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Technical report 09-006

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If you want to cite this report, please use the following reference instead:
W.P.M.H. Heemels, D. Lehmann, J. Lunze, and B. De Schutter, "Introduction to hybrid systems," Chapter 1 in *Handbook of Hybrid Systems Control – Theory, Tools, Applications* (J. Lunze and F. Lamnabhi-Lagarrigue, eds.), Cambridge, UK: Cambridge University Press, ISBN 978-0-521-76505-3, pp. 3–30, 2009.

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Introduction to hybrid systems

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This chapter gives an informal introduction to hybrid dynamical systems and illustrates by simple examples the main phenomena that are encountered due to the interaction of continuous and discrete dynamics. References to numerous applications show the practical importance of hybrid systems theory.

1.1 What is a hybrid system?

Wherever continuous and discrete dynamics interact, hybrid systems arise. This is especially profound in many technological systems, in which logic decision making and embedded control actions are combined with physical processes. To capture the evolution of these systems, mathematical models are needed that combine in one way or another the dynamics of the continuous parts of the system with the dynamics of the logic and discrete parts. These mathematical models come in all kinds of variations, but basically consist of some form of differential or difference equations on the one hand and automata or other discrete-event models on the other hand. The collection of analysis and synthesis techniques based on these models forms the research area of hybrid systems theory, which plays an important role in the multi-disciplinary design of many technological systems that surround us.

1.1.1 Three reasons to study hybrid systems

The reasons to study hybrid systems can be quite diverse. Here we will provide three sources of motivation, which are related to (i) design of technological systems, (ii) networked control systems, and (iii) physical processes exhibiting non-smooth behavior.

Challenges of multi-disciplinary design. When designing a technological system (Fig. 1.1) such as a wafer stepper, electron microscope, copier, robotic system, fast component mounter, medical system, etc., multiple disciplines need to make the overall design in close co-operation. For instance, the electronic design, mechanical design, and software design together have to result in a consistent, functioning machine. The designs are typically made in parallel by multiple groups of people, where the communication between these groups is often hampered by lack of common understanding and common models. The lack of common models complicates the making of cross-disciplinary design decisions that may have advantages for one discipline, but disadvantages for others. To make a good trade-off, the overall effect of such a design decision has to be evaluated as early as possible. As the complexity of a technological system with typically millions of lines of code and tens of thousands of mechanical components gives rise to many cross-disciplinary design decisions, a framework is required that supports efficient evaluation of design decisions incorporating quantitative information and models from multiple disciplines.

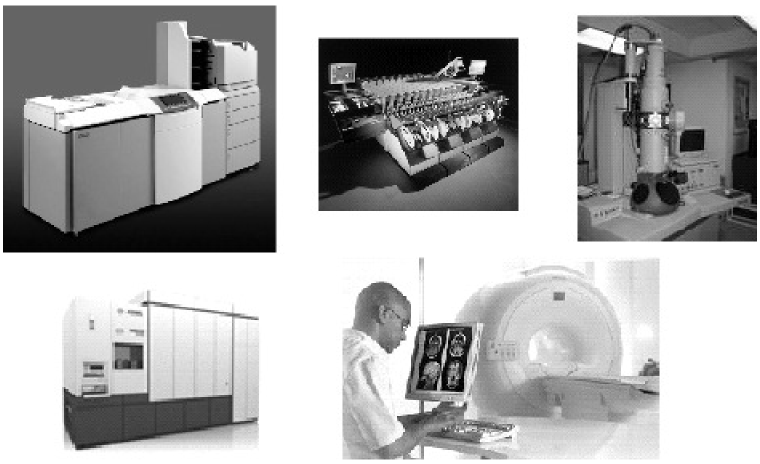

Fig. 1.1. Examples of technological systems.

Hybrid systems theory studies the behavior of dynamical systems, including the above mentioned technological systems, the modeling formalisms that involve both continuous models such as differential or difference equations describing the physical and mechanical part, and discrete models such as finite state machines or Petri nets that describe the software and logical behavior.

This theory is one of the few scientific research directions that aim at approaching the design problem of technological systems in a rigorous manner and at developing a complete design framework. As such, hybrid systems theory combines ideas originating in the computer science and the software engineering disciplines on one hand, and systems theory and control engineering on the other. This mixed character explains the terminology “hybrid systems”, which was used in this context for the first time by Witsenhausen in 1966 [Witsenhausen, 1966].

Hybrid systems theory is a relatively young research field as opposed to the more conventional mono-disciplinary research areas such as mechanical, electrical, or software engineering. The urgent need for multi-disciplinary design and development methods for technological systems has spurred the growth of hybrid systems theory in recent years. However, due to the inherent complexity of hybrid systems, many issues still remain unsolved at present, at least at the scale needed for industrial applications. The current status of hybrid systems theory is surveyed in this handbook, which can be used as a starting point for future developments in this appealing and challenging research domain.

Adding communication: Networked control systems. Besides merging software (discrete) and physical (continuous) aspects of systems, another important aspect of many technological systems is communication. Within one single system, many subsystems interact through communication networks. For systems-of-systems the coordination plays an even larger role, resulting in extremely complicated networks of communication. One might think of examples such as automated highways [Lygeros et al., 1998] and air-traffic management [Tomlin et al., 1997]. As the many control, computation, communication, sensing, or actuation actions take place through shared network or processor resources, another dimension is added to the design of these systems. Within the context of these *networked control systems* the asynchronous and event-driven nature of the data transmission caused by varying delays, varying sampling intervals, package loss, etc., and the implementation of the networks and protocols complicate their analysis and design even further. Also in this domain hybrid systems theory plays an essential role as a foundation to understand the behavior of these complex systems.

Physical processes modeled as hybrid systems. In the above mentioned technological and networked systems, the digital and logic (embedded control) aspects are typically brought in by design in order to control the physics and mechanics of the system. However, hybrid system theory is not only useful within these domains. Many physical processes exhibiting both fast and slow changing behaviors, can often be well described by using (simple) hybrid models. For instance, in non-smooth mechanics [Brogliato, 1996], the evolution of impacting rigid bodies can be captured in hybrid models. Indeed, as the impacts occur at a much smaller time scale than the unconstrained motion, the behavior can be described well by introducing discrete events and actions in a smooth model. The bouncing ball is a simple example demonstrating this. Also the vector fields defining the behavior of the system might be different over time as they depend crucially on the fact whether a contact is active or not. The dynamics of a robot arm moving freely in space is completely different from the situation in which it is striking the surface of an object. Other examples in mechanics with hybrid behavior include motion systems with friction models that distinguish between stick and slip modes, backlash in gears, and dead zones in cog wheels.

Examples are not only found in the mechanical domain. Nowadays switches such as thyristors and diodes are used in electrical networks for a wide variety of applications in both power engineering and signal processing. Examples include switched-capacitor filters, modulators, analog-to-digital converters, power converters, and choppers. In the ideal case, diodes are considered as elements with two (discrete) modes: the blocking mode and the conducting mode. Mode transitions for diodes are governed by state events, where currents or voltages change their sign. This indicates that hybrid modeling and analysis offer an attractive perspective on these switched circuits [Heemels et al., 2002]. The DC-DC converter discussed in Section 1.3.3 forms a simple example of this.

Also many biological and chemical systems can often be efficiently described by hybrid models. For example, simulating moving bed processes, which are a special kind of chromatographic separation processes, have to be switched regularly among different structures in order to avoid that the separation process will eventually stop. Like in DC-DC converters, the switching is an integral part of the physical principle utilized in such processes. For the analysis of these systems and for control design, the model has to be switched accordingly, which demonstrates the necessity to extend continuous models towards hybrid models.

1.1.2 Behavior of hybrid systems

The previous section indicates that multi-disciplinary design of technological systems and the study of several non-smooth physical processes require the understanding of the complex interaction between discrete dynamics and continuous dynamics. To provide some insight in this interaction, let us consider the following example.

