |z\rangle\langle z|)d|z\rangle\}. \tag{123}$$

Since $|z\rangle = \exp(-\frac{1}{2}|z|^2)\exp(za^\dagger)|0\rangle$ by (6),

$$d|z\rangle = \left\{(a^\dagger - \frac{\bar{z}}{2})dz - \frac{z}{2}d\bar{z}\right\}|z\rangle = \left\{a^\dagger dz - \frac{1}{2}(\bar{z}dz + zd\bar{z})\right\}|z\rangle = \left\{a^\dagger dz - \frac{1}{2}d(|z|^2)\right\}|z\rangle,$$

so that

$$(\mathbf{1} - |z\rangle\langle z|)d|z\rangle = (\mathbf{1} - |z\rangle\langle z|)a^\dagger|z\rangle dz = (a^\dagger - \langle z|a^\dagger|z\rangle)|z\rangle dz = (a - z)^\dagger dz|z\rangle$$

because $(\mathbf{1} - |z\rangle\langle z|)|z\rangle = \mathbf{0}$. Similarly $d\langle z|(\mathbf{1} - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z}$.
Let us summarize :

$$(\mathbf{1} - |z\rangle\langle z|)d|z\rangle = (a - z)^\dagger dz|z\rangle, \quad d\langle z|(\mathbf{1} - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z}\ . \tag{124}$$

Now we are in a position to determine the curvature form (123).

$$d\langle z|(\mathbf{1}-|z\rangle\langle z|)d|z\rangle = \langle z|(a-z)(a-z)^\dagger|z\rangle d\bar{z}\wedge dz = d\bar{z}\wedge dz$$

after some algebra. Therefore

$$\mathbf{\Omega} = PdP\wedge dP = |z\rangle\langle z|d\bar{z}\wedge dz\ . \tag{125}$$

¿From this result we know

$$\frac{\mathbf{\Omega}}{2\pi i} = |z\rangle\langle z|\frac{dx\wedge dy}{\pi}$$

when $z = x + iy$. This just gives the resolution of unity in (9).

7.2 Generalized Coherent States Based on $su(1,1)$

In this case $\langle w|w\rangle = 1$, so our projector is very simple to be

$$P(w) = |w\rangle\langle w|. \tag{126}$$

In this case the calculation of curvature is relatively simple. ¿From (119) we have

$$PdP\wedge dP = |w\rangle\{d\langle w|(\mathbf{1}_K - |w\rangle\langle w|)d|w\rangle\}\langle w| = |w\rangle\langle w|\{d\langle w|(\mathbf{1}_K - |w\rangle\langle w|)d|w\rangle\}, \tag{127}$$

where $d = dw\frac{\partial}{\partial w} + d\bar{w}\frac{\partial}{\partial \bar{w}}$. Since $|w\rangle = (1-|\zeta|^2)^K\exp(\zeta K_+)|K,0\rangle$ by (29),

$$d|w\rangle = \left\{d\zeta K_+ + Kd\log(1-|\zeta|^2)\right\}|w\rangle \tag{128}$$

by some calculation, so that

$$\begin{aligned}(\mathbf{1}_K - |w\rangle\langle w|)d|w\rangle &= (\mathbf{1}_K - |w\rangle\langle w|)K_+|w\rangle d\zeta = (K_+ - \langle w|K_+|w\rangle)|w\rangle d\zeta \\ &= \left(K_+ - \frac{2K\bar{\zeta}}{1-|\zeta|^2}\right)d\zeta|w\rangle\end{aligned} \tag{129}$$

because $(\mathbf{1}_K - |w\rangle\langle w|)|w\rangle = \mathbf{0}$. Similarly we have

$$d\langle w|(\mathbf{1}_K - |w\rangle\langle w|) = \langle w|\left(K_- - \frac{2K\zeta}{1-|\zeta|^2}\right)d\bar{\zeta} \tag{130}$$

Now we are in a position to determine the curvature form (127).

$$
\begin{aligned}
& d\langle w|(\mathbf{1}_K - |w\rangle\langle w|)d|w\rangle \\
&= \langle w|\left(K_- - \frac{2K\zeta}{1-|\zeta|^2}\right)\left(K_+ - \frac{2K\bar{\zeta}}{1-|\zeta|^2}\right)|w\rangle d\bar{\zeta}\wedge d\zeta \\
&= \left\{\langle w|K_-K_+|w\rangle - \frac{2K\bar{\zeta}}{1-|\zeta|^2}\langle w|K_-|w\rangle - \frac{2K\zeta}{1-|\zeta|^2}\langle w|K_+|w\rangle + \frac{4K^2|\zeta|^2}{(1-|\zeta|^2)^2}\right\} d\bar{\zeta}\wedge d\zeta \\
&= \frac{2K}{(1-|\zeta|^2)^2} d\bar{\zeta}\wedge d\zeta \qquad (131)
\end{aligned}
$$

after some algebra with (40) and (41). Therefore

$$
\mathbf{\Omega} = PdP\wedge dP = |w\rangle\langle w|\frac{2Kd\bar{\zeta}\wedge d\zeta}{(1-|\zeta|^2)^2}. \qquad (132)
$$

¿From this result we know

$$
\frac{\mathbf{\Omega}}{2\pi i} = \frac{2K}{\pi}\frac{d\zeta_1\wedge d\zeta_2}{(1-|\zeta|^2)^2}|w\rangle\langle w| = \frac{2K}{\pi}\frac{d\zeta_1\wedge d\zeta_2}{(1-|\zeta|^2)^2}|\zeta\rangle\langle\zeta|
$$

by (29) when $\zeta = \zeta_1 + \sqrt{-1}\zeta_2$. If we define a constant

$$
C_K = \frac{2K-1}{2K}, \qquad (133)
$$

then we have

$$
C_K\frac{\mathbf{\Omega}}{2\pi i} = \frac{2K-1}{\pi}\frac{d\zeta_1\wedge d\zeta_2}{(1-|\zeta|^2)^2}|\zeta\rangle\langle\zeta|. \qquad (134)
$$

This gives the resolution of unity in (31). But the situation is a bit different from [11] in which the constant corresponding to C_K is just one.

Problem What is a (deep) meaning of C_K ?

7.3 Generalized Coherent States Based on $su(2)$

In this case $\langle v|v\rangle = 1$, so our projector is very simple to be

$$
P(v) = |v\rangle\langle v|. \qquad (135)
$$

In this case the calculation of curvature is relatively simple. ¿From (119) we have

$$
PdP\wedge dP = |v\rangle\{d\langle v|(\mathbf{1}_J - |v\rangle\langle v|)d|v\rangle\}\langle v| = |v\rangle\langle v|\{d\langle v|(\mathbf{1}_J - |v\rangle\langle v|)d|v\rangle\}, \qquad (136)
$$

where $d = dv\frac{\partial}{\partial v} + d\bar{v}\frac{\partial}{\partial \bar{v}}$. Since $|v\rangle = (1+|\eta|^2)^{-J}\exp(\eta J_+)|J,0\rangle$ by (61),

$$d|v\rangle = \left\{d\eta J_+ - Jd\log(1+|\eta|^2)\right\}|v\rangle \tag{137}$$

by some calculation, so that

$$\begin{aligned}(\mathbf{1}_J - |v\rangle\langle v|)d|v\rangle &= (\mathbf{1}_J - |v\rangle\langle v|)J_+|v\rangle d\eta = (J_+ - \langle v|J_+|v\rangle)|v\rangle d\eta \\ &= \left(J_+ - \frac{2J\bar{\eta}}{1+|\eta|^2}\right)d\eta|v\rangle\end{aligned} \tag{138}$$

because $(\mathbf{1}_J - |v\rangle\langle v|)|v\rangle = \mathbf{0}$. Similarly we have

$$d\langle v|(\mathbf{1}_J - |v\rangle\langle v|) = \langle v|\left(J_- - \frac{2J\eta}{1+|\eta|^2}\right)d\bar{\eta} \tag{139}$$

Now we are in a position to determine the curvature form (136).

$$\begin{aligned}&d\langle v|(\mathbf{1}_J - |v\rangle\langle v|)d|v\rangle \\ &= \langle v|\left(J_- - \frac{2J\eta}{1+|\eta|^2}\right)\left(J_+ - \frac{2J\bar{\eta}}{1+|\eta|^2}\right)|v\rangle d\bar{\eta}\wedge d\eta \\ &= \left\{\langle v|J_-J_+|v\rangle - \frac{2J\bar{\eta}}{1+|\eta|^2}\langle v|J_-|v\rangle - \frac{2J\eta}{1+|\eta|^2}\langle v|J_+|v\rangle + \frac{4J^2|\eta|^2}{(1+|\eta|^2)^2}\right\}d\bar{\eta}\wedge d\eta \\ &= \frac{2J}{(1+|\eta|^2)^2}d\bar{\eta}\wedge d\eta\end{aligned} \tag{140}$$

after some algebra with (70) and (71). Therefore

$$\mathbf{\Omega} = PdP\wedge dP = |v\rangle\langle v|\frac{2Jd\bar{\eta}\wedge d\eta}{(1+|\eta|^2)^2}. \tag{141}$$

¿From this result we know

$$\frac{\mathbf{\Omega}}{2\pi i} = \frac{2J}{\pi}\frac{d\eta_1\wedge d\eta_2}{(1+|\eta|^2)^2}|v\rangle\langle v| = \frac{2J}{\pi}\frac{d\eta_1\wedge d\eta_2}{(1+|\eta|^2)^2}|\eta\rangle\langle\eta|$$

by (61) when $\eta = \eta_1 + \sqrt{-1}\eta_2$. If we define a constant

$$C_J = \frac{2J+1}{2J}, \tag{142}$$

then we have

$$C_J\frac{\mathbf{\Omega}}{2\pi i} = \frac{2J+1}{\pi}\frac{d\eta_1\wedge d\eta_2}{(1+|\eta|^2)^2}|\eta\rangle\langle\eta|. \tag{143}$$

This gives the resolution of unity in (64). But the situation is a bit different from [11] in which the constant corresponding to C_J is just one.

Problem What is a (deep) meaning of C_J ?

8 Holonomic Quantum Computation

In this section we introduce the concept of Holonomic Quantum Computation, see [23]. Our method is based on non–abelian Berry phase [14]. By the way we have Geometric Quantum Computation based on abelian Berry phase. However we don't make a comment on this topic, see for example [27], [28] and their references.

Let $\mathcal{M}$ be a parameter space and we denote by λ its element. Let $\lambda_{\mathbf{0}}$ be a fixed reference point of $\mathcal{M}$. Let H_λ be a family of Hamiltonians parameterized by $\mathcal{M}$ which act on a Fock space $\mathcal{H}$. We set $H_0 = H_{\lambda_{\mathbf{0}}}$ for simplicity and assume that this has a m-fold degenerate vacuum :

$$H_0 v_j = \mathbf{0}, \quad j = 1 \sim m. \tag{144}$$

These v_j's form a m-dimensional vector space. We may assume that $\langle v_i | v_j \rangle = \delta_{ij}$. Then $(v_1, \cdots, v_m) \in St_m(\mathcal{H})$ and

$$F_0 \equiv \left\{ \sum_{j=1}^{m} x_j v_j | x_j \in \mathbf{C} \right\} \cong \mathbf{C}^m.$$

Namely, F_0 is a vector space associated with o.n.basis $(v_1, \cdots, v_m)$.

Next we assume for simplicity that a family of unitary operators parameterized by $\mathcal{M}$

$$W : \mathcal{M} \to U(\mathcal{H}), \quad W(\lambda_{\mathbf{0}}) = \mathrm{id}. \tag{145}$$

is given and H_λ above is given by the following isospectral family

$$H_\lambda \equiv W(\lambda) H_0 W(\lambda)^{-1}. \tag{146}$$

In this case there is no level crossing of eigenvalues. Making use of $W(\lambda)$ we can define a projector

$$P : \mathcal{M} \to Gr_m(\mathcal{H}), \quad P(\lambda) \equiv W(\lambda) \left(\sum_{j=1}^{m} v_j {v_j}^{\dagger} \right) W(\lambda)^{-1} \tag{147}$$

and have the pullback bundles over $\mathcal{M}$ from (109) and (110)

$$\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\right\}, \quad \left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, \mathcal{M}\right\}. \tag{148}$$

For the latter we set

$$|vac\rangle = (v_1, \cdots, v_m). \tag{149}$$

In this case a canonical connection form $\mathcal{A}$ of $\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\right\}$ is given by

$$\mathcal{A} = \langle vac|W(\lambda)^{-1}dW(\lambda)|vac\rangle, \tag{150}$$

where d is a usual differential form on $\mathcal{M}$, and its curvature form by

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{151}$$

see [14] and [13].

