3. Hale/Kocak: Dynamics and Bifurcations.
Jerrold E. Marsden    Tudor S. Ratiu

Introduction to Mechanics and Symmetry

A Basic Exposition of Classical Mechanical Systems

Second Edition

With 54 Illustrations
To Barbara and Lilian for their love and support
Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

*TAM* will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.
Preface

Symmetry and mechanics have been close partners since the time of the founding masters: Newton, Euler, Lagrange, Laplace, Poisson, Jacobi, Hamilton, Kelvin, Routh, Riemann, Noether, Poincaré, Einstein, Schrödinger, Cartan, Dirac, and to this day, symmetry has continued to play a strong role, especially with the modern work of Kolmogorov, Arnold, Moser, Kirillov, Kostant, Smale, Souriau, Guillemin, Sternberg, and many others. This book is about these developments, with an emphasis on concrete applications that we hope will make it accessible to a wide variety of readers, especially senior undergraduate and graduate students in science and engineering.

The geometric point of view in mechanics combined with solid analysis has been a phenomenal success in linking various diverse areas, both within and across standard disciplinary lines. It has provided both insight into fundamental issues in mechanics (such as variational and Hamiltonian structures in continuum mechanics, fluid mechanics, and plasma physics) and provided useful tools in specific models such as new stability and bifurcation criteria using the energy–Casimir and energy–momentum methods, new numerical codes based on geometrically exact update procedures and variational integrators, and new reorientation techniques in control theory and robotics.

Symmetry was already widely used in mechanics by the founders of the subject, and has been developed considerably in recent times in such diverse phenomena as reduction, stability, bifurcation and solution symmetry breaking relative to a given system symmetry group, methods of finding explicit solutions for integrable systems, and a deeper understanding of spe-
cial systems, such as the Kowalewski top. We hope this book will provide a reasonable avenue to, and foundation for, these exciting developments.

Because of the extensive and complex set of possible directions in which one can develop the theory, we have provided a fairly lengthy introduction. It is intended to be read lightly at the beginning and then consulted from time to time as the text itself is read.

This volume contains much of the basic theory of mechanics and should prove to be a useful foundation for further, as well as more specialized, topics. Due to space limitations we warn the reader that many important topics in mechanics are not treated in this volume. We are preparing a second volume on general reduction theory and its applications. With luck, a little support, and yet more hard work, it will be available in the near future.

Solutions Manual. A solution manual is available for instructors. It contains complete solutions to many of the exercises, as well as other supplementary comments. For further information, see 

http://www cds.caltech.edu/~marsden/books/.

Internet Supplements. To keep the size of the book within reason, we are making some material available (free) on the Internet. These are a collection of sections whose omission does not interfere with the main flow of the text. See http://www cds.caltech.edu/~marsden/books/. Updates and information about the book can also be found at this website.

What Is New in the Second Edition? In this second edition, the main structural changes are the creation of a solutions manual (along with many more exercises in the text) and the Internet supplements. The Internet supplements contain, for example, the material on the Maslov index that was not needed for the main flow of the book. As for the substance of the text, much of the book was rewritten throughout to improve the flow of material and to correct inaccuracies. Some examples: The material on the Hamilton–Jacobi theory was completely rewritten, a new section on Routh reduction (§8.9) was added, Chapter 9 on Lie groups was substantially improved and expanded. The presentation of examples of coadjoint orbits (Chapter 14) was improved by stressing matrix methods throughout.

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