Example 1.1 *Thermostat*

As a textbook example of a simple hybrid system consider the regulation of the temperature in a house. In a simplified description, the heating system is assumed either to work at its maximum power or to be turned off completely. This is a system that can operate in two modes: "on" and "off". In each mode of operation (given by the discrete state $q \in \{\text{on, off}\}$) the evolution of the temperature T can be described by a different differential equation. This is illustrated in Fig. 1.2 in which each mode corresponds to a node of a directed graph, while the edges indicate the possible discrete state transitions. As such, this system has a hybrid state (q, T) consisting of a discrete state q taking the discrete values "on" and "off" and a continuous state T taking values in the real numbers.

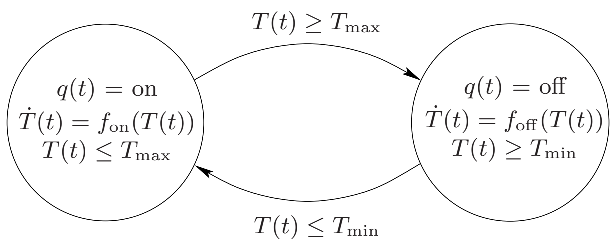

Fig. 1.2. Model of a temperature control system

Clearly, the value of the discrete state q affects the evolution of the continuous state T as a different vector field is active in each mode. Vice versa, the switching between the two modes of operations is controlled by a logical device (the embedded controller) called the *thermostat* and depends on the value of the continuous state T. The mode is changed from "on" to "off" whenever the temperature T reaches the value $T_{\max}$ (determined by the desired temperature). Vice versa, when the temperature T reaches a minimum value $T_{\min}$, the heating is switched "on."

This example already shows some of the main features of hybrid systems:

- The thermostat is a hybrid system because its state consists of a discrete state q and a continuous state T.
- The continuous behavior of the system depends on the discrete state, i.e. depending on whether the mode is "on" or "off" a different dynamics $\dot{T}(t) = f_{\text{on}}(T(t))$ or $\dot{T}(t) = f_{\text{off}}(T(t))$, respectively, governs the evolution of the temperature T.
- The changes of the discrete state q are determined by the continuous state T and different conditions on T might trigger the change of the discrete state (e.g. when the discrete state is "on," $T = T_{\max}$ triggers the mode change, while $T = T_{\min}$ triggers the change when the discrete state is "off.") □

Although the thermostat example is rather simple, it already contains some of the basic ingredients that are needed to properly model hybrid systems. A proper modeling format must involve (at least) the description of the evolution of both continuous-valued signals (temperatures, positions, velocities, currents, voltages, etc.) and discrete-valued signals (operation mode, position of switch, alarm on or off, etc.) over time and their mutual influence (see Fig. 1.3 for an abstraction of this perspective).

The system depicted in Fig. 1.3 has six types of signals:

- $y(t)$ – a continuous output signal
- $w(t)$ – a discrete output signal

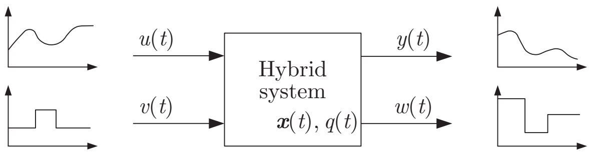

Fig. 1.3. Hybrid dynamical system

- $\boldsymbol{x}(t)$ – a continuous (n-dimensional) state vector
- $q(t)$ – a discrete state
- $u(t)$ – a continuous input signal
- $v(t)$ – a discrete input signal.

The input and output signals may be scalar or vector-valued, but for explaining the main idea of hybrid systems this distinction is not important.

Whereas the discrete signals (think about the "on" and "off" modes of the thermostat example) are typically piecewise constant, the continuous signals may change their value continuously or discontinuously. In the thermostat example the continuous signal representing the temperature is only changing continuously. There are no jumps (discontinuities) in the temperature. The state of the hybrid system is described by the pair $(\boldsymbol{x}, q)$ consisting of the continuous state vector $\boldsymbol{x}$ and the discrete state q. An important characteristic of hybrid systems lies in the fact that this pair influences the future behavior of the system. Moreover, the evolution of the system may also be influenced by a continuous as well as a discrete input, which are denoted by u and v, respectively, and one may receive some information on the hybrid state $(\boldsymbol{x}, q)$ from the discrete and continuous outputs w and y, respectively.

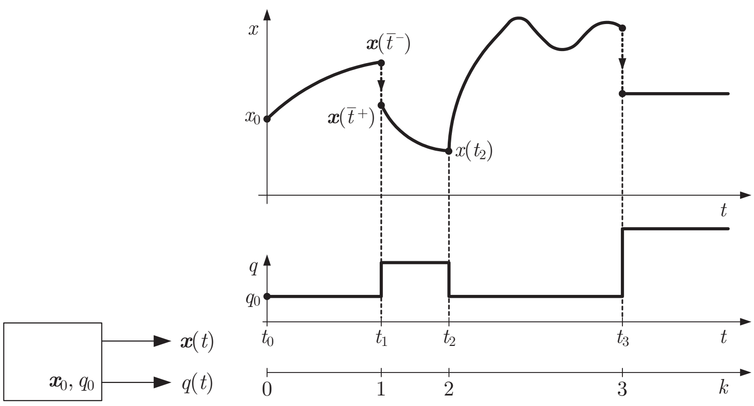

Fig. 1.4. Behavior of an autonomous hybrid system

Figure 1.4 displays some typical behavior of an autonomous hybrid system (i. e. an hybrid system without an input), where the scalar continuous state x and the discrete state q are identical to the outputs. It shows that the evolution consists of smooth phases in which the discrete state remains constant and the continuous state changes continuously. At the transition times $t_1, t_2, t_3, \ldots$ the discrete state changes from its current value to a new value. Simultaneously, the continuous state may jump as shown in the figure for the time t_1. At time t_1 the state changed abruptly from $x(t_1^-)$ to $x(t_1^+)$, where $x(t_1^-)$ and $x(t_1^+)$ denote the (limit) values of x just before and just after the state jump, respectively. It is important to realize that the transition times are not necessarily prescribed by some clock (time events), but usually depend on both the discrete and the continuous state. For instance, in the thermostat example these transition times were determined by the temperature T reaching the values $T_{\min}$ or $T_{\max}$ (state events). In summary:

> The trajectory of hybrid systems is partitioned into several time intervals. At the interval borders, the discrete state changes and/or jumps of the continuous state occur, whereas within all intervals the continuous signals change smoothly and the discrete state remains constant.

For hybrid system with inputs, the behavior also depends upon the input signals. In this case the time instant at which the discrete state jumps, the new discrete and continuous states that are assumed afterwards as well as the continuous state evolution are all affected by these inputs.

1.1.3 Hybrid dynamical phenomena

Appropriate models for hybrid systems are often obtained by adding new dynamical phenomena to the classical description formats of the mono-disciplinary research areas. Indeed, continuous models represented by differential or difference equations, as adopted by the dynamics and control community, have to be extended to be suitable for describing hybrid systems. On the other hand, the discrete models used in computer science such as automata or finite state machines, need to be extended by concepts like time, clocks, and continuous evolution in order to capture the mixed discrete and continuous evolution in hybrid systems. The hybrid system models explained in Part I of this handbook combine both ideas. Here we will describe the phenomena one has to add to the continuous models based on the differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{1.1}$$

Roughly speaking, as also argued in the previous discussion, four new phenomena that are typical for hybrid systems are required to extend the dynamics of purely continuous systems as in (1.1):

- autonomous switching of the dynamics
- autonomous state jumps
- controlled switching of the dynamics
- controlled state jumps.

These phenomena are first explained for autonomous hybrid systems.