Let γ be a loop in $\mathcal{M}$ at $\lambda_{\mathbf{0}}$.,

$$\gamma : [0,1] \longrightarrow \mathcal{M}, \quad \gamma(0) = \gamma(1).$$

For this γ a holonomy operator $\Gamma_{\mathcal{A}}$ is defined as the path–ordered integral of $\mathcal{A}$ along γ :

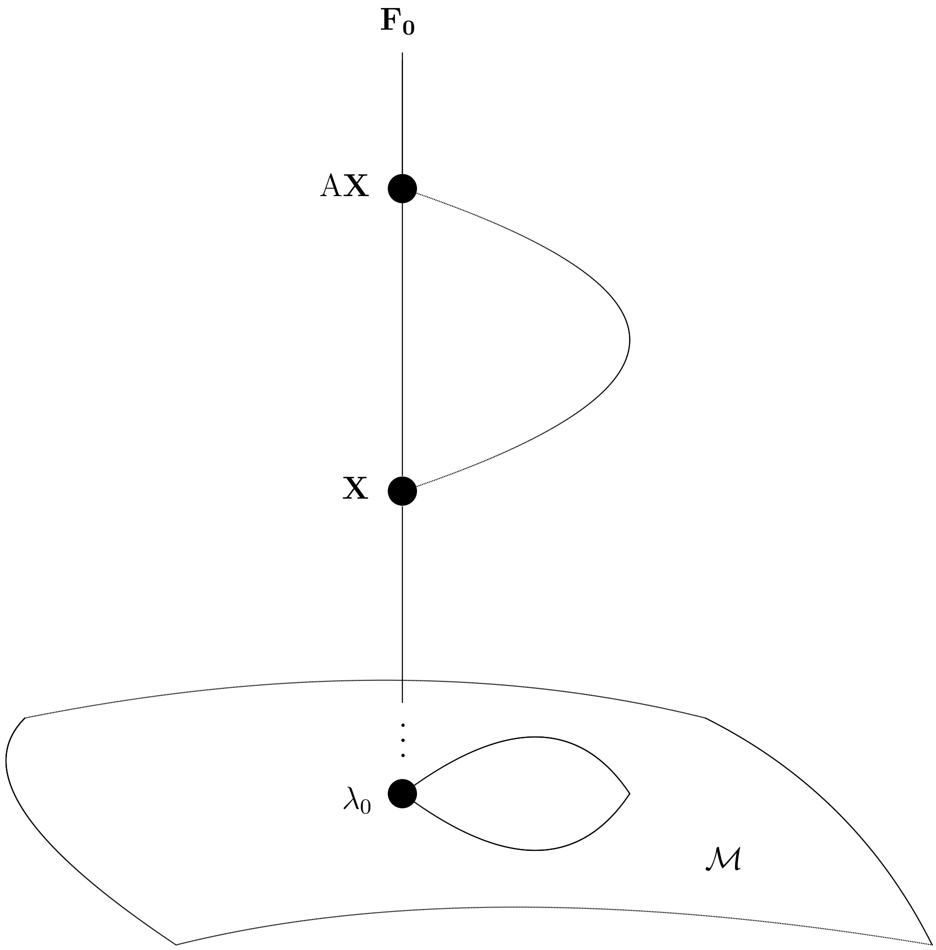
$$\Gamma_{\mathcal{A}}(\gamma) = \mathcal{P}\text{exp}\left\{\oint_\gamma \mathcal{A}\right\} \in U(m), \tag{152}$$

where $\mathcal{P}$ means path-ordered. See [13].

This acts on the fiber F_0 at $\lambda_{\mathbf{0}}$ of the vector bundle $\left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, M\right\}$ as follows : $\mathbf{x} \rightarrow \Gamma_{\mathcal{A}}(\gamma)\mathbf{x}$. The holonomy group $Hol(\mathcal{A})$ is in general subgroup of $U(m)$. In the case of $Hol(\mathcal{A}) = U(m)$, $\mathcal{A}$ is called irreducible, see [13].

In the Holonomic Quantum Computation we take

$$\begin{aligned} &\text{Encoding of Information} \Longrightarrow \mathbf{x} \in F_0, \\ &\text{Processing of Information} \Longrightarrow \Gamma_{\mathcal{A}}(\gamma) : \mathbf{x} \rightarrow \Gamma_{\mathcal{A}}(\gamma)\mathbf{x}. \end{aligned} \tag{153}$$



Quantum Computational Bundle

8.1 One–Qubit Case

Let H_0 be a Hamiltonian with nonlinear interaction produced by a Kerr medium., that is

$$H_0 = \hbar \mathrm{X} N(N-1), \tag{154}$$

where X is a certain constant, see [2] and [26]. The eigenvectors of H_0 corresponding to 0 is $\{|0\rangle, |1\rangle\}$, so its eigenspace is $\mathrm{Vect}\{|0\rangle, |1\rangle\} \cong \mathbf{C}^2$. The vector space $\mathrm{Vect}\{|0\rangle, |1\rangle\}$ is called 1-qubit (**qu**antum **bit**) space and we set

$$F_0 = \mathrm{Vect}\{|0\rangle, |1\rangle\} \quad \text{and} \quad |vac\rangle = (|0\rangle, |1\rangle).$$

Now we consider the following isospectral family of H_0 :

$$H_{(\alpha,\beta)} = W(\alpha,\beta)H_0W(\alpha,\beta)^{-1}, \tag{155}$$

$$W(\alpha,\beta) = D(\alpha)S(\beta). \tag{156}$$

In this case

$$\mathcal{M} = \left\{(\alpha,\beta) \in \mathbf{C}^2\right\} \tag{157}$$

and we want calculate

$$\mathcal{A} = \langle vac|W^{-1}dW|vac\rangle \tag{158}$$

where

$$d = d\alpha\frac{\partial}{\partial\alpha} + d\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}} + d\beta\frac{\partial}{\partial\beta} + d\bar{\beta}\frac{\partial}{\partial\bar{\beta}}. \tag{159}$$

Since $\mathcal{A}$ is anti–hermitian ($\mathcal{A}^\dagger = -\mathcal{A}$), we can write

$$\mathcal{A} = A_\alpha d\alpha + A_\beta d\beta - {A_\alpha}^\dagger d\bar{\alpha} - {A_\beta}^\dagger d\bar{\beta} \tag{160}$$

where

$$A_\alpha = \langle vac|W^{-1}\frac{\partial W}{\partial\alpha}|vac\rangle \qquad A_\beta = \langle vac|W^{-1}\frac{\partial W}{\partial\beta}|vac\rangle.$$

The calculation of A_α and A_β is as follows ([19]) :

$$A_\alpha = \frac{\bar{\alpha}}{2}L + cosh(|\beta|)F + \frac{\bar{\beta}sinh(|\beta|)}{|\beta|}E, \tag{161}$$

$$A_\beta = \frac{\bar{\beta}(-1+cosh(2|\beta|))}{4|\beta|^2}\left(K + \frac{1}{2}L\right), \tag{162}$$

where

$$E = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix},\ F = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix},\ K = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix},\ L = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

Then making use of these ones we can show that the holonomy group generated by (158) is irreducible in $U(2)$, namely just $U(2)$, see [25] and [19]. This is very crucial fact to Holonomic Quantum Computation.

8.2 Two–Qubit Case

We consider the system of two particles, so the Hamiltonian that we treat in the following is

$$H_0 = \hbar \mathrm{X} N_1(N_1 - 1) + \hbar \mathrm{X} N_2(N_2 - 1). \tag{163}$$

The eigenspace of 0 of this Hamiltonian becomes therefore

$$F_0 = \mathrm{Vect}\,\{|0\rangle, |1\rangle\} \otimes \mathrm{Vect}\,\{|0\rangle, |1\rangle\} = \mathrm{Vect}\,\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\} \cong \mathbf{C}^4. \tag{164}$$

We set $|vac\rangle = (|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle)$.

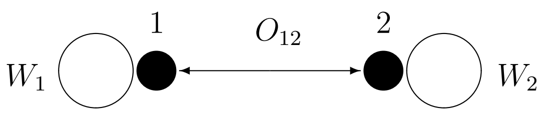
Next we consider the following isospectral family of H_0 :

$$H_{(\alpha_1,\beta_1,\lambda,\mu,\alpha_2,\beta_2)} = W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) H_0 W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2)^{-1}, \tag{165}$$

$$W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) = W_1(\alpha_1, \beta_1) O_{12}(\lambda, \mu) W_2(\alpha_2, \beta_2). \tag{166}$$

where

$$O_{12}(\lambda, \mu) = U_J(\lambda) U_K(\mu), \quad W_j(\alpha_j, \beta_j) = D_j(\alpha_j) S_j(\beta_j) \quad \text{for} \quad j = 1, 2. \tag{167}$$



In this case

$$\mathcal{M} = \left\{ (\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) \in \mathbf{C}^6 \right\} \tag{168}$$

and we have only to calculate the following

$$\mathcal{A} = \langle vac | W^{-1} dW | vac \rangle, \tag{169}$$

where

$$\begin{aligned} d = {} & d\alpha_1 \frac{\partial}{\partial \alpha_1} + d\bar{\alpha_1} \frac{\partial}{\partial \bar{\alpha_1}} + d\beta_1 \frac{\partial}{\partial \beta_1} + d\bar{\beta}_1 \frac{\partial}{\partial \bar{\beta}_1} + d\lambda \frac{\partial}{\partial \lambda} + d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} + d\mu \frac{\partial}{\partial \mu} + d\bar{\mu} \frac{\partial}{\partial \bar{\mu}} \\ & + d\alpha_2 \frac{\partial}{\partial \alpha_2} + d\bar{\alpha_2} \frac{\partial}{\partial \bar{\alpha_2}} + d\beta_2 \frac{\partial}{\partial \beta_2} + d\bar{\beta}_2 \frac{\partial}{\partial \bar{\beta}_2} \ . \end{aligned} \tag{170}$$

The calculation of (169) is not easy, but we can determine it, see [19], [20] and [22] for the details. But we cannot determine its curvature form which is necessary to look for the holonomy group (Ambrose–Singer theorem) owing to too complication.

Then the essential point is

Problem Is the connection form (169) irreducible in $U(4)$?

Our analysis in [22] shows that the holonomy group generated by $\mathcal{A}$ may be $SU(4)$ not $U(4)$. To obtain $U(4)$ a sophisticated trick $\cdots$ higher dimensional holonomies [52] $\cdots$ may be necessary.

A comment is in order. After submitting this paper to quant–ph the author found the papers by Lucarelli [29], [30]. In them he solved the problem above by using the another method.

9 Geometric Construction of Bell States

In this section we introduce the geometric constraction of Bell states by making use of coherent states based on $su(2)$, [48]. One of purpose of Quantum Information Theory is to clarify a role of **entanglement** of states, so that we would like to look for geometric meaning of entanglement.

The famous Bell states ([53], [54]) given by

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \tag{171}$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \tag{172}$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \tag{173}$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \tag{174}$$

are typical examples of entanglement. It is very interesting that these play an essential role in Quantum Teleportation, see [15]. We would like to reconstruct these states in a geometric manner.

9.1 Review on General Theory

Let us make a review of [50] and rewrite the result with our method. Let G be a compact linear Lie group (for example $G = U(n)$) and consider a coherent state representation of G whose parameter space is a compact complex manifold $S = G/H$, where H is a subgroup of G. For example $G = U(n)$ and $H = U(k) \times U(n-k)$, then $S = U(n)/U(k) \times U(n-k) \cong G_k(\mathbf{C}^n)$, which is called a complex Grassmann manifold, [4], [10]. Let Z be a local coordinate on S and $|Z\rangle$ a generalized coherent state in some representation space V ($\cong \mathbf{C}^K$ for some high $K \in \mathbf{N}$). Then we have, by definition, the measure $d\mu(Z, Z^\dagger)$ that satisfies the resolution of unity

$$\int_S d\mu(Z, Z^\dagger)|Z\rangle\langle Z| = \mathbf{1}_V \quad \text{and} \quad \int_S d\mu(Z, Z^\dagger) = \text{dim}V \ . \tag{175}$$

Next we define an anti-automorphism $\flat : S \longrightarrow S$. We call $Z \longrightarrow Z^\flat$ an anti-automorphism if and only if

$$\text{(i)} \ \ Z \longrightarrow Z^\flat \text{ induces an automorphism of } S, \tag{176}$$

$$\text{(ii)} \ \ \flat \text{ is an anti-map, namely} \quad \langle Z^\flat|W^\flat\rangle = \langle W|Z\rangle. \tag{177}$$

Now let us redefine the generalized Bell state in [50] as follows :

Definition The generalized Bell state is defined as

$$||B\rangle\rangle = \frac{1}{\sqrt{\text{dim}V}} \int_S d\mu(Z, Z^\dagger)|Z\rangle \otimes |Z^\flat\rangle. \tag{178}$$

Then we have

$$\begin{aligned}
\langle\langle B||B\rangle\rangle &= \frac{1}{\text{dim}V}\int_S\int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(\langle Z| \otimes \langle Z^\flat|)(|W\rangle \otimes |W^\flat\rangle) \\
&= \frac{1}{\text{dim}V}\int_S\int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)\langle Z|W\rangle\langle Z^\flat|W^\flat\rangle \\
&= \frac{1}{\text{dim}V}\int_S\int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)\langle Z|W\rangle\langle W|Z\rangle \\
&= \frac{1}{\text{dim}V}\int_S d\mu(Z, Z^\dagger)\langle Z|Z\rangle \\
&= \frac{1}{\text{dim}V}\int_S d\mu(Z, Z^\dagger) = 1,
\end{aligned}$$

where we have used (175) and (177).

