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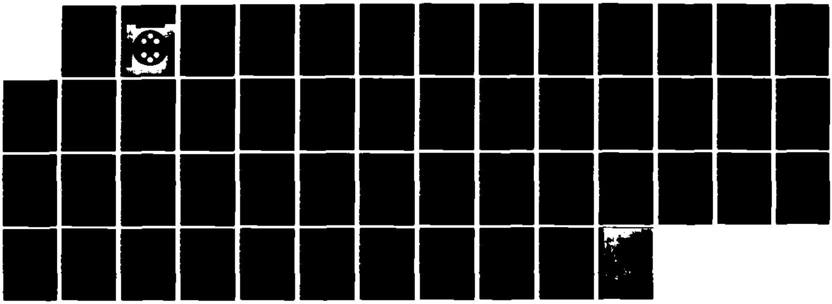

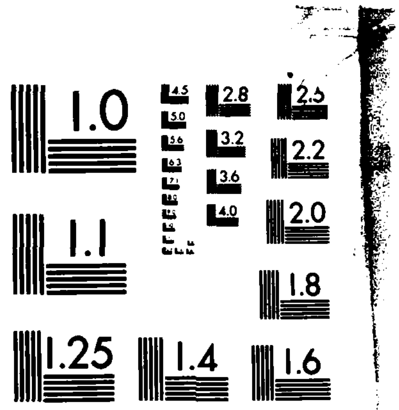

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Dynamic bifurcation theory in differential equations is concerned with the changes that occur in the structure of the limit sets of solutions as parameters in the vector field are varied. For example, if the vector field is the gradient of a function with a finite number of critical points, then the ω-limit set of each orbit is an equilibrium point. Thus, one must be concerned with how the number of equilibrium points changes with the parameters (this is usually called static bifurcation theory), how the stability properties of the equilibrium points change and the manner in which the equilibrium points are connected to each other by orbits. If the vector field is not the gradient of a function, then other types of limiting motions can occur; for example, periodic orbits, invariant tori, homoclinic and heteroclinic orbits. The purpose of these notes is to give an introduction to the methods used in determining how these more complicated limit sets change as parameters vary.

INTRODUCTION TO DYNAMIC BIFURCATION

by

Jack K. Hale

May 1983 LCDS Report #83-16

INTRODUCTION TO DYNAMIC BIFURCATION*

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Introduction. Dynamic bifurcation theory in differential equations is concerned with the changes that occur in the structure of the limit sets of solutions as parameters in the vector field are varied. For example, if the vector field is the gradient of a function with a finite number of critical points, then the ω-limit set of each orbit is an equilibrium point. Thus, one must be concerned with how the number of equilibrium points changes with the parameters (this is usually called static bifurcation theory), how the stability properties of the equilibrium points change and the manner in which the equilibrium points are connected to each other by orbits. If the vector field is not the gradient of a function, then other types of limiting motions can occur; for example, periodic orbits, invariant tori, homoclinic and heteroclinic orbits. Important questions in bifurcation theory are concerned with the manner in which these more complicated limit sets change as parameters vary.

A person being introduced for the first time to bifurcation theory may have the impression that it consists of a collection of isolated results without any unifying principles. Furthermore, since bifurcations are a rare occurrence, perhaps they could be avoided if one were clever enough. Neither of these statements are true. As an illustration, suppose one has a one parameter family of vector fields X_λ depending on a parameter λ, $0 \le \lambda \le 1$, with the property that the phase portraits of the flows for $\lambda = 0$ and $\lambda = 1$ are completely different. Then there must be some point λ_0 in $(0,1)$ where the structure of the flow changes in a neighborhood of λ_0; that is, a bifurcation must occur. This shows bifurcations cannot be avoided. The underlying principle in bifurcation theory for this illustration with one parameter familes of vector fields is the following. Among all of the one parameter families X_λ, $0 \le \lambda \le 1$, of vector fields, characterize those for which the bifurcations are the most elementary. By most elementary, one generally means that a perturbation of the one parameter family will have the same type of bifurcations as the unperturbed family. This implies that such a family is "transversal" to all of the bifurcation surfaces in the class of all vector fields. If the family X_λ depends on two parameters $\lambda = (\lambda_1, \lambda_2)$, $0 \le \lambda_j \le 1$, then one can attempt in the same way to

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classify those which are "transversal" to all the bifurcation surfaces. Two manifolds are transversal if the tangent spaces span the whole space. In Figure 1, we have schematically indicated one and two parameter families X_λ which are transversal to the bifurcation surface S.

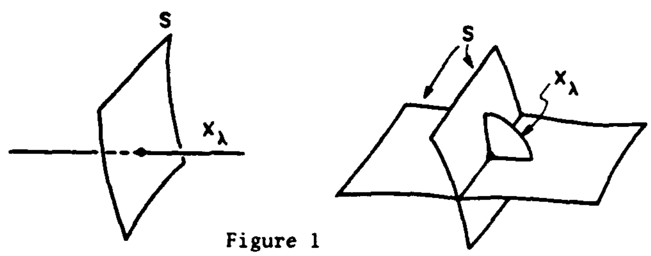

Figure 1

In the following, when we use the term codimension one (codimension k) singularity, we mean an elementary bifurcation point for a one (k) parameter family of vector fields. If the reader keeps this idea in mind as he studies the subject, he will recognize that specific theorems are precise mathematical descriptions of the above imprecise remarks. Of course, it should be clear that the same remarks apply to vector fields which depend on two or more parameters.

The purpose of these lectures is to introduce the reader to some of the basic ideas in bifurcation. The first lectures deal with applications of the Fredholm alternative and the method of Liapunov-Schmidt to bifurcation near equilibrium and the existence of homoclinic orbits. To illustrate the more global aspects of the theory, we summarize the codimension one singularities in the plane and give some examples of codimension two singularities.

Throughout the notes $C^k(X,Y)$ denotes the set of functions from X to Y which are continuous together with derivatives up through order k. The space $C^k_b(X,Y)$ is the set in $C^k(X,Y)$ with all derivatives up through order k bounded with the norm being the sup of all derivatives up through order k.

1. <u>The Fredholm alternative and Liapunov-Schmidt</u>. Many problems in bifurcation theory lead to the study of the zeros of a function in the neighborhood of a given point. Often, the analysis consists of the following steps: firstly, analyze the nonhomogeneous linear equation (referred to as the Fredholm alternative); secondly, use this information to reduce the original problem to one of lower dimension by obtaining a bifurcation function (referred to as the method of alternative problems or the method of Liapunov-Schmidt); thirdly, analyze the bifurcation equation; fourthly, relate the analysis to dynamical behavior.

The purpose of this section is to give an abstract version of the first two steps.

If P is a continuous projection on any Banach space X, we let X_P denote the range of P. If X,Z are Banach spaces, we let $\mathcal{L}(X,Z)$ denote the space of bounded linear operators from X to Z. If $A \in \mathscr{L}(X,Z)$, we let $\mathscr{N}(A) = \{x : Ax = 0\}$, $\mathscr{R}(A) = \{z \in Z : \exists x \in X \ni Ax = z\}$. We shall also use the notation $\mathscr{N}(A) = X_U$, $\mathscr{R}(A) = Z_E$ to denote that there are continuous projections U,E such that these equalities hold. The assertion that $\mathscr{R}(A) = Z_E$ is an important restriction on A if the space is infinite dimensions.

<u>Lemma 1.1.</u> <u>If</u> $A \in \mathscr{L}(X,Z)$, $\mathscr{N}(A) = X_U$, $\mathscr{R}(A) = Z_E$, <u>then there exists a right inverse</u> $K \in \mathscr{L}(Z_E, X_{I-U})$ <u>of</u> A, $AK = I$ <u>on</u> Z_E, $KA = I - U$ <u>on</u> X. (<u>see Fig.</u> 1.1)

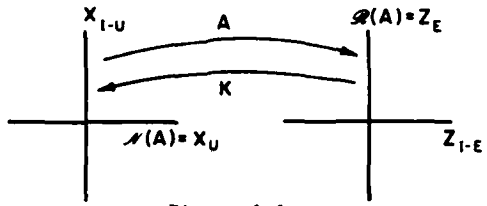

Figure 1.1

<u>Proof:</u> Since A is one-to-one from X_{I-U} onto Z_E, the existence of K is clear. The fact that K is bounded follows from the open mapping theorem.

Suppose A is a linear operator as in Lemma 1.1, Λ is a Banach space denoting the parameter space and $N : X \times \Lambda \to Z$ is a C^1 function satisfying

$$N(0,0) = 0, \quad D_x N(0,0) = 0 \tag{1.1}$$

We want to discuss the solutions of the equation

$$Ax = N(x,\lambda) \tag{1.2}$$

for (x,λ) in a neighborhood of (0.0).

Using the projection operator E in Lemma 1.1, we can rewrite (1.2) in the equivalent form

$$EAx = EN(x,\lambda), \quad (I-E)Ax = (I-E)N(x,\lambda).$$

If we let $x = y + z$, $y \in X_U$, $z \in X_{I-U}$, and use the fact that $EA = A, (I-E)A=0$, $Ax = Az$ and K is a right inverse of A on Z_E, we obtain the equivalent equations,

$$z = KEN(y+z,\lambda) \tag{1.3a}$$

$$0 = (I-E)N(y+z,\lambda). \tag{1.3b}$$

One can use the Implicit Function Theorem to obtain a unique solution $z^*(y,\lambda)$ of (1.3a) in a neighborhood of zero, $z^*(0,0) = 0$, $D_y z^*(0,0) = 0$. For $x = y + z^*(y,\lambda)$ to be a solution of the equation (1.2), the pair (y,λ) must satisfy

$$\begin{aligned} G(y,\lambda) &= 0 \\ G(y,\lambda) &= (I-E)N(y+z^*(y,\lambda),\lambda). \end{aligned} \tag{1.4}$$

The function $G(y,\lambda)$ is known as the <u>bifurcation function</u>. The above procedure for obtaining solutions of (1.2) is an application of the alternative method and is known as the <u>method of Liapunov-Schmidt</u> (<u>LS method</u>). It is summarized in

Lemma 1.2. There is a neighborhood U of $(x,\lambda) = (0,0)$ such that every solution of (1.2) has the form $x = y + z^*(y,\lambda)$ where $z^*(y,\lambda)$ is the solution of (1.3a) and (y,λ) satisfy (1.4).

Several specific illustrations of Lemma 1.2 will be given in these notes. At the same time, we will discuss the solutions of the bifurcation equation and relate the analysis to dynamical behavior.

An operator $A : X \to Z$ with closed range and having $\dim \mathcal{N}(A) < \infty$, $\operatorname{codim} \mathcal{R}(A) < \infty$, is called a Fredholm operator of index $\dim \mathcal{N}(A) - \operatorname{codim} \mathcal{R}(A)$.

In the applications, Equation (1.2) often arises in the following manner. Suppose $M : X \times \Lambda \to Z$ is a given smooth function and suppose it is known that the equation $M(x,\lambda) = 0$ has a solution $x = \varphi(\lambda)$ for λ in some open set. One can study the solutions of $M(x,\lambda) = 0$ near $\varphi(\lambda)$ by letting $x \leftrightarrow \varphi(\lambda) + x$ to obtain a new function which we again call M such that $M(0,\lambda) = 0$ for λ in an open set. The Taylor series for M is then

$$M(x,\lambda) = D(\lambda)x + \tilde{M}(x,\lambda), \quad \tilde{M}(x,\lambda) = o(|x|) \text{ as } |x| \to 0.$$

If the operator $D(\lambda_0)$ has a bounded inverse, then the Implicit Function Theorem implies that $M(x,\lambda) = 0$ has a unique solution $x^*(\lambda)$ in a neighborhood of $(0,\lambda_0)$, $x^*(\lambda_0) = 0$. Thus, no bifurcation can occur. If $D(\lambda_0)$ does not have an inverse, then there is the possibility of bifurcation near $(0,\lambda_0)$. In this case, $A = D(\lambda_0)$, $N(x,\lambda) = [D(\lambda)-D(\lambda_0)]x + \tilde{M}(x,\lambda)$.

An important special case arises when λ is a scalar parameter and $D(\lambda) = B - \lambda C$, where B,C are bounded linear operators. The values λ_0 where $D(\lambda_0)$ is singular are then eigenvalues of the pair of operators (B,C). For later reference, we say λ_0 is a simple eigenvalue of (B,C) if $B - \lambda_0 C$ is Fredholm of index 0 with $\dim \mathcal{N}(B-\lambda_0 C) = 1 = \operatorname{codim} \mathcal{R}(B-\lambda_0 C)$ and $C\mathcal{N}(B-\lambda_0 C) \oplus \mathcal{R}(B-\lambda_0 C) = Z$.

2. Stable and unstable manifolds. In this section, we show how the classical method of obtaining stable and unstable manifolds for an hyperbolic equilibrium point is a special case of the LS method for a Fredholm operator with $A : X \to Z$ with $\dim \mathcal{N}(A) < \infty$ and $\mathcal{R}(A) = Z$.

Suppose A_0 is an $n \times n$ constant matrix with $\operatorname{Re} \sigma(A_0) \neq 0$ where $\sigma(A_0)$ is the spectrum of A_0, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function with $f(0) = 0$, $f_x(0) = 0$. In order to construct the local unstable manifold of

$$(Ax)(t) = f(x(t)), \quad (Ax)(t) = \dot{x}(t) - A_0 x(t), \qquad (2.1)_f$$

we consider the set U_f = {initial values of solutions of (2.1) which are defined and remain in a "small" neighborhood of zero for $t \in (-\infty,0]$}. The local stable manifold S_f is defined similarly on $[0,\infty)$. Let X^0 = {bounded, uniformly continuous functions on $(-\infty,0]$ to $\mathbb{R}^n$} with the sup topology and let $X^1 = \{g \in X^0 : \dot{g} \in X^0\}$ with the C^1 topology. Then A in $(3.1)_f$ takes X^1 to X^0 and is continuous and linear.

If U_0, S_0 are the stable and unstable manifolds of $(2.1)_0$, $\mathbb{R}^n = U_0 \oplus S_0$, $\pi : \mathbb{R}^n \to U_0$, $I - \pi : \mathbb{R}^n \to S_0$ are projections, then $A : X^1 \longrightarrow X^0$

$$\mathcal{N}(A) = \{g \in X^1 : g(t) = e^{A_0 t}\pi g(0),\ t \le 0\}.$$

Thus, $\dim \mathcal{N}(A) < \infty$. One can define a projection $\tilde{\pi}$ of X^1 onto $\mathcal{N}(A)$ by the relation $(\tilde{\pi}g)(t) = e^{A_0}\pi g(0)$, $t \le 0$. It is a classical result in differential equations and easy to prove that $\mathcal{R}(A) = X^0$. If $x = y + z$, $y = e^{A_0}\xi \in X^1_{\tilde{\pi}}$, $z \in X^1_{I-\tilde{\pi}}$, $y(0) = \xi$, then the LS method implies there is a unique solution $x^*_f(\xi,\cdot) = y + z^*_f(\xi,\cdot) \in X^1$ of $(2.1)_f$ in a sufficiently small neighborhood of zero and this function is C^1 in ξ, $z^*_f(0,\cdot) = 0$, $D_\xi z^*_f(0,\cdot) = 0$. The manifold U_f is defined by $\{x : x = \xi + z^*_f(\xi,0),$ ξ small$\}$. It can be shown that any solution with data in U_f approaches zero as $t \to -\infty$ and thus, represents the local unstable manifold. A similar argument gives the stable manifold S_f.

Note that this is a good example of the application of the LS method, but there is no bifurcation equation since codim $\mathcal{R}(A) = 0$, index $A = \dim \mathcal{N}(A)$.

3. Equilibrium bifurcation. In this section, we consider the (n+1)-dimensional system

$$\dot{z} = Cz + f(z,\lambda) \tag{3.1}$$

where λ is a parameter in a Banach space Λ, $f : \mathbb{R}^{n+1} \times \Lambda \to \mathbb{R}^{n+1}$ is continuous together with derivatives up through order $k \ge 1$,

$$f(0,0) = 0, \quad \partial f(0,0)/\partial z = 0 \tag{3.2}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \tag{3.3}$$

and B is an $n \times n$ matrix with Re $\sigma(B) \ne 0$, where $\sigma(B)$ is the spectrum of B. If $z = (x,y)$, $f = (g,h)$, with $x \in \mathbb{R}$, $g \in \mathbb{R}$, Eq. (1.1) can be written as

$$\begin{aligned} \dot{x} &= g(x,y,\lambda) \\ \dot{y} &= By + h(x,y,\lambda). \end{aligned} \tag{3.4}$$

Our objective is to discuss the bifurcation and stability of equilibrium solutions of Eq. (3.1) in a neighborhood of $(z,\lambda) = (0,0)$. The equilibrium solutions are easily obtained by the LS method applied to the equation $Cz + f(z,\lambda) = 0$. This is equivalent to applying the Implicit Function Theorem to obtain a solution $\varphi(x,\lambda)$ of the equation

$$B\varphi + h(x,\varphi,\lambda) = 0 \tag{3.5}$$

in a neighborhood of $(x,\lambda) = (0,0)$ which satisfies $\varphi(0,0) = 0$, $\partial\varphi(0,0)/\partial x = 0$. The equilibrium points of Eq. (3.1) in a neighborhood of $(z,\lambda) = (0,0)$ are then given by $(x,\varphi(x,\lambda))$ where (x,λ) satisfy the bifurcation equation

$$G(x,\lambda) = 0$$
$$G(x,\lambda) = g(x,\varphi(x,\lambda),\lambda). \tag{3.6}$$

We remark that the function $\varphi(x,\lambda)$ depends only upon the vector field $h(x,y,\lambda)$ [not on $g(x,y,\lambda)$] and has the same smoothness properties as $h(x,y,\lambda)$.

The equilibrium points of Eq. (3.1) can also be expressed in terms of the center manifold. In fact, there is a center manifold $M_\lambda = \{(x,y) : y = \psi(x,\lambda)$, x in a neighborhood of zero$\}$ for λ in a neighborhood of zero. The flow on M_λ is given by $(x(t),y(t)) = (x(t),\psi(x(t),\lambda))$ where $x(t)$ satisfies the equation

$$\dot{x} = \tilde{G}(x,\lambda)$$
$$\tilde{G}(x,\lambda) = g(x,\psi(x,\lambda),\lambda). \tag{3.7}$$

The equilibrium points of Eq. (2.1) in a neighborhood of $(z,\lambda) = (0,0)$ are given by $(x,\psi(x,\lambda))$ where $\tilde{G}(x,\lambda) = 0$.

We remark that the function $\psi(x,\lambda)$ depends on both of the vector fields $g(x,y,\lambda)$, $h(x,y,\lambda)$ and, generally, is not as smooth as g and h. In particular, ψ will generally not be C^∞ or analytic even when g,h are C^∞ or analytic.