Autonomous switching of the dynamics reflects the fact that the vector field $\boldsymbol{f}$ that occurs in (1.1) is changed discontinuously. The switching may be invoked by a clock if the vector field $\boldsymbol{f}$ depends explicitly on the time t:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t).$$

For instance, if periodic switching between two different modes of operation is used with period $2T$, we would have

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t) := \begin{cases} f_1(\boldsymbol{x}(t)), & \text{if } t \in [2kT, (2k+1)T) \text{ for some } k \in \mathbb{N}, \\ f_2(\boldsymbol{x}(t)), & \text{if } t \in [(2k+1)T, (2k+2)T) \text{ for some } k \in \mathbb{N}. \end{cases}$$

This is an example of *time-driven switching*.

The switching can also be invoked when the continuous state $\boldsymbol{x}$ reaches some *switching set* $\mathcal{S}$. As the situation $\boldsymbol{x}(t) \in \mathcal{S}$ is considered to be a state event, this kind of switching is said to be *event-driven*. The thermostat example provided an illustration of event-driven switching as the transition from the "on" mode to the "off" mode was triggered by the temperature reaching the value $T_{\max}$.

The following example also illustrates event-driven switching.

Example 1.2 *Hybrid tank system*

The tank systems shown in Fig. 1.5 illustrate two situations in which the dynamics of a process changes in dependence upon the state (liquid level). The tank in the left part of the figure is filled by the pump, which is assumed to deliver a constant flow Q_{P}, and emptied by two outlet pipes, whose outflows $Q_1(t)$ and $Q_2(t)$ depend upon the level $h(t)$. As the flow $Q_2(t)$ vanishes if the liquid level is below the threshold h_{p} given by the position of the upper pipe, the dynamical properties of the tank change if the level $h(t)$ exceeds this threshold.

The dependence of the vector field upon the state can be simply written down. For $h(t) < h_{\mathrm{v}}$, the differential equation is given by

$$\dot{h}(t) = \frac{1}{A}(Q_{\mathrm{P}} - \sqrt{2gh(t)}) = f_1(h(t)),$$

where g denotes the gravity constant. For $h(t) \geq h_{\mathrm{v}}$ this equation changes towards

$$\dot{h}(t) = \frac{1}{A}(Q_{\mathrm{P}} - \sqrt{2gh(t)} - \sqrt{2g(h(t) - h_{\mathrm{v}})}) = f_2(h(t)).$$

Hence, the model can be written as

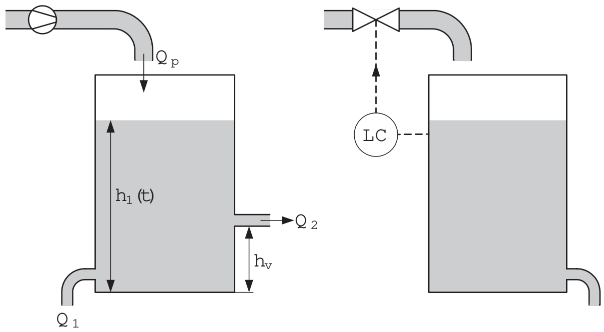

Fig. 1.5. Hybrid tank systems

$$\dot{h}(t) = \begin{cases} f_1(h(t)) & \text{if } h(t) < h_\mathrm{v} \\ f_2(h(t)) & \text{if } h(t) \geq h_\mathrm{v}, \end{cases}$$

which shows that the vector field switches between two different functions f_1 and f_2 in dependence upon the state $h(t)$ with the switching surface

$$\mathcal{S} = \{h \in \mathbb{R} \,|\, h = h_\mathrm{v}\}.$$

Now assume that the pump is switched on and off at different time instances t_1 and t_2. Then the function f occurring in the differential equation changes at these time points but this switching does not depend upon the state $h(t)$, but is time-driven.

The tank in the right part of Fig. 1.5 illustrates that autonomous switching is a typical phenomenon introduced by safety measures. In the tank system the level controller is equipped with a safety switch-off. If the liquid level is below the corresponding threshold, the dynamics is given by the controlled tank. If the level exceeds the threshold, the pump is switched off, which brings about a corresponding switching of the differential equation of the tank. □

Switching among different dynamics has important consequences for the behavior of the hybrid system. For instance, switching between two linear stable vector fields can result in an unstable overall system.

Autonomous state jumps constitute the second hybrid phenomenon. At some time $\bar{t}$, the state may jump from the value $\boldsymbol{x}(\bar{t}^-)$ towards the value $\boldsymbol{x}(\bar{t}^+)$. An illustrative example is a bouncing ball. If the ball touches the ground at time $\bar{t}$, then its velocity is instantaneously reversed.

A simple representation of state jumps is given as follows. An *autonomous jump set* is a set $\mathcal{S}$ on which a state jump is invoked (Fig. 1.6). Some relation R, which often is called a *reset map*, determines where the state jumps goes to:

$$(\boldsymbol{x}(\bar{t}^-), \boldsymbol{x}(\bar{t}^+)) \in \mathcal{R}.$$

Here $\bar{t}$ is the time instant at which the trajectory $\boldsymbol{x}(\cdot)$ reaches the set $\mathcal{S}$:

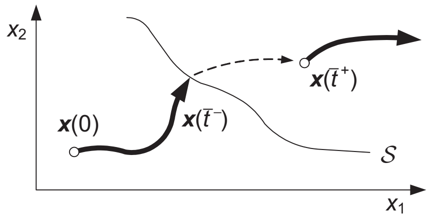

Fig. 1.6. Autonomous state jump

$$\boldsymbol{x}(\bar{t}) \in \mathcal{S}.$$

The reset map may depend on the discrete state $q(\bar{t}^-)$ of the hybrid system just before the reset. Including such state jumps in a continuous system described by the differential equation (1.1) results in the extended model

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t)), && \text{for } \boldsymbol{x}(t) \notin \mathcal{S} \\ (\boldsymbol{x}(t^-), \boldsymbol{x}(t^+)) &\in \mathcal{R}(q(\bar{t}^-)), && \text{for } \boldsymbol{x}(t) \in \mathcal{S}. \end{aligned}$$

Example 1.3 *Reset oscillator*

Consider the reset oscillator described by the affine state space model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2\delta \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

together with the reset map defined by

$$x_1(\bar{t}^+) = -x_1(\bar{t}^-),$$

where $\bar{t}$ denotes any time instant at which the state is on the switching set

$$\mathcal{S} = \{x \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 < 0\}.$$

Such reset systems find application, for instance, in data transmission over highly disturbed communication channels and reset control systems.

Figure 1.7 shows the behavior of the reset oscillator for $\delta = 0.1$. The left plot includes the trajectory in the state space for the short time interval $t \in [0, 10]$. The trajectory starts in the initial state $\boldsymbol{x}(0) = (0.2\ \ 0)^{\mathrm{T}}$ depicted by the small circle. The switching surface $\mathcal{S}$ is hit in the point $(-0.504, 0)^{\mathrm{T}}$ as indicated by the left diamond. Next, a state jump occurs that brings the state to the right diamond. The right plot shows the oscillator behavior for a longer time horizon.

The state jump has two consequences:

- Although the oscillator has an affine state space model, the behavior of the reset oscillator is chaotic.
- Although the oscillator without state jump is an unstable system (with eigenvalues $\lambda_{1,2} = 0.1 \pm \mathrm{j}0.995$) the reset oscillator state remains bounded.