9.2 Review on Projective Space

We make a review of complex projective spaces, [13], [9] and [22]. For $N \in \mathbf{N}$ the complex projective space $\mathbf{C}P^N$ is defined as follows : For $\boldsymbol{\zeta}$, $\boldsymbol{\mu} \in \mathbf{C}^{N+1} - \{\mathbf{0}\}$ $\boldsymbol{\zeta}$ is equivalent to $\boldsymbol{\mu}$ ($\boldsymbol{\zeta} \sim \boldsymbol{\mu}$) if and only if $\boldsymbol{\zeta} = \lambda\, \boldsymbol{\mu}$ for $\lambda \in \mathbf{C} - \{0\}$. We show the equivalent relation class as $[\boldsymbol{\zeta}]$ and set $\mathbf{C}P^N \equiv \mathbf{C}^{N+1} - \{\mathbf{0}\}/\sim$. When $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \cdots, \zeta_N)$ we write usually as $[\boldsymbol{\zeta}] = [\zeta_0 : \zeta_1 : \cdots : \zeta_N]$. Then it is well–known that $\mathbf{C}P^N$ has $N+1$ local charts, namely

$$\mathbf{C}P^N = \bigcup_{j=0}^{N} U_j\ , \quad U_j = \{[\zeta_0 : \cdots : \zeta_j : \cdots : \zeta_N] \mid \zeta_j \neq 0\}. \tag{179}$$

Since

$$(\zeta_0, \cdots, \zeta_j, \cdots, \zeta_N) = \zeta_j \left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, 1, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j}\right),$$

we have the local coordinate on U_j

$$\left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j}\right). \tag{180}$$

However the above definition of $\mathbf{C}P^N$ is not tractable, so we use the well–known expression by projections

$$\mathbf{C}P^N \cong G_1(\mathbf{C}^{N+1}) = \{P \in M(N+1;\mathbf{C}) \mid P^2 = P,\ P^\dagger = P \text{ and } \mathrm{tr}P = 1\} \tag{181}$$

and the correspondence

$$[\zeta_0 : \zeta_1 : \cdots : \zeta_N] \Longleftrightarrow \frac{1}{|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_N|^2} \begin{pmatrix} |\zeta_0|^2 & \zeta_0\bar{\zeta}_1 & \cdot & \cdot & \zeta_0\bar{\zeta}_N \\ \zeta_1\bar{\zeta}_0 & |\zeta_1|^2 & \cdot & \cdot & \zeta_1\bar{\zeta}_N \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \zeta_N\bar{\zeta}_0 & \zeta_N\bar{\zeta}_1 & \cdot & \cdot & |\zeta_N|^2 \end{pmatrix} \equiv P\ . \tag{182}$$

If we set

$$|\boldsymbol{\zeta}\rangle = \frac{1}{\sqrt{\sum_{j=0}^{N} |\zeta_j|^2}} \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \cdot \\ \cdot \\ \zeta_N \end{pmatrix}, \tag{183}$$

then we can write the right hand side of (182) as

$$P = |\boldsymbol{\zeta}\rangle\langle\boldsymbol{\zeta}| \quad \text{and} \quad \langle\boldsymbol{\zeta}|\boldsymbol{\zeta}\rangle = 1. \tag{184}$$

For example on U_1

$$(z_1, z_2, \cdots, z_N) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}, \cdots, \frac{\zeta_N}{\zeta_0}\right),$$

we have

$$\begin{aligned} P(z_1, \cdots, z_N) &= \frac{1}{1 + \sum_{j=1}^{N} |z_j|^2} \begin{pmatrix} 1 & \bar{z}_1 & \cdot & \cdot & \bar{z}_N \\ z_1 & |z_1|^2 & \cdot & \cdot & z_1\bar{z}_N \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ z_N & z_N\bar{z}_1 & \cdot & \cdot & |z_N|^2 \end{pmatrix} \\ &= |(z_1, z_2, \cdots, z_N)\rangle\langle(z_1, z_2, \cdots, z_N)| \ , \end{aligned} \tag{185}$$

where

$$|(z_1, z_2, \cdots, z_N)\rangle = \frac{1}{\sqrt{1 + \sum_{j=1}^{N} |z_j|^2}} \begin{pmatrix} 1 \\ z_1 \\ \cdot \\ \cdot \\ z_N \end{pmatrix}. \tag{186}$$

Let us give a more detail description for the cases $N = 1$ and 2.

(a) $N = 1$:

$$\begin{aligned} P(z) &= \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} = |z\rangle\langle z|, \\ &\text{where } |z\rangle = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z = \frac{\zeta_1}{\zeta_0}, \quad \text{on } U_1 \ , \end{aligned} \tag{187}$$

$$\begin{aligned} P(w) &= \frac{1}{|w|^2 + 1} \begin{pmatrix} |w|^2 & w \\ \bar{w} & 1 \end{pmatrix} = |w\rangle\langle w|, \\ &\text{where } |w\rangle = \frac{1}{\sqrt{|w|^2 + 1}} \begin{pmatrix} w \\ 1 \end{pmatrix}, \quad w = \frac{\zeta_0}{\zeta_1}, \quad \text{on } U_2 \ . \end{aligned} \tag{188}$$

(b) $N=2$:

$$P(z_1,z_2) \quad = \quad \frac{1}{1+|z_1|^2+|z_2|^2}\begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & z_1\bar{z}_2 \\ z_2 & z_2\bar{z}_1 & |z_2|^2 \end{pmatrix} = |(z_1,z_2)\rangle\langle(z_1,z_2)|,$$

$$\text{where} \quad |(z_1,z_2)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}\begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \quad (z_1,z_2) = \left(\frac{\zeta_1}{\zeta_0},\frac{\zeta_2}{\zeta_0}\right) \quad \text{on } U_1 \ , \tag{189}$$

$$P(w_1,w_2) \quad = \quad \frac{1}{|w_1|^2+1+|w_2|^2}\begin{pmatrix} |w_1|^2 & w_1 & w_1\bar{w}_2 \\ \bar{w}_1 & 1 & \bar{w}_2 \\ w_2\bar{w}_1 & w_2 & |w_2|^2 \end{pmatrix} = |(w_1,w_2)\rangle\langle(w_1,w_2)|,$$

$$\text{where} \quad |(w_1,w_2)\rangle = \frac{1}{\sqrt{|w_1|^2+1+|w_2|^2}}\begin{pmatrix} w_1 \\ 1 \\ w_2 \end{pmatrix}, \quad (w_1,w_2) = \left(\frac{\zeta_0}{\zeta_1},\frac{\zeta_2}{\zeta_1}\right) \quad \text{on } U_2 , \tag{190}$$

$$P(v_1,v_2) \quad = \quad \frac{1}{|v_1|^2+|v_2|^2+1}\begin{pmatrix} |v_1|^2 & v_1\bar{v}_2 & v_1 \\ v_2\bar{v}_1 & |v_2|^2 & v_2 \\ \bar{v}_1 & \bar{v}_2 & 1 \end{pmatrix} = |(v_1,v_2)\rangle\langle(v_1,v_2)|,$$

$$\text{where} \quad |(v_1,v_2)\rangle = \frac{1}{\sqrt{|v_1|^2+|v_2|^2+1}}\begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}, \quad (v_1,v_2) = \left(\frac{\zeta_0}{\zeta_2},\frac{\zeta_1}{\zeta_2}\right) \quad \text{on } U_3 \ . \tag{191}$$

9.3 Bell States Revisited

In this subsection we show that (178) coinsides with the Bell states (171)–(174) by choosing anti-automorphism $\flat$ suitably.

We treat first of all the case of spin $\frac{1}{2}$. From here we identify

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we have

$$|\eta\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|0\rangle + \eta|1\rangle) = \frac{1}{\sqrt{1+|\eta|^2}}\begin{pmatrix} 1 \\ \eta \end{pmatrix}. \tag{192}$$

In this case we consider the following four anti-automorphisms (176) and (177) :

$$(1)\ \ \eta^\flat = \bar{\eta} \qquad (2)\ \ \eta^\flat = -\bar{\eta} \qquad (3)\ \ \eta^\flat = \frac{1}{\bar{\eta}} \qquad (4)\ \ \eta^\flat = \frac{-1}{\bar{\eta}}\ . \tag{193}$$

Now by making use of these we define

Definition

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle \otimes |\bar{\eta}\rangle, \tag{194}$$

$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle \otimes |-\bar{\eta}\rangle, \tag{195}$$

$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle \otimes |1/\bar{\eta}\rangle, \tag{196}$$

$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle \otimes |-1/\bar{\eta}\rangle, \tag{197}$$

where we have put for simplicity

$$d\mu(\eta,\bar{\eta}) = \frac{2}{\pi}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\ .$$

It is easy to see from (187) and (188)

$$(1)\ \ |\eta^\flat\rangle = |\bar{\eta}\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|0\rangle + \bar{\eta}|1\rangle), \tag{198}$$

$$(2)\ \ |\eta^\flat\rangle = |-\bar{\eta}\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|0\rangle - \bar{\eta}|1\rangle), \tag{199}$$

$$(3)\ \ |\eta^\flat\rangle = |1/\bar{\eta}\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(\bar{\eta}|0\rangle + |1\rangle), \tag{200}$$

$$(4)\ \ |\eta^\flat\rangle = |-1/\bar{\eta}\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(-\bar{\eta}|0\rangle + |1\rangle), \tag{201}$$

Then making use of elementary facts

$$\frac{2}{\pi}\int_{\mathbf{C}}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\frac{1}{1+|\eta|^2} = \frac{2}{\pi}\int_{\mathbf{C}}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\frac{|\eta|^2}{1+|\eta|^2} = 1,$$

$$\frac{2}{\pi}\int_{\mathbf{C}}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\frac{\eta}{1+|\eta|^2} = \frac{2}{\pi}\int_{\mathbf{C}}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\frac{\bar{\eta}}{1+|\eta|^2} = 0,$$

we obtain easily

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle), \tag{202}$$
$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle), \tag{203}$$
$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle), \tag{204}$$
$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle). \tag{205}$$

We just recovered the Bell states (171)– (174) !! We can say that four $||B\rangle\rangle$ in **Definition** are overcomplete expression (making use of generalized coherent states) of the Bell states. This is an important point of view.

