# Introduction to Operator Space Theory 

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## CONTENTS

0 . Introduction ..... 1
Part I. INTRODUCTION TO OPERATOR SPACES

1. Completely bounded maps ..... 17
2. The minimal tensor product. Ruan's theorem. Basic operations ..... 28
2.1. Minimal tensor product ..... 28
2.2. Ruan's theorem ..... 34
2.3. Dual space ..... 40
2.4. Quotient space ..... 42
Quotient by a subspace ..... 42
Quotient by an ideal ..... 43
2.5. Bidual. Von Neumann algebras ..... 47
2.6. Direct sum ..... 51
2.7. Intersection, sum, complex interpolation ..... 52
2.8. Ultraproduct ..... 59
2.9. Complex conjugate ..... 63
2.10. Opposite ..... 64
2.11. Ruan's theorem and quantization ..... 65
2.12. Universal objects ..... 67
2.13. Perturbation lemmas ..... 68
3. Minimal and maximal operator space structures ..... 71
4. Projective tensor product ..... 81
5. The Haagerup tensor product ..... 86
Basic properties ..... 86
Multilinear factorization ..... 92
Injectivity/projectivity ..... 93
Self-duality ..... 93
Free products ..... 98
Factorization through $R$ or $C$ ..... 101
Symmetrized Haagerup tensor product ..... 102
Complex interpolation ..... 106
6. Characterizations of operator algebras ..... 109
7. The operator Hilbert space ..... 122
Hilbertian operator spaces ..... 122
Existence and unicity of OH. Basic properties ..... 122
Finite-dimensional estimates ..... 130
Complex interpolation ..... 135
Vector-valued $L_{p}$-spaces, either commutative or noncommutative ..... 138
8. Group $C^{*}$-algebras. Universal algebras and unitization for an operator space ..... 148
9. Examples and comments ..... 165
9.1. A concrete quotient: Hankel matrices ..... 165
9.2. Homogeneous operator spaces ..... 172
9.3. Fermions. Antisymmetric Fock space. Spin systems ..... 173
9.4. The Cuntz algebra $O_{n}$ ..... 175
9.5. The operator space structure of the classical $L_{p}$-spaces ..... 178
9.6. The $C^{*}$-algebra of the free group with $n$ generators ..... 182
9.7. Reduced $C^{*}$-algebra of the free group with $n$ generators ..... 183
9.8. Operator space generated in the usual $L_{p}$-space by Gaussain random variables or by the Rademacher functions ..... 191
9.9. Semi-circular systems in Voiculescu's sense ..... 200
9.10. Embeddings of von Neumann algebras into ultraproducts ..... 210
9.11. Dvoretzky's theorem ..... 215
10. Comparisons ..... 217
Part II. OPERATOR SPACES AND $C^{*}-T E N S O R ~ P R O D U C T S ~$
11. $C^{*}$-norms on tensor products. Decomposable maps. Nuclearity ..... 227
12. Nuclearity and approximation properties ..... 240
13. $C^{*}\left(\mathbb{F}_{\infty}\right) \otimes B(H)$ ..... 252
14. Kirchberg's theorem on decomposable maps ..... 261
15. The Weak Expectation Property (WEP) ..... 267
16. The Local Lifting Property (LLP) ..... 275
17. Exactness ..... 285
18. Local reflexivity ..... 303
Basic properties ..... 303
A conjecture on local reflexivity and OLLP ..... 305
Properties $C, C^{\prime}$, and $C^{\prime \prime}$. Exactness versus local reflexivity ..... 309
19. Grothendieck's theorem for operator spaces ..... 316
20. Estimating the norms of sums of unitaries: Ramanujan graphs, property $T$, random matrices ..... 324
21. Local theory of operator spaces. Nonseparability of $O S_{n}$ ..... 334
22. $B(H) \otimes B(H)$ ..... 348
23. Completely isomorphic $C^{*}$-algebras ..... 354
24. Injective and projective operator spaces ..... 356
Part III. OPERATOR SPACES AND NON-SELF-ADJOINT OPERATOR ALGEBRAS
25. Maximal tensor products and free products of operator algebras ..... 365
26. The Blecher-Paulsen factorization. Infinite Haagerup tensor products ..... 384
27. Similarity problems ..... 396
28. The Sz.-Nagy-Halmos similarity problem ..... 407
Solutions to the exercises ..... 418
References ..... 457
Subject index ..... 477
Notation index ..... 479

## Chapter 1. Completely Bounded Maps

Let us start by recalling the definition and a few facts on $C^{*}$-algebras:
Definition 1.1. A $C^{*}$-algebra is a Banach *-algebra satisfying the identity

$$
\left\|x^{*} x\right\|=\|x\|^{2}
$$

for any element $x$ in the algebra.
The simplest example is the space

$$
B(H)
$$

of all bounded operators on a Hilbert space $H$, equipped with the operator norm. More generally, any closed subspace

$$
A \subset B(H)
$$

stable under product and involution is a $C^{*}$-algebra.
By classical results (Gelfand and Naimark) we know that every $C^{*}$-algebra can be realized as a closed self-adjoint subalgebra of $B(H)$. Moreover, we also know that every commutative unital $C^{*}$-algebra can be identified with the space $C(T)$ of all complex-valued continuous functions $f: T \rightarrow \mathbb{C}$ on some compact space $T$. If $A$ has no unit, $A$ can be identified with the space $C_{0}(T)$ of all complex-valued continuous functions, vanishing at infinity, on some locally compact space $T$.