Since a center manifold has a hyperbolic structure (each point on M_λ looks like a saddle point), the flow on the center manifold gives a complete description of the flow defined by Eq. (3.1) in a neighborhood of $(z,\lambda) = (0,0)$. Even though the bifurcation function $G(x,\lambda)$ in (3.6) was constructed without mentioning dynamical behavior, it is an interesting fact (stated precisely in Theorem 3.1 below) that the flow defined by the equation

$$\dot{x} = G(x,\lambda) \tag{3.8}$$

is equivalent to the flow defined by Eq. (3.7). Thus, the qualitative properties of the flow can be determined without knowing the center manifold. The advantage of this observation lies in the fact that $G(x,\lambda)$ is as smooth as the original vector field and is easier to calculate approximately. Up to this point, the parameter λ has played no role and it plays no role in the following theorem. However, it will be used in a significant way in the applications of Theorem 3.1.

Theorem 3.1. The vector fields $G(\cdot,\lambda)$, $\tilde{G}(\cdot,\lambda)$ are equivalent in a neighborhood of zero for λ in a neighborhood of zero; that is, there is a homeomorphism of a neighborhood of $x = 0$ mapping the orbits of (3.7) onto the orbits of (3.8) preserving the sense of direction in time.

Proof: The functions $G(x,\lambda)$, $\tilde{G}(x,\lambda)$ must have the same set of zeros in a neighborhood of $(x,\lambda) = (0,0)$. The essential element of the proof of the theorem and the only part that will be given is to show that $G(x,\lambda)$, $\tilde{G}(x,\lambda)$ have the same sign between zeros. Suppose this is not the case; that is, there is an x_0 such that $G(x_0,\lambda)\,\tilde{G}(x_0,\lambda) < 0$. By making a small perturbation of the vector $f(\cdot,\lambda)$ in Eq. (3.1), one obtains new functions $G_1(x,\lambda)$, $\tilde{G}_1(x,\lambda)$ such that $G_1(x_0,\lambda)\,\tilde{G}_1(x_0,\lambda) < 0$ and the first zero x_1 before x_0 is simple. Now make another perturbation of $f(\cdot,\lambda)$ in Eq. (3.1) by replacing g by $g+\varepsilon$ with ε small > 0. The new bifurcation function $G_2(x,\lambda,\varepsilon) = G_1(x,\lambda) + \varepsilon$. Also, using the theory of center manifolds, one can show that the vector field on the center manifold $\tilde{G}_2(x,\lambda,\varepsilon) = \tilde{G}_1(x,\lambda)$, + $\sigma(x,\lambda,\varepsilon)\varepsilon$ where $\sigma(x,\lambda,\varepsilon) > 0$. Since x_1 is a simple zero of $G_1(x,\lambda)$, $\tilde{G}_1(x,\lambda)$, this implies there are functions $x_2(\lambda,\varepsilon)$, $\tilde{x}_2(\lambda,\varepsilon)$, $x_2(\lambda,0) = x_1 = \tilde{x}_2(\lambda,0)$, such that $G_2(x_2(\lambda,\varepsilon),\lambda,\varepsilon) = 0$, $\tilde{G}_2(\tilde{x}_2(\lambda,\varepsilon),\lambda,\varepsilon) = 0$ and such that $[dx_2(\lambda,0)/d\varepsilon]\cdot[d\tilde{x}_2(\lambda,0)/d\varepsilon] < 0$, which implies $x_2(\lambda,\varepsilon) \neq \tilde{x}_2(\lambda,\varepsilon)$, $\varepsilon > 0$. But this is a contradiction since the bifurcation function and the vector field on the center manifold must always have the same zeros. This implies $G(x,\lambda)$ and $\tilde{G}(x,\lambda)$ have the same sign between zeros.

As a first application, we give a result on <u>bifurcation from a simple eigenvalue</u> for one parameter families of vector fields. Suppose $\lambda \in \mathbb{R}$, $f(z,\lambda)$ is C^1 and write $f(z,\lambda)$ as

$$\begin{aligned} &f(z,\lambda) = C_1(\lambda)z + F(z,\lambda) \\ &F(0,\lambda) = 0,\ \partial F(0,\lambda)/\partial z = 0 \quad \text{for all } \lambda \end{aligned} \tag{3.9}$$

where $C_1(\lambda)$ is an $(n+1)\times(n+1)$ matrix, $C_1(0) = 0$. Suppose that

$$C_1(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & B_1(\lambda) \end{bmatrix} \tag{3.10}$$

and $B_1(\lambda)$ is an $n\times n$ matrix. The bifurcation function $G(x,\lambda)$ for this special case has the form

$$G(x,\lambda) = \alpha(\lambda)x + \overline{G}(x,\lambda), \tag{3.11}$$

where $\overline{G}(x,\lambda) = O(|x|^2)$ as $x \to 0$, $\alpha(0) = 0$, $\alpha'(0) = 1$ from (3.9), (3.10). The Implicit Function Theorem and Theorem 3.1 imply

<u>Theorem 3.2.</u> <u>If</u> $\lambda \in \mathbb{R}$, f is C^2 <u>and satisfies</u> (3.9), (3.10), <u>then there is a neighborhood</u> V <u>of</u> $z = 0$, W <u>of</u> $\lambda = 0$, U <u>of</u> $x = 0$ <u>and a</u> C^1-<u>function</u> $\lambda^* : U \to \mathbb{R}$ <u>such that</u> $\lambda^*(0) = 0$ <u>and, for</u> $\lambda \in W$, <u>the equilibrium solutions of</u> (3.1) <u>in</u> V <u>are given by</u> $\{0\} \cup \{(x,\varphi(x,\lambda)) : \lambda^*(x) = \lambda\}$. <u>A solution</u> $(\bar{x},\varphi(\bar{x},\lambda))$ <u>is asymptotically stable (unstable) on the center manifold if</u> $(x-\bar{x})G(x,\lambda) < 0$ (>0) <u>near</u> $\bar{x}$.

The stability properties are easy to determine from the curve $\Gamma : \lambda = \lambda^*(x)$, $x \in U$. In fact, since $\partial G(0,\lambda)/\partial x = \lambda + o(|\lambda|)$ as $\lambda \to 0$, it follows that, for a fixed $\lambda \neq 0$, sign $xG(x,\lambda)$ = sign λ for x near zero. This implies that the sign of G is as indicated in Figure 3.1. This yields stability of all equilibrium points immediately with the situation as shown in Figure (3.1). At the bifurcation points, there is always a transfer of stability on the center manifold.

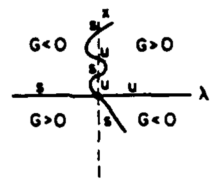

Figure 3.1

As another illustration suppose f in (3.1) is C^2 and λ is a parameter in Λ and there is a $\beta \neq 0$ such that

$$G(x,0) = \beta x^2 + o(|x|^2) \text{ as } x \to 0. \tag{3.12}$$

Since $\varphi(0,0) = 0$, $\partial\varphi(0,0)/\partial x = 0$, it follows that β is given explicitly by $2\beta = \partial^2 g(0,0,0)/\partial x^2$. The Implicit Function Theorem implies there is a unique C^1-function $\alpha(\lambda)$, $\alpha(0) = 0$, such that $\partial G(\alpha(\lambda),\lambda)/\partial x = 0$ for λ small. Then

$$G(\alpha(\lambda) + \bar{x},\lambda) = \gamma_0(\lambda) + \gamma_2(\lambda)\bar{x}^2 + o(|\bar{x}|^2) \quad \text{as} \quad |x| \to 0$$

where $\gamma_0(0) = 0$ and $\gamma_2(0) = \beta \neq 0$. Theorem 3.1 therefore implies, near $(z,\lambda) = (0,0)$, that

(i) $\gamma_0(\lambda) > 0 \Rightarrow$ no equilibrium solution of (3.1).

(ii) $\gamma_0(\lambda) = 0 \Rightarrow$ one equilibrium solution of (3.1) stable from one side and unstable from the other side on the center manifold.

(iii) $\gamma_0(\lambda) < 0$ implies two equilibrium solutions of (3.1), hyperbolic with one stable and the other unstable on the center manifold.

The surface in Λ defined by $\Gamma = \{\lambda : \gamma_0(\lambda) = 0\}$ is called the <u>bifurcation surface</u>. This bifurcation is referred to as the <u>saddle-node bifurcation</u>. The name comes from the fact that, for z in $\mathbb{R}^2$, a saddle and node coalesce and disappear as Γ is crossed.

As another illustration, suppose the vector field in (3.1) is C^3 and there is a $\beta \neq 0$ such that

$$G(x,0) = \beta x^3 + o(|x|^3) \quad \text{as} \quad |x| \to 0. \tag{3.13}$$

Then there is a unique point $\alpha(\lambda)$, $\alpha(0) = 0$, where $\partial^2 G(x,\lambda)/\partial x^2 = 0$ in a neighborhood of zero. If $x = \alpha(\lambda) + \bar{x}$, then

$$G(\alpha(\lambda) + \bar{x},\lambda) = \gamma_0(\lambda) + \gamma_1(\lambda)\bar{x} + \gamma_3(\lambda)\bar{x}^3 + \bar{G}(\bar{x},\lambda) \tag{3.14}$$

where $\bar{G}(\bar{x},\lambda) = o(|\bar{x}|^3)$ as $\bar{x} \to 0$, $\gamma_0(0) = \gamma_1(0) = 0$, $\gamma_3(0) = \beta \neq 0$.

It is no loss in generality to assume $\gamma_3(\lambda) = 1$ in discussing the solutions of (3.14) in a neighborhood of $(x,\lambda) = (0,0)$. Even in the case when $\bar{G}(\bar{x},\lambda) = 0$ in (3.14), a complete discussion of the zeros of the cubic requires two parameters.

This corresponds to a codimension two bifurcation. A generic two parameter family of parameters $\lambda = (\lambda_0, \lambda_1)$ should satisfy $\det[\partial(\gamma_0, \gamma_1)/\partial(\lambda_0, \lambda_1)] \neq 0$ at $\lambda = 0$. If we assume this is the case, then we can introduce new parameters $\mu = (\mu_0, \mu_1)$ instead of λ to obtain the equation

$$H(\bar{x}, \mu) = \mu_0 + \mu_1 \bar{x} + \bar{x}^3 + \bar{H}(\bar{x}, \mu) = 0 \tag{3.15}$$

where $\bar{H}(x, \mu) = o(|\bar{x}|^3)$ as $\bar{x} \to 0$. The bifurcation curves in μ-space correspond to multiple solutions of (3.15); that is, $H(\bar{x}, \mu) = 0$, $\partial H(\bar{x}, \mu)/\partial \bar{x} = 0$. In a neighborhood of zero, this determines μ_0, μ_1 as functions of $\bar{x}$, $\mu_1 = -3\bar{x}^2 + o(|\bar{x}|^2)$, $\mu_0 = 2\bar{x}^3 + o(|\bar{x}|^3)$ as $\bar{x} \to 0$. Eliminating $\bar{x}$ from these equations, one obtains a cusp in μ-space which is approximately given by $\mu_1^3 = -(27/4)\mu_0^2$. The bifurcation diagrams for Equation (3.15) are shown in Fig. 3.2 with the number of solutions as indicated.

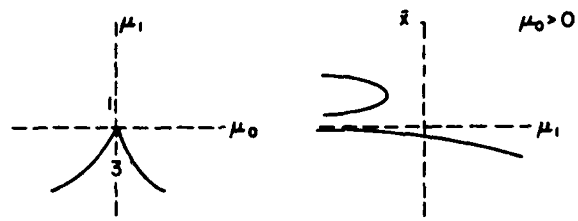

Figure 3.2

For some problems, one may have a parameter $\lambda \in \Lambda$ for which rank $[\partial(\gamma_0, \gamma_1)/\partial\lambda] < 2$ at $\lambda = 0$. In this case, the bifurcation diagram will be different since the set $\{(\gamma_0(\lambda), \gamma_1(\lambda)), \lambda \in \Lambda\}$ will be a curve in (μ_0, μ_1)-space. The number of solutions and the bifurcations for Equation (3.14) will be determined by how this curve intersects the bifurcation curve in Fig. 3.2 as λ varies.

If $G(x,0) = \beta x^q + o(|x|^q)$ as $|x| \to 0$, where $\beta \neq 0$, then the two previous examples show that only one parameter is needed to describe all bifurcations when $q = 2$ and two are needed to describe the bifurcations when $q = 3$. For $q > 3$, one can show that q parameters are needed. The bifurcation surfaces are extremely complicated and will not be discussed here.

Theorem 3.1 is valid for several types of partial differential equations and functional differential equations. As is often the case, the theoretical results are very simple to state, but there are many technical problems in making the applications. We briefly indicate how this can be done for retarded functional differential equations and for abstract evolutionary equations which include parabolic equations.

Let A be the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$, on a Banach space X. Choose b real so that $\mathrm{Re}\,\sigma(\tilde{A}) > 0$, $\tilde{A} = A + bI$. One can define fractional powers $\tilde{A}^\alpha$ of $\tilde{A}$ for any $\alpha \in \mathbb{R}$. For $\alpha \leq 0$, $\tilde{A}^\alpha$ is one-to-one and a bounded linear operator on X. For $\alpha \geq 0$, consider the set $X^\alpha \subseteq X$ consisting of all $x \in X$ for which $|\tilde{A}^\alpha x| < \infty$. For $x \in X$, define $|x|_\alpha = |x| + |\tilde{A}^\alpha x|$. Then

$|\cdot|_\alpha$ is a norm on X^α and X^α is a Banach space with this norm. Also, the inclusion map taking X^α into X is continuous. Now suppose that $f : X^\alpha \times \Lambda \to X$ is a smooth function for some $0 \le \alpha < 1$ and consider the evolutionary equation

$$\dot{u} + Au = f(u,\lambda). \tag{3.16}$$

Suppose that Eq. (3.16) generates a strongly continuous semigroup $T_\lambda(t)$, $t \ge 0$, on X^α. The choice of α depends upon the specific function f.

Suppose $0 \in \sigma(A)$ is a simple eigenvalue of A and there is a $\delta > 0$ such that $|\mathrm{Re}[\sigma(A) \setminus \{0\}]| \ge \delta$. Then the space X^α can be decomposed as a direct sum $X^\alpha = V \oplus W$ with V,W invariant under $T_\lambda(t)$ where V is one dimensional and spanned by a unit vector u_0 satisfying $Au_0 = 0$. If $u = v + w$, $v \in V$, $w \in W$, $f = g + h$, $g \in V$, $h \in W$, then (3.16) is equivalent to

$$\begin{aligned} \dot{v} &= g(v+w,\lambda) \\ \dot{w} &= -Aw + h(v+w,\lambda). \end{aligned} \tag{3.17}$$

The operator A restricted to W has a bounded inverse which we write as A^{-1} taking X into X. Since $D(A) \subseteq D(\tilde{A}^\alpha)$ for $0 \le \alpha < 1$, it follows that the equation

$$-A\varphi + h(v+\varphi,\lambda) = 0, \qquad \varphi \in W, \tag{3.18}$$

is equivalent to the equation

$$\varphi + A^{-1}h(v+\varphi,\lambda) = 0, \qquad \varphi \in W. \tag{3.19}$$

Consider the operator A^{-1} as mapping W into X^α, $0 \le \alpha < 1$. Then A^{-1} is a bounded linear operator from W to X^α since $\tilde{A}^{-1+\alpha}$ is a bounded linear operator on X $(-1+\alpha<0)$ and $|A^{-1}u|_\alpha = |A^{-1}u| + |\tilde{A}^\alpha A^{-1}u| + |A^{-1}u| + |\tilde{A}^{-1+\alpha}(I+bA^{-1})u|$ for any $u \in D(\tilde{A}^\alpha) \cap W$.

The Implicit Function Theorem implies there is a unique function $\varphi(v,\lambda)$, $v(0,0) = 0$, satisfying (3.19). The equilibrium points of (3.16) are therefore given by $u = v + \varphi(v,\lambda)$, where u,λ must satisfy the bifurcation equation

$$G(v,\lambda) = 0, \; G(v,\lambda) = g(v+\varphi(v,\lambda),\lambda).$$

The flow on the center manifold of (3.16) is equivalent to the flow defined by the scalar ordinary differential equation $\dot{v} = G(v,\lambda)$.

As a specific example, consider the equation

$$\begin{aligned} u_t &= u_{xx} + \mu u - u^3 \quad , \quad 0 < x < \pi \\ u &= 0 \quad \text{at} \quad x = 0,\pi. \end{aligned} \tag{3.20}$$

Let $X = L^2(0,\pi)$, $W_0^{1,2} = \{\varphi \in X : \varphi_x \in X,\ \varphi(0) = \varphi(\pi) = 0\}$ and define $A\varphi = -\varphi_{xx}$ with $D(A) = W_0^{1,2} \cap W^{2,2}$. One can show that the operator A generates an analytic semigroup on X. It is also possible to show that $D(A^{1/2}) = W_0^{1,2} = X^{1/2}$, the operator A has compact resolvent and $\sigma(A)$ consists of simple eigenvalues $\{1,2^2,\ldots,n^2,\ldots\}$ with the eigenfunction $(2/\pi)^{1/2}\sin nx$ corresponding to n^2. If $F(\varphi,\mu)(x) = \mu\varphi(x) - \varphi^3(x)$, $0 \le x \le \pi$, then one can show that $F : X^{1/2} \times \mathbb{R} \to X$ is an analytic function of φ,μ. This is enough to conclude that, for any $\varphi \in X^{1/2}$, there is a function $u(t,x,\varphi)$, defined on some interval $0 \le t < t_\varphi$, $u(t,\cdot,\varphi) \in X^{1/2}$, $0 \le t < t_\varphi$, $u(0,\cdot,\varphi) = \varphi$, $u(t,x,\varphi)$ is continuous in (t,x,φ) and satisfies (3.20) for $0 \le t < t_\varphi$.

For Equation (3.20), there is a Liapunov function

$$V(\varphi) = \int_0^\pi [\tfrac{1}{2}\varphi_x^2(x) - \lambda H(\varphi(x))]dx$$

for $\varphi \in X^{1/2}$ and $H(u) = \mu u^2/2 - u^4/4$. In fact,

$$\dot V(u(t,\cdot)) = \frac{d}{dt} V(u(t,\cdot)) = -\int_0^\pi u_t^2(t,x)dx \le 0.$$

One can use this inequality to show that $u(t,x,\varphi)$ is defined for $t \ge 0$; that is, $t_\varphi = \infty$. If one now defines $T_\mu(t)\varphi = u(t,\cdot,\varphi)$, then $T_\mu(t)$, $t \ge 0$, is a strongly continuous semigroup on $X^{1/2}$.