Since we consider the case of higher spin J, we write $|\eta\rangle$ as

$$|\eta\rangle_J = \frac{1}{\left(1+|\eta|^2\right)^J}\sum_{k=0}^{2J}\sqrt{{}_{2J}C_k}\ \eta^k|k\rangle \tag{206}$$

to emphasize the dependence of spin J. Here we have set $|k\rangle = |J,k\rangle$ for simplicity. ¿From the above result it is very natural to define **Bell states with spin** J as follows because the parameter space is the same $\mathbf{C}P^1$:

Definition

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle_J\otimes|\bar{\eta}\rangle_J, \tag{207}$$
$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle_J\otimes|-\bar{\eta}\rangle_J, \tag{208}$$
$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle_J\otimes|1/\bar{\eta}\rangle_J, \tag{209}$$
$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\int_{\mathbf{C}} d\mu(\eta,\bar{\eta})|\eta\rangle_J\otimes|-1/\bar{\eta}\rangle_J, \tag{210}$$

where

$$d\mu(\eta,\bar{\eta}) = \frac{2J+1}{\pi}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\ .$$

Let us calculate $|\bar{\eta}\rangle_J$, $|-\bar{\eta}\rangle_J$, $|1/\bar{\eta}\rangle_J$ and $|-1/\bar{\eta}\rangle_J$. It is easy to see

$$(1)\ \ |\bar{\eta}\rangle_J = \frac{1}{\left(1+|\eta|^2\right)^J}\sum_{k=0}^{2J}\sqrt{{}_{2J}C_k}\ \bar{\eta}^k|k\rangle, \tag{211}$$

$$(2)\ \ |-\bar{\eta}\rangle_J = \frac{1}{\left(1+|\eta|^2\right)^J}\sum_{k=0}^{2J}\sqrt{{}_{2J}C_k}\ (-1)^k\bar{\eta}^k|k\rangle, \tag{212}$$

$$(3)\ \ |1/\bar{\eta}\rangle_J = \frac{1}{\left(1+|\eta|^2\right)^J}\sum_{k=0}^{2J}\sqrt{{}_{2J}C_k}\ \bar{\eta}^k|2J-k\rangle, \tag{213}$$

$$(4)\ \ |-1/\bar{\eta}\rangle_J = \frac{1}{\left(1+|\eta|^2\right)^J}\sum_{k=0}^{2J}\sqrt{{}_{2J}C_k}\ (-1)^k\bar{\eta}^k|2J-k\rangle. \tag{214}$$

¿From this lemma and the elementary facts

$$\frac{2J+1}{\pi}\int_{\mathbf{C}}\frac{[d^2\eta]}{(1+|\eta|^2)^2}\frac{|\eta|^{2k}}{(1+|\eta|^2)^{2J}} = \frac{1}{{}_{2J}C_k} \quad \text{for} \quad 0 \le k \le 2J\ ,$$

we can give explicit forms to the Bell states with spin J :

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\sum_{k=0}^{2J}|k\rangle\otimes|k\rangle, \tag{215}$$

$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\sum_{k=0}^{2J}(-1)^k|k\rangle\otimes|k\rangle, \tag{216}$$

$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\sum_{k=0}^{2J}|k\rangle\otimes|2J-k\rangle, \tag{217}$$

$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2J+1}}\sum_{k=0}^{2J}(-1)^k|k\rangle\otimes|2J-k\rangle. \tag{218}$$

We obtained the Bell states with spin J which are a natural extension of usual ones ($J = 1/2$).

A comment is in order. For the case $J = 1$:

$$\begin{aligned}
&(1)\ \ \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle+|2\rangle\otimes|2\rangle),\\
&(2)\ \ \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle-|1\rangle\otimes|1\rangle+|2\rangle\otimes|2\rangle),\\
&(3)\ \ \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle+|1\rangle\otimes|1\rangle+|2\rangle\otimes|0\rangle),\\
&(4)\ \ \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle-|1\rangle\otimes|1\rangle+|2\rangle\otimes|0\rangle).
\end{aligned}$$

It is easy to see that they are not linearly independent, so that only this case is very special (peculiar).

Comment We cannot give a geometric construction of Bell states by making use of generalized coherent states based on $su(1,1)$ (Lie algebra of non–compact Lie group).

Because the parameter space in this case is a Poincare disk $D = \{\zeta \in \mathbf{C} \mid |\zeta| < 1\}$ and the measure on it is given by

$$d\mu(\zeta,\zeta) = \frac{2K-1}{\pi}\frac{[d^2\zeta]}{(1-|\zeta|^2)^2}\ ,$$

see (31). Therefore we have

$$\int_D d\mu(\zeta,\zeta) = (2K-1)\int_0^1 dr\frac{1}{(1-r)^2} = \infty\ !$$

Compare this with (175). This is a reason why we cannot determine a normalization.

10 Topics in Quantum Information Theory

In this section we don't introduce a general theory of quantum information theory (see for example [15]), but focus our attention to special topics of it, that is,

- swap of coherent states
- cloning of coherent states

Because this is just a good one as examples of applications of coherent and generalized coherent states and our method developed in the following may open a new possibility.
First let us define a swap operator :

$$S : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}, \quad S(a\otimes b) = b\otimes a \quad \text{for any } a, b \in \mathcal{H} \tag{219}$$

where $\mathcal{H}$ is the Fock space in Section 2.
It is not difficult to construct this operator in a universal manner, see Appendix C. But for coherent states we can construct a better one by making use of generalized coherent operators in the preceding section.

Next let us introduce no cloning theorem, [51]. For that we define a cloning (copying) operator C which is unitary

$$C : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}, \quad C(h\otimes|0\rangle) = h\otimes h \quad \text{for any } h \in \mathcal{H}\ . \tag{220}$$

It is very known that there is no cloning theorem

"No Cloning Theorem" We have no C above.

The proof is very easy (almost trivial). Because $2h = h + h \in \mathcal{H}$ and C is a linear operator, so

$$C(2h \otimes |0\rangle) = 2C(h \otimes |0\rangle). \tag{221}$$

The LHS of (221) is

$$C(2h \otimes |0\rangle) = 2h \otimes 2h = 4(h \otimes h),$$

while the RHS of (221)

$$2C(h \otimes |0\rangle) = 2(h \otimes h).$$

This is a contradiction. This is called no cloning theorem.

Let us return to the case of coherent states. For coherent states $|\alpha\rangle$ and $|\beta\rangle$ the superposition $|\alpha\rangle + |\beta\rangle$ is no longer a coherent state, so that coherent states may not suffer from the theorem above.

Problem Is it possible to clone coherent states ?

At this stage it is not easy, so we will make do with approximating it (imperfect cloning in our terminology) instead of making a perfect cloning.

We write notations once more.

$$\text{Coherent States} \quad |\alpha\rangle = D(\alpha)|0\rangle \quad \text{for} \quad \alpha \in \mathbf{C}$$

$$\text{Squeezed–like States} \quad |\beta\rangle = S(\beta)|0\rangle \quad \text{for} \quad \beta \in \mathbf{C}$$

10.1 Some Useful Formulas

We list and prove some useful formulas in the following. Now we prepare some parameters α, ϵ, κ in which ϵ, κ are free ones, while α is unknown one in the cloning case. Let us unify the notations as follows.

$$\alpha : \text{(unknown)} \quad \alpha = |\alpha| \mathrm{e}^{i\chi}, \tag{222}$$

$$\epsilon : \text{known} \qquad \epsilon = |\epsilon| \mathrm{e}^{i\phi}, \tag{223}$$

$$\kappa : \text{known} \qquad \kappa = |\kappa| \mathrm{e}^{i\delta}, \tag{224}$$

Let us start.

(i) First let us calculate

$$S(\epsilon) D(a) S(\epsilon)^{-1}. \tag{225}$$

For that we show

$$S(\epsilon) a S(\epsilon)^{-1} = cosh(|\epsilon|) a - \mathrm{e}^{i\phi} sinh(|\epsilon|) a^{\dagger}. \tag{226}$$

Proof is as follows. For $X = (1/2)\{\epsilon (a^{\dagger})^2 - \bar{\epsilon} a^2\}$ we have easily $[X, a] = -\epsilon a^{\dagger}$ and $[X, a^{\dagger}] = -\bar{\epsilon} a$, so

$$\begin{aligned} S(\epsilon) a S(\epsilon)^{-1} &= \mathrm{e}^{X} a \mathrm{e}^{-X} = a + [X, a] + \frac{1}{2!}[X, [X, a]] + \frac{1}{3!}[X, [X, [X, a]]] + \cdots \\ &= a - \epsilon a^{\dagger} + \frac{|\epsilon|^2}{2!} a - \frac{\epsilon |\epsilon|^2}{3!} a^{\dagger} + \cdots \\ &= \left\{ 1 + \frac{|\epsilon|^2}{2!} + \cdots \right\} a - \frac{\epsilon}{|\epsilon|} \left\{ |\epsilon| + \frac{|\epsilon|^3}{3!} + \cdots \right\} a^{\dagger} \\ &= cosh(|\epsilon|) a - \frac{\epsilon sinh(|\epsilon|)}{|\epsilon|} a^{\dagger} = cosh(|\epsilon|) a - \mathrm{e}^{i\phi} sinh(|\epsilon|) a^{\dagger}. \end{aligned}$$

¿From this it is easy to check

$$\begin{aligned} S(\epsilon) D(\alpha) S(\epsilon)^{-1} &= D\left(\alpha S(\epsilon) a^{\dagger} S(\epsilon)^{-1} - \bar{\alpha} S(\epsilon) a S(\epsilon)^{-1} \right) \\ &= D\left(cosh(|\epsilon|) \alpha + \mathrm{e}^{i\phi} sinh(|\epsilon|) \bar{\alpha} \right). \end{aligned} \tag{227}$$

Therefore

$$S(\epsilon) D(\alpha) S(\epsilon)^{-1} = \begin{cases} D(\mathrm{e}^{|\epsilon|} \alpha) & \text{if} \quad \phi = 2\chi \\ D(\mathrm{e}^{-|\epsilon|} \alpha) & \text{if} \quad \phi = 2\chi + \pi \end{cases} \tag{228}$$

By making use of this formula we can change a scale of α.

(ii) Next le us calculate

$$S(\epsilon) S(\alpha) S(\epsilon)^{-1}. \tag{229}$$

¿From the definition

$$S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\epsilon)\mathrm{exp}\left\{\frac{1}{2}\left(\alpha(a^\dagger)^2 - \bar{\alpha}a^2\right)\right\}S(\epsilon)^{-1} \equiv \mathrm{e}^{Y/2}$$

where

$$Y = \alpha\left(S(\epsilon)a^\dagger S(\epsilon)^{-1}\right)^2 - \bar{\alpha}\left(S(\epsilon)aS(\epsilon)^{-1}\right)^2.$$

¿From (226) and after some calculations we have

$$\begin{aligned}
Y &= \left\{cosh^2(|\epsilon|)\alpha - \mathrm{e}^{2i\phi}sinh^2(|\epsilon|)\bar{\alpha}\right\}(a^\dagger)^2 - \left\{cosh^2(|\epsilon|)\bar{\alpha} - \mathrm{e}^{-2i\phi}sinh^2(|\epsilon|)\alpha\right\}a^2 \\
&+ \frac{(-\mathrm{e}^{-i\phi}\alpha + \mathrm{e}^{i\phi}\bar{\alpha})}{2}sinh(2|\epsilon|)(a^\dagger a + aa^\dagger) \\
&= \left\{cosh^2(|\epsilon|)\alpha - \mathrm{e}^{2i\phi}sinh^2(|\epsilon|)\bar{\alpha}\right\}(a^\dagger)^2 - \left\{cosh^2(|\epsilon|)\bar{\alpha} - \mathrm{e}^{-2i\phi}sinh^2(|\epsilon|)\alpha\right\}a^2 \\
&+ (-\mathrm{e}^{-i\phi}\alpha + \mathrm{e}^{i\phi}\bar{\alpha})sinh(2|\epsilon|)(a^\dagger a + \frac{1}{2}) \quad (\Longleftarrow [a, a^\dagger] = 1),
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{2}Y &= \left\{cosh^2(|\epsilon|)\alpha - \mathrm{e}^{2i\phi}sinh^2(|\epsilon|)\bar{\alpha}\right\}K_+ - \left\{cosh^2(|\epsilon|)\bar{\alpha} - \mathrm{e}^{-2i\phi}sinh^2(|\epsilon|)\alpha\right\}K_- \\
&+ (-\mathrm{e}^{-i\phi}\alpha + \mathrm{e}^{i\phi}\bar{\alpha})sinh(2|\epsilon|)K_3
\end{aligned} \tag{230}$$

with $\{K_+, K_-, K_3\}$ in (32). This is our formula.

Now

$$-\mathrm{e}^{-i\phi}\alpha + \mathrm{e}^{i\phi}\bar{\alpha} = |\alpha|(-\mathrm{e}^{-i(\phi-\chi)} + \mathrm{e}^{i(\phi-\chi)}) = 2i|\alpha|sin(\phi - \chi),$$

so if we choose $\phi = \chi$, then $\mathrm{e}^{2i\phi}\bar{\alpha} = \mathrm{e}^{2i\chi}\mathrm{e}^{-i\chi}|\alpha| = \alpha$ and

$$cosh^2(|\epsilon|)\alpha - \mathrm{e}^{2i\phi}sinh^2(|\epsilon|)\bar{\alpha} = \left(cosh^2(|\epsilon|) - sinh^2(|\epsilon|)\right)\alpha = \alpha$$

, and finally

$$Y = \alpha(a^\dagger)^2 - \bar{\alpha}a^2.$$

That is,

$$S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\alpha) \Longleftrightarrow S(\epsilon)S(\alpha) = S(\alpha)S(\epsilon).$$

The operators $S(\epsilon)$ and $S(\alpha)$ commute if the phases of ϵ and α coincide.