Of course the object of $C^{*}$-algebra theory (as developed in the last 50 years; cf. [KaR, Ta3]) is the classification of $C^{*}$-algebras. Similarly, the object of Banach space theory is the classification of Banach spaces.

In the last 25 years, it is their classification up to isomorphism (and NOT up to isometry) that has largely predominated (cf., e.g., [LT1-3, P4]).

This already indicates one major difference between these two fields since, if $A_{1}$ and $A_{2}$ are two $C^{*}$-algebras,

$$
A_{1} \text { isomorphic to } A_{2} \Rightarrow A_{1} \text { isometric to } A_{2}
$$

In particular, a $C^{*}$-algebra admits a unique $C^{*}$-norm. So there is no "isomorphic theory" of $C^{*}$-algebras. However, in recent years, operator algebraists have found the need to relax the structure of $C^{*}$-algebras and consider more general objects called operator systems. These are subspaces of $B(H)$ containing the unit that are stable under the involution but not under the product. The theory of operator systems was developed using the order structure repeatedly, and it is still mostly an isometric theory. The natural morphisms here are the "completely positive" maps (cf. [St, Ar1]). We refer the reader to a survey by Effros [E1] and a series of papers by Choi and Effros (especially [CE3]). Even more recently, operator algebraists have done a radical simplification and considered just "operator spaces":

Definition 1.2. An operator space is a closed subspace of $B(H)$.
Equivalently, since we can think of $C^{*}$-algebras as closed self-adjoint subalgebras of $B(H)$, we can think of operator spaces as closed subspaces of $C^{*}$-algebras.

Operator space theory can be considered as a merger of $C^{*}$-algebra theory and Banach space theory.

It is important to immediately observe that any Banach space can appear as a closed subspace of a $C^{*}$-algebra. Indeed, for any Banach space $X$ (with the dual unit ball denoted by $B_{X^{*}}$ ), if we let

$$
T=\left(B_{X^{*}}, \sigma\left(X^{*}, X\right)\right)
$$

then $T$ is compact and we have an isometric embedding

$$
X \subset C(T)
$$

Hence, since $C(T)$ is a $C^{*}$-algebra (and $C(T) \subset B(H)$ with $H=\ell_{2}(T)$ ), X also appears among operator spaces. So operator spaces are just ordinary Banach spaces $X$ but equipped with an extra structure in the form of an embedding

$$
X \subset B(H)
$$

The main difference between the category of Banach spaces and that of operator spaces lies not in the spaces but in the morphisms. We need morphisms that somehow keep track of the extra information contained in the data of the embedding $X \subset B(H)$; the maps that do just this are the completely bounded maps.

Definition 1.3. Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces and consider a map


For any $n \geq 1$, let

$$
M_{n}(E)=\left\{\left(x_{i j}\right)_{i j \leq n} \mid x_{i j} \in E\right\}
$$

be the space of $n \times n$ matrices with entries in $E$. In particular, we have a natural identification

$$
M_{n}(B(H)) \simeq B\left(\ell_{2}^{n}(H)\right)
$$

where $\ell_{2}^{n}(H)$ means $\underbrace{H \oplus H \oplus \cdots \oplus H}_{n \text { times }}$. Thus, we may equip $M_{n}(B(H))$ and a fortiori its subspace

$$
M_{n}(E) \subset M_{n}(B(H))
$$

with the norm induced by

$$
B\left(\ell_{2}^{n}(H)\right)
$$

Then, for any $n \geq 1$, the linear map $u: E \rightarrow F$ allows us to define a linear map

$$
u_{n}: M_{n}(E) \longrightarrow M_{n}(F)
$$

defined by

$$
u_{n}\left(\begin{array}{ccc} 
& \vdots & \\
\ldots & x_{i j} & \ldots \\
& \vdots &
\end{array}\right)=\left(\begin{array}{ccc} 
& \vdots & \\
\ldots & u\left(x_{i j}\right) & \ldots \\
& \vdots &
\end{array}\right)
$$

A map $u: E \rightarrow F$ is called completely bounded (in short c.b.) if

$$
\sup _{n \geq 1}\left\|u_{n}\right\|_{M_{n}(E) \rightarrow M_{n}(F)}<\infty
$$

We define

$$
\|u\|_{c b}=\sup _{n \geq 1}\left\|u_{n}\right\|_{M_{n}(E) \rightarrow M_{n}(F)}
$$

and we denote by

$$
C B(E, F)
$$

the Banach space of all c.b. maps from $E$ into $F$ equipped with the c.b. norm.
This space will replace the space $B(E, F)$ of all bounded operators from $E$ into $F$. (We will see later on that it can be equipped with an operator space structure.)