Let us now determine the nature of the bifurcation near the simple eigenvalue $\mu_n = n^2$. Let $\mu = n^2 + \lambda$ and, for $u \in X^{1/2}$,

$$u(x) = (\tfrac{2}{\pi})^{1/2} v \sin nx + w(x), \qquad v \overset{\text{def}}{=} (\tfrac{2}{\pi})^{1/2}\int_0^\pi u(x)\sin nx\, dx$$

$$\int_0^\pi w(x)\sin nx\, dx = 0.$$

If $Uu = v \sin nx$, then U is a projection onto $\mathcal{N}(A-\mu_n I)$ and $(I-U)$ is a projection onto $\mathcal{R}(A-\mu_n I)$. With this decomposition, the equilibrium solutions of (3.20), that is $u_t = 0$, must satisfy the equations

$$\lambda v \sin nx - U[(2/\pi)^{1/2} v \sin n\cdot + w(\cdot)]^3 = 0$$

$$w_{xx} + n^2 w + \lambda w - (I-U)[(2/\pi)^{1/2} \sin n\cdot + w(\cdot)]^3 = 0,$$

with $w = 0$ at $x = 0,\pi$. This last equation has the solution $w(v,\lambda) \in X^{1/2}$ satisfying $w(0,0) = 0$, $\partial w(0,0)/\partial v = 0$, $\int_0^\pi w(v,\lambda)(x)\sin nx\, dx = 0$. Therefore, the equilibrium solutions of (3.20) near $(u,\mu) = 0,0)$ are given by

$$u(x) = (\tfrac{2}{\pi})^{1/2} v \sin nx + w(v,\lambda)(x)$$

with (v,λ) satisfying the bifurcation equation

$$G(v,\lambda) = 0$$

$$G(v,\lambda) = \lambda v - (\tfrac{2}{\pi})^{1/2}\int_0^\pi (\sin nx)[(\tfrac{2}{\pi})^{1/2} v \sin nx + w(v,\lambda)(x)]^3 dx.$$

The function $G(v,\lambda) = \gamma_0(\lambda)v - \gamma_1(\lambda)v^3 + o(|v|^3)$ as $v \to 0$ where $\gamma_0'(0) = 1$, $\gamma_1(0) = 3/\pi^2$. Thus, the bifurcation curve $\lambda = \lambda^*(v)$ satisfying $G(v,\lambda^*(v)) = 0$ is given by $\lambda = (3/\pi^2)v^2 + o(v^2)$ as $v \to 0$ (see Fig. 3.3). At each bifurcation, there

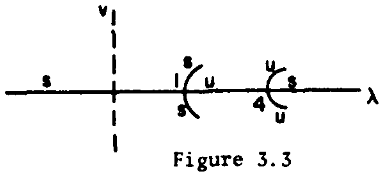

Figure 3.3

is a transfer of stability on the center manifold. However, for the complete flow, the only stable equilibrium points are the ones that occur at the first bifurcation. This is because the zero solution has an unstable manifold of dimension > 1 at the other points.

As another illustration, we consider a retarded functional differential equation. Suppose $r \geq 0$ is a given constant, $C = C([-r,0], \mathbb{R}^n)$, $L : C \to \mathbb{R}^n$ is a continuous linear operator, $f : C \times \Lambda \to \mathbb{R}^n$ is a given C^1-function, $f(0,0) = 0$, $D_\varphi f(\varphi,\lambda) = 0$ when $(\varphi,\lambda) = (0,0)$. For a given function $x : [-r,A) \to \mathbb{R}^n$ and a fixed $t \in [0,A)$, we let x_t designate the function from $[-r,0]$ to $\mathbb{R}^n$ given by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$, and consider the retarded functional differential equation

$$\dot{x}(t) = Lx_t + f(x_t,\lambda). \tag{3.21}$$

If $\varphi \in C$, let $x(\varphi)(t)$ be the solution of (3.21) with $x_0(\varphi) = \varphi$. We suppose all solutions are defined on $[-r,\infty)$ and define $T_\lambda(t)\varphi = x_t(\varphi)$. Then $T_\lambda(t)$, $t \geq 0$, is a strongly continuous semigroup.

An <u>equilibrium point</u> for (3.21) is a solution x defined for all $t \in \mathbb{R}$, $x_0 = \psi$, and $T_\lambda(t)\psi = \psi$ for all $t \in \mathbb{R}$. This is equivalent to $x(t+\theta,\psi) = \psi(\theta)$ for all $t \in \mathbb{R}$. Since $x(t,\psi)$ is continuously differentiable for all $t \in \mathbb{R}$ (this is a consequence of the fact that it exists for $t \in \mathbb{R}$), this implies $\dot{x}(t) = 0$ for all $t \in \mathbb{R}$; that is, $x(t) = b$, a constant, for all $t \in \mathbb{R}$ and $\psi(\theta) = b$, $-r \leq \theta \leq 0$. Therefore, the equilibrium points of (3.21) are constant functions ψ such that

$$\begin{aligned} &L\psi + f(\psi,\lambda) = 0 \\ &\dot{\psi}(\theta) = 0, \ -r \leq \theta \leq 0. \end{aligned} \tag{3.22}$$

The linear equation

$$\dot{x}(t) = L(x_t) \tag{3.23}$$

also generates a strongly continuous linear semigroup $S(t)$, $t \geq 0$, on C. It can be shown that the infinitesimal generator A of $S(t)$ is given by $(A\varphi)(\theta) = \dot{\varphi}(\theta)$, $-r \leq \theta \leq 0$, with $D(A) = \{\varphi \in C : \dot{\varphi} \in C, \ \dot{\varphi}(0) = L(\varphi)\}$. Also, $\sigma(A)$ consists only of point spectrum, and, if $L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\psi(\theta)$, then $\mu \in \sigma(A)$ if and only if

$$\det \Delta(\mu) = 0, \ \Delta(\mu) = \mu I - \int_{-r}^{0} e^{\mu\theta} d\eta(\theta). \tag{3.24}$$

Also, if $\mu \in \sigma(A)$, then $\mathcal{N}(A-\mu I)$ is the span of the functions $e^{\mu\theta}b$, $-r \leq \theta \leq 0$, where $\Delta(\mu)b = 0$.

Suppose $0 \in \sigma(A)$ is simple and $\text{Re}\,\mu \neq 0$ for $\mu \in \sigma(A) \setminus \{0\}$. Then it follows immediately from (3.23) that there is a $\delta > 0$ such that $|\text{Re}\,\mu| \geq \delta$ for $\mu \in \sigma(A) \setminus \{0\}$. Our first objective is to determine the equilibrium points of (3.21) near $x = 0$ for λ near zero.

To characterize the equilibrium solutions of (3.21) as the solutions of an operator equation, we need to enlarge the space of functions that are being considered. Let X_0 be the $n \times n$ matrix function on $[-r,0]$ defined by $X_0(\theta) = 0$, $-r \leq \theta < 0$, $X_0(0) = I$, the identity, and define $PC = C \oplus \langle X_0 \rangle$ where $\langle X_0 \rangle$ denotes the space of X_0. A function $\psi \in PC$ is represented by a pair (φ, b) where $\varphi \in C$ and $b \in \mathbb{R}^n$; the function ψ is uniformly continuous on $[-r,0)$, $\psi(\theta) = \varphi(\theta)$, $\theta \in [-r,0)$, and $\psi(0) = \varphi(0) + b$; that is, ψ has a jump of b at 0. Consider A as a map from C to PC with $D(A) = \{\varphi \in C : \dot{\varphi} \in C\}$ and

$$(A\varphi)(\theta) = \dot{\varphi}(\theta), \quad -r \leq \theta < 0$$
$$(A\varphi)(0) = L\varphi.$$

It is now easy to see from (3.22) that ψ is an equilibrium point of (3.21) if and only if

$$A\psi + X_0 f(\psi,\lambda) = 0 \tag{3.25}$$

We now apply the LS method to Eq. (3.25) under the assumption that 0 is a simple eigenvalue of A. We know that $\mathcal{N}(A) = \{\varphi \in C : \varphi = \beta\varphi_0,\ \varphi_0(\theta) = a,\ -r \leq \theta \leq 0,$ where a is a constant n-vector, $\Delta(0)a = 0\}$. To characterize the range of A, we solve the equation $A\varphi = \psi$ for $\psi \in PC$. This relation is equivalent to $\dot{\varphi}(\theta) = \psi(\theta)$, $-r \leq \theta < 0$, and $L\varphi = \psi(0)$; that is,

$$\varphi(\theta) = \varphi(0) + \int_0^\theta \psi(s)ds$$
$$\Delta(0)\varphi(0) = \psi(0) - \int_{-r}^0 [d\eta(\theta)]\int_0^\theta \psi(s)ds.$$

Let b be any nonzero row vector such that $b\Delta(0) = 0$. Then $\psi \in \mathcal{R}(A)$ if and only if

$$\begin{aligned} &(\alpha_0,\psi) = 0, \qquad \alpha_0(\theta) = b, \qquad -r \leq \theta \leq 0 \\ &(\alpha_0,\psi) = b\psi(0) - \int_{-r}^0 b[d\eta(\theta)]\int_0^\theta \psi(s)ds. \end{aligned} \tag{3.26}$$

Choose b so that $(\alpha_0,\varphi_0) = 1$ with α_0 in (3.26) and φ_0 a basis for $\mathcal{N}(A)$. The space PC can be decomposed as

$$\begin{aligned} &PC = \mathcal{N}(A) \oplus \mathcal{R}(A) \\ &\psi \in PC,\ \psi = \beta\varphi_0 + \eta,\ \beta = (\alpha_0,\psi) \in \mathbb{R},\ (\alpha_0,\eta) = 0. \end{aligned} \tag{3.27}$$

These same formulas also give a decomposition of C as $C = P \oplus Q$, $P = \{\varphi = \beta\varphi_0, \beta \in \mathbb{R}\}$, $Q = \{\varphi : (\alpha_0, \varphi) = 0\}$.

The computations above show that A^{-1} exists and is bounded as a map from $\mathcal{R}(A)$ into Q. We can now apply the LS method.

If $X_0^1 = \varphi_0\delta$, $\delta = (\alpha_0, X_0)$, $X_0^2 = X_0 - \varphi_0\delta$, then (3.25) implies that $\psi = \beta\varphi_0 + \eta$, $(\alpha_0, \eta) = 0$, is an equilibrium point of (3.21) if and only if

$$bf(\varphi_0\beta + \eta, \lambda) = 0$$

$$\eta + A^{-1}X_0^Q f(\varphi_0\beta + \eta, \lambda) = 0.$$

The last equation has a unique solution $\eta(\beta,\lambda)$ in a neighborhood of $(\beta,\lambda) = (0,0)$, $\eta(0,0) = 0$, $\partial\eta(0,0)/\partial\beta = 0$. The equilibrium solutions of (3.21) are given by $\psi = \varphi_0\beta + \eta(\beta,\lambda)$ where (β,λ) satisfy the bifurcation equation

$$G(\beta,\lambda) = bf(\varphi_0\beta + \eta(\beta,\lambda), \lambda) = 0. \tag{3.28}$$

One can also show that the flow on the center manifold of (3.21) is equivalent to the flow defined by $\dot{\beta} = G(\beta,\lambda)$.

As an example, consider the linear equation

$$\dot{x}(t) = -x(t) + x(t-1)$$

which has $\Delta(\lambda) = \lambda - 1 - \exp(-\lambda)$. Zero is a simple eigenvalue and $(\alpha_0, \psi) = b[\psi(0) + \int_{-1}^{0} \psi(s)ds]$. If we choose $\varphi_0(\theta) = 1$, $-r \le \theta \le 0$, then $(\alpha_0, \varphi_0) = 1$ if $b = 1/2$. If we choose $f(\varphi,\lambda) = \lambda + \varphi^2(0)$, then the bifurcation equation (3.28) is equivalent to $\lambda - (\beta + \eta(\beta,\lambda)(0))^2 = 0$. Thus, there will be two equilibrium points for $\lambda > 0$ and no equilibrium points for $\lambda < 0$. The same remark is true if $f(\varphi,\lambda) = \lambda - \varphi^2(-1)$.

4. Bifurcation from two purely imaginary roots. In this section, we consider the equation

$$\dot{z} = Cz + Z(z,\mu) \tag{4.1}$$

where $\mu \in E$ is a parameter in a Banach space E, $z \in \mathbb{R}^{n+2}$, Z is a C^k-function, $k \ge 2$, C is an $(n+2) \times (n+2)$ matrix satisfying

$$Z(0,\mu) = 0, \ \partial Z(0,0)/\partial z = 0, \ \text{for all } \mu,$$

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.2}$$

with the eigenvalues of B having nonzero real parts (for much of the discussion $(e^{2\pi B} - I)$ being nonsingular is sufficient).

Our objective is to discuss the periodic orbits of (4.1) in a neighborhood of $(z,\mu) = (0,0)$. These orbits must have period approximately 2π and be close to the

2π-periodic solutions of the linear equation $\dot{z} = Cz$. If $t \longmapsto \beta t$, where $2\pi/\beta$ is the period of the solutions to be determined, then β should be close to 1 and the new equation for z is

$$\dot{z} = \beta Cz + \beta Z(z,\mu). \tag{4.3}$$

We will apply the LS method for 2π-periodic solutions of this equation. If $z = (x,y)$, $Z = (X,Y)$, $x,X \in \mathbb{R}^2$, then (4.3) is equivalent to

$$\begin{aligned} \dot{x} - Ax &= (\beta-1)Ax + \beta X(x,y,\mu) \\ \dot{y} - By &= (\beta-1)By + \beta Y(x,y,\mu). \end{aligned} \tag{4.4}$$

Let $p^0_{2\pi}$ be the space of 2π-periodic (n+2-vector functions which are continuous with the sup topology and $p^1_{2\pi}$ be those 2π-periodic (n+2)-vector functions in $p^0_{2\pi}$ whose first derivatives are also in $p^0_{2\pi}$ with the C^1 topology. If $\mathcal{A}: p^1_{2\pi} \to p^0_{2\pi}$ is defined on functions $f = (g,h)$ by

$$\mathcal{A}f = (\dot{g}-Ag, \dot{h}-Bh)$$

then $\mathcal{A}$ is a continuous, linear operator and $\mathcal{N}(\mathcal{A}) = \{(e^{At}b,0),\ 0 \in \mathbb{R}^n,\ b \in \mathbb{R}^2\}$. The classical Fredholm alternative for linear periodic systems implies $\mathcal{R}(\mathcal{A}) = \{f=(g,h) : \int_0^{2\pi} e^{-At} g(t)dt = 0\}$. Define $E = \mathrm{diag}(\bar{E},I)$, $\bar{E}g = e^{At}D^{-1}\int_0^{2\pi} e^{-At}g(t)dt$, $D = 2\pi I$. Then $\mathcal{R}(\mathcal{A}) = (I-E)p^0_{2\pi}$.

One can now apply the LS method. Before doing this, it is convenient to observe that $\mathcal{N}(\mathcal{A})$ is determined from a phase shift on the one dimensional subspace $a(\cos t,-\sin t,0)$, $a \in \mathbb{R}$. Since (4.4) is autonomous, we may therefore fix the element in $\mathcal{N}(A)$ as $a(\cos t,-\sin t,0)$. An application of the LS method yields a $z^*(\beta,a,\mu) \in p^1_{2\pi}$ of the form

$$\begin{aligned} z^* &= z^*(\beta,a,\mu) = (a\varphi+x^*(\beta,a,\mu),y^*(\beta,a,\mu)) \\ \varphi(t) &= (\cos t,-\sin t),\ \int_0^{2\pi} e^{-At}x^*(\beta,a,\mu)(t)dt = 0 \end{aligned} \tag{4.5}$$

and $x^* = x^*(\beta,a,\mu)$, $y^* = y^*(\beta,a,\mu)$ satisfying $x^*(\beta,0,\mu) = 0$, $y^*(\beta,0,\mu) = 0$, $\partial x^*(1,0,0)/\partial a = 0$, $\partial y^*(1,0,0)/\partial a = 0$ and the equations

$$\begin{aligned} \dot{x} - Ax &= (I-\bar{E})[(\beta-1)A(a\varphi+x) + \beta X(a\varphi+x,y,\mu) \\ \dot{y} - By &= (\beta-1)By + \beta Y(a\varphi+x,y,\mu). \end{aligned} \tag{4.6}$$

The function z^* will satisfy (4.1) if and only if $\bar{E}[(\beta-1)A(a\varphi+x^*) + \beta X(a\varphi+x^*,y^*,\mu)]=0$. This latter equation is equivalent to

$$G(\beta,a,\mu) \overset{\mathrm{def}}{=} \int_0^{2\pi} e^{-At}[(\beta-1)A(a\varphi(t)+x^*(t))+\beta X(a\varphi(t)+x^*(t),y^*(t),\mu)]dt = 0. \tag{4.7}$$

This represents two equations in the parameters (β,a,μ). However, it is easy to see that one of these equations can always be solved for β (this determines the period) as a function of a,μ. In fact a simple calculation shows that $e^{-At}A\varphi(t) = (0,-1)$. Thus, if $G = (G_1,G_2)$ in (4.7), then $G_2(\beta,a,\mu)$ satisfies $G_2(\beta,0,\mu) = 0$, $\partial G_2(1,0,0)/\partial a = 0$ and $\partial^2 G_2(1,0,0)/\partial a\partial\beta = -1$. This implies that the function $G_2(\beta,a,\mu)/a$ has a unique solution $\beta = \beta^*(a,\mu)$ in a neighborhood of $(\beta,a,\mu) = (1,0,0)$, $\beta^*(0,0) = 1$. Thus, equations (4.7) are equivalent to the scalar equation

$$G(a,\mu) = 0$$
$$G(a,\mu) = G_1(\beta^*(a,\mu),a,\mu). \tag{4.8}$$

We refer to the function $G(a,\mu)$ as the <u>bifurcation function for periodic orbits of (4.1) near</u> $(z,\mu) = (0,0)$.

Summarizing the above remarks, we see there is a periodic orbit of (4.1) near $(z,\mu) = (0,0)$ if and only if

$$z = (a + x^*(\beta^*(a,\mu),a,\mu), y^*(\beta^*(a,\mu),a,\mu)) \tag{4.9}$$

where $x^*(\beta,a,\mu), y^*(\beta,a,\mu)$ satisfy (4.6), and (a,μ) satisfy (4.8). The period of z is $2\pi/\beta^*(a,\mu)$. All functions can be approximated to any accuracy desired by using successive approximations.

It is useful to observe and not difficult to show that $\beta^*(a,\mu) = \beta^*(-a,\mu)$ and $G(a,\mu) = -G(-a,\mu)$ for all a.