(iii) Third formula is : For $V(t) = \mathrm{e}^{itN}$ where $N = a^\dagger a$ (a number operator)

$$V(t)D(\alpha)V(t)^{-1} = D(\mathrm{e}^{it}\alpha). \tag{231}$$

The proof is as follows.

$$V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha V(t)a^\dagger V(t)^{-1} - \bar{\alpha}V(t)aV(t)^{-1}\right).$$

It is easy to see

$$\begin{aligned} V(t)aV(t)^{-1} = \mathrm{e}^{itN}a\mathrm{e}^{-itN} &= a + [itN, a] + \frac{1}{2!}[itN, [itN, a]] + \cdots \\ &= a + (-it)a + \frac{(-it)^2}{2!}a + \cdots \\ &= \mathrm{e}^{-it}a. \end{aligned}$$

Therefore we obtain

$$V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha\mathrm{e}^{it}a^\dagger - \bar{\alpha}\mathrm{e}^{-it}a^\dagger\right) = D(\mathrm{e}^{it}\alpha).$$

This formula is often used as follows.

$$|\alpha\rangle \longrightarrow V(t)|\alpha\rangle = V(t)D(\alpha)V(t)^{-1}V(t)|0\rangle = D(\mathrm{e}^{it}\alpha)|0\rangle = |\mathrm{e}^{it}\alpha\rangle, \tag{232}$$

where we have used

$$V(t)|0\rangle = |0\rangle$$

becase $N|0\rangle = 0$. That is, we can add a phase to α by making use of this formula.

10.2 Swap of Coherent States

The purpose of this section is to construct a swap operator satifying

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle. \tag{233}$$

Let us remember $U_J(\kappa)$ once more

$$U_J(\kappa) = \mathrm{e}^{\kappa a_1^\dagger a_2 - \bar{\kappa}a_1 a_2^\dagger} \quad \text{for} \quad \kappa \in \mathbf{C}.$$

We note an important property of this operator :

$$U_J(\kappa)|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle. \tag{234}$$

The construction is as follows.

$$\begin{aligned} U_J(\kappa)|\alpha_1\rangle \otimes |\alpha_2\rangle &= U_J(\kappa)D(\alpha_1) \otimes D(\alpha_2)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}U_J(\kappa)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \quad \text{by} \quad (234), \end{aligned} \tag{235}$$

and

$$\begin{aligned} &U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1} = U_J(\kappa)\exp\left\{\alpha_1 a_1^\dagger - \bar{\alpha}_1 a_1 + \alpha_2 a_2^\dagger - \bar{\alpha}_2 a_2\right\} U_J(\kappa)^{-1} \\ &= \exp\left\{\alpha_1 (U_J(\kappa)a_1 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_1 U_J(\kappa)a_1 U_J(\kappa)^{-1} \right. \\ &\qquad \left. +\alpha_2 (U_J(\kappa)a_2 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_2 U_J(\kappa)a_2 U_J(\kappa)^{-1}\right\} \\ &\equiv \exp(X). \end{aligned} \tag{236}$$

¿From (88) and (89) we have

$$\begin{aligned} X = &\left\{cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right\} a_1^\dagger - \left\{cos(|\kappa|)\bar{\alpha}_1 + \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\bar{\alpha}_2\right\} a_1 \\ &+ \left\{cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right\} a_2^\dagger - \left\{cos(|\kappa|)\bar{\alpha}_2 - \frac{\kappa sin(|\kappa|)}{|\kappa|}\bar{\alpha}_1\right\} a_2, \end{aligned}$$

so

$$\begin{aligned} \exp(X) &= D_1\left(cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right) D_2\left(cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right) \\ &= D\left(cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right) \otimes D\left(cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right). \end{aligned}$$

Therefore we have from (236)

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \ \longrightarrow \ |cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\rangle \otimes |cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\rangle.$$

If we write κ as $|\kappa|\mathrm{e}^{i\delta}$, then the above formula reduces to

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \ \longrightarrow \ |cos(|\kappa|)\alpha_1 + \mathrm{e}^{i\delta} sin(|\kappa|)\alpha_2\rangle \otimes |cos(|\kappa|)\alpha_2 - \mathrm{e}^{-i\delta} sin(|\kappa|)\alpha_1\rangle. \tag{237}$$

Here if we choose for example $\kappa = |\kappa| = \pi/2$, then

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \;\longrightarrow\; |\alpha_2\rangle \otimes |-\alpha_1\rangle = |\alpha_2\rangle \otimes |\mathrm{e}^{-i\pi}\alpha_1\rangle.$$

Now by operating the operator $V = \mathbf{1} \otimes \mathrm{e}^{i\pi N}$ where $N = a^\dagger a$ from the left (see (232)) we obtain the swap

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \;\longrightarrow\; |\alpha_2\rangle \otimes |\alpha_1\rangle.$$

A comment is in order. In the formula above we set $\alpha_1 = \alpha$ and $\alpha_2 = 0$, then the formula reduces to

$$U_J(\kappa)D_1(\alpha)U_J(\kappa)^{-1} = D_1(cos(|\kappa|)\alpha)D_2(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha). \tag{238}$$

This will be used in the next subsection.

10.3 Imperfect Cloning of Coherent States

We cannot clone coherent states in a perfect manner like

$$|\alpha\rangle \otimes |0\rangle \longrightarrow |\alpha\rangle \otimes |\alpha\rangle \quad \text{for } \alpha \in \mathbf{C}. \tag{239}$$

Then our question is : is it possible to approximate ? We show that we can at least make an "imperfect cloning" in our terminology against the statement of [45].

Let us start. The method is almost same with one in the preceding subsection, but we repeat it once more. Operating the operator $U_J(\kappa)$ on $|\alpha\rangle \otimes |0\rangle$

$$\begin{aligned} U_J(\kappa)|\alpha\rangle \otimes |0\rangle =& D_1(cos(|\kappa|)\alpha)D_2(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \quad \text{by (238)} \\ =& \left\{D(cos(|\kappa|)\alpha) \otimes D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)\right\}|0\rangle \otimes |0\rangle. \end{aligned}$$

Operating the operator $\mathbf{1} \otimes \mathrm{e}^{i(\delta+\pi)N}$ on the last equation

$$\begin{aligned} & D(cos(|\kappa|)\alpha) \otimes \mathrm{e}^{i(\delta+\pi)N}D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \\ =& D(cos(|\kappa|)\alpha) \otimes \mathrm{e}^{i(\delta+\pi)N}D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)\mathrm{e}^{-i(\delta+\pi)N}|0\rangle \otimes |0\rangle \\ =& D(cos(|\kappa|)\alpha) \otimes D(sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \quad \text{by (231)} \\ =& |cos(|\kappa|)\alpha\rangle \otimes |sin(|\kappa|)\alpha\rangle. \end{aligned}$$

Namely we have constructed

$$|\alpha\rangle \otimes |0\rangle \longrightarrow |cos(|\kappa|)\alpha\rangle \otimes |sin(|\kappa|)\alpha\rangle. \tag{240}$$

This is an "imperfect cloning" what we have called. When $cos(|\kappa|) = sin(|\kappa|) = 1/\sqrt{2}$, we have

$$|\alpha\rangle \otimes |0\rangle \longrightarrow |\frac{\alpha}{\sqrt{2}}\rangle \otimes |\frac{\alpha}{\sqrt{2}}\rangle.$$

For the "imperfect cloning" of general quantum states see Appendix D.

A comment is in order. The authors in [45] state that the "perfect cloning" (in their terminology) for coherent states is possible. But it is not correct as shown below. Their method is very interesting, so let us introduce it.

Before starting let us prepare a notation for simplicity (227) :

$$S(\epsilon)D(\alpha)S(\epsilon)^{-1} = D(\tilde{\alpha}), \quad \tilde{\alpha} \equiv cosh(|\epsilon|)\alpha + \mathrm{e}^{i\phi}sinh(|\epsilon|)\bar{\alpha}.$$

Operating the operator $S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})$ from the left

$$\begin{aligned}
S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})|\alpha\rangle \otimes |0\rangle &= \left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}\{D(\alpha) \otimes \mathbf{1}\}\,|0\rangle \otimes |0\rangle \\
&= S(\epsilon)D(\alpha) \otimes S(\mathrm{e}^{-2i\delta})|0\rangle \otimes |0\rangle \\
&= S(\epsilon)D(\alpha)S(\epsilon)^{-1}S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})|0\rangle \otimes |0\rangle \\
&= D(\tilde{\alpha})S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})|0\rangle \otimes |0\rangle \\
&= \{D(\tilde{\alpha}) \otimes \mathbf{1}\}\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}|0\rangle \otimes |0\rangle \\
&= D_1(\tilde{\alpha})\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}|0\rangle \otimes |0\rangle.
\end{aligned}$$

Operating the operator $U_J(\kappa)$ (remember that $\kappa = |\kappa|\mathrm{e}^{i\delta}$) from the left

$$\begin{aligned}
&U_J(\kappa)D_1(\tilde{\alpha})\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}|0\rangle \otimes |0\rangle \\
=&U_J(\kappa)D_1(\tilde{\alpha})\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}U_J(\kappa)^{-1}U_J(\kappa)|0\rangle \otimes |0\rangle \\
=&U_J(\kappa)D_1(\tilde{\alpha})\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta})\right\}U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \qquad \text{by (234)} \\
=&U_J(\kappa)D_1(\tilde{\alpha})U_J(\kappa)^{-1}U_J(\kappa)\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta}\epsilon)\right\}U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \\
=&D_1(cos(|\kappa|)\tilde{\alpha})D_2(-\mathrm{e}^{-i\delta}sin(|\kappa|)\tilde{\alpha})\left\{S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta}\epsilon)\right\}|0\rangle \otimes |0\rangle \quad \text{by (99) and (238)}
\end{aligned}$$

$$= \left\{ D(cos(|\kappa|)\tilde{\alpha}) \otimes D(-\mathrm{e}^{-i\delta} sin(|\kappa|)\tilde{\alpha}) \right\} \left\{ S(\epsilon) \otimes S(\mathrm{e}^{-2i\delta}\epsilon) \right\} |0\rangle \otimes |0\rangle$$
$$= D(cos(|\kappa|)\tilde{\alpha}) S(\epsilon) \otimes D(-\mathrm{e}^{-i\delta} sin(|\kappa|)\tilde{\alpha}) S(\mathrm{e}^{-2i\delta}\epsilon)|0\rangle \otimes |0\rangle.$$
$$= D(cos(|\kappa|)\tilde{\alpha}) S(\epsilon) \otimes D(-isin(|\kappa|)\tilde{\alpha}) S(-\epsilon)|0\rangle \otimes |0\rangle,$$

where we have chosen in the last step

$$\mathrm{e}^{-i\delta} = i \quad (\Longleftarrow \text{ for example } \quad \delta = -\frac{\pi}{2}). \tag{241}$$

Operating the operator $S(-\epsilon) \otimes S(\epsilon)$ from the left

$$\{S(-\epsilon) \otimes S(\epsilon)\} \, D(cos(|\kappa|)\tilde{\alpha}) S(\epsilon) \otimes D(-isin(|\kappa|)\tilde{\alpha}) S(-\epsilon)|0\rangle \otimes |0\rangle,$$
$$= S(-\epsilon) D(cos(|\kappa|)\tilde{\alpha}) S(\epsilon) \otimes S(\epsilon) D(-isin(|\kappa|)\tilde{\alpha}) S(\epsilon)^{-1}|0\rangle \otimes |0\rangle.$$

Here let us calculate the last term :

$$S(-\epsilon) D(cos(|\kappa|)\tilde{\alpha}) S(\epsilon) = D(cos(|\kappa|)\alpha) \tag{242}$$

and we obtain

$$S(\epsilon) D(-isin(|\kappa|)\tilde{\alpha}) S(\epsilon)^{-1} = D(-isin(|\kappa|)\alpha) \tag{243}$$

against the equation (38) in [45]

$$S(\epsilon) D(-isin(|\kappa|)\tilde{\alpha}) S(\epsilon)^{-1} = D(-isin(|\kappa|)\tilde{\tilde{\alpha}}) \tag{244}$$

where

$$\tilde{\tilde{\alpha}} = cosh(2|\epsilon|)\alpha + \mathrm{e}^{i\phi} sinh(2|\epsilon|)\bar{\alpha}.$$

Therefore one cannot follow their method from this stage.