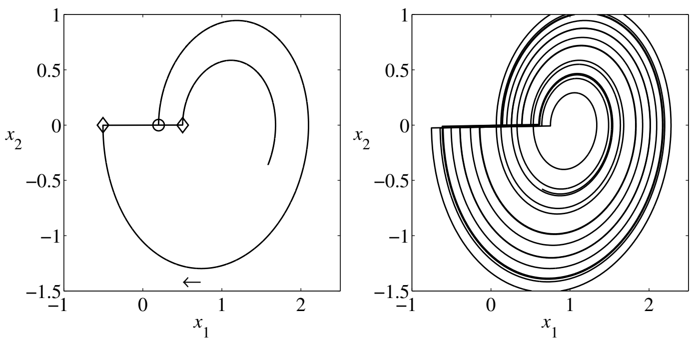

Fig. 1.7. Behavior of the reset oscillator

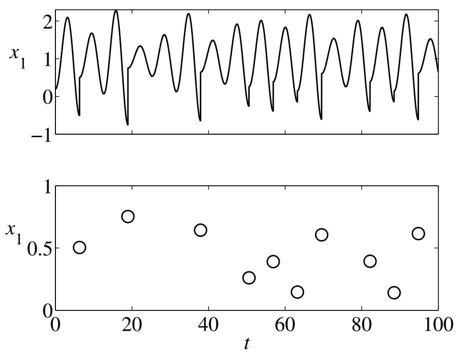

Fig. 1.8. Trajectory of the reset oscillator: state trajectory $x_1(t)$ (upper part) and destination state sequence $x_1(\bar{t}_k^+)$ of the state jumps (lower part)

The irregular (chaotic) behavior can be seen in Fig. 1.8, where in the upper plot the state trajectory x_1 is shown. The lower plot extracts the state jumps from the evolution. The circles depict the points $x_1(\bar{t}_k^+)$ just after the occurrence of the state jumps at the time instants $\bar{t}_k^+$, $(k = 0, 1, ...)$. Neither the temporal distance $\bar{t}_{k+1}^+ - \bar{t}_k^+$ between these jumps nor the endpoints $x_1(\bar{t}_k^+)$ show a regular behavior. □

The above example shows that state jumps may considerably change the dynamical properties of a system in comparison to the same system without state jumps.

Controlled switching occurs if the system has a discrete input v that is used to invoke the switching among different continuous dynamics. If the value of the discrete input is changed at time $\bar{t}$, then the vector field $\boldsymbol{f}(\boldsymbol{x}(t), v(t))$ changes abruptly at time $\bar{t}$ as well.

Systems with discrete control inputs represent a relevant system class from a practical point of view. The DC-DC converter is a simple example of such systems that will be used as running example throughout this handbook (cf. Section 1.3.3).

Controlled state jumps are discontinuities in the state trajectory that occur as a response to a control command. An example in which such a state jump is necessary for satisfying performance requirements is the automatic gearbox described in Section 1.3.2. A state jump in the gearbox controller must be invoked whenever the gearing is changed in order to avoid a jump in the acceleration of the vehicle.

1.2 Models of hybrid systems

Although many different models have been proposed in literature as will be seen in the next chapters, the model ingredients (including the main dynamical phenomena as seen in the previous section) are basically the same.

1.2.1 Model ingredients

The structure of hybrid systems introduced so far shows that every model of a hybrid system has to define at least the following elements (Fig. 1.9):

- $\mathcal{X}$ – the continuous state space, for which often $\mathcal{X} = \mathbb{R}^n$ holds,
- $\mathcal{Q}$ – the discrete state space, for example $\mathcal{Q} = \{0, 1, 2, ..., Q\}$,
- $\boldsymbol{f}$ – a set of vector fields describing the continuous dynamics for all $q \in \mathcal{Q}$,
- Init – a set of initial values $(q_0, \boldsymbol{x}_0)$ of the hybrid state,
- δ – the discrete state transition function,
- $\mathcal{G}$ – a set of guards prescribing when a discrete state transition occurs.

To simplify the considerations, hybrid systems without external inputs are investigated here.

The model elements lead to a graphical representation of the hybrid system. The discrete part of the dynamics is modeled by means of a graph whose vertices represent the discrete states (also called *operation modes*) and whose edges represent state transitions. Every vertex is associated with the vector field

$$\boldsymbol{f} : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$$

that belongs to the corresponding value q of the discrete state. It describes the evolution of the continuous state if the discrete state is $q(t)$:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(q(t), \boldsymbol{x}(t)).$$

The trajectories that can be obtained for all possible initial continuous states is also called the set of *activities*.

Whereas the discrete state q influences the continuous dynamics by selecting a specific vector field $\boldsymbol{f}(q, .)$, the influence of the continuous dynamics

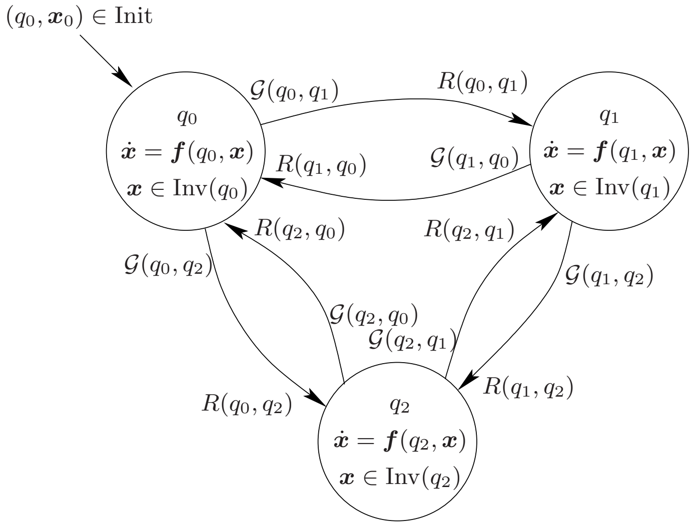

Fig. 1.9. Schematic representation of a hybrid automaton with three discrete states. Each node of the directed graph represents a mode (operating point) given by a system of differential (or difference) equations. The arrows indicate the possible discrete transitions that correspond to a change of the mode.

on the discrete state evolution is represented by a set of guards. A guard describes a region in the state space $\mathcal{X}$. If the state $\boldsymbol{x}$ is in this region, a discrete state transition *may* occur. For example, in Fig. 1.9, the guard $\mathcal{G}(q_0, q_1)$ poses a condition on the state $\boldsymbol{x}$ that has to be satisfied in order to invoke the discrete state transition $q_0 \to q_1$.

The change of the discrete state is described by the state transition function δ, which determines the discrete successor state q' if the system is in a given discrete state q. This function is graphically represented by the arrows among the discrete states in Fig. 1.9. As the figure shows, the question of which successor state is assumed depends upon the guard condition $\mathcal{G}$ that is satisfied by the continuous state x at the switching time.

The model explained above can be extended by the following elements in order to include state jumps and to improve the representation of the interaction between the continuous and the discrete dynamics:

- $\mathcal{R}$ – Reset map defining the state jumps,
- Inv – Invariants.

Each mode has an invariant associated to it, which describes the conditions that the continuous state has to satisfy at this mode. Invariants and guards play complementary roles: whereas invariants describe when a transition *must* take place (namely when otherwise the motion of the continuous state as described in the set of activities would lead to violation of the conditions given by the invariant), the guards serve as "enabling conditions" that describe when a particular transition *may* take place.

The reset map is, in general, a set-valued function that specifies how new continuous states are related to previous continuous states for a particular transition.

1.2.2 Model behavior

To provide insight in the evolution of the dynamical system defined above, we give a short, rather informal description. The initial hybrid state $(q_0, \boldsymbol{x}_0)$ of "trajectories" of a hybrid automaton lies in the initial set Init. From this hybrid state the continuous state $\boldsymbol{x}$ evolves according to the differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(q_0, \boldsymbol{x}) \quad \text{with } \boldsymbol{x}(0) = \boldsymbol{x}_0$$

and the discrete state q remains constant: $q(t) = q_0$. The continuous evolution can go on as long as $\boldsymbol{x}$ stays in $\mathrm{Inv}(q_0)$. If at some point the continuous state $\boldsymbol{x}$ reaches the guard $\mathcal{G}(q_0, q_1)$, we say that the transition (q_0, q_1) is enabled. The discrete state may then change to q_1, and the continuous state jumps from the current value $\boldsymbol{x}^-$ to a new value $\boldsymbol{x}^+$ with $(\boldsymbol{x}^-, \boldsymbol{x}^+) \in R(q_0, q_1)$. After this transition, the continuous evolution resumes according to the mode q_1 and the whole process is repeated. Note that the invariants and guards are related to the switching sets and jumps sets introduced earlier, as all these concepts are related to triggering discrete actions such as resets of the continuous states or changes in the discrete state.