If $G \subset B(L)$ is another operator space and if $v: F \rightarrow G$ is c.b., then the compositon vu: $E \rightarrow G$ clearly remains c.b. and we have

$$
\|v u\|_{c b} \leq\|v\|_{c b}\|u\|_{c b}
$$

Of course, when $n=1,1 \times 1$ matrices are just elements of $E$, so $u_{1}: M_{1}(E) \rightarrow$ $M_{1}(F)$ is nothing but $u$ itself. In particular we have

$$
\|u\| \leq\|u\|_{c b}
$$

and

$$
C B(E, F) \subset B(E, F)
$$

When $\|u\|_{c b} \leq 1$, we say that $u$ is "completely contractive" (or "a complete contraction").

The notion of isometry is replaced by that of "complete isometry": A map $u: E \rightarrow F$ is called a complete isometry ( $=u$ is completely isometric) if

$$
u_{n}: M_{n}(E) \rightarrow M_{n}(F)
$$

is an isometry for all $n \geq 1$.
Similarly, a map $u: E \rightarrow F$ is called completely positive (in short c.p.) if $u_{n}: M_{n}(E) \rightarrow M_{n}(F)$ is positive for all $n$ (in the order structure induced by the $C^{*}$-algebras $M_{n}(B(H))$ and $M_{n}(B(K))$. Moreover, we should emphasize

Definition 1.4. Two operator spaces $E, F$ are called completely isomorphic if there is a linear isomorphism $u: E \rightarrow F$ such that $u$ and $u^{-1}$ are c.b.

We will say that $E, F$ are completely isometric if there is a linear isomorphism $u: E \rightarrow F$ that is a complete isometry (or, equivalently, that satisfies $\|u\|_{c b}=\left\|u^{-1}\right\|_{c b}=1$ ). In that case, we will often identify these spaces, although this might sometimes be abusive.

Proposition 1.5. Let $A_{1} \subset B\left(H_{1}\right), A_{2} \subset B\left(H_{2}\right)$ be two $C^{*}$-algebras; let $E_{1} \subset A_{1}, E_{2} \subset A_{2}$ be two operator spaces; let $\pi: A_{1} \rightarrow A_{2}$ be a representation such that $\pi\left(E_{1}\right) \subset E_{2}$; and let $u: E_{1} \rightarrow E_{2}$ be the restriction of $\pi$. Then $u$ is completely bounded and $\|u\|_{c b} \leq 1$. Moreover, if $\pi$ is injective, $u$ is completely isometric.

Proof. It is well known that a $C^{*}$-algebra representation $\pi$ automatically has norm at most 1 and a closed range (cf. [Ta3, p. 21-22]). Therefore, $\|\pi\| \leq 1$, but since $\pi_{n}: M_{n}\left(A_{1}\right) \rightarrow M_{n}\left(A_{2}\right)$ also is a $C^{*}$-algebra representation, we again have $\left\|\pi_{n}\right\| \leq 1$ for all $n$, and hence $\|u\|_{c b} \leq 1$. Moreover, if a representation $\pi$ is injective, it is necessarily isometric (since its inverse must also have norm at most 1), and hence $\pi_{n}$ itself is isometric for all $n$.

We can measure the "c.b. distance" between $E$ and $F$ by setting

$$
d_{c b}(E, F)=\inf \left\{\|u\|_{c b}\left\|u^{-1}\right\|_{c b} \mid u: E \rightarrow F \text { complete isomorphism }\right\} .
$$

If $E, F$ are not completely isomorphic, we will set

$$
d_{c b}(E, F)=\infty
$$

Examples. When $E, F$ are Banach spaces we can view them as operator spaces via the embeddings

$$
E \subset C\left(B_{E^{*}}\right), \quad F \subset C\left(B_{F^{*}}\right)
$$

This is of course not a very interesting operator space structure, but it shows that - to some extent - Banach space theory can be viewed as embedded into operator space theory, since for a map

we have necessarily

$$
u \text { bounded } \Leftrightarrow u \quad \text { c.b. }
$$

and

$$
\|u\|=\|u\|_{c b} .
$$

Actually (see Proposition 1.10), this remains true when $E$ is an arbitrary operator space, assuming only that $F$ is equipped with its "commutative structure" as above. Moreover, it is easy to check that $\|u\|=\|u\|_{c b}$ for any rank one mapping $u$ between operator spaces. This implies of course that if $\operatorname{dim}(E)=1$, then its commutative operator space structure is the only possible one on $E$.