But as stated above their method is simple and very interesting, so it may be possible to modify that more subtly by making use of (228).

Problem Is it possible to make a "perfect cloning" in the sense of [45] ?

10.4 Swap of Squeezed–like States

We would like to construct an operator like

$$|\beta_1\rangle \otimes |\beta_2\rangle \longrightarrow |\beta_2\rangle \otimes |\beta_1\rangle. \tag{245}$$

In this case we also use the operator $U_J(\kappa)$.

Similar to (235)

$$\begin{aligned} U_J(\kappa)|\beta_1\rangle \otimes |\beta_2\rangle &= U_J(\kappa)S(\beta_1) \otimes S(\beta_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)S_1(\beta_1)S_2(\beta_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle. \end{aligned} \tag{246}$$

On the other hand by (96)

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = \mathrm{e}^X,$$

where

$$\begin{aligned} \mathrm{X} = &\frac{1}{2}\left\{cos^2(|\kappa|)\beta_1 + \frac{\kappa^2 sin^2(|\kappa|)}{|\kappa|^2}\beta_2\right\}(a_1^\dagger)^2 - \frac{1}{2}\left\{cos^2(|\kappa|)\bar{\beta}_1 + \frac{\bar{\kappa}^2 sin^2(|\kappa|)}{|\kappa|^2}\bar{\beta}_2\right\}a_1^2 \\ &+ \frac{1}{2}\left\{cos^2(|\kappa|)\beta_2 + \frac{\bar{\kappa}^2 sin^2(|\kappa|)}{|\kappa|^2}\beta_1\right\}(a_2^\dagger)^2 - \frac{1}{2}\left\{cos^2(|\kappa|)\bar{\beta}_2 + \frac{\kappa^2 sin^2(|\kappa|)}{|\kappa|^2}\bar{\beta}_1\right\}a_2^2 \\ &+ (\beta_2\kappa - \beta_1\bar{\kappa})\frac{sin(2|\kappa|)}{2|\kappa|}a_1^\dagger a_2^\dagger - (\bar{\beta}_2\bar{\kappa} - \bar{\beta}_1\kappa)\frac{sin(2|\kappa|)}{2|\kappa|}a_1 a_2\ . \end{aligned}$$

Here an extra term containing $a_1^\dagger a_2^\dagger$ appeared. To remove this we must set $\beta_2\kappa - \beta_1\bar{\kappa} = 0$, but in this case we meet

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = S_1(\beta_1)S_2(\beta_2)$$

by (99). That is, there is no change.

Therefore from the beginning we set for example $\kappa = |\kappa| = \pi/2$, then

$$\mathrm{X} = \frac{1}{2}\left\{\beta_2(a_1^\dagger)^2 - \bar{\beta}_2 a_1^2\right\} + \frac{1}{2}\left\{\beta_1(a_2^\dagger)^2 - \bar{\beta}_1 a_2^2\right\},$$

so we just have

$$U_J(\pi/2)S_1(\beta_1)S_2(\beta_2)U_J(\pi/2)^{-1} = S_1(\beta_2)S_2(\beta_1) \tag{247}$$

or equivarently (245)

$$|\beta_1\rangle \otimes |\beta_2\rangle \longrightarrow |\beta_2\rangle \otimes |\beta_1\rangle.$$

A comment is in order. The operator $U_J(\kappa)$ plays a central role in the swap of quantum states. For the swap of general quantum states see the Appendix C.

10.5 A Comment

We have used in the process of proofs both a displacement operator $D(\alpha)$ and a squeezed one $S(\epsilon)$

$$D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a), \quad S(\epsilon) = \exp\frac{1}{2}\left(\epsilon (a^\dagger)^2 - \bar{\epsilon} a^2\right)$$

as a product operator

$$S(\epsilon)D(\alpha). \tag{248}$$

We note that this product operator with the parameter space $\{(\alpha, \epsilon) \in \mathbf{C}^2\}$ plays a crucial role in our Holonomic Quantum Computation (Computer), see section 8.1.
Similarly, the product operator

$$U_K(w)U_J(v) \tag{249}$$

with the parameter space $\{(v, w) \in \mathbf{C}^2\}$ also plays a crucial role in it, see section 8.2.
We believe that Holonomic Quantum Computation and our geometric method (involving swap or imperfect cloning) in Quantum Information Theory are well–matched.

11 Path Integral on A Quantum Computer

In this section we present a very important problem (at least to the author) about the possibility of calculation of path integral on a Quantum Computer.

The path integral method plays an essential role in Quantum Mechanics or Quantum Field Theory. But it is, in general, not easy to calculate it except for Gaussian cases. Some specialists must, in a perturbation theory, calculate many Feynman's graphs by making use of a classical computer(s). This is a hard and painful task.
Now let us present our general problem.

Problem Is it possible to calculate a path integral in polynomial times by making use of a quantum computer ?

For this subject refer [46] and its references. But our method or interest is a bit different from [46]. To match our method with path integrals we should use **coherent state path integral method**, see [8], [9], [10]. [55] is also recommended.
To calculate a physical quantity such as a trace formula of the Hamiltonian we, for example, give a coherent state path integral expression to it. We want to calculate it, but it is usally not easy to do so. Therefore we have to make do with some approximations (WKB approximation, etc). Then our next problem is

Problem Is it possible to give it in polynomial times with Holonomic Quantum Computer ?

For the readers who are not familiar with coherent state path integral method let us show a simple, but very instructive example, [47].
Let us consider the Hamiltonian of harmonic oscillator

$$H = \omega N = \omega a^\dagger a, \tag{250}$$

where we have omitted the constant term for simplicity. The eigenvalues of H are well–known to be $\{n\omega \mid n = 0, 1, \cdots\}$ and its trace formula is given as

$$\text{tr } \mathrm{e}^{-iTH} = \sum_{n=0}^{\infty} \mathrm{e}^{-in\omega T} = \frac{1}{1 - \mathrm{e}^{-i\omega T}} \quad \text{(Abel sum)}. \tag{251}$$

Let us give a coherent state path integral expression to this trace formula. Making use of the resolution of unity (9) we obtain

$$\text{tr } \mathrm{e}^{-iTH} = \text{tr } \mathbf{1}\mathrm{e}^{-iTH} = \text{tr } \int_{\mathbf{C}} \frac{[d^2 z]}{\pi} |z\rangle\langle z|\mathrm{e}^{-iTH} = \int_{\mathbf{C}} \frac{[d^2 z]}{\pi} \langle z|\mathrm{e}^{-iTH}|z\rangle \ . \tag{252}$$

This is just an analytical expression of the trace formula. It is easy to calculate this directly, but we give this a path integral expression. Noting

$$\mathrm{e}^X = \lim_{N\to\infty}\left(1+\frac{X}{N}\right)^N ,$$

we have

$$\text{RHS of (252)} = \lim_{N\to\infty}\int_{\mathbf{C}}\frac{[d^2 z]}{\pi}\langle z|\left(1-i\Delta tH\right)^N|z\rangle, \tag{253}$$

where we have set $\Delta t = T/N$.

By inserting the resolution of unity (9) at each step likely

$$\left(1-i\Delta tH\right)^N = \left(1-i\Delta tH\right)\mathbf{1}\left(1-i\Delta tH\right)\mathbf{1}\cdots\left(1-i\Delta tH\right)\mathbf{1}\left(1-i\Delta tH\right)$$
$$\mathbf{1} = \int_{\mathbf{C}}\frac{[d^2 z_j]}{\pi}|z_j\rangle\langle z_j| \quad \text{for any} \quad 1\le j\le N-1 \ ,$$

we have

$$\text{RHS of (252)} = \lim_{N\to\infty}\int_{PBC}\prod_{j=1}^{N}\frac{[d^2 z_j]}{\pi}\prod_{j=1}^{N}\langle z_j|1-i\Delta tH|z_{j-1}\rangle, \tag{254}$$

where PBC (periodic boundary condition) means $z_N = z_0 = z$. We note that the choice of $\{z_1, z_2, \cdots, z_{N-1}\}$ is random.

Let us calculate the term $\langle z_j|1-i\Delta tH|z_{j-1}\rangle$:

$$\begin{aligned}
\langle z_j|1-i\Delta tH|z_{j-1}\rangle &= \langle z_j|z_{j-1}\rangle - i\Delta t\langle z_j|H|z_{j-1}\rangle \\
&= \langle z_j|z_{j-1}\rangle\left\{1-i\Delta t\frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle}\right\} \\
&= \langle z_j|z_{j-1}\rangle \mathrm{exp}\left\{-i\Delta t\frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle}\right\} \quad \text{up to } \mathrm{O}((\Delta t)^2).
\end{aligned}$$

On the other hand from (10) and (5)

$$\langle z_j|z_{j-1}\rangle = \exp\left(-\frac{1}{2}|z_j|^2-\frac{1}{2}|z_{j-1}|^2+\bar{z}_j z_{j-1}\right),$$
$$\langle z_j|H|z_{j-1}\rangle = \omega\langle z_j|a^\dagger a|z_{j-1}\rangle = \omega\bar{z}_j z_{j-1}\langle z_j|z_{j-1}\rangle, \quad \frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle} = \omega\bar{z}_j z_{j-1},$$

so we have

$$\langle z_j|1-i\Delta tH|z_{j-1}\rangle = \exp\left\{-\frac{1}{2}|z_j|^2-\frac{1}{2}|z_{j-1}|^2+\bar{z}_j z_{j-1} \ -i\omega\Delta t\bar{z}_j z_{j-1}\right\}$$

after some algebra. Here taking the periodic bountary condition $z_N = z_0$ it is easy to see

$$\prod_{j=1}^{N} \langle z_j | 1 - i\Delta t H | z_{j-1} \rangle = \exp\left\{ -\sum_{j=1}^{N} \left\{ \bar{z}_j (z_j - z_{j-1}) + i\omega\Delta t \bar{z}_j z_{j-1} \right\} \right\}.$$

Therefore from this we reach

$$\text{RHS of (252)} = \lim_{N\to\infty} \int_{PBC} \prod_{j=1}^{N} \frac{[d^2 z_j]}{\pi} \exp\left\{ -\sum_{j=1}^{N} \left\{ \bar{z}_j (z_j - z_{j-1}) + i\omega\Delta t \bar{z}_j z_{j-1} \right\} \right\}. \quad (255)$$

This is just the coherent state path integral expression of trace formula of the harmonic oscillator. As for calculation of (255) see Appendix E.

12 Discussion and Dream

We in this paper discussed a geometric method to Quantum Information Theory which is mostly based on the author's work. We used several properties of coherent states or generalized coherent ones based on Lie algebras $su(2)$ and $su(1,1)$. It is not difficult to extend these to Lie algebras $su(n+1)$ and $su(n,1)$ for general $n \geq 2$, see [9], [10], [42], [43].

The parameter spaces of these generalized coherent states are usally (famous) homogeneous spaces such as complex projective spaces, Grassmann manifolds (compact cases), or Poincare disks, Siegel Domains (non– compact cases) in Geometry. Therefore we can use many tools developed in Global Analysis via generalized coherent states.

However the method is of course not sufficent to give a geometric method to QIT. There might be several geometric methods (models) in QIT which are unfamiliar to the author. For example we have not made comments on geometric methods of quantum algorithms. Please visit arXiv (quant–ph) and look for them.

Geometric understanding of several concepts in QIT is very important because we can view them from global point of view. This is indispensable for us to understand QIT more deeply.

Here let us state our dream once more.

Geometric Quantum Information Theory.

For example,

- **Geometric** Quantum Computer (Computation)
- **Geometric** Quantum Cryptgraphy
- **Geometric** Quantum Teleportation

We believe that we have taken a first step towards this dream. The author expects strongly that graduate students and/or young researchers in Sciences will take part in our dream.