This framework leads to the behavior of a hybrid system as depicted in Fig 1.4: continuous phases separated by events at which maybe multiple discrete actions (jumps of the continuous state $\boldsymbol{x}$ and/or changes in the discrete state q) take place. It is obvious that these systems may switch between many operating modes where each mode is governed by its own vector field (Fig. 1.9). Mode transitions are triggered by variables crossing specific thresholds (state events) and by the elapse of certain time periods (time events) due to the invariants and guards. With a change of mode, discontinuities in the continuous variables may occur as given by the reset map.

1.2.3 Hybrid automata

The model ingredients introduced above directly lead to one of the main modeling formalisms used in hybrid systems theory: the hybrid automaton. Below we provide an informal definition, where $2^{\mathcal{X}}$ denotes the power set of $\mathcal{X}$, i.e. the collection of all subsets of $\mathcal{X}$:

A hybrid automaton H is an 8-tuple

$$H = (\mathcal{Q}, \mathcal{X}, \boldsymbol{f}, \mathrm{Init}, \mathrm{Inv}, \mathcal{E}, \mathcal{G}, \mathcal{R})$$

with

- $\mathcal{Q} = \{q_1, \ldots, q_k\}$ is a finite set of *discrete states (control locations)*;
- $\mathcal{X}$ is the continuous state space;
- $\boldsymbol{f} : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector field;
- $\mathrm{Init} \subset \mathcal{Q} \times \mathbb{R}^n$ is the set of *initial states*;
- $\mathrm{Inv} : \mathcal{Q} \to 2^{\mathbb{R}^n}$ describe the *invariants* of the locations;
- $\mathcal{E} \subseteq \mathcal{Q} \times \mathcal{Q}$ is the *transition relation*;
- $\mathcal{G} : \mathcal{E} \to 2^{\mathbb{R}^n}$ is the *guard condition*;
- $\mathcal{R} : \mathcal{E} \to 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n}$ is the *reset map*;

The hybrid state of the system H is given by $(q, \boldsymbol{x}) \in Q \times \mathcal{X}$.

Based on the description of this general hybrid system model various ramifications and extensions can be created as well as other more specific models of hybrid systems such as piecewise affine systems, mixed logical dynamical systems, complementarity systems, and so on. This handbook will provide an overview of the available results for all these model classes and will also pinpoint various open issues for future research. Before doing so, we will introduce some running examples that will be used throughout the handbook to illustrate the main ideas.

1.3 Running examples

This section introduces three simple examples that illustrate the main new phenomena that are introduced by the interaction of continuous and discrete dynamics. These examples will return frequently later on in this handbook.

1.3.1 Two-tank system

Process description. The two-tank system is a hybrid system with autonomous switching. The main control task is to stabilize its state. The system represents a simplified version of systems that are widely used in the process industry to provide a costumer with a continuous liquid flow by maintaining the liquid levels of the tanks at prescribed values.

This example consists of two coupled cylindrical tanks T_1 and T_2 connected by pipes (Fig. 1.10). The water flow between the tanks and out of the tanks can be controlled by the valves V_1, V_2, V_3, $V_{1\mathrm{L}}$, and $V_{2\mathrm{L}}$, each of which can only be completely opened or closed (on/off valves). The connection pipes between the tanks are placed at the bottom of the tanks (with valve V_2) and at the height h_0 above the bottom (with valve V_1).

The maximum water level of each tank is denoted by $h_{\max}$. All tanks have the same cross-sectional area A and are located at the same level.

In a typical situation, the valves V_1, V_2, and V_3 are opened and the valves $V_{1\mathrm{L}}$ and $V_{2\mathrm{L}}$ closed. Liquid is filled into the left tank by the pump P_1. Measurements concern the levels $h_1(t)$ and $h_2(t)$ in tanks T_1 and T_2 respectively.

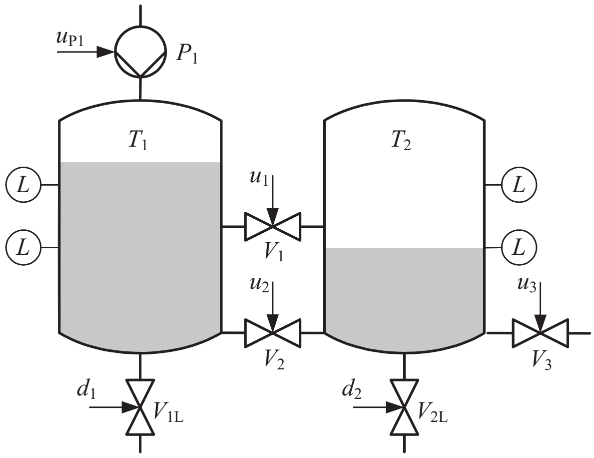

Fig. 1.10. Two-tank system

Discrete sensors (denoted by L in the figure) yield a qualitative characterization of the liquid levels as *low*, *medium*, and *high*.

The system has both continuous and discrete inputs. The continuous input is the inflow through the pump $u_{P1}(t) = Q^{P_1}(t)$ and the discrete inputs are the positions of the valves V_1, V_2, and V_3, so that

$$\boldsymbol{u}(t) = (u_{P1}(t) \;\; u_1(t) \;\; u_2(t) \;\; u_3(t))^{\mathrm{T}}$$

holds. Disturbances affecting the system can be induced by changing the positions of the valves V_{1L} and V_{2L}.

Hybrid phenomena. The two-tank is a typical hybrid system, as it has a continuous dynamics with state-dependent and controlled switching. If the valve positions remain constant the continuous dynamics switches autonomously between four discrete modes $q(t)$ depending on whether or not the liquid levels exceed the height h_0 of the upper connection pipe. The discrete system behavior is represented by the automaton shown in Fig. 1.11, where each node represents one discrete operation mode.

Dynamical model. The two-tank system has two continuous state variables

$$\boldsymbol{x}(t) = (h_1(t) \;\; h_2(t))^{\mathrm{T}} \; , \; h_i \in \mathbb{R}$$

and four discrete states

$$q(t) \in \{1, 2, 3, 4\}$$

that depend on the levels as shown in Table 1.1. The nonlinear dynamics follows from Torricelli's law:

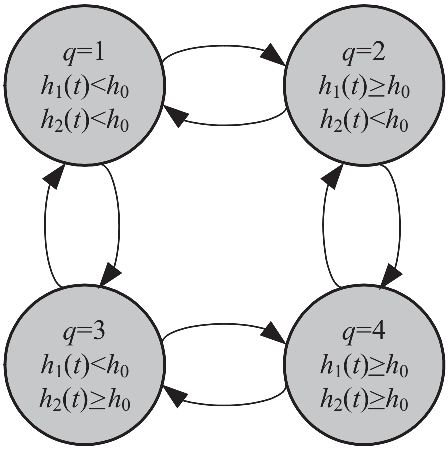

Fig. 1.11. Discrete behavior of the two-tank system

Table 1.1. Discrete modes in dependence of the continuous states

$q(t)$	$h_1(t)$	$h_2(t)$
1	$< h_0$	$< h_0$
2	$\geq h_0$	$< h_0$
3	$< h_0$	$\geq h_0$
4	$\geq h_0$	$\geq h_0$

$$Q_{ij}^{\mathrm{V}_l}(t) = c \cdot \mathrm{sgn}(h_i(t) - h_j(t)) \cdot \sqrt{2g \cdot \mid h_i(t) - h_j(t) \mid} \cdot u_l(t),$$

where $Q_{ij}^{\mathrm{V}_l}(t)$ is the water flow from tank T_i into tank T_j through the pipe with valve V_l, c the flow constant of the valves, $u_l(t) \in\{0,1\}$ the position of valve V_L (0 means the valve is closed and 1 the valve is opened), and g the gravity constant.