Here are more interesting examples:
In $B\left(\ell_{2}\right)$ consider the column Hilbert space

$$
\begin{equation*}
C=\overline{\operatorname{span}}\left\{e_{i 1} \mid i \in \mathbb{N}\right\} \tag{1.1}
\end{equation*}
$$

and the row Hilbert space

$$
\begin{equation*}
R=\overline{\operatorname{span}}\left\{e_{1 j} \mid j \in \mathbb{N}\right\} \tag{1.2}
\end{equation*}
$$

We will also need their finite-dimensional versions:

$$
\begin{aligned}
& C_{n}=\operatorname{span}\left\{e_{i 1} \mid 1 \leq i \leq n\right\} \\
& R_{n}=\operatorname{span}\left\{e_{1 j} \mid 1 \leq j \leq n\right\}
\end{aligned}
$$

Then, as Banach spaces, $R$ and $C$ are indistinguishable, since they are both isometric to $\ell_{2}$, that is, we have

$$
\begin{equation*}
\forall x \in \ell_{2} \quad\left\|\sum x_{i} e_{i 1}\right\|_{B\left(\ell_{2}\right)}=\left(\sum\left|x_{i}\right|^{2}\right)^{1 / 2}=\left\|\sum x_{j} e_{1 j}\right\|_{B\left(\ell_{2}\right)} \tag{1.3}
\end{equation*}
$$

However, as operator spaces, they are not isomorphic. Actually they are extremely far apart, since we have (see [Mat1-2])

$$
\begin{equation*}
d_{c b}\left(R_{n}, C_{n}\right)=n, \tag{1.4}
\end{equation*}
$$

which is the maximal distance possible between any two $n$-dimensional operator spaces. Actually, it can be shown (cf., e.g., [P5, p. 270], [ER4]) that for any

$$
u: R \rightarrow C \quad(\text { or } u: C \rightarrow R)
$$

we have ( $H S$ stands for Hilbert-Schmidt)

$$
\begin{equation*}
\|u\|_{c b}=\|u\|_{H S} \tag{1.5}
\end{equation*}
$$

For the proof, see the solution to Exercise 1.1. It follows that, for any isomorphism $u: R_{n} \rightarrow C_{n}$, we have

$$
n=\operatorname{tr}\left(u^{-1} u\right) \leq\|u\|_{H S}\left\|u^{-1}\right\|_{H S}=\|u\|_{c b}\left\|u^{-1}\right\|_{c b}
$$

which implies $d_{c b}\left(R_{n}, C_{n}\right) \geq n$. For the converse it suffices to observe that the map $u: R_{n} \rightarrow C_{n}$ taking $e_{1 j}$ to $e_{j 1}$ (=transposition) satisfies $\|u\|_{c b}=$ $\|u\|_{H S}=\sqrt{n}$ and $\left\|u^{-1}\right\|_{c b}=\left\|u^{-1}\right\|_{H S}=\sqrt{n}$.

Letting $n \rightarrow \infty$, this gives us a simple example of an isometric map from $R$ to $C$ that is not c.b. A fortiori, the transposition $x \rightarrow{ }^{t} x$ is isometric but is not c.b. either on $B\left(\ell_{2}\right)$ or on $\mathcal{K}$. More precisely, let $\tau_{n}: M_{n} \rightarrow M_{n}$ denote the transposition of matrices. Then one can prove (see Exercise 1.2)

$$
\begin{equation*}
\left\|\tau_{n}\right\|_{c b}=n \tag{1.6}
\end{equation*}
$$

These examples $R$ and $C$ are fundamental. Indeed, using the Haagerup tensor product (denoted by $\otimes_{h}$ ) presented in Chapter 5 , one can reconstruct the whole of $B\left(\ell_{2}\right)$ or $B(H)$ using $R$ and $C$ as the basic "building blocks" more precisely, we have $M_{n}=C_{n} \otimes_{h} R_{n}, K\left(\ell_{2}\right)=C \otimes_{h} R$, and of course $B\left(\ell_{2}\right)=K\left(\ell_{2}\right)^{* *}$.

More generally, let $H_{1}, H_{2}$ be two Hilbert spaces and let $\mathcal{H}=H_{1} \oplus H_{2}$. The mapping

$$
x \rightarrow\left(\begin{array}{ll}
0 & 0 \\
x & 0
\end{array}\right)
$$

is an isometric embedding of $B\left(H_{1}, H_{2}\right)$ into $B(\mathcal{H})$. Using this, we can view $B\left(H_{1}, H_{2}\right)$ as an operator space. Note that the norm induced on $M_{n}\left(B\left(H_{1}\right.\right.$, $\left.H_{2}\right)$ ) by $M_{n}(B(\mathcal{H}))$ coincides with the norm of the space $B\left(\ell_{2}^{n}\left(H_{1}\right), \ell_{2}^{n}\left(H_{2}\right)\right)$. In particular, we will often use the following:
Notation. Let $H$ be an arbitrary Hilbert space. For any $h \in H$, we denote by $h_{c} \in B(\mathbb{C}, H)$ and $h_{r} \in B\left(H^{*}, \mathbb{C}\right)$ the isometric embeddings defined by

$$
\begin{aligned}
\forall \lambda \in \mathbb{C} & h_{c}(\lambda)=\lambda h \\
\forall \xi \in H^{*} & h_{r}(\xi)=\langle\xi, h\rangle
\end{aligned}
$$

We will denote by $H_{c}$ and $H_{r}$ the resulting operator space structures on $H$. Recall that the dual $H^{*}$ can be canonically identified with the complex conjugate Hilbert space $\bar{H}$.