Even though the function $G(a,\mu)$ was constructed only to obtain periodic orbits, it contains also information about the dynamic behavior of (4.1). To see this, we recall that there is a two dimensional center manifold of (4.1) given by $y = \psi(x,\mu)$. The flow on the center manifold is given by the equation

$$\dot{x} = Ax + X(x,\psi(x,\mu),\mu). \tag{4.10}$$

Any periodic orbit of (4.1) must lie on this center manifold and is in one-to-one correspondence with the periodic orbits of (4.10). Formulas (4.9) also give a one-to-one correspondence of the periodic orbits of (4.1) with the zeros of $G(a,\mu)$. One can now state

<u>Theorem 4.1.</u> <u>Let</u> (a_0,μ) <u>satisfy</u> $G(a_0,\mu) = 0$ <u>and let</u> $\psi_\mu(t)$ <u>be the corresponding periodic orbit of</u> (4.10). <u>Then the stability properties of</u> $\psi_\mu(t)$ <u>as a solution of</u> (4.10) <u>are the same as the stability properties of the solution</u> a_0 <u>of the scalar equation</u>

$$\dot{a} = G(a,\mu). \tag{4.11}$$

<u>Proof</u>: We only indicate the proof. Since $x = 0$ is a solution of (4.10), we can introduce polar coordinates $x = (\rho\cos\theta, -\rho\sin\theta)$ and eliminate t to obtain

$$\frac{d\rho}{d\theta} = R(\theta,\rho,\mu) \tag{4.12}$$

where $R(\theta,0,\mu) = 0$, $\partial R(\theta,0,0)/\partial\rho = 0$. One can apply the LS method to this equation for 2π-periodic solutions to obtain

$$\rho = a + \rho^*(a,\mu), \qquad \int_0^{2\pi} \rho^*(a,\mu)(t)dt = 0$$

$$\dot{\rho}^* = R(\theta,a+\rho^*,\mu) - \frac{1}{2\pi}\int_0^{2\pi} R(\theta,a+\rho^*(\theta),\mu)d\theta$$

and the bifurcation function $\tilde{G}(a,\mu) = \frac{1}{2\pi}\int_0^{2\pi} R(\theta,a+\rho^*(a,\mu)(\theta),\mu)d\theta$. The 2π-periodic solutions of (4.12) are in one-to-one correspondence with the solutions of $\tilde{G}(a,\mu) = 0$. For the given function $\rho^*(a,\mu)$ make the change of variables $\rho = r + \rho^*(r,\mu)(\theta)$ in (4.12). Then

$$\frac{dr}{d\theta} = [1+ \frac{\partial\rho^*}{\partial r}(r,\mu)(\theta)]^{-1}\tilde{G}(r,\mu).$$

Since the coefficient of $\tilde{G}(r,\mu)$ is positive for r,μ small, it follows that the stability properties of the periodic orbits of (4.12) are the same as the corresponding equilibrium point of $\dot{r} = \tilde{G}(r,\mu)$. The remainder of the proof follows along the same lines of the proof of Theorem 3.1.

It is now easy to give interesting applications of Theorem 4.1. For example, suppose $\mu \in \mathbb{R}$ and

$$Z(z,\mu) = C_1(\mu)z + O(|z|^2)$$
$$C(\mu) = C + C_1(\mu)$$

and $C(\mu)$ has two eigenvalues $\alpha(\mu) \pm i\beta(\mu)$, $\alpha(0) = 0$, $\beta(0) = 1$, and $d\alpha(0)/d\mu \neq 0$. This latter condition is referred to as the Hopf condition and says that the curves $\{\alpha(\mu) \pm i\beta(\mu), \mu \text{ small}\}$ are transversal to the imaginary axis in the complex plane. Using the formula (4.7), one easily shows that

$$G(0,\mu) = 0, \ \partial G(0,0)/\partial\mu = 0, \ \partial^2 G(0,0)/\partial a\partial\mu = d\alpha(0)/d\mu.$$

Thus, the Implicit Function Theorem implies there is a unique function $\mu^*(a)$ defined in a neighborhood of zero, $\mu^*(0) = 0$, such that $a^{-1}G(a,\mu^*(a)) = 0$. Thus, for each a_0, equation (4.1) for $\mu = \mu^*(a_0)$ has a periodic orbit given by (4.9) with $\mu = \mu^*(a_0)$, $a = a_0$. The stability properties of this orbit are determined by the sign of $G(a,\mu^*(a))$ near a_0. The general situation is the same as the one in Fig. 3.1 if one assumes $\alpha(\mu) < 0$ for $\mu < 0$, $\alpha(\mu) > 0$ for $\mu > 0$. The result just stated is referred to as the Hopf Bifurcation Theorem.

The Hopf Bifurcation Theorem is a consequence of an hypothesis on the linear terms in z in (4.1). The specific form of the curve $\mu^*(a)$ depends on the nonlinear terms in a in the function $G(a,\mu)$. Since $G(a,\mu)$ is odd in a, the simplest situation is

$$G(a,\mu) = \gamma_0(\mu)a + \gamma_1(\mu)a^3 + \cdots$$
$$\gamma_1(0) \neq 0.$$

If we again assume μ is a scalar and $\gamma_0'(0) = \alpha'(0)$, where $\alpha(\mu)$ is the same as before, the curve $\mu^*(a)$ is given approximately by $\mu^*(a) = [\alpha'(0)]^{-1}\gamma_1(0)a^2 + \cdots$ and there is a unique periodic orbit near $a = 0$. This parabola opens to the left (right) if $[\alpha'(0)]^{-1}\gamma_1(0) < 0$ (> 0). Now suppose Re $\lambda B < 0$ and $\alpha'(0) > 0$. Then the solution $z = 0$ of (4.1) is stable for $\mu < 0$ and unstable for $\mu > 0$. If $\gamma_1(0) < 0$, then the periodic orbit exists for $\mu < 0$ and is unstable (this is called the subcritical case). If $\gamma_1(0) > 0$, the periodic orbit exists for $\mu > 0$ and is stable (this is called the supercritical case). This result is referred to as the <u>generic Hopf bifurcation theorem</u>.

For one parameter families of vector fields, the generic Hopf bifurcation is typical in the sense that the parameter is used to make the coefficient of a in the Taylor expansion of $G(a,\mu)$ vanish at some value of the parameter. For two parameter families of vector fields, it is possible to use the parameters μ to make the first and third terms in the Taylor expansion vanish at say $\mu = 0$. It is then natural to assume that $G(a,0) = \beta a^5 + \cdots$, $\beta \neq 0$. The analysis of the number of periodic orbits (two at most) and their stability is easily discussed as above. The one and two parameter bifurcation diagrams for $\beta a^3 + \mu a + \cdots$ and $\beta a^5 + \mu_1 a^3 + \mu_2 + \cdots$ are illustrated, respectively, in Figures 4.1 and 4.2 with the number of periodic orbits labeled. The coordinate a represents the amplititude of the orbit.

We remark that the same type of proof as given for Theorem 3.1 for the ordinary differential equation (4.1) holds as well for the parabolic equations and functional differential equations mentioned in Section 1.

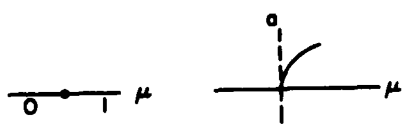

Figure 4.1

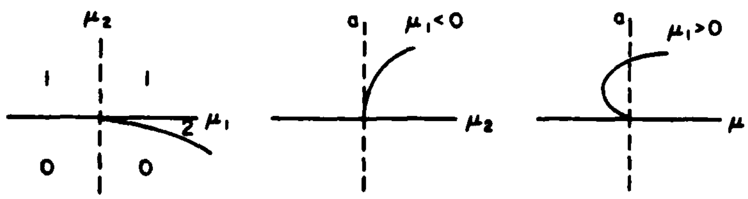

Figure 4.2

5. <u>Homoclinic and heteroclinic orbits</u>. Suppose $g \in C^k(\mathbb{R}^n,\mathbb{R}^n)$, $k \geq 1$, and the equation

$$\dot{x} = g(x) \tag{5.1}$$

has an orbit Γ connecting an hyperbolic equilibrium point x_0 to an hyperbolic equilibrium point x_1; that is, $g(x_0) = 0 = g(x_1)$, $\text{Re}\,\sigma(g_x(x_0)) \neq 0$, $\text{Re}\,\sigma(g_x(x_1)) \neq 0$, and there is a nonconstant solution $p(t)$ of (5.1) such that $p(t) \to x_0$ as $t \to -\infty$,

$p(t) \to x_1$ as $t \to +\infty$. The orbit Γ is called an heteroclinic orbit if $x_0 \neq x_1$ and an homoclinic orbit if $x_0 = x_1$. If E is a Banach space, $h \in C^k(\mathbb{R} \times \mathbb{R}^k \times E, \mathbb{R}^n)$, $k \geq 1$, $h(t,x,0) = 0$, and h is bounded in t, then the problem to be discussed in this section is to determine the behavior of the solutions of the equation

$$\dot{x} = g(x) + h(t,x,\mu) \tag{5.2}$$

for $t \in \mathbb{R}$ and (x,μ) in a neighborhood of $\Gamma \times \{0\}$.

Since x_0, x_1 are hyperbolic, there are solutions $x_0(t,\mu)$, $x_1(t,\mu)$, bounded for $t \in \mathbb{R}$, existing for μ small such that $x_0(t,0) = x_0$, $x_1(t,0) = x_1$ and $x_0(t,\mu)$, $x_1(t,\mu)$ are hyperbolic. Let $\mathscr{U}_{0\mu}(\mathscr{U}_{1\mu}) \subset \mathbb{R} \times \mathbb{R}^n$ be the unstable manifold for $x_0(\cdot,\mu)$ $(x_1(\cdot,\mu))$ and $\mathscr{S}_{0\mu}(\mathscr{S}_{1\mu})$ the corresponding stable manifolds. For $\mu = 0$, the hypothesis that Γ connects x_0 to x_1 implies $\mathscr{U}_{00} \cap \mathscr{S}_{10} \neq \phi$. In fact, since the equation for $\mu = 0$ is autonomous, this intersection contains an orbit of (5.1). We want to determine the nature of the set $\mathscr{U}_{0\mu} \cap \mathscr{S}_{1\mu}$ for $\mu \neq 0$.

Several problems motivate this type of investigation. For example, if the perturbation $h(t,x,\mu)$ is independent of t, then $x_0(\mu)$, $x_1(\mu)$ are constant and one often wants to determine conditions which ensure there is an orbit connecting $x_0(\mu)$ to $x_1(\mu)$. This is the typical problem of traveling waves in parabolic equations.

If $x_0 = x_1$ and the perturbations are periodic in t of period τ, and $U_{t\mu} = \{x : (t,x) \in \mathscr{U}_{0\mu}\}$, that is $U_{t\mu}$ is the cross section of $\mathscr{U}_{0,\mu}$ at t, then $U_{t,\mu} = U_{t+\tau,\mu}$. Similarly, the cross section $S_{t\mu}$ of $\mathscr{S}_{0\mu}$ is τ-periodic in t. If $\mathscr{U}_{0\mu} \cap \mathscr{S}_{0\mu} \neq \phi$, then there must be a point q homoclinic to the point $x_0(0,\mu)$ for the Poincaré map π, which takes points in $\mathbb{R}^n$ to the value of the solution of (5.2) at time τ. The sets $U_{0\mu}(S_{0\mu})$ are the stable (unstable) manifolds of $x_0(0,\mu)$ for the map π and have nonempty intersection q. If they intersect transversally at q, then it is well known that the dynamics near q can be described by the left shift automorphism on doubly infinite sequences on a finite number of symbols (see Section 6).

These two applications are sufficient motivation to investigate conditions on h to ensure that $\mathscr{U}_{0\mu} \cap \mathscr{S}_{0\mu}$ is nonempty. To carry out this investigation, we make a transformation from the continuous functions on $\mathbb{R}$ with range in a neighborhood of Γ to continuous functions on $\mathbb{R}$ with range in a neighborhood of zero. More specifically, let

$$x(t) = p(t+\alpha) + z(t+\alpha), \ \alpha \in \mathbb{R},$$

and choose $z(0)$ to be orthogonal to $\dot{p}(0)$. If $x(t)$ is a solution of (5.2) and t is replaced by $t - \alpha$, then $z(t)$ satisfies

$$\begin{aligned} &\dot{z} = A(t)z + f(t,z,\mu,\alpha) \\ &A(t) = g_x(p(t)), \\ &f(t,z,\mu,\alpha) = g(p(t)+z) - g_x(p(t))z - g(p(t)) + h(t-\alpha,p(t)+z,\mu) \end{aligned} \tag{5.3}$$

We consider (5.3) in a neighborhood of $(z,\mu) = (0,0)$ as a perturbation of the linear equation

$$(Lx)(t) = 0 \qquad Lx = d/dt - A(\cdot). \tag{5.4}$$

From the definition of $A(t)$, we have $A(t) \to A^+ = g_x(x_1)$ as $t \to +\infty$, $A(t) \to A^- = g_x(x_0)$ as $t \to -\infty$ with $\mathrm{Re}\,\sigma(A^+) \neq 0$, $\mathrm{Re}\,\sigma(A^-) \neq 0$.

Note that $\dot{p}(t)$ is a nontrivial solution of (5.4), bounded on $\mathbb{R}$.

We consider the operator L as a continuous linear operator from $C_b^1(\mathbb{R},\mathbb{R}^n)$ to $C_b^0(\mathbb{R},\mathbb{R}^n)$. Our first objective is to characterize $\mathcal{N}(L)$, $\mathcal{R}(L)$. To do this, the following concept is useful.

Let $X(t,s)$, $X(t,t) = I$, be the principal matrix solution (solution operator) of (5.4). Equation (5.4) is said to have an <u>exponential dichotomy on an interval</u> J with constants K,α if there are projections $P(s)$, $s \in J$, continuous in s, such that, if $Q(s) = I - P(s)$, then

$$\begin{aligned} &\text{(i)} \quad X(t,s)P(s) = P(t)X(t,s) && t, s \in J. \\ &\text{(ii)} \quad |X(t,s)P(s)| \le Ke^{-\alpha(t-s)}, && t \ge s \text{ in } J \\ &\text{(iii)} \quad |X(t,s)Q(s)| \le Ke^{-\alpha(s-t)}, && s \ge t \text{ in } J. \end{aligned} \tag{5.5}$$

The operator $P(t)$ is called the <u>projection matrix function of the dichotomy</u>.

This concept is equivalent to the existence of a projection P_0 and constants K,α such that

$$\begin{aligned} |X(t,0)P_0X^{-1}(s,0)| &\le Ke^{-\alpha(t-s)}, \quad t \ge s \text{ in } J \\ |X(t,0)Q_0X^{-1}(s,0)| &\le Ke^{-\alpha(s-t)}, \quad s \ge t \text{ in } J. \end{aligned} \tag{5.6}$$

In fact, if (5.5) is satisfied, let $P_0 = P(0)$ and observe that (i) implies

$$X(t,0)P_0X^{-1}(s,0) = X(t,0)X^{-1}(s,0)P(s) = X(t,s)P(s).$$

Thus, (5.5) implies (5.6). Conversely, if (5.6) is satisfied and one defines $P(t) = X(t,0)P_0X^{-1}(t,0)$, then one easily verifies (5.5).

<u>Remark</u>. Note that the projection $P(t)$ is uniquely determined if $J = \mathbb{R}$, but is not unique in other cases. Also, if J is finite, then there is always an exponential dichotomy on J. One can choose any projection P_0 on $\mathbb{R}^n$ and define $P(t) = X(t,0)P_0X^{-1}(t,0)$.

<u>Remark</u>. Definition (5.5) may be modified to apply to dynamical systems in infinite dimensional spaces for which the solution operator $X(t,s)$ is only defined for $t \ge s$. Condition (i) is replaced by

(i)' $X(t,s)P(s) = P(t)X(t,s)$, $t \ge s$ in J.

(i)" The restriction $T(t,s)\mathcal{R}(Q(s))$, $t \ge s$, is an isomorphism of $\mathcal{R}(Q(s))$ onto $\mathcal{R}(Q(t))$ and we define $T(s,t)$ as the inverse mapping.

The most interesting cases for the interval J in an exponential dichotomy are

$\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (-\infty,0]$ and $\mathbb{R}$.

Relation (5.5) has a very simple geometric interpretation. In fact, if $J = \mathbb{R}_+$, then, for each fixed s, there is a finite dimensional subspace of $\mathbb{R}^n$ given by $\mathscr{R}(P(s))$ and called the stable subspace at S such that solutions with initial values in $\mathscr{R}(P(s))$ at s tend to zero uniformly and exponentially as $t \to \infty$. If $J = \mathbb{R}_-$ then $\mathscr{R}(Q(s))$ is the unstable manifold at s with solutions through points here tending to zero uniformly and exponentially as $t \to -\infty$. The fact that $|P(t)| \leq K$ for all t implies the angle between the subspaces $\mathscr{R}(P(t))$, $\mathscr{R}(Q(t))$ is bounded. The angle $\alpha(Y,Z)$, $0 \leq \alpha(Y,Z) \leq \pi/2$, between two subspaces Y,Z in $\mathbb{R}^n$, $Y \cap Z = \{0\}$, $Y,Z \neq \{0\}$ is defined as

$$\cos \alpha(Y,Z) = \sup\left\{\frac{|y\cdot z|}{|y||z|} : y \in Y \setminus \{0\},\ z \in Z \setminus \{0\}\right\}.$$

If A is a constant $n \times n$ matrix with $\mathrm{Re}\,\sigma(A) \neq 0$, then $\dot{x} - Ax = 0$ has an exponential dichotomy on $\mathbb{R}$ with projection $P_A = P(0)$ given by

$$I - P_A = \frac{1}{2\pi i}\int_\gamma (\lambda I - A)^{-1} d\lambda$$

where γ is a closed curve in $\mathbb{C}$ enclosing the eigenvalues of A with positive real parts.

We need several fundamental lemmas. The first one is elementary and stated without proof.

Lemma 5.1. Let A(t) be an $n \times n$ matrix function defined and continuous on $\mathbb{R}$. Then the equation (5.4) has an exponential dichotomy on $\mathbb{R}$ if and only if it has an exponential dichotomy on both $[0,\infty)$ and $(-\infty,0]$ and $\mathbb{R}^n$ in the direct sum of the stable and unstable subspaces at zero.

Lemma 5.2. Let J be either $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$. If $\dot{x} = A(t)x$ has an exponential dichotomy on J and B(t) is a continuous $n \times n$ matrix function on J with $|B(t)| \leq \delta$, then

$$\dot{x} = (A(t) + B(t))x \tag{5.7}$$

has an exponential dichotomy on J if δ is sufficiently small.

Sketch of Proof. Consider first $J = \mathbb{R}_+$. For any $x \in \mathbb{R}^n$ and any $y \in C_b^0(\mathbb{R}_+,\mathbb{R}^n)$, define

$$(\mathscr{T}y)(t) = X(t,s)P(s)x + \int_s^t X(t,u)P(u)B(u)y(u)du - \int_t^\infty X(t,u)Q(u)B(u)y(u)du.$$

The operator $\mathscr{T}y$ is motivated by the following consideration. If $f \in C_b^0(\mathbb{R}_+,\mathbb{R}^n)$, then one can show that the solutions x(t) of the nonhomogeneous equation $\dot{x} = A(t)x + f(t)$ which are bounded on $\mathbb{R}_+$ must be given by

$$x(t) = X(t,s)P(s)x + \int_s^t X(t,u)P(u)f(u)du - \int_t^\infty X(t,u)Q(u)f(u)du.$$

For δ sufficiently small, it is easy to show that the operator $\mathcal{T}$ has a unique fixed point in $C_b^0(\mathbb{R},\mathbb{R}^n)$ which is a solution of (5.7) on $\mathbb{R}_+$, is continuous in t,s,x and, for each fixed t,s, is linear in x. If the value of this fixed point at t = s is denoted by $\tilde{P}(s)x$, then $\tilde{P}(s) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous linear operator. If $X_B(t,s)$ is the principal matrix solution of (5.7), then the fixed point of $\mathcal{T}$ is given by $X_B(t,s)\tilde{P}(s)x$. We will show that (5.7) has an exponential dichotomy on $\mathbb{R}_+$ with projection matrix $\tilde{P}(s)$.