The change of water volume $V(t)$ in a tank can be described by

$$\dot{V}(t) = \dot{h}(t) \cdot A = \sum Q_{\mathrm{in}}(t) - \sum Q_{\mathrm{out}}(t),$$

where $\sum Q_{\mathrm{in}}(t)$ is the sum of all inflows into the tank and $\sum Q_{\mathrm{out}}(t)$ the sum of all outflows. By applying this equation to the two tanks, the following nonlinear differential equations are obtained:

$$\dot{h}_1(t) = \frac{u_{\mathrm{P}_1}(t) - Q_{12}^{\mathrm{V}_1}(t) - Q_{12}^{\mathrm{V}_2}(t) - Q_{\mathrm{L}}^{\mathrm{V}_{1\mathrm{L}}}(t)}{A}$$
$$\dot{h}_2(t) = \frac{Q_{12}^{\mathrm{V}_1}(t) + Q_{12}^{\mathrm{V}_2}(t) - Q_{\mathrm{L}}^{\mathrm{V}_{2\mathrm{L}}}(t) - Q_{\mathrm{N}}^{\mathrm{V}_3}(t)}{A}.$$

The flow $Q_{12}^{\mathrm{V}_1}(t)$ depends on the mode $q(t)$ as follows:

$$Q_{12}^{\mathrm{V}_1}(t) = \begin{cases} 0, & q(t) = 1 \\ c \cdot \mathrm{sgn}(h_1(t) - h_0) \cdot \sqrt{2g \cdot \mid h_1(t) - h_0 \mid} \cdot u_1(t), & q(t) = 2 \\ c \cdot \mathrm{sgn}(h_0 - h_2(t)) \cdot \sqrt{2g \cdot \mid h_0 - h_2(t) \mid} \cdot u_1(t), & q(t) = 3 \\ c \cdot \mathrm{sgn}(h_1(t) - h_2(t)) \cdot \sqrt{2g \cdot \mid h_1(t) - h_2(t) \mid} \cdot u_1(t), & q(t) = 4. \end{cases}$$

The following equations hold in all four modes:

$$\begin{aligned} Q_{12}^{\mathrm{V}_2}(t) &= c \cdot \mathrm{sgn}(h_1(t) - h_2(t)) \cdot \sqrt{2g \cdot \mid h_1(t) - h_2(t) \mid} \cdot u_2(t) \\ Q_{\mathrm{N}}^{\mathrm{V}_3}(t) &= c \cdot \sqrt{2g \cdot h_2(t)} \cdot u_3(t) \\ Q_{\mathrm{L}}^{\mathrm{V}_{i\mathrm{L}}} &= c \cdot \sqrt{2g \cdot h_i(t)} \cdot d_i(t), \quad i = 1, 2 \end{aligned}$$

where $Q_{\mathrm{N}}^{\mathrm{V}_3}(t)$ is the water flow exiting from tank T_2 through the pipe with valve V_3, and $Q_{\mathrm{L}}^{\mathrm{V}_{i\mathrm{L}}}$ is the water flow exiting from tank T_i through the pipe with valve $V_{i\mathrm{L}}$. If these differential and algebraic equations are associated with the discrete model shown in Fig. 1.11, a hybrid automaton results as overall model.

All relevant parameter values are given in Table 1.2.

Table 1.2. Parameter values and ranges of the two-tank system

Parameter	Value
c	$3.6\mathrm{e}^{-5}$ m^2
h_0	0.3 m
$h_{\max}$	0.6 m
A	0.0154 m^2
g	9.81 m s^{-2}
$Q_{\max}$	$0.1\mathrm{e}^{-3}$ m^3 s^{-1}
Qualitative value	**Range**
low	[0...20] cm
medium	[20...25] cm
high	[25...60] cm
State	**Value**
$h_1,\ h_2$	$\in \mathbb{R}$ cm
q	$\in \{1, 2, 3, 4\}$
Input	**Possible value/range**
$u_1,\ u_2,\ u_3$	$\in \{0,1\}$
$u_{\mathrm{P}1} = Q^{\mathrm{P}_1}$	$\in [0,\ Q_{\max}]$ m^3 s^{-1}
Disturbance	**Value**
$d_1,\ d_2$	$\in \{0,1\}$

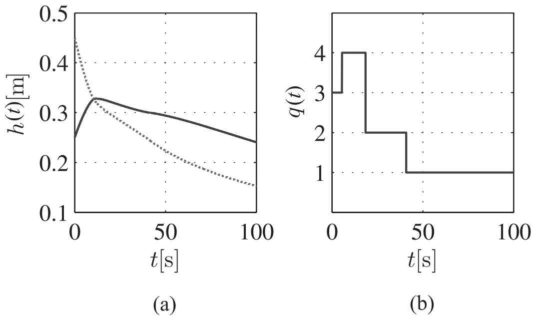

Fig. 1.12. Simulation results of the two-tank system

Hybrid behavior. Figure 1.12 shows the behavior of the two-tank system with the initial state $\boldsymbol{x}_0 = (0.25 \quad 0.45)^{\mathrm{T}}$ and constant inflow $u_{\mathrm{P1}}(t) = 0.03\ \mathrm{m}^3\ \mathrm{s}^{-1}$. Figure 1.12(a) depicts the trajectories of the tank levels and Fig. 1.12(b) the related modes demonstrating the autonomous switching of the system, when the height h_0 is reached.

The two-tank system offers various possibilities to illustrate the main analysis and design concepts presented in the handbook. One potential question is whether or not the system state depicted in Fig. 1.13 can be reached by an appropriate control input.

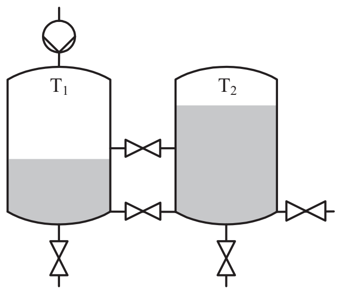

Fig. 1.13. Is this a potential state of the two-tank system?

1.3.2 Automatic gearbox

Process description. The automatic gearbox is a switched system with a discontinuous evolution of the continuous state. State jumps occur together with controlled switching.

Automatic gearboxes are used to change gear ratios automatically. This example presents an automatic transmission with four gears. It consists of the gearbox and a controller comprised of a continuous and a discrete-event part as depicted in Fig. 1.14.

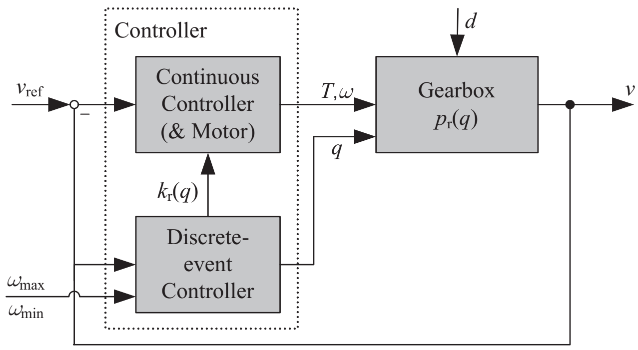

Fig. 1.14. Automatic gearbox

The gearbox and its controller interact dependent on the vehicle velocity $v(t)$. The continuous inputs of the gearbox are the torque $u(t) = T(t)$ and the angular velocity $\omega(t)$ of the motor. Disturbances $d(t)$ are induced by the road, e.g. by different coefficients of friction.

Hybrid phenomena and dynamical model. The automatic gearbox has four discrete modes $q(t)$

$$q(t) \in \{1, 2, 3, 4\}$$

that affect the continuous dynamics by changing the transmission ration $p_r(q)$. Table 1.3 presents all modes with their specific parameters $p_r(q)$ and $k_r(q)$, where $p_r(1) > p_r(2) > p_r(3) > p_r(4)$ holds. The mode is automatically changed by the discrete inputs selected by the controller.