In particular, we have

$$
C=\left(\ell_{2}\right)_{c} \quad \text { and } \quad R=\left(\ell_{2}\right)_{r}
$$

Let $a: H_{1} \rightarrow H_{1}$ and $b: H_{2} \rightarrow H_{2}$ be bounded operators and let $u_{a b}$ : $B\left(H_{1}, H_{2}\right) \rightarrow B\left(H_{1}, H_{2}\right)$ be defined by $u_{a b}(T)=b T a$. Clearly, $u_{a b}$ is c.b. and $\left\|u_{a b}\right\|_{c b} \leq\|a\|\|b\|$. Taking either $H_{1}$ or $H_{2}$ one-dimensional, this implies immediately for any Hilbert space $H$

$$
\forall u: H_{c} \rightarrow H_{c} \quad\|u\|_{c b}=\|u\| \quad \text { and } \quad \forall v: H_{r} \rightarrow H_{r} \quad\|v\|_{c b}=\|v\|
$$

Indeed, we have $u\left(h_{c}\right)=[u(h)]_{c}$ and analogously for $r$. In particular,

$$
\begin{equation*}
\forall u: C \rightarrow C \quad\|u\|_{c b}=\|u\| \quad \text { and } \quad \forall v: R \rightarrow R \quad\|v\|_{c b}=\|v\| . \tag{1.7}
\end{equation*}
$$

The theory of c.b. maps clearly is the basis for operator space theory. It emerged in the early 1980s through the works of Wittstock [Wit1-2], Haagerup [H3], and Paulsen [Pa3], who proved (independently) a fundamental factorization and extension theorem for c.b. maps. This factorization is a generalization of earlier important work by Stinespring and Arveson ([St, Ar1]) who proved a factorization/extension theorem for completely positive maps.

Theorem 1.6. (Fundamental Factorization/Extension Theorem.) Consider a c.b. map


Then there is a Hilbert space $\widehat{H}$, a representation

$$
\pi: B(H) \longrightarrow B(\widehat{H})
$$

and operators $V_{1}: K \rightarrow \widehat{H}, V_{2}: \widehat{H} \rightarrow K$ such that $\left\|V_{1}\right\|\left\|V_{2}\right\|=\|u\|_{c b}$ and

$$
\begin{equation*}
\forall x \in E \quad u(x)=V_{2} \pi(x) V_{1} \tag{1.8}
\end{equation*}
$$

Conversely, if (1.8) holds then $u$ is c.b. and $\|u\|_{c b} \leq\left\|V_{1}\right\|\left\|V_{2}\right\|$ (in addition, if $V_{1}=V_{2}^{*}$, then $u$ is completely positive). Moreover, $u$ admits a c.b. extension $\widetilde{u}: B(H) \rightarrow B(K)$

such that $\|\widetilde{u}\|_{c b}=\|u\|_{c b}$.
For a proof, see either [Pa1], [P10], or [P5]; the latter extends to the case when $H$ and $K$ are Banach spaces.

This theorem explains the claim that c.b. maps keep track of the operator space structure. Indeed, it shows that (as explained in the Introduction) every c.b. map is the restriction of the composition of a representation and a two-sided multiplication.

For emphasis and for later reference, we state as separate corollaries parts of Theorem 1.6 that will be used frequently in the sequel. The first is the extension property of $B(K)$, which can be viewed as an operator-valued version of the Hahn-Banach Theorem:

Corollary 1.7. Let $E, \widetilde{E}$ be operator spaces so that $E \subset \widetilde{E} \subset B(H)$. Then any c.b. map $u: E \rightarrow B(K)$ admits a c.b. extension $\widetilde{u}: \widetilde{E} \rightarrow B(K)$ with $\|\widetilde{u}\|_{c b}=\|u\|_{c b}$.

Proof. We simply let $\widetilde{u}$ be the restriction of $x \mapsto V_{2} \pi(x) V_{1}$ to $\widetilde{E}$.
The second is the dilation property of unital complete contractions:
Corollary 1.8. Let $E \subset B(H)$ be an operator space containing $I$. Consider a map $u: E \rightarrow B(K)$. If $u(I)=I$ and $\|u\|_{c b}=1$, then there is a Hilbert space $\widehat{H}$ with $K \subset \widehat{H}$ and a representation $\pi: B(H) \rightarrow B(\widehat{H})$ such that

$$
\forall x \in E \quad u(x)=P_{K} \pi(x)_{\mid K}
$$

In particular, $u$ is completely positive.