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Appendix

A Formula on Associated Laguerre Polynomials

Here we show an interesting formula on associated Laguerre polynomials by making use of (13) and (14), (15).
¿From

$$D(z+w)=\mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\ D(z)D(w)$$

we take a matrix element

$\langle n|D(z+w)|n\rangle=\mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\ \langle n|D(z)D(w)|n\rangle$

$$= \mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)} \sum_{k=0}^{\infty} \langle n|D(z)|k\rangle\langle k|D(w)|n\rangle$$

$$= \mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)} \left\{\sum_{k=0}^{n} \langle n|D(z)|k\rangle\langle k|D(w)|n\rangle + \sum_{k=n+1}^{\infty} \langle n|D(z)|k\rangle\langle k|D(w)|n\rangle\right\} \tag{256}$$

¿From (14), (15)

$$\text{LHS of (256)} = \mathrm{e}^{-\frac{1}{2}|z+w|^2} L_n(|z+w|^2) = \mathrm{e}^{-\frac{1}{2}(z\bar{w}+\bar{z}w)}\mathrm{e}^{-\frac{1}{2}(|z|^2+|w|^2)} L_n(|z+w|^2)\ . \tag{257}$$

On the other hand

$$\begin{aligned}
&\text{RHS of (256)}\\
&= \mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)} \left\{\sum_{k=0}^{n} \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{k!}{n!}} z^{n-k} L_k^{(n-k)}(|z|^2)\ \mathrm{e}^{-\frac{1}{2}|w|^2}\sqrt{\frac{k!}{n!}}(-\bar{w})^{n-k} L_k^{(n-k)}(|w|^2)\right.\\
&\qquad \left. + \sum_{k=n+1}^{\infty} \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{n!}{k!}}(-\bar{z})^{k-n} L_n^{(k-n)}(|z|^2)\ \mathrm{e}^{-\frac{1}{2}|w|^2}\sqrt{\frac{n!}{k!}} w^{k-n} L_n^{(k-n)}(|w|^2)\right\}\\
&= \mathrm{e}^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\mathrm{e}^{-\frac{1}{2}(|z|^2+|w|^2)} \left\{\sum_{k=0}^{n} \frac{k!}{n!}(-z\bar{w})^{n-k} L_k^{(n-k)}(|z|^2) L_k^{(n-k)}(|w|^2)\right.\\
&\qquad \left. + \sum_{k=n+1}^{\infty} \frac{n!}{k!}(-\bar{z}w)^{k-n} L_n^{(k-n)}(|z|^2) L_n^{(k-n)}(|w|^2)\right\}.
\end{aligned} \tag{258}$$

Comparing (258) with (257) we have an interesting formula :

$$\begin{aligned}
&L_n(|z+w|^2)\\
&= \mathrm{e}^{\bar{z}w} \left\{\sum_{k=0}^{n} \frac{k!}{n!}(-z\bar{w})^{n-k} L_k^{(n-k)}(|z|^2) L_k^{(n-k)}(|w|^2) + \sum_{k=n+1}^{\infty} \frac{n!}{k!}(-\bar{z}w)^{k-n} L_n^{(k-n)}(|z|^2) L_n^{(k-n)}(|w|^2)\right\}.
\end{aligned} \tag{259}$$

This is considered as a kind of additivity formula. We don't know whether this formula has been known or not. For the more generalization see [58].

B Proof of Disentangling Formulas

Here we prove the disentangling formulas (87) and (86) for generalized coherent operators based on Lie algebras $su(1,1)$ and $su(2)$.

In general a representation of Lie algebra cannot be lifted to the representation of its Lie group if a Lie group is not simply connected. We note that $SU(1,1)$ is not simply connected because $\pi_1(SU(1,1)) = \pi_1(U(1)) = \mathcal{Z}$.

First we start under the assumption that there is a representation of Lie group $SU(1,1)$. Namely, let ρ be a representation of Lie group $SU(1,1) \subset SL(2,\mathbf{C})$

$$\rho : SL(2,\mathbf{C}) \longrightarrow U(\mathcal{H} \otimes \mathcal{H}) \tag{260}$$

and

$$K_+ = d\rho(k_+), \quad K_- = d\rho(k_-), \quad K_3 = d\rho(k_-) \tag{261}$$

where

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{262}$$

It is easy to see

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3 \quad \text{but} \quad {k_+}^\dagger = -k_-.$$

In this case

$$\begin{aligned} &\exp\left(wK_+ - \bar{w}K_-\right) = \exp\left(d\rho(wk_+ - \bar{w}k_-)\right) \\ &= \exp\left(d\rho\begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}\right) = \rho\left(\exp\begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}\right) \equiv \rho(\mathrm{e}^A). \end{aligned} \tag{263}$$

¿From

$$A^2 = |w|^2 E$$

we have

$$\mathrm{e}^A = \cosh(|w|)E + \frac{\sinh(|w|)}{|w|}A = \begin{pmatrix} \cosh(|w|) & \frac{\sinh(|w|)}{|w|}w \\ \frac{\sinh(|w|)}{|w|}\bar{w} & \cosh(|w|) \end{pmatrix}. \tag{264}$$

For $\mathrm{e}^A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad - bc = 1)$, the Gauss decomposition of this matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}. \tag{265}$$

Since ρ is a representation of Lie group (not Lie algebra !) we have

$$
\begin{aligned}
&\rho\left(\begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}\right) \\
&= \rho\left(\begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix}\right)\rho\left(\begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix}\right)\rho\left(\begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}\right) \\
&= \rho\left(\exp\begin{pmatrix} 0 & \frac{b}{d} \\ 0 & 0 \end{pmatrix}\right)\rho\left(\exp\begin{pmatrix} -\log d & 0 \\ 0 & \log d \end{pmatrix}\right)\rho\left(\exp\begin{pmatrix} 0 & 0 \\ \frac{c}{d} & 0 \end{pmatrix}\right) \\
&= \exp\left(d\rho\begin{pmatrix} 0 & \frac{b}{d} \\ 0 & 0 \end{pmatrix}\right)\exp\left(d\rho\begin{pmatrix} -\log d & 0 \\ 0 & \log d \end{pmatrix}\right)\exp\left(d\rho\begin{pmatrix} 0 & 0 \\ \frac{c}{d} & 0 \end{pmatrix}\right) \\
&= \exp\left(\frac{b}{d}d\rho(k_+)\right)\exp(-2\log d\ d\rho(k_3))\exp\left(-\frac{c}{d}d\rho(k_-)\right) \\
&= \exp\left(\frac{b}{d}K_+\right)\exp\left(-2\log d K_3\right)\exp\left(-\frac{c}{d}K_-\right) \\
&= \exp\left(\frac{b}{d}K_+\right)\exp\left(\log\left(\frac{1}{d^2}\right)K_3\right)\exp\left(-\frac{c}{d}K_-\right)
\end{aligned}
\tag{266}
$$

where

$$
\begin{aligned}
\frac{b}{d} &= \frac{\frac{\sinh(|w|)}{|w|}w}{\cosh(|w|)} = \frac{\tanh(|w|)w}{|w|} \\
\frac{c}{d} &= \frac{\tanh(|w|)\bar{w}}{|w|} \\
\frac{1}{d^2} &= \frac{1}{\cosh^2(|w|)} = 1 - \tanh^2(|w|).
\end{aligned}
\tag{267}
$$

If we set

$$
\zeta = \frac{\tanh(|w|)w}{|w|} \quad \Longrightarrow \quad |\zeta| = \tanh(|w|)
$$

then we have (87). That is, we could prove (87) under the assumption. To remove this we needs some tricks. We define

$$
f(t) = \exp\left\{t(wK_+ - \bar{w}K_-)\right\} \tag{268}
$$

$$
g(t) = \exp\{\zeta(t)K_+\}\exp\{\log(1-|\zeta(t)|^2)K_3\}\exp\{-\bar{\zeta}(t)K_-\}, \quad \text{where} \quad \zeta(t) = \frac{w\tanh(t|w|)}{|w|}. \tag{269}
$$

Then

$$f(0) = \mathbf{1}, \quad \frac{d}{dt}f(t) = (wK_+ - \bar{w}K_-)f(t). \tag{270}$$

On the other hand

$$\begin{aligned}
g(0) &= \mathbf{1}, \\
\frac{d}{dt}g(t) &= \zeta'(t)K_+g(t) + \frac{d}{dt}\log(1-|\zeta(t)|^2)e^{\zeta(t)K_+}K_3e^{\log(1-|\zeta(t)|^2)K_3}e^{-\bar{\zeta}(t)K_-} - \bar{\zeta}'(t)g(t)K_- \\
&= \zeta'(t)K_+g(t) + \frac{d}{dt}\log(1-|\zeta(t)|^2)e^{\zeta(t)K_+}K_3e^{-\zeta(t)K_+}g(t) - \bar{\zeta}'(t)g(t)K_-g(t)^{-1}g(t) \\
&= \left\{\zeta'(t)K_+ + \frac{d}{dt}\log(1-|\zeta(t)|^2)e^{\zeta(t)K_+}K_3e^{-\zeta(t)K_+} - \bar{\zeta}'(t)g(t)K_-g(t)^{-1}\right\}g(t).
\end{aligned} \tag{271}$$

Then it is not difficult to see

$$\begin{aligned}
e^{\zeta(t)K_+}K_3e^{-\zeta(t)K_+} &= K_3 - \zeta(t)K_+\ , \\
g(t)K_-g(t)^{-1} &= e^{\zeta(t)K_+}e^{\log(1-|\zeta(t)|^2)K_3}K_-e^{-\log(1-|\zeta(t)|^2)K_3}e^{-\zeta(t)K_+} \\
&= e^{-\log(1-|\zeta(t)|^2)}e^{\zeta(t)K_+}K_-e^{-\zeta(t)K_+} \\
&= \frac{1}{1-|\zeta(t)|^2}\left\{K_- - 2\zeta(t)K_3 + \zeta(t)^2K_+\right\}\ ,
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{d}{dt}g(t) \\
&= \left\{\zeta'(t)K_+ - \frac{\zeta'(t)\bar{\zeta}(t) + \zeta(t)\bar{\zeta}'(t)}{1-|\zeta(t)|^2}(K_3 - \zeta(t)K_+) - \frac{\bar{\zeta}'(t)}{1-|\zeta(t)|^2}\left(K_- - 2\zeta(t)K_3 + \zeta(t)^2K_+\right)\right\}g(t).
\end{aligned}$$

If we notice

$$\begin{aligned}
&|\zeta(t)| = \tanh(t|w|), \\
&\zeta'(t)\bar{\zeta}(t) = |w|(1-\tanh^2(t|w|))\tanh(t|w|) = \zeta(t)\bar{\zeta}'(t), \\
&\frac{\zeta'(t)}{1-|\zeta(t)|^2} = w, \ \frac{\bar{\zeta}'(t)}{1-|\zeta(t)|^2} = \bar{w},
\end{aligned}$$

then we reach after some algebra

$$\frac{d}{dt}g(t) = (wK_+ - \bar{w}K_-)g(t). \tag{272}$$

Comparing (272) with (270) we obtain (87).

Similar method is still valid for a representation of Lie group $SU(2)$ to prove (86). Since we don't repeat here, so we leave it to the readers.

C Universal Swap Operator

Let us construct the swap operator in a universal manner

$$U : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} \ , \quad U(a \otimes b) = b \otimes a \quad \text{for any } a,\ b \in \mathcal{H}$$

where $\mathcal{H}$ is an infinite–dimensional Hilbert space. Before constructing it we show in the finite–dimensional case.

For $a,\ b \in \mathbf{C}^2$ then

$$a \otimes b = \begin{pmatrix} a_1 b \\ a_2 b \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}, \quad b \otimes a = \begin{pmatrix} b_1 a_1 \\ b_1 a_2 \\ b_2 a_1 \\ b_2 a_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_1 \\ a_1 b_2 \\ a_2 b_2 \end{pmatrix},$$

so it is easy to see

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_1 \\ a_1 b_2 \\ a_2 b_2 \end{pmatrix}.$$

That is, the swap operator is

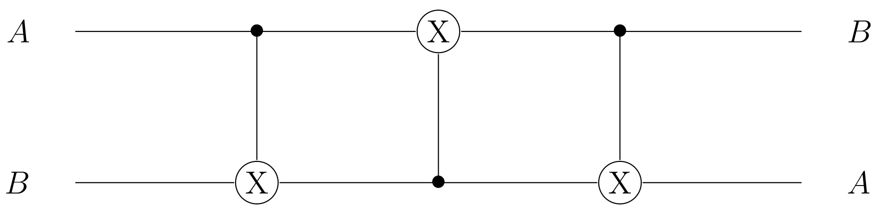
$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{273}$$

This matrix can be written as follows by making use of three Controlled–NOT matrices

(gates)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{274}$$

or graphically



See for example [18] and [56].

A comment is in order. In this case we can happen to write U as

$$U = \frac{1}{2}\left(\mathbf{1} \otimes \mathbf{1} + \sum_{j=1}^{3} \sigma_j \otimes \sigma_j\right), \tag{275}$$

where $\{\sigma_1, \sigma_2, \sigma\}$ are Pauli matrices. But unfortunately we cannot extend this formula further.