We need several elementary observations to prove that $\tilde{P}(s)$ is a projection. Since $P^2(s) = P(s)$ and the fixed point of $\mathcal{T}$ is unique, it follows that $\tilde{P}(s)P(s) = \tilde{P}(s)$. From the definition of $X_B(t,s)\tilde{P}(s)x$ and $\tilde{P}(s)x$, we have

$$\tilde{P}(s)x = P(s)x - \int_s^\infty X(s,u)Q(u)B(u)X_B(u,s)\tilde{P}(s)x\,du.$$

Operate with $X(t,s)P(s)$ to obtain $X(t,s)P(s)\tilde{P}(s)x = X(t,s)P(s)x$. For t = s, this implies $P(s)\tilde{P}(s) = P(s)$ for all s. Operating on this last relation with $\tilde{P}(s)$ and using the fact that $\tilde{P}(s)P(s) = \tilde{P}(s)$, one obtains $\tilde{P}^2(s) = \tilde{P}(s)$ and $\tilde{P}(s)$ is a projection.

Let $\tilde{Q}(s) = I - \tilde{P}(s)$. Using the fact that $X_B(t,s)\tilde{P}(s)x$ is a fixed point of $\mathcal{T}$, the variation of constants formula for $X_B(t,s)$ and the fact that $X(t,s)P(s) = P(t)X(t,s)$, $\tilde{P}(t)P(t) = \tilde{P}(t)$, $\tilde{P}(t)[I-P(t)] = 0$, one obtains $\tilde{P}(t)X_B(t,s)\tilde{P}(s)x = \tilde{P}(t)X_B(t,s)x$. This implies $\tilde{P}(t)X_B(t,s)\tilde{Q}(s)x = 0$ for all x. But then the fact that $X_B(t,s)\tilde{P}(s)x \in \mathcal{R}(\tilde{P}(t))$ implies that

$$\begin{aligned} X_B(t,s)\tilde{P}(s)x &= \tilde{P}(t)X_B(t,s)\tilde{P}(s)x + \tilde{Q}(t)X_B(t,s)\tilde{P}(s)x = \\ &= \tilde{P}(t)X_B(t,s)\tilde{P}(s)x = \tilde{P}(t)X_B(t,s)x. \end{aligned}$$

Thus, $X_B(t,s)\tilde{P}(s) = \tilde{P}(t)X_B(t,s)$ for all t,s and (i) in (5.5) is satisfied.

After a few computations, one obtains

$$\begin{aligned} &X_B(t,s)\tilde{Q}(s)x \\ &= X(t,s)Q(s)\tilde{Q}(s)x + \int_s^t X(t,u)P(u)B(u)X_B(u,s)\tilde{Q}(s)x - \int_t^s X(t,u)Q(u)B(u)X_B(u,s)\tilde{Q}(s)ds. \end{aligned}$$

If $\theta = \alpha^{-1}K\delta < 1/2$, $\beta = \alpha(1-2\theta)^{1/2}$, $\rho = \theta^{-1}[1-(1-2\theta)^{1/2}]$, then one can obtain the following estimate (it is nontrivial):

$$|X_B(t,s)\tilde{P}(s)| \le \rho K e^{-\beta(t-s)}, \quad t \ge s \ge 0.$$

$$|X_B(t,s)\tilde{Q}(s)| \le \rho K e^{-\beta(s-t)}, \quad s \ge t \ge 0.$$

Thus, estimates (ii), (iii) hold. This proves the lemma for $J = \mathbb{R}_+$. The same type of argument applies to $\mathbb{R}_-$ and $\mathbb{R}$.

Remark. In the proof of Lemma 5.2, one obtains the estimate

$$|\tilde{P}(s) - P(s)| \le \delta\rho K^2/(\alpha+\beta)$$

which $\to 0$ as $\delta \to 0$ uniformly in s.

We state the following lemma without proof since it is similar to the proof of Lemma 5.2.

Lemma 5.3. If $A(t) \to A^{\pm}$ as $t \to \pm\infty$, $\mathrm{Re}\,\sigma(A^{\pm}) \neq 0$, then (5.4) has an exponential dichotomy on $\mathbb{R}_+$ ($\mathbb{R}_-$) with projection matrix $P^+(t)$ ($P^-(t)$) satisfying $P^{\pm}(t) \to P^{\pm}$ as $t \to \pm\infty$.

Lemma 5.4. Let $A(t)$ be an $n \times n$ matrix function, bounded and continuous on $\mathbb{R}$ such that equation (5.4) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projection matrices $P^+(t), P^-(t)$, respectively. Then $L : C_b^1(\mathbb{R},\mathbb{R}^n) \to C_b^0(\mathbb{R},\mathbb{R}^n)$ is Fredholm of index $\dim \mathcal{R}(P^+(0)) + \dim \mathcal{R}(P^-(0)) - n$,

$$\mathcal{N}(L) = \left\{ g \in C_b^1(\mathbb{R},\mathbb{R}^n) : \dot g(t) - A(t)g(t) = 0,\ t \in \mathbb{R},\ g(0) \in \mathcal{R}(P^+(0)) \cap \mathcal{R}(I - P^-(0)) \right\}$$

$$\mathcal{R}(L) = \left\{ f \in C_b^0(\mathbb{R},\mathbb{R}^n) : \int_{-\infty}^{\infty} \psi^*(t) f(t)\,dt = 0,\ \ * = \text{transpose}, \right.$$

$$\left. \text{for all } \psi \in C_b^1(\mathbb{R},\mathbb{R}^n) \text{ satisfying the adjoint equation } \dot x + A^*(t)x = 0 \right\}.$$

Remark. If $A(t)$ satisfies the conditions of Lemma 5.3 and P_{A^+}, P_{A^-} are the projection operators respectively for the dichotomies of $\dot x - A_+ x = 0$, $\dot x - A_- x = 0$, then $\dim \mathcal{R}(P_{A^+}) = \dim \mathcal{R}(P^+(0))$, $\dim \mathcal{R}(P_{A^-}) = \dim \mathcal{R}(P^-(0))$ and L in (5.4) has index $\dim \mathcal{R}(P_{A^+}) + \dim \mathcal{R}(P_{A^-}) - n$.

Sketch of proof of Lemma 5.4. If $X(t,s)$ is the solution operator of (5.4), then $X^{-1}(t,s) = X(s,t)$ is the solution operator of the adjoint equation

$$(L^* x)(t) = 0, \qquad L^* = d/dt + A^*(\cdot). \tag{5.8}$$

This implies (5.8) has an exponential dichotomy on $\mathbb{R}_+$ with projection matrix $Q^{+*}(t) = I - P^{+*}(t)$ and on $\mathbb{R}_-$ with projection matrix $Q^{-*}(t) = I - P^{-*}(t)$. The fact that $\mathcal{N}(L) = \left\{ g \in C_b^1(\mathbb{R},\mathbb{R}^n) : \dot g(t) - A(t)g(t) = 0,\ t \in \mathbb{R},\ g(0) \in V \cap W \right\}$ where $V = \mathbb{R}(P^+(0))$, $w = \mathbb{R}(Q^-(0))$, $Q^- = I - P^-$, is clear. Also, $\mathcal{N}(L^*) = \left\{ g \in C_b^1(\mathbb{R},\mathbb{R}^n) : \dot g(t) + A^*(t)g(t) = 0,\ t \in \mathbb{R},\ g(0) \in V^{\perp} \cap W^{\perp} \right\}$ where $V^{\perp} = \mathcal{R}(Q^{+*}(0))$, $W^{\perp} = \mathcal{R}(P^{-*}(0))$.

If $f \in \mathcal{R}(L)$, then straightforward calculations show that $\int_{-\infty}^{\infty} \psi^*(t) f(t)\,dt = 0$ for all $\psi \in \mathcal{N}(L^*)$. Conversely, suppose f satisfies this orthogonality condition for all $\psi \in \mathcal{N}(L^*)$. A solution of $Lx(t) = f(t)$, $f \in C^0(\mathbb{R},\mathbb{R}^n)$, is bounded on $\mathbb{R}$ if and only if there is a $\xi \in \mathbb{R}^n$ such that

$$x(t) = X(t,0)P^+(0)\xi + \int_0^t X(t,s)P^+(s)f(s)\,ds - \int_t^{\infty} X(t,s)Q^+(s)f(s)\,ds, \quad t \geq 0$$

$$x(t) = X(t,0)Q^-(0)\xi + \int_0^t X(t,s)Q^-(0)f(s)\,ds + \int_{-\infty}^t X(t,s)P^-(s)f(s)\,ds, \quad t \leq 0$$

that is, if and only if

$$[P^+(0) - Q^-(0)]\xi = \int_{-\infty}^0 X^{-1}(s,0)P^-(s)f(s)\,ds + \int_0^{\infty} X^{-1}(s,0)Q^+(s)f(s)\,ds, \quad \text{or}$$

$$[P^+(0)-Q^-(0)]\xi = \int_{-\infty}^{0} P^-(0)X^{-1}(s,0)f(s)ds + \int_0^{\infty} Q^+(0)X^{-1}(s,0)f(s)ds.$$

For this equation to have a solution, one must have the right hand side orthogonal to all vectors $\eta \in \mathbb{R}^n$ such that $\eta^*[P^+(0)-Q^-(0)] = 0$; that is, $P^{+*}(0)\eta = Q^{-*}(0)\eta$. But, $\psi \in \mathcal{N}(L^*)$ if and only if

$$\begin{aligned}\psi(t) &= X^{*-1}(t,0)(I-P^{+*}(0))\eta \quad t \geq 0\\ &= X^{*-1}(t,0)P^{-*}(0)\eta \qquad t \leq 0\end{aligned}$$

with $P^{+*}(0)\eta = Q^{-*}(0)\eta$. This proves that $\mathcal{R}(L)$ is as stated in the lemma.

Thus L is Fredholm of index $\dim\mathcal{N}(L) - \dim(V^\perp \cap W^\perp)$. But $\dim(V^\perp \cap W^\perp) = n-\dim(V+W) = n-\dim V - \dim W - \dim V \cap W$. This proves the lemma.

We can now use Lemma 5.4 to apply the LS method to obtain the bifurcation equations for bounded solutions of (5.3). In fact, let $\Phi = (\dot{p},\varphi_2,\ldots,\varphi_q)$ be a basis for $\mathcal{N}(L)$. Since $z(0)$ is required to be orthogonal to $\dot{p}(0)$, this implies the projection of z onto $\mathcal{N}(L)$ must have value at zero given by $\Phi(0)b$, $b = (0,a)$, $a \in \mathbb{R}^{q-1}$. If we let $\Psi = (\psi_1,\ldots,\psi_p)$ be a basis for $\mathcal{N}(L^*)$, $(L^*\psi)(t) = \dot{x}(t) - A^*(t)x(t)$, $D = \int_{-\infty}^{\infty} \Psi^*(t)\Psi(t)dt$, $(I-E)f = \Psi D^{-1}\int_{-\infty}^{\infty} \Psi^*(t)f(t)dt$ for $f \in C_b^0(\mathbb{R},\mathbb{R}^n)$, then E is a continuous projection on $C_b^0(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{R}(L) = \mathcal{R}(E)$. Fix $a \in \mathbb{R}^{q-1}$ and apply the LS method to obtain a function $z(a,\mu,\alpha)$ in $C_b^1(\mathbb{R},\mathbb{R}^n)$ satisfying the equation

$$\begin{aligned}&\dot{z} - A(t)z = Ef(\cdot,z,\mu,\alpha)\\ &z^*(a,0,\alpha)(0) = \Phi(0)b, \quad b = (0,a).\end{aligned} \tag{5.9}$$

This function $z(a,\mu,\alpha)$ will be a solution of (5.3) if and only if (a,μ,α) satisfy $(I-E)f(\cdot,z(a,\mu,\alpha),\mu,\alpha) = 0$ which is equivalent to

$$\begin{aligned}&G(a,\mu,\alpha) = 0\\ &G(a,\mu,\alpha) = \int_{-\infty}^{\infty} \Psi^*(t)f(t,z(a,\mu,\alpha)(t),\mu,\alpha)dt = 0.\end{aligned} \tag{5.10}$$

For a fixed μ, this represents p equations for the q parameters (a,α).

As an illustration, consider the equation

$$\begin{aligned}&\dot{x}_1 = x_2\\ &\dot{x}_2 = -g(x_1) - \mu_1 x_2 + \mu_2 f(t)\end{aligned} \tag{5.11}$$

where $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$ is a parameter, $f(t) = f(t+1)$ is a continuous function, $g(0) = 0$, $g'(0) < 0$. These conditions imply that zero is a saddle point for the system

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -g(x_1). \tag{5.12}$$

Suppose there is an orbit $\Gamma = \{(p(t),\dot{p}(t)),\ t \in \mathbb{R}\}$ of (5.12) such that

$(p(t),\dot{p}(t)) \to (0,0)$ as $t \to \pm\infty$; that is, Γ is an homoclinic orbit through $x = 0$.

Since zero is a saddle point for (5.12), there is a unique hyperbolic periodic solution $\varphi(t,\mu)$ of (5.11) of period 1 for $|\mu|$ small, $\varphi(t,0) = 0$. Our objective is to give necessary and sufficient conditions on (5.11) in order that there is a homoclinic point to $\varphi(0,\mu)$ in a small neighborhood of $\{0\} \times \Gamma$ for μ in a small neighborhood of zero. From the definition, we must determine a solution x of (5.11) which remains in a small neighborhood of Γ for μ near zero with the property that $x(t) - \varphi(t,\mu) \to 0$ as $t \to \pm\infty$. The trajectory $\{(t,\varphi(t,\mu)), t \in \mathbb{R}\}$ being a hyperbolic saddle implies that we need only look for solutions of (5.11) which remain in a small neighborhood of $\mathbb{R} \times \Gamma$ for μ near zero. Thus, for any solution $x(t)$ of (5.11), we let $x(t) = p(t+\alpha) + z(t+\alpha)$, replace t by $t - \alpha$ to obtain the equation

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -g'(p(t))z_1 - \mu_1 z_2 - \mu_1\dot{p}(t) + \mu_2 f(t-\alpha) - g(p(t)+z_1) + g(p(t)) + g'(p(t))z_1. \end{aligned} \tag{5.13}$$

One can now apply the previous theory for small solutions $z(t)$ of (5.13) for μ in a neighborhood of zero.

The linear variational equation around Γ is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -g'(p(t))x_2. \tag{5.14}$$

The only solutions of this equation which are bounded on $\mathbb{R}$ must be multiples of $(\dot{p}(t),\ddot{p}(t))$. This is proved by using the fact that a principal matrix solution has determinant equal to one for all t. The adjoint equation

$$\dot{x}_1 = g'(p(t))x_2, \quad \dot{x}_2 = -x_1$$

has the solution $(-\ddot{p}(t),\dot{p}(t))$ bounded on $\mathbb{R}$ and all other solutions bounded on $\mathbb{R}$ are multiples of this one. In the terminology of the general setting for equation (5.1), (5.4), we have shown that $\mathscr{N}(L)$ is spanned by $(\dot{p}(t),\ddot{p}(t))$, $\mathscr{N}(L^*)$ is spanned by $(-\ddot{p}(t),\dot{p}(t))$ and L has index zero. Thus, when we apply the LS method, $q = p = 1$ and the vector a is not needed in (5.9). The bifurcation function in (5.10) is given approximately by

$$G(\mu,\alpha) = \int_{-\infty}^{\infty} \dot{p}(t)[-\mu_1\dot{p}(t)+\mu_2 f(t-\alpha)]dt + o(|\mu|) \tag{5.15}$$

as $\mu \to 0$. This says that μ_1/μ_2 should be given approximately by

$$\begin{aligned} \eta \frac{\mu_1}{\mu_2} &= h(\alpha) \\ \eta = \int_{-\infty}^{\infty} \dot{p}^2(t)dt, &\quad h(\alpha) = \int_{-\infty}^{\infty} \dot{p}(t)f(t-\alpha)dt. \end{aligned} \tag{5.16}$$

The function $h(\alpha)$ is periodic of period 1. If $\eta\mu_1/\mu_2$ is given and satisfies $\min_\alpha h(\alpha) < \eta\mu_1/\mu_2 < \max_\alpha h(\alpha)$, and $h(\alpha_0) = \eta\mu_1/\mu_2$, then, if μ is sufficiently small there always exist an $\alpha(\mu),\alpha(0) = \alpha_0$, such that equation (5.11) has a solution which approaches the periodic solution $\varphi(t,\mu)$ as $t \to \pm\infty$. This value $\alpha(\mu)$ corresponds to the initial data near $(p(\alpha(\mu)),\dot{p}(\alpha(\mu)))$ for a solution x of (5.11) which lies on the stable and unstable manifolds of the periodic solution $\varphi(t,\mu)$ of (5.11) near zero. Furthermore, one can show that the intersection of these manifolds is transversal if $h'(\alpha_0) \neq 0$. For the Poincaré map π, this implies that the flow has a behavior similar to the one shown in Fig. 5.1, where U_μ, S_μ are the unstable and stable manifolds of the fixed point P_μ of π near zero.

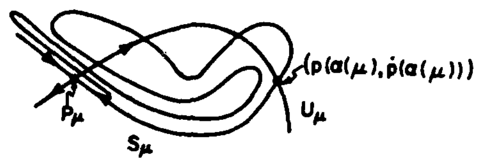

Figure 5.1

In the next section, we discuss the implications of the existence of a transverse homoclinic point.

If we make some further hypotheses on the function $h(\alpha)$ in (5.16), we can discuss the existence of homoclinic orbits for a full neighborhood of $\mu = (\mu_1,\mu_2) = (0,0)$. Suppose $h(\alpha)$ has an absolute maximum (minimum) at $\alpha_M(\alpha_m)$ and

$$h''(\alpha_M) < 0, \qquad h''(\alpha_m) > 0. \tag{5.17}$$

Then one can prove that there are C^1-curves C_M, C_m in μ-space with tangents at $\mu = 0$ respectively given by $h(\alpha_M)/\eta$, $h(\alpha_m)/\eta$ which divide a neighborhood of $\mu = 0$ into sectors S_1,S_2 as in Fig. 5.2, such that there are no homoclinic orbits in S_1 and homoclinic orbits in S_2.