Proof. By Theorem 1.6, we have $u(\cdot)=V_{2} \pi(\cdot) V_{1}$. By homogeneity, we may assume $\left\|V_{1}\right\|=\left\|V_{2}\right\|=1$. Since $I=u(I)=V_{2} \pi(I) V_{1}=V_{2} V_{1}, V_{1}$ must be an isometric embedding of $K$ into $\widehat{H}$. Identifying $K$ with $V_{1}(K), u(\cdot)=V_{2} \pi(\cdot) V_{1}$ becomes $u(\cdot)=P_{K} \pi(\cdot)_{\mid K}$.

Finally, the third corollary is the decomposability of c.b. maps as linear combinations of c.p. maps:

Corollary 1.9. Any c.b. map $u: E \rightarrow B(K)$ can be decomposed as $u=$ $u_{1}-u_{2}+i\left(u_{3}-u_{4}\right)$, where $u_{1}, u_{2}, u_{3}, u_{4}$ are $c . p$. maps with $\left\|u_{j}\right\|_{c b} \leq\|u\|_{c b}$.

Proof. By Theorem 1.6, we have $u(\cdot)=V_{2} \pi(\cdot) V_{1}$. Let us denote $V=V_{1}$ and $V_{2}=W^{*}$, so that $u(\cdot)=W^{*} \pi(\cdot) V$. Then the result simply follows from the polarization formula: We define $u_{1}, u_{2}, u_{3}, u_{4}$ by

$$
\begin{array}{ll}
u_{1}(\cdot)=4^{-1}(V+W)^{*} \pi(\cdot)(V+W), & u_{2}(\cdot)=4^{-1}(V-W)^{*} \pi(\cdot)(V-W) \\
u_{3}(\cdot)=4^{-1}(V+i W)^{*} \pi(\cdot)(V+i W), & u_{4}(\cdot)=4^{-1}(V-i W)^{*} \pi(\cdot)(V-i W)
\end{array}
$$

Then $\left\|u_{j}\right\|_{c b} \leq 1$ for $j=1,2,3,4$ and $u=u_{1}-u_{2}+i\left(u_{3}-u_{4}\right)$. (Note that actually $\left\|u_{1}+u_{2}\right\|_{c b} \leq 1$ and $\left\|u_{3}+u_{4}\right\|_{c b} \leq 1$ ).

Proposition 1.10. Let $F \subset B(H)$ be an operator space. Let $A_{F}$ be the $C^{*}$-algebra generated by $F$.
(i) For any $n \geq 1$ and any $x$ in $M_{n}(F)$ we have

$$
\begin{aligned}
&\|x\|_{M_{n}(F)} \geq \sup \left\{\left.\left\|\sum \lambda_{i} \mu_{j} x_{i j}\right\|_{F}\left|\lambda_{i} \in \mathbb{C}, \mu_{j} \in \mathbb{C}, \sum\right| \lambda_{i}\right|^{2} \leq 1\right. \\
&\left.\sum\left|\mu_{j}\right|^{2} \leq 1\right\}
\end{aligned}
$$

(ii) Assume either $A_{F}$ commutative or $\operatorname{dim}(F)=1$. Then we have equality in (i). Moreover, in either case, if $E$ is an arbitrary operator space, any bounded map $u: E \rightarrow F$ is c.b. and satisfies $\|u\|_{c b}=\|u\|$.
(iii) For any $E, F$, every finite-rank map $u: E \rightarrow F$ is c.b.

Proof. (i) is an easy exercise. When $A_{F}$ is commutative, we can assume $A_{F}=C_{0}(\Omega)$ and also $M_{n}\left(A_{F}\right)=C_{0}\left(\Omega ; M_{n}\right)$ for some locally compact space $\Omega$. Then equality in (i) is very simple to check. When $\operatorname{dim}(F)=1$, the verification is again an easy exercise. The second assertion in (ii) then follows by applying (i) in $E$ and the equality case in $F$. Thus any map of rank one is c.b., which implies the same for any finite-rank map.

Note that (ii) implies that (not too surprisingly!) there is only one abstract operator space structure on $\mathbb{C}$.
Remark 1.11. Let $E_{1}, E_{2}$ be two Banach spaces. Consider an element $x=\sum a_{i} \otimes b_{i}$ in the algebraic tensor product $E_{1} \otimes E_{2}$. The "injective" tensor norm (in Grothendieck's sense) is defined as

$$
\begin{aligned}
\|x\|_{\vee} & =\sup \left\{\left|\left\langle\xi_{1} \otimes \xi_{2}, x\right\rangle\right| \mid \xi_{1} \in B_{E_{1}^{*}}, \xi_{2} \in B_{E_{2}^{*}}\right\} \\
& =\sup \left\{\left|\sum \xi_{1}\left(a_{i}\right) \xi_{2}\left(b_{i}\right)\right| \mid \xi_{1} \in B_{E_{1}^{*}}, \xi_{2} \in B_{E_{2}^{*}}\right\}
\end{aligned}
$$