Table 1.3. Modes and specific parameter

$q(t)$	Transmission ration	Controller gain
1	$p_r(1)$	$k_r(1)$
2	$p_r(2)$	$k_r(2)$
3	$p_r(3)$	$k_r(3)$
4	$p_r(4)$	$k_r(4)$

The continuous part of the controller consists of a PI-controller with the integrator state $T_I(t)$. To obtain a comfortable ride, restrictions are imposed on the derivative of the acceleration $\ddot{v}(t)$, which make it necessary to switch the controller parameters $k_\mathrm{r}(q)$ dependent on the gearing $q(t)$ and to impose state jumps in the integrator state at the times when the gear is changed. In Fig. 1.15 the switching scheme of the automatic gearbox is depicted.

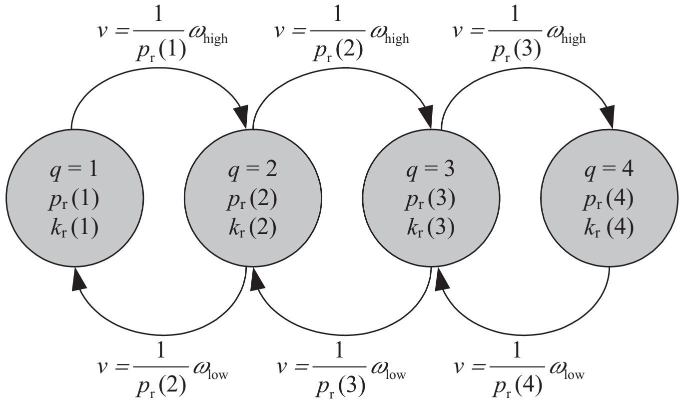

Fig. 1.15. Hybrid automaton of the automatic gearbox

Dynamical model. The gearbox is modeled here with only the velocity as a continuous state variable

$$\boldsymbol{x}(t) = (v(t) \;\; T_\mathrm{I}(t))^\mathrm{T}.$$

The transmission influences the torque $T(t)$ and the angular velocity $\omega(t)$ of the motor according to

$$\begin{aligned} T_\mathrm{w}(t) &= p(q)\, T(t) = p(q)\, u(t) \\ \omega_\mathrm{w}(t) &= \frac{1}{p(q)}\omega(t), \end{aligned}$$

where $T_\mathrm{w}(t)$, $\omega_\mathrm{w}(t)$ are respectively the torque and the velocity of the wheels, and $p(q)$ denotes the transmission ratio of the gearbox, which is obtained from the ratio $p_\mathrm{r}(q)$ used in Table 1.5 by

$$p_\mathrm{r}(q) = \frac{p(q)}{r},$$

where r is the wheel radius.

The relations between the torque and the force $F(t)$ accelerating the vehicle and between the velocity of the vehicle and the angular velocity are given by

$$\begin{aligned} F(t) &= \frac{T_\mathrm{w}(t)}{r} \\ v(t) &= r\, \omega_\mathrm{w}(t). \end{aligned}$$

Applying Newton's law of motion leads to the vehicle dynamics

$$m\dot{v}(t) = F(t) - F_{\mathrm{l}}(t)$$

and

$$\dot{v}(t) = \frac{p_{\mathrm{r}}(q)u(t)}{m} - \frac{c}{m}v(t)^2 \mathrm{sign}\, v(t) - g\,\sin(d(t)),$$

where m is the mass of the vehicle and the latter two terms represent the load force $F_{\mathrm{l}}(t)$ which is assumed to be proportional to the square of the velocity. The disturbance $d(t)$ is considered as the road angle.

Control tasks. A gear change should occur if $\omega(t)$ reaches a *high* or *low* limit ω_{high} and ω_{low}, respectively (Fig. 1.15). According to the relation $\omega(t) = p_{\mathrm{r}}(q)v(t)$ the limits correspond to certain velocities of the vehicle. The mode shifts are given by the switching sets

$$\begin{aligned} S_{q,q+1} &= \{v \in \mathbb{R} \mid v = \frac{1}{p_{\mathrm{r}}(q)}\omega_{\mathrm{high}}\} \\ S_{q+1,q} &= \{v \in \mathbb{R} \mid v = \frac{1}{p_{\mathrm{r}}(q+1)}\omega_{\mathrm{low}}\}, \end{aligned}$$

where $S_{q,q+1}$ denotes a mode shift from mode q to $q+1$ and $S_{q+1,q}$ a mode shift from mode $q+1$ to q.

The continuous PI-control law with a compensation of the nonlinearities is given by

$$u(t) = T_{\mathrm{P}}(t) + T_{\mathrm{I}}(t) + \frac{c}{p_{\mathrm{r}}(q)}v(t)^2 \mathrm{sign}\, v(t),$$

with

$$\begin{aligned} T_{\mathrm{P}}(t) &= k_{\mathrm{r}}(q)(v_{\mathrm{ref}}(t) - v(t)) \\ \dot{T}_{\mathrm{I}}(t) &= \frac{k_{\mathrm{r}}(q)}{T_{\mathrm{R}}}(v_{\mathrm{ref}}(t) - v(t)) \end{aligned}$$

where T_{R} is the integration time constant. Every time a new value of the set point $v_{\mathrm{ref}}(t)$ is fixed by the driver, the integrator state $T_{\mathrm{I}}(t)$ is put to zero (controlled state jump).

The ride is comfortable if the acceleration is limited, which causes a restriction on the gain $k_{\mathrm{r}}(q)$, and if restrictions on the derivative of the acceleration are imposed. If $k_{\mathrm{r}}(q)$ takes a value out of the set

$$k_{\mathrm{r}}(q) \in \{k_{\mathrm{r}}(1), k_{\mathrm{r}}(2), k_{\mathrm{r}}(3), k_{\mathrm{r}}(4)\}$$

no abrupt changes of $\ddot{v}(t)$ and $\dot{v}(t)$ occur due to a mode shift if

$$\begin{aligned} p_{\mathrm{r}}(q)k_{\mathrm{r}}(q) &= p_{\mathrm{r}}(q+1)k_{\mathrm{r}}(q+1) \\ p_{\mathrm{r}}(q)T_{\mathrm{I}}(\bar{t}^-) &= p_{\mathrm{r}}(q+1)T_{\mathrm{I}}(\bar{t}^+), \quad \text{mode change } q \rightarrow q+1 \\ p_{\mathrm{r}}(q)T_{\mathrm{I}}(\bar{t}^-) &= p_{\mathrm{r}}(q-1)T_{\mathrm{I}}(\bar{t}^+), \quad \text{mode change } q \rightarrow q-1. \end{aligned}$$

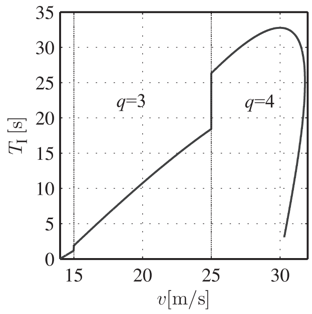

Fig. 1.16. Trajectory in the state space

This is called *bumpless* transfer.

Hybrid behavior. Figure 1.16 illustrates a trajectory in the state space including jumps in the integrator state at the time the gear changes. The desired velocity $v_{\mathrm{ref}}(t)$ is set to 30 m/s and the limits ω_{high} and ω_{low} are equal to 500 rad/s or 230 rad/s, respectively. Table 1.4 contains all relevant parameter values.

Table 1.4. Parameter values of the automatic gearbox

Parameter	Value
$p_{\mathrm{r}}(1)$, $p_{\mathrm{r}}(2)$, $p_{\mathrm{r}}(3)$, $p_{\mathrm{r}}(4)$	50, 32, 20, 14
c	0.7 kg m^{-1}
m	1500 kg
g	10 m s^{-2}
$k_{\mathrm{r}}(1)$, $k_{\mathrm{r}}(2)$, $k_{\mathrm{r}}(3)$, $k_{\mathrm{r}}(4)$,	3.75, 5.86, 9.375, 13.39 N s
T_{R}	40 s
State	**Value**
v, T_{I}	$\in \mathbb{R}$ m s^{-1}, Nm
$v(0)$, $T_{\mathrm{I}}(0)$	14 m s^{-1}, 0 Nm
q	$\in \{1, 2, 3, 4\}$
$q(0)$	2
Input	**Value**
u	$\in \mathbb{R}$ Nm
v_{ref}	30 m s^{-1}
ω_{high}	500 rad s^{-1}
ω_{low}	230 rad s^{-1}
Disturbance	**Value**
d	$\in [-\frac{\pi}{2}, ..., \frac{\pi}{2}]$ rad

1.3.3 DC-DC converter

Process description. The DC-DC converter is a switched system, where the controlled switching is necessary for retaining its function of stabilizing the output voltage. The system has to be stabilized at a limit cycle rather than in an equilibrium state.