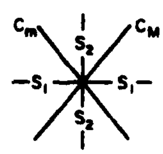

Figure 5.2

6. <u>Transverse homoclinic points</u>. To say more about the flow near a transverse homoclinic point, we need some further results on exponential dichotomies. We consider only periodic systems

$$\dot{x} = f(t,x) \tag{6.1}$$

where $f(t,x)$ is T-periodic in t. If π is the Poincaré map taking points $x_0 \in \mathbb{R}^n$ into the solution through X_0 at time T, then fixed points of π correspond to T-periodic solutions of (6.1). Let $\varphi(t)$ be a T-periodic solution of (6.1). It is hyperbolic if no characteristic exponents of the linear variational equation

$$\dot{x} = f_x(t,\varphi(t))x \tag{6.2}$$

have zero real parts. This is equivalent to the statement that no eigenvalues of

$d\pi\varphi(0)/dx$ are on the unit circle. From the Floquet theory, it is clear that φ is hyperbolic if and only if (6.2) has an exponential dichotomy. Let $W^u(\varphi)$, $W^s(\varphi)$ be the cross section of the stable and unstable manifolds of φ at $t = 0$. Any $\xi \in W^u(\varphi) \cap W^s(\varphi)$ is a homoclinic point of $\varphi(0)$ and it is transversal if $W^u(\varphi)$ intersects $W^s(\varphi)$ transversally at ξ. If $\xi \in W^u(\varphi) \cap W^s(\varphi)$ and $\psi(t,\xi)$ is the solution of (6.1) through ξ at zero, then $\psi(t,\xi) - \varphi(t) \to 0$ as $t \to \pm\infty$. Lemma 5.3 implies that the equation

$$\dot{x} = f_x(t,\psi(t,\xi))x \tag{6.3}$$

has an exponential dichotomy on $\mathbb{R}_-$ and $\mathbb{R}_+$. Furthermore, Lemma 5.1 implies that (6.3) has an exponential dichotomy or $\mathbb{R}$ if and only if the stable and unstable subspace at $t = 0$ intersect transversally. We have proved the following

Proposition 6.1. If $\varphi(\cdot)$ is a hyperbolic T-periodic solution of (6.1), then ξ is a transversal homoclinic point of $\varphi(0)$ if and only if equation (6.3) has an exponential dichotomy.

We also need the following

Lemma 6.2. For each integer k, let $A_k(t)$ be a bounded continuous $n \times n$ matrix function such that the system

$$\dot{x} = A_k(t)x \tag{6.4}$$

has an exponential dichotomy on an interval $[t_{k-1},t_k]$ with constants K,α (independent of k) and projection matrix function $P_k(t)$. Let $A(t) = A_k(t)$, $t \in [t_{k-1},t_k]$. Also, suppose $|A_k(t)| \le M$ for all t,k. Then there exist $\tau_0 = \tau_0(K,\alpha)$, $\delta_0 = \delta_0(K,\alpha)$ such that the equation

$$\dot{x} = A(t)x \tag{6.5}$$

has an exponential dichotomy on $\mathbb{R}$ if $t_k - t_{k-1} \ge \tau_0$ and $|P_k(t_{k-1}) - P_{k-1}(t_{k-1})| \le \delta_0$.

Proof: Only the ideas will be given. The first step of the proof is to construct $n \times n$ matrix functions $B_k(t)$ which are close to $A_k(t)$ and such that $\dot{x} = B_k(t)x$ has an exponential dichotomy on $[t_{k-1},t_k]$ with constants $3K,\alpha$ and projection matrix function $R_k(t)$ with $R_k(t_{k-1}) = R_{k-1}(t_{k-1})$. If $B(t) = B_k(t)$ and $R(t) = R_k(t)$ on $[t_{k-1},t_k]$, and $Y(t,s)$ is the principal matrix solution of $\dot{x} = B(t)x$, then $Y(t,s)R(s) = R(t)Y(t,s)$. One then shows that $t_k - t_{k-1} \ge 2\alpha^{-1}\ln 3K$ implies there is an exponential dichotomy for this equation with constants $9K^2,\alpha/2$ and matrix $R(t)$. The equation (6.5) then can be considered as a perturbation of $\dot{x} = B(t)x$ to obtain an exponential dichotomy of (6.5).

Let $X_k(t,s)$ be the principal matrix solution of $\dot{x} = A_k(t)x$, $t_{k-1} \le t \le t_k$. We construct the $B_k(t)$ by finding a nonsingular transformation of variables $S_k(t)$ which is close to the identify such that $Y_k(t,0) = S_k(t)X_k(t,0)$ is a fundamental matrix of solutions for the equation $\dot{x} = B_k(t)x$ on $[t_{k-1},t_k]$ and let $R_k(t) = Y_k(t,0)P_k(0)Y_{k-1}^{-1}(t,0)$. The simplest form for $S_k(t)$ is

$$S_k(t) = I + (t_k-t_{k-1})^{-1}(t-t_{k-1})(J_{k+1}-I), \quad t_{k-1} \le t \le t_k,$$

where J_{k+1} is nonsingular. Using the fact that $X_k(t,0)P_k(0) = P_k(t)X_k(t,0)$, then one sees that $R_k(t_{k-1}) = R_{k-1}(t_{k-1})$ if and only if $P_k(t_{k-1})J_k = J_kP_{k-1}(t_{k-1})$ and J_k is nonsingular. The operator

$$J_k = P_k(t_{k-1})P_{k-1}(t_{k-1}) + (I-P_k(t_{k-1}))(I-P_{k-1}(t_{k-1}))$$

is a simple choice for J_k. To show J_k has an inverse, observe that

$$I - J_k = [P_k(t_{k-1}) - (I-P_k(t_{k-1}))][P_k(t_{k-1}) - P_{k-1}(t_{k-1})].$$

Thus, $|I-J_k| \le 1/2$ if $\delta_0 = 1/4K$ and J_k has an inverse.

With the notation above, let $B(t) = B_k(t)$, $R(t) = R_k(t)$, $t \in [t_{k-1},t_k)$ and let $Y(t,s)$, $Y(s,s) = I$, be the principal matrix solution of $\dot{x} = B(t)x$. We show that $\dot{x} = B(t)x$ has an exponential dichotomy with constants $9K^2,\alpha/2$ and projection matrix function $Q(t)$ if $t_{k-1} - t_k$ is $\ge 2\alpha^{-1}\log 3K$. Let $Q(0) = Q$. For $s \le t$, there are integers $k \le j$ such that $t_{k-1} \le s < t_k$, $t_{j-1} \le t < t_j$ and the following estimates hold

$$\begin{aligned} &|Y(t,s)Q(s)| \\ &\quad \le |Y(t,t_{j-1})Q(t_{j-1})| \prod_{i=k}^{j-2} |Y(t_{i+1},t_i)(Q(t_i)| \cdot |Y(t_k,s)Q(s)| \\ &\quad \le (3K)^{j-k+1}e^{-\alpha(t-s)} \le 9K^2e^{-\alpha(t-s)/2} \end{aligned}$$

since $t - s \ge t_{j-1} - t_k \ge (j-k-1)2\alpha^{-1}\log 3K$. If one supplies the details of these computations, the proof is complete.

Theorem 6.3. *Suppose* $f \in C_b^0(\mathbb{R} \times \mathbb{R}^n,\mathbb{R}^n)$, $f_x \in C_b^0(\mathbb{R} \times \mathbb{R}^n,\mathbb{R}^n)$ *and* f_x *is continuous in* x *uniformly in* t,x. *For each integer* k, *suppose that the system*

$$\dot{x} = f(t,x) \tag{6.6}$$

has a solution $w_k(t)$ *defined on* $[t_{k-1},t_k]$ *such that*

$$\dot{x} = f_x(t,w_k(t))x \tag{6.7}$$

has an exponential dichotomy on $[t_{k-1},t_k]$ *with constants* K,α *and projection matrix function* $P_k(t)$ *and the following conditions are satisfied*:

(i) $|w_{k-1}(t_{k-1}) - w_k(t_{k-1})| < \delta$

(ii) $|P_{k-1}(t_{k-1}) - P_k(t_{k-1})| < \delta$

(iii) $t_k - t_{k-1} \ge \tau$.

Then there are positive constants ϵ_0,τ_0 *and a function* $\delta_0(\epsilon)$ *such that, if* $\tau \ge \tau_0$,

<u>$0 < \varepsilon \le \varepsilon_0$ and $\delta \le \delta_0(\varepsilon)$, then</u> (6.6) <u>has a unique solution</u> $x(t)$ <u>satisfying</u> $|x(t)-w_k(t)| \le \varepsilon$ <u>for</u> $t_{k-1} \le t \le t_k$ <u>for all</u> k.

<u>Proof</u>: Only an outline of the proof is given. If $w(t) = w_k(t)$, $t \in [t_{k-1}, t_k]$, then w is continuous except with small jumps β_k at t_k. Lemma 6.2 implies the equation $\dot{x} = f_x(t,w(t))x$ has an exponential dichotomy on $\mathbb{R}$ if τ is large enough and δ is small enough. The next idea is to approximate $w(t)$ by a continuous function $z(t)$ by linear interpolation

$$z(t) = w(t) + (t_k-t_{k-1})^{-1}(t-s_k)\beta_{k-1} \quad \text{if} \quad t_{k-1} \le t \le s_k$$
$$z(t) = w(t) + (t_k-t_{k-1})^{-1}(t-s_k)\beta_k \quad \text{if} \quad s_k \le t < t_k$$

where $s_k = (t_{k-1}+t_k)/2$. Then $|z(t)-w(t)| \le \delta, |\dot{z}(t)-\dot{w}(t)| \le \tau^{-1}\delta$, except at the points t_k. Since $z(t)$ is close to $w(t)$ and $f_x(t,x)$ is continuous in x uniformly with respect to t,x, the equation $\dot{x} = f_x(t,z(t))x$ has an exponential dichotomy on $\mathbb{R}$. One now considers the solutions of (6.6) as variations from z by letting $x(t) = z(t) + v(t)$ to obtain

$$\begin{aligned} \dot{v} &= f_x(t,z(t))v + g(t,v) \\ g(t,v) &= [f(t,z(t))-\dot{z}(t)] + [f(t,z(t)+v) - f(t,z(t)) - f_x(t,z(t))v]. \end{aligned} \tag{6.8}$$

Now

$$|g(t,0)| \le |f(t,z(t)) - f(t,w(t))| + |\dot{w}(t) - \dot{z}(t)| \le (\text{const})\delta$$

except at the points t_k. Also,

$$|g_v(t,v)| = |f_x(t,z(t)+v) - f_x(t,z(t))| \le \omega(|v|)$$

where ω is the uniform modulus of continuity of f_x. We have assumed that $\omega(s) \to 0$ as $s \to 0$. Since $\dot{v} = f_x(t,z(t))v$ has an exponential dichotomy on $\mathbb{R}$, the equation

$$\dot{v} = f_x(t,z(t))v + g(t)$$

where g is bounded on $\mathbb{R}$ and continuous except at the t_k, has a unique solution Kg bounded on $\mathbb{R}$ and $\sup_t|(Kg)(t)| \le (\text{const}) \sup_t|g(t)|$. If we let $F : \mathbb{R} \times C_b^0(\mathbb{R},\mathbb{R}^n) \to C_b^0(\mathbb{R},\mathbb{R}^n)$ be defined by the relation $F(t,v) = v(t) - K(g(\cdot,v(\cdot))(t)$, then equation (6.8) has a solution in $C_b^0(\mathbb{R},\mathbb{R}^n)$ if and only if $F(\cdot,v) = 0$. The function $F(\cdot,v)$ is continuous together with its first derivative $D_vF(\cdot,v)$ in a neighborhood U of $v = 0$ since $f_x(t,x)$ is continuous in x uniformly in (t,x). Furthermore, $D_vF(\cdot,0) = I$. Also, $F(\cdot,\varphi) = 0$, where $\varphi = K[f(\cdot,z(\cdot)) - \dot{z}(\cdot)]$. Since $|\varphi| \le (\text{constant})\cdot\delta$, the Implicit Function Theorem will imply that, for δ sufficiently small, there is a unique solution $v^* \in C_b^0(\mathbb{R},\mathbb{R}^n)$ of $F(\cdot,v) = 0$ in a neighborhood of $v = 0$ which is $O(|\varphi|)$ as $|\varphi| \to 0$. The function $x^*(t) = v^*(t) + z(t)$ is continuous and satisfies $\dot{x}^*(t) = f(t,x^*(t))$ except perhaps at $t = t_k$. But, since $f(t,x)$ is continuous, it also must satisfy the equation at t_k. This proves the theorem.

Theorem 6.4. Suppose $f(t,x)$ satisfies the smoothness properties in Theorem 6.3 and, in addition, is T-periodic in t. Suppose there is a doubly infinite sequence $\{u_k(t)\}$ of hyperbolic T-periodic solutions of (6.6) and another sequence $\{v_k(t)\}$ of bounded solutions such that $v_k(t) - u_{k-1}(t) \to 0$ as $t \to -\infty$, $v_k(t) - u_k(t) \to 0$ as $t \to +\infty$ and the equation

$$\dot{x} = f_x(t,v_k(t))x \tag{6.9}$$

has an exponential dichotomy on $\mathbb{R}$ with constants K,α independent of k. Then there are $\varepsilon_0 > 0$ and a function $M_0(\varepsilon)$ such that, for any $0 < \varepsilon \le \varepsilon_0$ and any positive integer $m \ge M_0(\varepsilon)$, system (6.6) has a unique solution $x(t)$ defined on $\mathbb{R}$ satisfying

$$|x(t+(2k-1)mT) - v_k(t)| \le \varepsilon$$

for $-mT \le t \le mT$ and all k.

Proof. Only an outline of the proof is given. Let $\tilde{P}_k(t)$ be the projection matrix function for the dichotomy of (6.9). Since $v_k(t) - u_k(t) \to 0$ as $t \to \infty$, the equation $\dot{x}(t) = f_x(t,u_k(t))x$ has an exponential dichotomy on $[0,\infty)$. Since $u_k(t)$ is periodic in t, the Floquet theory implies this equation has an exponential dichotomy on $\mathbb{R}$. Let $Q_k(t)$ be the corresponding projection matrix function. An extension of Lemma 5.3 implies $|\tilde{P}_k(t)-Q_k(t)| \to 0$ as $t \to \infty$, $|\tilde{P}_k(t)-Q_{k-1}(t)| \to 0$ as $t \to -\infty$ uniformly with respect to k. Let $t_k = 2kmT$, $w_k(t) = v_k(t-(2k-1)mT)$, $P_k(t) = \tilde{P}(t-(2k-1)mT)$, and apply Theorem 6.3 to complete the proof.

Figure 6.1 should assist the reader in understanding the meaning of the hypotheses in Theorem 6.4 and also to feel intuitively why the conclusion is true; that is, how one should be able to switch from comparing $x(t+(2k-1)mT)$ to $v_k(t)$ on $[-mT,mT]$ to comparing $x(t+(2(k+1)-1)mT)$ to $v_{k+1}(t)$ on $[-mT,mT]$; that is, $x(\tau+(2k-1)mT)$ to $v_{k+1}(-2mT+\tau)$ on $[-mT,mT]$. The hypotheses imply that $v_{k+1}(-2mT+\tau)|_{\tau=mT} = v_{k+1}(-mT)$ is close to $v_k(mT)$ if m is sufficiently large. Thus, if $x(\tau+(2k-1)mT)$ is close to $v_k(\tau)$ on $[-mT,mT]$, then $x(mT+(2k-1)mT) = x(-mT+(2(k+1)-1)mT)$ is close to $v_{k+1}(-mT)$.

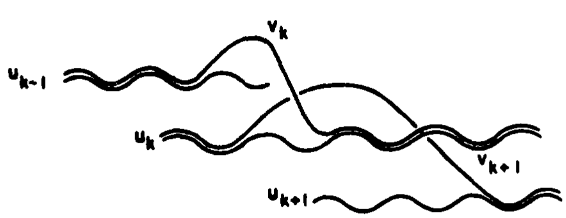

Figure 6.1

We now use Theorem 6.4 to obtain information about the flow near a transverse homoclinic point. Let N be a positive integer and let S_N be the set of all doubly infinite sequences $a = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ with $a_k \in \{0, 1, \dots, N-1\}$, and put the product topology on S_N, that is, a neighborhood basis of a point $b = (\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots)$ consists of the sets $U_j = \{a \in S_N : a_k = b_k \text{ for } |k| \le j\}$. The (right) Bernoulli shift β of S_N is defined as $(\beta(a))_k = a_{k+1}$.

Corollary 6.5. (Shadowing lemma). Suppose $f(t,x)$ satisfies the smoothness properties in Theorem 6.3, is T-periodic in t, has an hyperbolic T-periodic solution $u(t)$ and another solution $v(t)$ such that $v(t) - u(t) \to 0$ as $t \to \pm\infty$ and the equation

$$\dot{x} = f_x(t, v(t))x$$

has an exponential dichotomy on $\mathbb{R}$ (that is, $v(0)$ is transverse homoclinic to $u(0)$ by Proposition 6.1).

Then there exists an $\varepsilon_0 > 0$ and, for each positive integer N, a function $M_N(\varepsilon)$ such that, for any $0 < \varepsilon \le \varepsilon_0$ and any positive integer $m \ge M_N(\varepsilon)$ and any $a \in S_N$, equation (6.6) has a unique solution $x_a(t)$ satisfying

$$|x_a(t+(2k-1)mT) - v(t+a_kT)| \le \varepsilon$$

for $-mT \le t \le mT$ and all k.

The mapping $\varphi(a) = x_a(0)$ is a homeomorphism onto a compact set of $\mathbb{R}^n$ on which the $2m^{th}$ iterate π^{2m} of the Poincaré map π is invariant and $\pi^{2m} \circ \varphi = \varphi \circ \beta$, where β is the right Bernoulli shift on S_N.

Proof: Let $v_k(t) = v(t+a_kT)$, $u_k(t) = u(t)$, $k = 0, 1, \dots$ and apply Theorem 6.4 to obtain the existence of $x_a(t)$. Using the uniqueness of $x_a(t)$, one can prove after some computations that $\varphi(a)$ is a homeomorphism. To show that $\pi^{2m} \circ \varphi = \varphi \circ \beta$, observe that

$$|x_a(t+(2(k+1)-1)mT) - v(t+a_{k+1}T)| \le \varepsilon$$

for $-mT \le t \le mT$ for all t. Thus, uniqueness implies $x_a(t+2mT) = x_{\beta(a)}(t)$. Thus, $\varphi(\beta(a)) = x_a(2mT) = \pi^{2m}(\varphi(a))$.

Remark. The "shadowing lemma" for diffeomorphisms is usually proved by using horseshoes. In this type of proof, the symbols $\{0, 1, \dots, N-1\}$ occur as a specification of whether or not iterates of a point belong to certain intervals.

7. Codimension one bifurcations in the plane.

In previous sections, we have discussed various types of dynamic bifurcation for autonomous systems; for example, the saddle-node bifurcation in Section 3 and the generic Hopf bifurcation in Section 4. We also discussed some aspects of homoclinic bifurcation. For differential equations in the plane, much more information is available. In fact, one can completely characterize all of the codimension one bifurcations. To make this more precise, we need the concept of structural stability. We restrict the discussion to the interior Ω of a closed curve Γ without contact to any of the vector fields to be considered. Let $\mathcal{X}_2^r$ be the set of all such C^r vector fields. Two vector fields X,Y in $\mathcal{X}_2^r$, $r \geq 1$, are equivalent if there is a homeomorphism on $\Omega \cup \Gamma$ which maps orbits of one onto orbits of the other and preserves the sense of direction in time. This is an equivalence relation "$\sim$" among vector fields. X is structurally stable if every Y in a neighborhood of X is equivalent to X.