Note that we can write alternatively

$$
\|x\|_{\vee}=\sup _{\xi_{1} \in B_{E_{1}^{*}}}\left\{\left\|\sum \xi_{1}\left(a_{i}\right) b_{i}\right\|_{E_{2}}\right\}=\sup _{\xi_{2} \in B_{E_{2}^{*}}}\left\{\left\|\sum a_{i} \xi_{2}\left(b_{i}\right)\right\|_{E_{1}}\right\}
$$

We denote by $E_{1} \stackrel{\vee}{\otimes} E_{2}$ the completion of $E_{1} \otimes E_{2}$ for this norm, and we call it the injective tensor product of $E_{1}, E_{2}$.

In particular, for any Banach space $E$, we have for any $x \in M_{n} \otimes E$

$$
\begin{align*}
\|x\|_{M_{n} \stackrel{\vee}{\otimes} E} & =\sup \left\{\left.\left\|\sum \lambda_{i} \mu_{j} x_{i j}\right\|_{E}\left|\left(\lambda_{i}\right),\left(\mu_{j}\right) \in \mathbb{C}^{n}, \sum\right| \lambda_{i}\right|^{2} \leq 1, \sum\left|\mu_{j}\right|^{2} \leq 1\right\} \\
& =\sup \left\{\left\|\sum e_{i j} \xi\left(x_{i j}\right)\right\|_{M_{n}} \mid \xi \in B_{E^{*}}\right\} . \tag{1.9}
\end{align*}
$$

Note that for any locally compact space $\Omega$ and any Banach space $B$ (in particular for $B=M_{n}$ ) we have an isometric isomorphism

$$
C_{0}(\Omega, B)=C_{0}(\Omega) \stackrel{\vee}{\otimes} B .
$$

Remark. Let $\alpha(n)$ be the best constant $C$ such that, for any $E, F$, any map $u: E \rightarrow F$ of rank $n$ satisfies

$$
\|u\|_{c b} \leq C\|u\| .
$$

We will see in Theorem 3.8 later that $n / 2 \leq \alpha(n) \leq n$ and in Chapter 7 that $\alpha(n) \leq n / 2^{1 / 4}$ (due to Eric Ricard), but the exact value of $\alpha(n)$ does not seem to be known.

The following result due to R. Smith $[\mathrm{Sm} 2]$ is often useful.

Proposition 1.12. Consider $E \subset B(H)$ and $u: E \rightarrow M_{N}=B\left(\ell_{2}^{N}, \ell_{2}^{N}\right)$. Then we have

$$
\|u\|_{c b}=\left\|u_{N}\right\|_{M_{N}(E) \rightarrow M_{N}\left(M_{N}\right)}
$$

Proof. This can be proved using the fact that, if $x_{1}, \ldots, x_{n}$ is a finite subset of $\ell_{2}^{N}$ with $\sum_{1}^{n}\left\|x_{i}\right\|^{2} \leq 1$, then (we leave this as an exercise for the reader) there are an $n \times N$ scalar matrix $b=\left(b_{j k}\right)$ with $\left\|\left(b_{j k}\right)\right\| \leq 1$ and vectors $\widetilde{x}_{1}, \ldots, \widetilde{x}_{N}$ in $\ell_{2}^{N}$ such that $\sum_{1}^{N}\left\|\widetilde{x}_{i}\right\|^{2} \leq 1$ and

$$
\forall j \leq n \quad x_{j}=\sum_{k=1}^{N} b_{j k} \widetilde{x}_{k}
$$

Similarly, for any $y_{1}, \ldots, y_{n}$ in $\ell_{2}^{N}$ there are a scalar matrix $c=\left(c_{i l}\right)$ with $\left\|\left(c_{i l}\right)\right\| \leq 1$ and $\widetilde{y}_{1}, \ldots, \widetilde{y}_{N}$ in $\ell_{2}^{N}$ such that $\sum_{1}^{N}\left\|\widetilde{y}_{i}\right\|^{2} \leq 1$ and

$$
\forall i \leq n \quad y_{i}=\sum_{l=1}^{N} c_{i l} \widetilde{y}_{l}
$$

Hence for any $n \times n$ marix $\left(a_{i j}\right)$ in $M_{n}(E)$ we have

$$
\sum_{i, j=1}^{n}\left\langle u\left(a_{i j}\right) x_{j}, y_{i}\right\rangle=\sum_{k, l=1}^{N}\left\langle u\left(\alpha_{l k}\right) \widetilde{x}_{k}, \widetilde{y}_{l}\right\rangle
$$

where $\left(\alpha_{l k}\right) \in M_{N}(E)$ is defined by $\left(\alpha_{l k}\right)=c^{*} .\left(a_{i j}\right) . b$ (matrix product). Therefore:

$$
\begin{aligned}
\left\|\left(u\left(a_{i j}\right)\right)\right\|_{M_{n}\left(M_{N}\right)} \leq & \left\|\left(u\left(\alpha_{k l}\right)\right)\right\|_{M_{N}\left(M_{N}\right)} \leq\left\|u_{N}\right\|_{M_{N}(E) \rightarrow M_{N}\left(M_{N}\right)}\left\|\left(\alpha_{l k}\right)\right\|_{M_{N}(E)} \\
& \leq\left\|u_{N}\right\|_{M_{N}(E) \rightarrow M_{N}\left(M_{N}\right)}\left\|\left(a_{i j}\right)\right\|_{M_{N}(E)} .
\end{aligned}
$$