It is not easy for us to conjecture its general form from this swap operator. Let us try for $n = 3$. The result is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 \\ a_2b_1 \\ a_3b_1 \\ a_1b_2 \\ a_2b_2 \\ a_3b_2 \\ a_1b_3 \\ a_2b_3 \\ a_3b_3 \end{pmatrix}.$$

Here we rewrite the swap operator above as follows.

$$U = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}. \tag{276}$$

Now, from the above form we can conjecture the general form of the swap operator.

We note that

$$(\mathbf{1} \otimes \mathbf{1})_{ij,kl} = \delta_{ik}\delta_{jl}, \tag{277}$$

so after some trials we conclude

$$U : \mathbf{C}^n \otimes \mathbf{C}^n \longrightarrow \mathbf{C}^n \otimes \mathbf{C}^n$$

as

$$U = (U_{ij,kl}) \quad ; \quad U_{ij,kl} = \delta_{il}\delta_{jk}, \tag{278}$$

where $ij = 11, 12, \cdots, 1n, 21, 22, \cdots, 2n, \ \cdots, \ n1, n2, \cdots, nn.$

The proof is simple and as follows.

$$\begin{aligned}(a \otimes b)_{ij} = a_i b_j \longrightarrow \{U(a \otimes b)\}_{ij} &= \sum_{kl=11}^{nn} U_{ij,kl} a_k b_l = \sum_{kl=11}^{nn} \delta_{il}\delta_{jk} a_k b_l \\ &= \sum_{l=1}^{n} \delta_{il} b_l \sum_{k=1}^{n} \delta_{jk} a_k = b_i a_j = (b \otimes a)_{ij}.\end{aligned}$$

At this stage there is no problem to take a limit $n \to \infty$.

Let $\mathcal{H}$ be a Hilbert space with a basis $\{e_n\}$ $(n \geq 1)$. Then the universal swap operator is given by

$$U = (U_{ij,kl}) \quad ; \quad U_{ij,kl} = \delta_{il}\delta_{jk}, \tag{279}$$

where $ij = 11, 12, \cdots, \cdots$.

Problem Is it possible to construct this universal swap operator by making use of some techniques in Quantum Optics ?

We can in fact construct this by use of some techniques developed in this paper. Let us prove. Let $\mathcal{H}$ be the Fock space with a basis $\{|n\rangle\}$ $(n \geq 0)$ defined in Section 2, and $|X\rangle$ and $|Y\rangle$ be any quantum states in $\mathcal{H}$. Then we show the general formula of swap

$$|X\rangle \otimes |Y\rangle \longrightarrow |Y\rangle \otimes |X\rangle. \tag{280}$$

Let us recall that the typical feature of coherent states $|z\rangle$ $(z \in \mathbf{C})$ is the resolution of unity

$$\int_{\mathbf{C}} \frac{[d^2 z]}{\pi} |z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1},$$

see (9). From this we can expand $|X\rangle$ or $|Y\rangle$ by use of coherent states like

$$|M\rangle = \int_{\mathbf{C}} \frac{[d^2 z]}{\pi} \langle z|M\rangle |z\rangle \quad \text{for} \quad M = X,\ Y, \tag{281}$$

so we have

$$|X\rangle \otimes |Y\rangle = \int\int \frac{[d^2 z]}{\pi} \frac{[d^2 w]}{\pi} \langle z|X\rangle\langle w|Y\rangle |z\rangle \otimes |w\rangle. \tag{282}$$

However we have shown the swap $|z\rangle \otimes |w\rangle \longrightarrow |w\rangle \otimes |z\rangle$ in Section 10.2 by use of the operator $\left(\mathbf{1} \otimes \mathrm{e}^{i\pi N}\right) U_J(\pi/2)$, so that

$$\int\int \frac{[d^2 w]}{\pi} \frac{[d^2 z]}{\pi} \langle w|Y\rangle\langle z|X\rangle |w\rangle \otimes |z\rangle = |Y\rangle \otimes |X\rangle. \tag{283}$$

We finally obtain (280).

D Imperfect Cloning of Quantum States

Here let us state the general theory of our "imperfect cloning". Let $|X\rangle$ be any element in the Fock space $\mathcal{H}$. Then by (281) we can write $|X\rangle \otimes |0\rangle$ as

$$|X\rangle \otimes |0\rangle = \int \frac{[d^2 z]}{\pi} \langle z|X\rangle |z\rangle \otimes |0\rangle.$$

On the other hand we have constructed the "imperfect cloning" $|z\rangle \otimes |0\rangle \longrightarrow |\cos(|\kappa|)z\rangle \otimes |\sin(|\kappa|)z\rangle$ in Section 10.3 by use of the operator $\left(\mathbf{1} \otimes \mathrm{e}^{i\pi N}\right) U_J(|\kappa|)$, so

$$|X\rangle \otimes |0\rangle \longrightarrow \int \frac{[d^2 z]}{\pi} \langle z|X\rangle |\cos(|\kappa|)z\rangle \otimes |\sin(|\kappa|)z\rangle. \tag{284}$$

We have only to calculate the integral in the right hand side. The coherent state $|z\rangle$ is expanded as

$$|z\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle,$$

see (6). By expressing $|X\rangle$ as

$$|X\rangle = \sum_{n=0}^{\infty} x_n |n\rangle \quad \text{for} \quad x_n \in \mathbf{C}$$

and after some algebras we have

$$\text{RHS of (284)} = \sum_{n,m=0}^{\infty} \sqrt{\frac{(n+m)!}{n!m!}} \cos^n(|\kappa|) \sin^m(|\kappa|) x_{n+m} |n\rangle \otimes |m\rangle.$$

That is, we finally obtain the general formula of "imperfect cloning"

$$|X\rangle \otimes |0\rangle \longrightarrow \sum_{n,m=0}^{\infty} \sqrt{\frac{(n+m)!}{n!m!}} \cos^n(|\kappa|) \sin^m(|\kappa|) x_{n+m} |n\rangle \otimes |m\rangle. \tag{285}$$

In particular when $\cos(|\kappa|) = \sin(|\kappa|) = 1/\sqrt{2}$ we have

$$|X\rangle \otimes |0\rangle \longrightarrow \sum_{n,m=0}^{\infty} \sqrt{\frac{(n+m)!}{n!m!}} 2^{-(n+m)/2} x_{n+m} |n\rangle \otimes |m\rangle. \tag{286}$$

A comment is in order. The author doesn't know whether this general formula of "imperfect cloning" is really useful or not in Quantum Information Theory.

E Calculation of Path Integral

Let us calculate (255) explicitly. Noting $z_N = z_0$ and rewriting

$$\sum_{j=1}^{N} \left\{ \bar{z}_j (z_j - z_{j-1}) + i\omega\Delta t \bar{z}_j z_{j-1} \right\}$$

$$= \sum_{j=1}^{N} \left\{ \bar{z}_j z_j - \bar{z}_j (1 - i\omega\Delta t) z_{j-1} \right\} = (\bar{z}_N, \bar{z}_{N-1}, \cdots, \bar{z}_1) \mathbf{A} \begin{pmatrix} z_N \\ z_{N-1} \\ \cdot \\ \cdot \\ z_1 \end{pmatrix} \equiv \mathbf{Z}^\dagger \mathbf{A} \mathbf{Z},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & -(1-i\omega\Delta t) & 0 & \cdots & 0 & 0 \\ 0 & 1 & -(1-i\omega\Delta t) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -(1-i\omega\Delta t) \\ -(1-i\omega\Delta t) & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

we have

$$\begin{aligned} &\int_{PBC} \prod_{j=1}^{N} \frac{[d^2 z_j]}{\pi} \exp\left\{ -\sum_{j=1}^{N} \{\bar{z}_j(z_j - z_{j-1}) + i\omega\Delta t \bar{z}_j z_{j-1}\} \right\} \\ &= \int \prod_{j=1}^{N} \frac{[d^2 z_j]}{\pi} \exp\left(-\mathbf{Z}^\dagger \mathbf{A}\mathbf{Z}\right) = \frac{1}{\mathrm{det}\mathbf{A}}, \end{aligned}$$

where we have used that $\mathbf{A}$ is a normal matrix ($X^\dagger X = XX^\dagger$). Since it is easy to see

$$\mathrm{det}\mathbf{A} = 1 - (1 - i\omega\Delta t)^N,$$

we obtain

$$(255) = \lim_{N\to\infty} \frac{1}{1-(1-i\omega\Delta t)^N} = \lim_{N\to\infty} \frac{1}{1-(1-\frac{i\omega T}{N})^N} = \frac{1}{1-\mathrm{e}^{-i\omega T}}. \tag{287}$$

This is just (251).

F Representation from $SU(2)$ to $SO(3)$

We in this appendix give a useful expression to the well–known representation from $SU(2)$ to $SO(3)$. This result is no direct relation to the text of this paper, but may become useful in the near future. Now let us define

$$\rho : SU(2) \longrightarrow SO(3).$$

First of all we note a simple fact :

$$g = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \quad a,b,c,d \ \in \mathbf{R}$$

where

$$g \in SU(2) \Longleftrightarrow a^2+b^2+c^2+d^2=1.$$

Let us set $\{\sigma_1, \sigma_2, \sigma_3\}$ Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and set

$$\tau_j = \frac{1}{2}\sigma_j \quad \text{for} \quad j=1,2,3.$$

The representation ρ is given as follows : it is easy to see

$$\begin{aligned}
g^{-1}\tau_1 g &= (a^2-b^2-c^2+d^2)\tau_1 + 2(ab+cd)\tau_2 - 2(ac-bd)\tau_3, \\
g^{-1}\tau_2 g &= -2(ab-cd)\tau_1 + (a^2-b^2+c^2-d^2)\tau_2 + 2(ad+bc)\tau_3, \\
g^{-1}\tau_3 g &= 2(ac+bd)\tau_1 - 2(ad-bc)\tau_2 + (a^2+b^2-c^2-d^2)\tau_3,
\end{aligned}$$

so we have

$$\left(g^{-1}\tau_1 g, g^{-1}\tau_2 g, g^{-1}\tau_3 g\right) = (\tau_1, \tau_2, \tau_3)\,\rho(g)$$

where

$$G \equiv \rho(g) = \begin{pmatrix} a^2-b^2-c^2+d^2 & -2(ab-cd) & 2(ac+bd) \\ 2(ab+cd) & a^2-b^2+c^2-d^2 & -2(ad-bc) \\ -2(ac-bd) & 2(ad+bc) & a^2+b^2-c^2-d^2 \end{pmatrix}. \tag{288}$$

Here let us transform the above G. Noting that

$$\begin{aligned}
a^2-b^2-c^2+d^2 &= 1-2(b^2+c^2), \\
a^2-b^2+c^2-d^2 &= 1-2(b^2+d^2), \\
a^2+b^2-c^2-d^2 &= 1-2(c^2+d^2),
\end{aligned}$$

from $a^2+b^2+c^2+d^2=1$, we have

$$G = \begin{pmatrix} 1-2(b^2+c^2) & -2(ab-cd) & 2(ac+bd) \\ 2(ab+cd) & 1-2(b^2+d^2) & -2(ad-bc) \\ -2(ac-bd) & 2(ad+bc) & 1-2(c^2+d^2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2(b^2+c^2) & -2(ab-cd) & 2(ac+bd) \\ 2(ab+cd) & -2(b^2+d^2) & -2(ad-bc) \\ -2(ac-bd) & 2(ad+bc) & -2(c^2+d^2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2ab & 2ac \\ 2ab & 0 & -2ad \\ -2ac & 2ad & 0 \end{pmatrix} + \begin{pmatrix} -2(b^2+c^2) & 2cd & 2bd \\ 2cd & -2(b^2+d^2) & 2bc \\ 2bd & 2bc & -2(c^2+d^2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2a \begin{pmatrix} 0 & -b & c \\ b & 0 & -d \\ -c & d & 0 \end{pmatrix} + 2 \begin{pmatrix} -(b^2+c^2) & cd & bd \\ cd & -(b^2+d^2) & bc \\ bd & bc & -(c^2+d^2) \end{pmatrix}$$

If we define M as

$$M = \begin{pmatrix} 0 & -b & c \\ b & 0 & -d \\ -c & d & 0 \end{pmatrix} \tag{289}$$

then easily

$$M^2 = \begin{pmatrix} -(b^2+c^2) & cd & bd \\ cd & -(b^2+d^2) & bc \\ bd & bc & -(c^2+d^2) \end{pmatrix},$$

so we finally obtain

$$G = \mathbf{1} + 2aM + 2M^2. \tag{290}$$

This equation is very simple and interesting.

The author could not find standard books (not papers) in representation theory which write this equation.

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