The condition that the vector fields are nowhere tangent to Γ is very convenient since it makes the domain where the differential equations are being considered to be a compact set. It also avoids certain complications which can arise at the boundary. Our hypothesis does put restrictions on the vector fields. Since our objective is to present some of the basic ideas, the hypothesis seems justified in view of the technicalities that arise in considering the noncompact case or flows on manifolds.

The basic result on structural stability is the following.

Theorem 7.1. An $f \in \mathcal{X}_2^r$ is structurally stable if and only if every equilibrium point and every periodic orbit is hyperbolic and there are no connections between saddle points. The set of structurally stable systems is obviously open but is also dense in $\mathcal{X}_2^r$.

An $X \in \mathcal{X}_2^r$ is a bifurcation point (a vector field for which a perturbation could lead to a bifurcation) if X is structurally unstable; that is, not structurally stable. We now give an inductive definition of a bifurcation point of codimension k.

The vector field X is a bifurcation point of codimension 0 if it is structurally stable. X is a bifurcation point of codimension 1 if it is not codimension zero and there is a neighborhood of X which has only bifurcation points of codimension 0 or ones which are equivalent to X. It is a bifurcation point of codimension 2 if it is not of codimension zero or one and there is a neighborhood containing only bifurcation points of codimension 0 or 1, or, ones which are equivalent to X. Similarly, one defines bifurcation points of codimension k.

The following result is a classification of bifurcation points of codimension one in the plane.

Theorem 7.2. A vecotr field $f \in \mathcal{X}_2^r$, $r \geq 3$, is a bifurcation point of codimension 1 if and only if there is a neighborhood W of f and a submanifold Γ of codimension one in W such that $W \setminus \Gamma = U_1 \cup U_2$ where each $g \in U_j$ is structurally stable but $g \not\sim h$ if $g \in U_1, h \in U_2$. For $g \in \Gamma$, only one of the following situations prevails:

(i) $g \in \Gamma$ has an elementary saddle-node at x_0, there are no equilibrium points of g near x_0 if $g \in U_1$ and a saddle and node near x_0 if $g \in U_2$.

(ii) $g \in \Gamma$ has an elementary focus at x_0, there is no periodic orbit of g near x_0 if $g \in U_1$ and a periodic orbit near x_0 if $g \in U_2$ --the generic Hopf bifurcation.

(iii) $g \in \Gamma$ has a periodic orbit γ which is stable from one side, unstable from the other, $g \in U_1$ has no periodic orbit near γ and $g \in U_2$ has two hyperbolic periodic orbits near γ.

(iv) $\sigma_0 = \text{tr}\ \partial f(x_0)/\partial x \neq 0$, $g \in \Gamma$ has a homoclinic orbit containing a saddle point x_0, $g \in U_1$ has a saddle near x_0 and no periodic orbit near γ, $g \in U_2$ has a saddle point and a unique hyperbolic periodic orbit near γ which coalesce as $g \to \Gamma$.

(v) there is a connection between distinct saddle points.

Each of the cases (i)-(v) is shown in Figure 7.1.

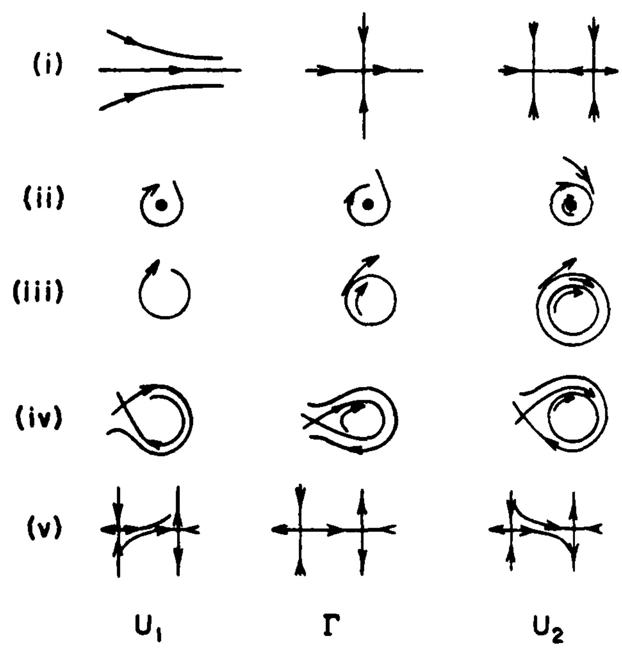

Figure 7.1

We do not give a proof of either Theorem 7.1 or 7.2. We only remark that we have discussed the bifurcations that occur in cases (i) and (ii) in Section 3 and 4. Case (iii) can be discussed using the methods in Section 4 after the introduction of a coordinate system $x \longmapsto (\theta,\rho)$, $x = p(\theta) + \rho v(\theta)$ where $p(t)$ is a non-hyperbolic periodic solution of a bifurcation point $f \in \mathscr{X}_2^\Gamma$ and $v(\theta)$ is orthogonal to $\dot{p}(\theta)$. Case (iv) is the most difficult. Let us only motivate the condition $\sigma_0 \neq 0$.

Let us suppose that we have a vector field $f(x,\mu)$ in R^2 depending on a scalar parameter μ with the property that there is a hyperbolic saddle point at 0 with the eigenvalues bounded away from zero for all μ. Also, suppose there is a hyperbolic periodic orbit P_μ of period $\omega(\mu)$ which has the property that $\mathrm{dist}(0,P_\mu) \to 0$ as $\mu \to 0$. This creates a homoclinic orbit Γ at $\mu = 0$. For such a situation to occur, the period $\omega(\mu)$ must approach ∞ as $\mu \to 0$. If this one parameter family of vector fields is generic, then one cannot expect to have other bad things happen as we change the vector field since we use the parameter to make $\omega(\mu) \to \infty$ as $\mu \to 0$. In particular, the rate of attraction or repulsion of the periodic orbit P_μ should be exponential and uniform in μ. If we keep this uniformity as $\mu \to 0$, then the orbit Γ should have the property that it is either asymptotically stable or unstable depending on whether P_μ is stable or unstable.

To find a quantitative expression for this uniformity, let us recall the formula for the characteristic exponents of the linear variational equation for a periodic orbit in the plane. If $P_\mu = \{p_\mu(t) : 0 \le t < \omega(\mu)\}$ and $x = p_\mu + y$, then the linear variational equation for p_μ is

$$\dot{y} = \frac{\partial f}{\partial x}(p_\mu(t),\mu)y.$$

The nontrivial periodic function $\dot{p}_\mu$ satisfies this equation and thus one characteristic exponent may be taken to be zero. Let $\lambda(\mu)$ be the other characteristic exponent. The sum $\lambda(\mu) + 0 = \lambda(\mu)$ of the characteristic exponents must be

$$\lambda(\mu) = \frac{1}{\omega(\mu)}\int_0^{\omega(\mu)} [\mathrm{tr}\ \partial f(p_\mu(t),\mu)/dx]dt.$$

Then one can show that

$$\lambda(\mu) = \frac{1}{\omega(\mu)}\int_{-\omega(\mu)/2}^{\omega(\mu)/2} [\mathrm{tr}\ \partial f(p_\mu(t),\mu)/\partial x]dt \to \mathrm{tr}\ \partial f(0,0)/\partial x$$

as $\mu \to 0$. Consequently, the rate of attraction or repulsion of each P_μ will be exponential and uniform in μ if

$$\sigma_0 \overset{\mathrm{def}}{=} \mathrm{tr}\ \frac{\partial f(0,0)}{\partial x} = 0.$$

The fact that only two possibilities arise in a neighborhood of a bifurcation point of codimension one suggests that this is the typical or generic situation that arises in the discussion of one parameter families of vector fields. This is, in fact, the case and one can prove

<u>Theorem 7.3.</u> <u>If</u> $\Phi^k = \{\varphi : [0,1] \to \mathcal{X}_2^k, \varphi \in C^k\}$, $k \ge 3$, <u>and</u> $\tilde{\Phi}^k = \{\varphi \in \Phi^k : \varphi(t)$ <u>is structurally stable except at a finite number (depending on</u> φ) <u>points</u> t_j <u>with</u> $\varphi(t_j)$ <u>a bifurcation point of codimension one</u>$\}$, <u>then</u> $\tilde{\Phi}^k$ <u>is a residual set in</u> Φ^k.

The analysis of bifurcations of codimension greater than one are generally much more difficult. For some cases, the ideas are clearly understood and it is mainly a technical problem to do the complete analysis. This remark applies to the situation where the linear variational equation near an equilibrium point has either a simple eigenvalue zero or a pair of purely imaginary eigenvalues on the imaginary axis. The ideas in Sections 3 and 4 apply to this situation. In the plane, the methods necessary to analyze the bifurcations of higher codimension resulting from the nonhyperbolicity of a periodic orbit are also clear. For other situations, special difficulties arise and each problem is a challenge in itself. In the next sections, we discuss some special codimension two bifurcations.

8. Two zero eigenvalues. In this section, we discuss a codimension two bifurcation. The unperturbed system is taken to be

$$\dot{x} = -y, \ \dot{y} = \alpha x^2 + \beta xy \tag{8.1}$$

where $\alpha = 0$, $\beta \neq 0$. Without loss in generality, one can assume $\alpha < 0$, $\beta > 0$. The perturbed system will be

$$\dot{x} = y, \ \dot{y} = \varepsilon_1 x + \varepsilon_2 y + \alpha x^2 + \beta xy \tag{8.2}$$

where $\varepsilon_1, \varepsilon_2$ are small parameters. The problem is to discuss the behavior of the solutions of (8.2) in a neighborhood of $(x,y) = (0,0)$ for $(\varepsilon_1, \varepsilon_2)$ in a neighborhood of $(0,0)$. We remark that the conclusions below are valid for some higher order perturbations of (8.2) and that we omit such terms only for simplicity in notation.

We only discuss $\varepsilon_1 \geq 0$, since the other case is less intersting. The scaling

$$\varepsilon_1 = \delta^2, \qquad \varepsilon_2 = \mu\delta^2, \qquad \delta > 0$$

$$t \longmapsto \delta^{-1}t, \quad x \longmapsto \delta^2|\alpha|^{-1}x, \quad y \longmapsto \delta^3|\alpha|^{-1}y$$

leads to new equations

$$\dot{x} = y, \ \dot{y} = x - x^2 + \mu\delta y + \delta\gamma xy \tag{8.3}$$

where $\gamma = \beta|\alpha|^{-1}$.

For $\delta = 0$, Equation (8.3) becomes the conservative system

$$\dot{x} = y, \ \dot{y} = x - x^2 \tag{8.4}$$

with first integral

$$V(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}. \tag{8.5}$$

The equilibrium point (0,0) is a saddle with a homoclinic orbit through it while the equilibrium point (1,0) is a center.

The parametrization of the phase space by the above scaling is suggested by the following. The orbits of the unperturbed equation (8.1) satisfy the equation $ydy = (\alpha x^2+\beta xy)dx$. The orbit which passes through the origin is given approximately by $y^2 = \alpha x^3$. Thus when one "blows up" the flow at the origin, it is natural to do it by parametrizing the phase space with cusps. The orbits for (8.2) satisfy

$$dy = [(\varepsilon_1 + \alpha x)xy^{-1} + (\varepsilon_2+\beta x)]dx.$$

For $\varepsilon_1 > 0$, it turns out that the appropriate parameterization in parameter space is to take $\varepsilon_1, \varepsilon_2, x$ of the same order. This leads to the scaling used above. Notice that the flows defined by (8.2) and (8.3) are equivalent for $\delta > 0$, but are not equivalent for $\delta = 0$.

Our next task is to determine the curves in the $(\varepsilon_1,\varepsilon_2)$-plane (that is, the values of δ,μ) at which the topological structure of the trajectories of Equation (8.3) changes. As we shall see, this structure can change only due to a homoclinic orbit or a change in stability of the equilibrium point (1,0).

Let $\Gamma = \{(q(t),\dot{q}(t)),\ t \in \mathbb{R}\} \cup \{(0,0)\}$ be the homoclinic orbit of (8.4). We may now apply the results of Section 5 to obtain the curve in (μ,δ)-space such that (8.3) has a homoclinic orbit. In particular, from Eq. (5.4), we have this curve corresponds to the zeros of the bifurcation function

$$G(\mu,\delta) = \mu + \gamma\nu + \tilde{G}(\mu,\delta)$$
$$\nu = \int_{-\infty}^{\infty} q\dot{q}^2 \Big/ \int_{-\infty}^{\infty} \dot{q}^2$$

where $\tilde{G}(\mu,0) = 0$. There is no α in $G(\mu,\delta)$ since the equation is autonomous. One can show that $\nu = 6/7$. The equation $G(\mu,\delta) = 0$ has a unique solution $(\mu(\delta),\delta)$, $0 \le |\delta| \le \delta_0$, $\delta_0 > 0$, $\mu(0) = \mu_0 = -\gamma\nu$. Finally, the curve C_∞ in $(\varepsilon_1,\varepsilon_2)$-space along which there is a homoclinic orbit is given by

$$C_\infty = \{(\varepsilon_1,\varepsilon_2) : \varepsilon_2 = \mu(\varepsilon_1^{1/2})\varepsilon_1,\ \ \mu(0) = -\gamma\nu\} .$$

On the curve $C_\infty \setminus \{0,0\}$ we have $\sigma_0 = \varepsilon_2 < 0$, where σ_0 is the number given in Theorem 7.2. From part (iv) of Theorem 7.2, this suggests there should be a periodic orbit near this curve.

We now discuss the periodic orbits of (8.3). This part of the analysis in problems of this type is the most difficult, especially the discussion of the number of periodic orbits that can exist. One can prove the following lemma.

Lemma 8.1. Every periodic orbit of Equation (8.3) must intersect the segment $(0,1) \times \{0\}$ in the (x,y)-plane. There is a continuous positive function $\delta_0 : (0,1) \to \mathbb{R}$ and a continuously differentiable function $\mu^*(b,\delta)$, $b \in (0,1)$, $|\delta| < \delta_0(b)$ such that there is a periodic orbit of Equation (8.3) through $(b,0)$ if and only if $\mu = \mu^*(b,\delta)$. Furthermore,

$$\mu^*(b,0) = -\gamma\beta(b)/\alpha(b)$$

$$\alpha(b) = \int_b^{c(b)} y\,dx, \qquad \beta(b) = \int_b^{c(b)} xy\,dx > 0$$

$$y \geq 0, \quad \frac{y^2}{2} = \frac{x^2-b^2}{2} - \frac{x^3-b^3}{3}$$

$$c(b) > 1, \quad \frac{c^2(b)-b^2}{2} - \frac{c^3(b)-b^3}{3} = 0$$

and $d\mu^*(b,0)/db < 0$. Also, $\mu^*(b,0) \to -\gamma$ as $b \to 1$, $\mu^*(b,0) \to -\frac{6}{7}\gamma$ as $b \to 0$. Finally, if $\mu = \mu^*(b,\delta)$ for a fixed $b \in (0,1)$ and $|\delta| < \delta_0(b)$, then the periodic orbit through $(b,0)$ is the only one corresponding to this μ,δ.

Remark: The assertion $d\mu^*(b,0)/db < 0$ implies, for any b_0, there is a $\delta_0(b_0)$ such that along the curve $\varepsilon_1 = \mu^*(b_0,\delta)\varepsilon_2$, $0 < \delta < \delta_0(b)$, there is a unique periodic orbit of (8.3) which approaches the periodic orbit of (8.4) through $(b_0,0)$ as $\delta \to 0$.

We give only an indication of the proof of Lemma 8.1. If $\dot{V}(x,y)$ is the derivative of V along the solutions of (8.3), then $\dot{V} = \delta(\mu+\gamma x)y^2$.

For a fixed $(b,0)$, $0 < b < 1$, and δ sufficiently small, there are numbers $\tau_1 = \tau_1(b,\delta,\mu) < 0 < \tau_2 = \tau_2(b,\delta,\mu)$ such that the solution through $(b,0)$ intersects the x-axis at time τ_j at a point larger than 1. Furthermore, for $t \in (\tau_1,\tau_2)$, the orbit intersects the x-axis only for $t = 0$. Let $\Gamma = \Gamma(b,\delta,\mu)$ be that part of the orbit through $(b,0)$ corresponding to $t \in [\tau_1,\tau_2]$. For Γ to be a periodic orbit, it is necessary and sufficient that $\int_\Gamma \dot{V}dt = 0$, which for $\delta \neq 0$, is equivalent to

$$F(b,\delta,\mu) \overset{\text{def}}{=} \int_\Gamma (\mu+\gamma x)y^2 = 0.$$

One can now apply the Implicit Function Theorem to this equation near the point $(b_0,0,\mu_0)$, $\mu_0 = -\gamma\beta(b_0)/\alpha(b_0)$. This will prove the first part of the lemma. The fact that $\mu^*(b,0)$ approaches the limits indicated above require only elementary computations.

The difficult part of the lemma is to show $d\mu^*(b,0)/db < 0$. If $2a = -b^2 + 2b^3/3$, then $0 < b < 1$ implies $-1 < 6a < 0$. Let $b(a)$ be the inverse of this transformation and put $\nu(a) = \beta(b(a)/\alpha(b(a))$. If we consider β, as functions of a and let " ' " be differentiation with respect to a, then the lemma is proved if one shows that $\nu' < 0$, $-1 < 6a < 0$. To carry out this proof, one exploits special properties of the elliptic integrals α,β. More precisely, one shows that α,β and α'',β'' can be expressed as linear combinations of α',β'. We are going to show that

$$\begin{aligned} &5\alpha = 6a\alpha' + \beta', \quad 35\beta = 6a\alpha' + 6(1+5a)\beta' \\ &6a(1+6a)\alpha'' = -6a\alpha' - \beta', \quad (1+6a)\beta'' = \beta' - \alpha'. \end{aligned} \tag{8.6}$$

We have already proved that $\beta'(a) = \int_b^c xy^{-1}dx$. Using the fact that $x = yy_x + x^2$, $y(b) = y(c) = 0$, one obtains $\beta'(a) = \int_b^c x^2y^{-1}dx$. Integrating by parts, using $yy_x = x - x^2$ and the formula for β', we have

$$\alpha = \int_b^c ydx = -\int_b^c xy_x dx = -\int_b^c x(x-x^2)y^{-1}dx = -\beta' + \int_b^c x^3y^{-1}dx. \qquad (8.7)$$

This relation and the formula for y imply that

$$\alpha = \int_b^c ydx = \int_b^c y^2y^{-1}dx = \int_b^c [x^2 - \tfrac{2}{3}x^3 + 2a]y^{-1}dx$$

$$= \beta' - \tfrac{2}{3}(\alpha+\beta') + 2a\alpha'$$

or, $\qquad 5\alpha = 6a\alpha' + \beta'$.