Power converters are widespread industrial devices used, for example, in variable-speed DC motor drives, computer power supply, cell phones, and cameras. The main functional principle lies in switching an electrical circuit among different structures in order to transform a constant or slowly varying DC voltage into a DC voltage that is independent of the load.

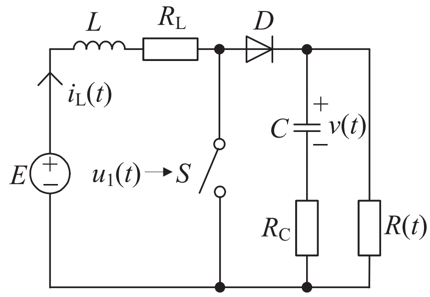

Fig. 1.17. Boost converter

This example deals with a boost converter, whose output voltage is higher than the input voltage. The topology of a DC-DC boost converter is depicted in Fig. 1.17. The circuit consists of a load R, a capacitor C, an inductor L, a diode D, and a switch S. It has a fixed input voltage E and a variable output voltage $v(t)$. Moreover, R_C and R_L represent the series resistors of the capacitor and the inductor.

In the *on-state* the switch is closed, resulting in an increase of the inductor current $i_L(t)$. In the *off-state* the switch is open. The only path for the current is through the flyback diode, the capacitor and the load, and then the current ramps down. This situation results in transferring the energy accumulated in the inductance during the on-state into the capacitor. The process repeats cyclically, whereas the boost converter operates with the switching period T_S and the duty cycle $d_1(t)$, which corresponds to the ratio of activation duration of an *on-state* mode to the period. The duty cycle is considered as the system input $u_1(t) = d_1(t) \in [0, 1]$.

Hybrid phenomena. The boost converter is a hybrid system with the three operation modes

$$q(t) \in \{1, 2, 3\}$$

summarized in Table 1.5. It is a second-order system with the continuous state variables $i_L(t)$ and $v(t)$

$$\boldsymbol{x}(t) = (i_L(t) \;\; v(t))^T.$$

The switching scheme of the converter expressed by an automaton is shown in Fig. 1.18, where n denotes the cycle index.

Table 1.5. Modes of the boost converter

$q(t)$	S	$i_L(t)$
1	closed	$i_L > 0$
2	opened	$i_L > 0$
3	opened	$i_L = 0$

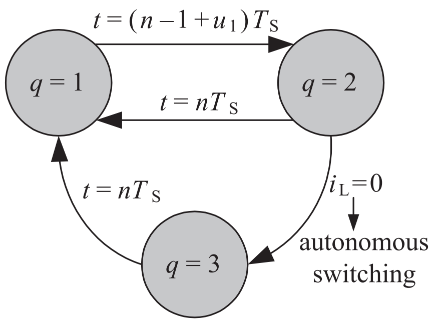

Fig. 1.18. Hybrid automaton of the boost converter

If the current through the inductor never falls to zero the boost converter operates in continuous conduction mode (CCM), in which the switch and the diode are turned on and off in a cyclic and complementary manner and merely the modes $q = 1$ and $q = 2$ are accessible. Switching to the third mode occurs autonomously if the current falls to zero (discontinuous conduction mode (DCM)).

Dynamical model. The affine state-space model of the converter

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(q)\boldsymbol{x}(t) + \boldsymbol{B}(q) \tag{1.2}$$

is given with the following matrices:

- $q(t) = 1$:

$$\boldsymbol{A}(1) = \begin{pmatrix} -\frac{R_{\mathrm{L}}}{L} & 0 \\ 0 & -\frac{1}{(R+R_{\mathrm{C}})C} \end{pmatrix}$$

$$\boldsymbol{B}(1) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} E$$

- $q(t) = 2:$

$$\boldsymbol{A}(2) = \begin{pmatrix} -\frac{1}{L}\left(R_{\mathrm{L}} + \frac{R_{\mathrm{C}}R}{R+R_{\mathrm{C}}}\right) & \frac{1}{L}\left(-1 + \frac{R}{R+R_{\mathrm{C}}}\right) \\ \frac{R}{(R+R_{\mathrm{C}})C} & -\frac{1}{(R+R_{\mathrm{C}})C} \end{pmatrix}$$

$$\boldsymbol{B}(2) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} E$$

- $q(t) = 3:$

$$\boldsymbol{A}(3) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{(R+R_{\mathrm{C}})C} \end{pmatrix}$$

$$\boldsymbol{B}(3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} E.$$

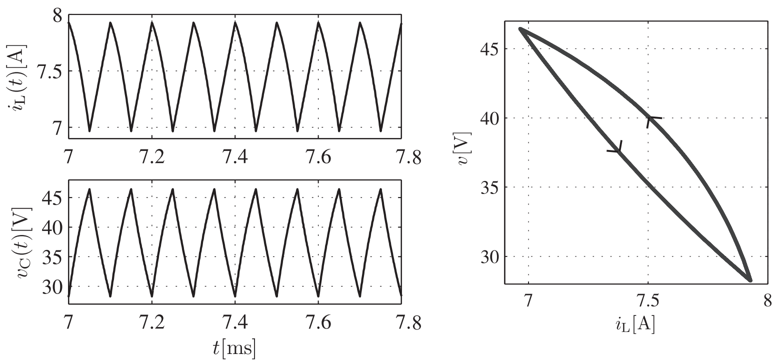

Fig. 1.19. Stationary behavior of the boost converter (left) and limit cycle (right)

Hybrid behavior. Figure 1.19 (left) shows the stationary behavior of $i_{\mathrm{L}}(t)$ and $v(t)$ of the boost converter operating in CCM with a fixed input $u_1(t) = 0.5$ and a constant disturbance $R(t) = 10\ \Omega$. In Fig. 1.19 (right) the stationary behavior represented by a limit cycle is shown. All parameter values of the converter are listed in Table 1.6.

Table 1.6. Values of circuit components

Component	Value
E	20 V
L	1 mH
R_{L}	0.1 Ω
C	10 μF
R_{C}	0.06 Ω
T_{S}	0.1 ms
State	Value
$i_{\mathrm{L}},\ v$	$\in \mathbb{R}$ A, V
q	$\in \{1, 2, 3\}$
Input	Value/range
u_1	$\in [0, 1]$
Disturbance	Values/ranges
R	$\in \mathbb{R}\ \Omega$ (10 Ω)

Bibliographical notes

There are several excellent introductions explaining the phenomena of hybrid dynamical systems, e.g. [Branicky, 2005].

Hybrid systems are dealt with in monographs, that all consider particular subclasses of hybrid models, such as [van der Schaft and Schumacher, 2002] focusing on complementarity systems, [Johansson, 2003] on piecewise linear systems, [Schrder, 2003] on quantized systems and [Liberzon, 2003] on switched systems. Collections of papers on hybrid systems, besides the proceedings of the annual conferences on hybrid systems, can be found in [Engell et al., 2002], [R. Johansson, 2002], [Hristu-Varsakelis and Levine, 2005], and [C. Cassandras, 2007].

Example 1.3 illustrates hybrid system behavior by means of a chaotic oscillator, which is described, for example, in [Saito, 1985; T. Tsubone, 1989].

The running examples represent rather simplified descriptions of real-world applications of hybrid systems, which are developed on the basis of the references [Blanke et al., 2006] for the two-tank system, [Middlebrook and Cuk, 1976] for the DC-DC converter.

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