Integrating β by parts and using the formula for yy_x, we have

$$\beta = \int_b^c xydx = -\int_b^c (x^2/2)y_x dx = -\frac{1}{2}\int_b^c x^3y^{-1}dx + \frac{1}{2}\int_b^c x^4y^{-1}dx.$$

Using (8.7), this implies $\beta + (1/2)(\alpha+\beta') = (1/2)\int_b^c x^4y^{-1}dx$. Using the formula for y^2, (8.7) and this latter relation, one obtains

$$\beta = \int_b^c xy^2y^{-1}dx = \int_b^c [x^3 - \tfrac{2}{3}x^4 + 2ax]y^{-1}dx = \alpha + \beta' - \tfrac{4}{3}\beta - \tfrac{2}{3}(\alpha+\beta') + 2a\beta' .$$

Simplifying this expression, one obtains $7\beta = \alpha + \beta' + 6a\beta'$. Using the previously obtained expression for α in terms of α',β', one finds that $35\beta = 6a\alpha' + 6(1+5a)\beta'$. The expressions for α'',β'' are obtained from the relations for α,β. This completes the proof of (8.6).

Using these relations, one now proves that, if $\nu'(a) = 0$ for some a, $-1 < 6a < 0$, then

$$-6a(1+6a)\alpha\nu''/\alpha' = -(\nu + 6a)^2 - 6a(1+6a) < 0;$$

that is, $\nu''(a) < 0$.

Next, one shows that $\nu'(a) = 0$ implies $7\nu^2(a) + 6(2a-1)\nu(a) - 6a = 0$. This implies that, if $\nu'(a) = 0$ and $\nu(a) = 1$, then $6a = -1$. Since $\nu(0) = 6/7$, we have $\nu'(a) = 0$ implies $0 < \nu(a) < 1$.

Using the fact that $\mu_0(-1/6) = 1$, $\mu_0(0) = 6/7$, one easily concludes that $\gamma'(a) < 0$, $-1 < 6a < 0$ and the lemma is proved.

Using Lemma 8.1 and the remarks about the homoclinic orbit before the statement of the lemma, Theorem 7.2 (iv) implies that for each point in the region below the curve C_∞, there is a unique periodic orbit. Next, we analyze the behavior of the solutions of (8.3) near the equilibrium point (1,0). This point is a stable focus if $\mu < -\gamma$, and an unstable focus if $\mu > -\gamma, \gamma = \beta|\alpha|^{-1}$. The curve $\mu = -\gamma$ is therefore a possible value for a Hopf bifurcation. One can apply the method of Section 4 for the periodic orbits near (1,0) and obtain a bifurcation function $G(a,\mu,\delta)$ for $|\delta| < \delta_0$, $|a| < a_0$, $|\mu + \gamma| < \eta$ for some constants $\delta_0 > 0$, $a_0 > 0$, $\eta > 0$. Since Equation (8.3) for $\delta = 0$ has a center at (1,0), it follows that $G(a,\mu,0) = 0$ for all a,μ. Thus, the appropriate bifurcation function to consider is $H(a,\mu,\delta) = G(a,\mu,\delta)/\delta$. This function satisfies $H(0,\mu,\delta) = 0$, $\partial H(0,\mu,0)/\partial a = (\mu + \gamma)/2$. Lemma 8.1 implies for each $\mu > -\gamma$ and sufficiently close to $-\gamma$, there is a unique periodic orbit of (8.3) through (b,0) with b near 1. Thus, if b is taken close enough to 1, this periodic orbit must correspond to a zero of the bifurcation function $H(a,\mu,\delta)$. This proves there is a Hopf bifurcation at $\mu = -\gamma$ and the periodic orbit is asymptotically stable.

One thus obtains the complete bifurcation diagrams as shown in Fig. 8.1 with the flow in each sector given in Fig. 8.2. We draw the curves in Fig. 8.1 as straight lines but this is really only the first approximation.

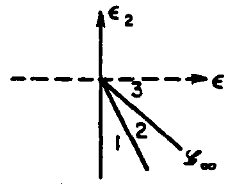

Figure 8.1

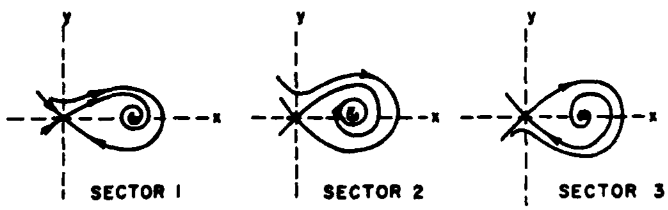

Figure 8.2

9. <u>Two zero roots with symmetry.</u> In a planar system where the matrix of the linear variational equation near an equilibrium point has both eigenvalues zero with non-simple elementary divisors, the analysis in the previous section showed that the vector field (8.1) with quadratic terms was a codimension two bifurcation. If there is some symmetry in the vector field; for example, it is odd in (x,y), then the quadratic terms in the Taylor expansion vanish. Thus, it becomes of interest to know what additional nonlinear terms are needed in order to obtain a codimension two bifurcation. In this section, we summarize some results with only brief indications of the proofs.

Consider the equation

$$\dot{x} = y, \quad \dot{y} = \varepsilon_1 x + \varepsilon_2 y + \alpha x^3 + \beta x^2 y \tag{9.1}$$

with $\alpha < 0$, $\beta < 0$ and $\varepsilon_1, \varepsilon_2$ small parameters. The problem is to analyze the behavior of the solutions of (9.1) in a neighborhood of $(x,y) = (0,0)$ for $(\varepsilon_1,\varepsilon_2)$ in a neighborhood of $(\varepsilon_1,\varepsilon_2) = (0,0)$.

The bifurcation diagram is shown in Fig. 9.1 with the flow in each sector given in Fig. 9.2.

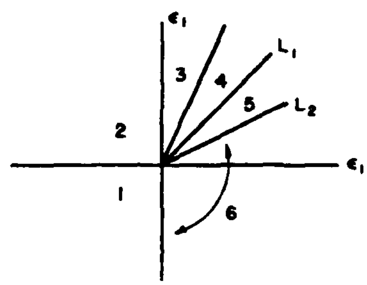

Figure 9.1

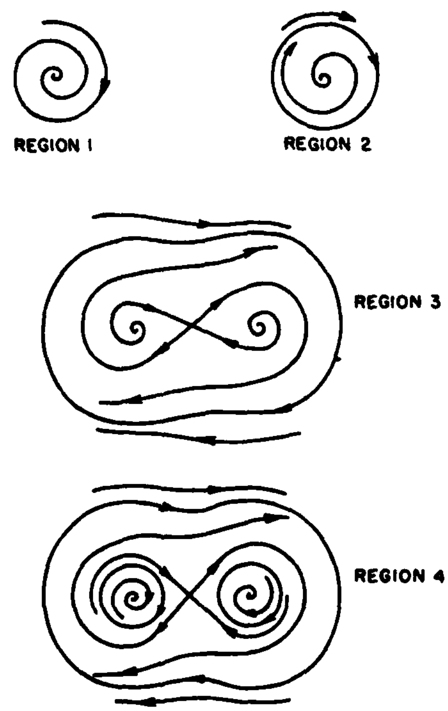

Figure 9.2

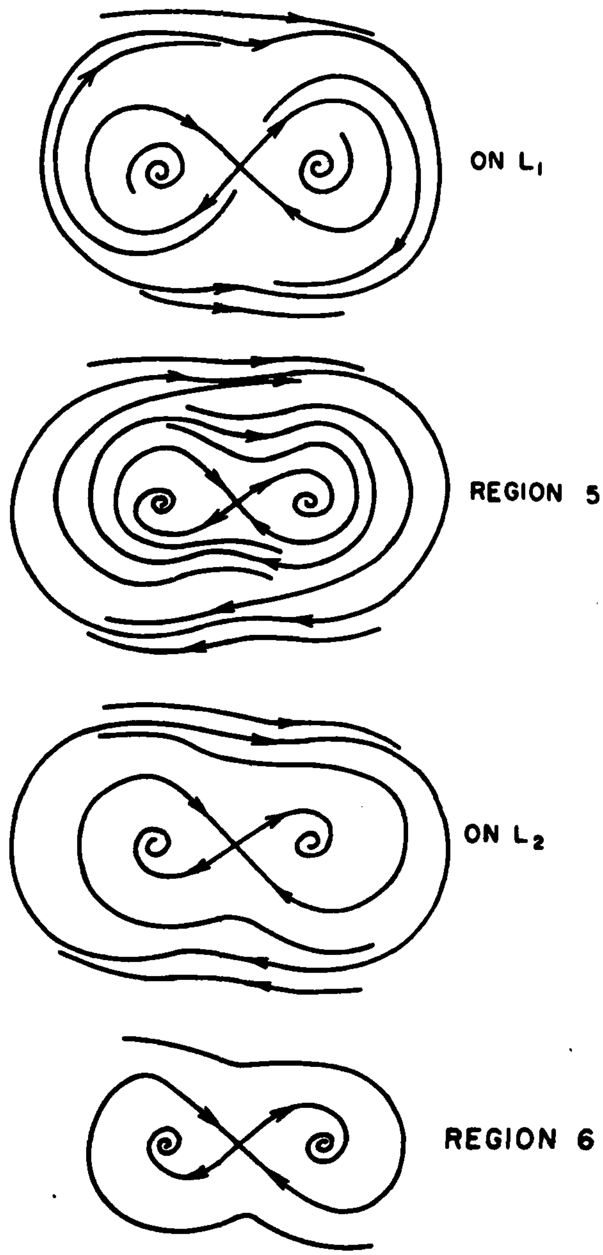

Figure 9.2 (cont.)

Let us give an idea of how these results are obtained for the case $\varepsilon_1 > 0$. Firstly, one introduces scaling

$$\delta = |\varepsilon_1 \alpha^{-1}|^{1/2},\ \varepsilon_2 = |\alpha|^{1/2}\delta\mu,\ x \longmapsto \delta x,\ y \longmapsto \delta^2|\alpha|^{1/2}y$$
$$t \longmapsto |\alpha|^{-1/2}\delta^{-1}\ t$$

to obtain

$$\dot{x} = y, \quad \dot{y} = x + \mu y - x^3 + \delta\gamma x^2 y \tag{9.2}$$

where $\gamma = \beta|\alpha|^{-1/2}$. For $\mu = 0$, $\delta = 0$, this equation has the first integral

$$H(x,y) = y^2/2 - x^2/2 + x^4/4. \tag{9.3}$$

Some of the level curves $H(x,y) = b$ of this function are shown in Fig. 9.3. For $b = 0$, the curve is a figure of eight and for $b > 0$, it is a closed curve through the point $(x,y) = (0,(2b)^{1/2})$. For $b < 0$, the set $H(x,y) = b$ consists of two closed

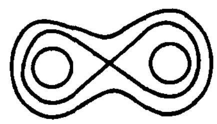

Figure 9.3

curves surrounding, respectively, the equilibrium points $(1,0)$, $(-1,0)$. These curves pass through the point $(0,c)$, $0 < c < 1$, $b = -c^2/2 + c^4/4$. The derivative $\dot{H}(x,y)$ along the solutions of (9.2) is given by $\dot{H}(x,y) = \mu y^2 + \delta\gamma x^2 y^2$.

The first step in the analysis is to determine the curve $\mu = \mu^*(\delta)$ so that (9.2) has a homoclinic orbit. This curve is obtained as in the example of Section 8 and is shown to be

$$\mu^*(\delta) = -(4/5)\delta\gamma + O(\delta^2) \quad \text{as} \quad \delta \to 0.$$

The next step is to analyze the periodic orbits. There is the possibility of two types: an orbit which contains only one equilibrium point in its interior or an orbit which contains three points in its interior. These orbits will be close to a curve $H(x,y) = b$ for some $b < 0$ in the first case and some $b > 0$ in the latter case. These two cases must be analyzed separately.

For $b > 0$, let $\tau_1 = \tau_1(\mu,b,\delta) > 0$ be the first positive value of t for which the solution through $(0,(2b)^{1/2})$ crosses the x-axis, say at $x_1(\mu,b,\delta)$. Let $\tau_2 = \tau_2(\mu,b,\delta)$ be the first negative value of t for which the solution through $(0,-(2b)^{1/2})$ crosses the x-axis, say at $x_2(\mu,b,\delta)$. From the symmetry in the equation, it follows that $(0,(2b)^{1/2})$ lies on a periodic orbit if and only if $H(x_1(\mu,b,\delta),0) = H(x_2(\mu,b,\delta),0)$. Using the expression for $\dot{H}(x,y)$, it is not difficult to show that this implies

$$\mu = \mu_1(b,\delta) = -\gamma P(b)\delta + O(\delta^2)$$
$$P(b) = \frac{\int_0^c x^2 y dx}{\int_0^c y dx} \tag{9.4}$$

where $c = c(b)$ is the positive solution of $4b = c^4 - 2c^2$ and $H(x,y) = b$. For $P(b)$, one can now prove the following basic result.

<u>Lemma 9.1.</u> $P(b) \to \infty$ <u>as</u> $b \to \infty$ <u>and there is a unique minimum of</u> $P(b)$ <u>at</u> $b = b_1$ <u>and</u> $P''(b_1) > 0$.

<u>Idea of the proof:</u> If $r(w) = (w^2 - w^4/2 + 2b)^{1/2}$, $\alpha(b) = \int_0^c r(w)dw$, $\beta(b) = \int_0^c w^2 r(w)dw$, $4b = c^4 - 2c^2$, then $P(b) = \beta(b)/\alpha(b)$. If $\alpha' = d\alpha/db, \beta' = d\beta/db$, then one shows that

$$\begin{aligned} 3\alpha &= 4b\alpha' + \beta' \\ 15\beta &= 4b\alpha' + (4+12b)\beta' \end{aligned} \tag{9.5}$$

after several computations.

Now, suppose that $P'(b_1) = 0$. Then $\alpha(b_1)P''(b_1) = \beta''(b_1) - P(b_1)\alpha''(b_1)$ and relations (9.5) imply that

$$4b_1(4b_1+1)[\beta''(b_1) - P(b_1)\alpha''(b_1)] = \alpha'(b_1)[P^2(b_1) + 8b_1P(b_1) - 4b_1].$$

Thus, $P''(b_1)$ has the same sign as $P^2(b_1)+8b_1P(b_1)-4b_1$.

On the other hand, $P'(b_1) = 0$ and relations (9.5) imply that

$$5P^2(b_1) + 8b_1P(b_1) - 4P(b_1) - 4b_1 = 0$$

which implies $P(b_1) < 1$ since $b_1 > 0$. Using the fact that $8b_1P(b_1) - 4b_1 = 4P(b_1) - 5P^2(b_1)$, we see that $P''(b_1)$ has the same sign as $P(b_1) - P^2(b_1$ which is > 0 since $P(b_1) < 1$.

It is not difficult to prove that $P(b) \to \infty$ as $b \to \infty$ and $P'(b) \to -\infty$ as $b \to 0$. This will complete the proof of the lemma.

The graph of $P(b)$ is illustrated in Fig. 9.4.

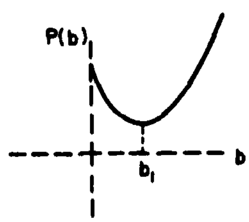

Figure 9.4

From (9.4), the periodic orbits with $b > 0$ are given approximately by $-\mu(\gamma\delta)^{-1} = P(b)$. Thus, approximately, there should be no periodic orbit encircling the three equilibrium points if $-\mu(\gamma\delta)^{-1} < \min P(b) = P(b_1)$, two periodic orbits if $P(b_1) < -\mu(\gamma\delta)^{-1} < P(0)$ and one periodic orbit if $P(0) < -\mu(\gamma\delta)^{-1}$. This can be made precise since $P''(b_1) > 0$ to confirm the part of the bifurcation diagram in Fig. 9.1 for the periodic orbits which encircle three equilibrium points.

The analysis of the periodic orbits encircling only one equilibrium point uses methods very similar to the ones in Section 8 and will not be given.

Notes

These notes are intended to relate the results stated in previous sections to existing literature. No claim is made toward completeness nor even original sources.

Section 1. The methods in this section are very special cases of a much more general global procedure for discussing the zeros of functions. This procedure often is called the alternative method and originated from some fundamental papers of Cesari in the early 1960's (for references and an historical discussion, see Cesari [6], Chow and Hale [7]).

Section 3. Theorem 3.1 can be found in deOliveira and Hale [13] and can also be obtained from a result in Golubitsky and Schaeffer [15]. Theorem 3.2 appeared in the paper of Crandall and Rabinowitz [11], [12]. The saddle-node and cusp bifurcations can be found in Andronov et al [1]. The complete discussion of the case $G(x,0) = \beta x^q + o(|x|^q)$, $\beta \neq 0$, belongs to the general theory of unfolding of singularities (see, for example, Golubitsky and Buillemin [14]). For a full treatment of the evolutionary equations of the form (3.17), see Henry [17]. For functional differential equations, see Hale [16].

Section 4. For a detailed discussion of the Hopf Bifurcation Theorem as well as references, see Marsden and McCracken [21] and Chow and Hale [7]. Results and references for parabolic equations may be found in Kielhöfer [20] and for functional differential equations in [16].

Section 5. An excellent discussion of dichotomies is contained in Coppel [10]. Exponential dichotomies for parabolic equations are contained in Henry [17] and for functional differential equations Pecelli [24]. Lemmas 5.3 and 5.4 are due to Palmer [23]. The use of the Fredholm alternative to discuss Example (5.11) was first given by Chow, Hale and Mallet-Paret [8]. The Mel'nikov function can also be used to discuss (5.11) (see, e.g. Holmes [18]). Lemma 5.4 and its application to obtain the function $G(a,\mu,\alpha)$ in (5.10) can be considered as a generalization of the Mel'nikov function to n-dimensions.

Section 6. The methods and results in this section are based on Palmer [23]. For other references and approaches to the "shadowing lemma" and the symbolic dynamics of Corollary 6.5, see Smale [27], Conley [9], Moser [22], Sil'nikov [26].

Section 7. Theorem 7.1 is due to Andronov and Pontrjagin [2] and Peixoto [25]. Theorem 7.2 is due to Andronov et al [1] and Sotomayor [28]. Theorem 7.3 is due to Sotomayor [28].

Section 8. The bifurcation diagram for Eq. (8.2) was considered by Howard and Koppell [19]. Arnol'd [3] and Bogdanov [4] considered an equivalent equation $\dot{x} = y$, $\dot{y} = \varepsilon_2 + \varepsilon_1 x + \alpha x^2 + \beta xy$. Bogdanov [4] has shown that every two parameter family of vector fields close to $\dot{x} = y$, $\dot{y} = \alpha x^2 + \beta xy$ in the C^3 topology is equivalent to a member of the above two parameter family.

Section 9. The results in this section are due to Takens [29,30] and Carr [5].

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