Remark 1.13. Consider $a_{1}, \ldots, a_{n}$ and $b_{1}, \ldots, b_{n}$ in $B(H)$. Let $a \in M_{n}$ $(B(H))\left(\right.$ resp. $\left.b \in M_{n}(B(H))\right)$ be the $n \times n$ matrix that has $a_{1}, \ldots, a_{n}$ (resp. $b_{1}, \ldots, b_{n}$ ) on its first column (resp. row) and zero elsewhere; that is, we have

$$
a=\left(\begin{array}{cc}
a_{1} & \\
\vdots & \\
\vdots & \\
a_{n} &
\end{array}\right) \quad b=\left(\begin{array}{ccc}
b_{1} & \ldots & b_{n} \\
& \bigcirc &
\end{array}\right)
$$

Then

$$
\begin{equation*}
\|a\|=\left\|\sum a_{i}^{*} a_{i}\right\|_{B(H)}^{1 / 2} \quad \text { and } \quad\|b\|=\left\|\sum b_{i} b_{i}^{*}\right\|_{B(H)}^{1 / 2} \tag{1.10}
\end{equation*}
$$

Indeed, we have $\|a\|=\left\|a^{*} a\right\|^{1 / 2}$ and $\|b\|=\left\|b b^{*}\right\|^{1 / 2}$. Moreover, we have $\|b a\| \leq\|b\|\|a\|$, and hence

$$
\begin{equation*}
\left\|\sum b_{i} a_{i}\right\| \leq\left\|\sum b_{i} b_{i}^{*}\right\|_{B(H)}^{1 / 2}\left\|\sum a_{i}^{*} a_{i}\right\|_{B(H)}^{1 / 2} \tag{1.11}
\end{equation*}
$$

More generally, for any $x=\left(x_{i j}\right)$ in $M_{n}(B(H))$ we have $\|b x a\| \leq\|b\|\|x\|\|a\|$, and hence

$$
\begin{equation*}
\left\|\sum_{i, j} b_{i} x_{i j} a_{j}\right\| \leq\left\|\sum b_{i} b_{i}^{*}\right\|_{B(H)}^{1 / 2}\|x\|_{M_{n}(B(H))}\left\|\sum a_{j}^{*} a_{j}\right\|_{B(H)}^{1 / 2} \tag{1.12}
\end{equation*}
$$

Note that it is easy to extend this remark to $n=\infty$.

## Exercises

Exercise 1.1. Prove (1.5).
Exercise 1.2. Prove (1.6).
Exercise 1.3. Let $u: E \rightarrow F$ be a mapping between operator spaces. Show that for any $a_{1}, \ldots, a_{n}$ in $E$ we have

$$
\begin{aligned}
&\left\|\sum u\left(a_{j}\right)^{*} u\left(a_{j}\right)\right\|^{1 / 2} \leq\|u\|_{c b}\left\|\sum a_{j}^{*} a_{j}\right\|^{1 / 2} \text { and } \| \sum u\left(a_{j}\right) u\left(a_{j}\right)^{*} \|^{1 / 2} \\
& \leq\|u\|_{c b}\left\|\sum a_{j} a_{j}^{*}\right\|^{1 / 2}
\end{aligned}
$$

Exercise 1.4. Let $u: E \rightarrow F$ be a mapping between operator spaces. Show that

$$
\|u\|_{c b}=\sup _{n \geq 1}\left\{\|v u\|_{c b} \mid v: F \rightarrow M_{n} \quad\|v\|_{c b} \leq 1\right\} .
$$

Exercise 1.5. (Schur Multipliers) (i) Let $\left\{x_{i} \mid i \leq n\right\}$ and $\left\{y_{j} \mid j \leq n\right\}$ be elements in the unit ball of a Hilbert space $K$. Then the mapping $u: M_{n} \rightarrow$ $M_{n}$ defined by $u\left(\left[a_{i j}\right]\right)=\left[a_{i j}\left\langle x_{i}, y_{j}\right\rangle\right]$ is a complete contraction. In addition, if $x_{i}=y_{i}$ for all $i$, then $u$ is completely positive.
(ii) More generally, let $S, T$ be arbitrary sets. We will identify an element of $B\left(\ell_{2}(T), \ell_{2}(S)\right)$ with a matrix $\{a(s, t) \mid(s, t) \in S \times T\}$ in the usual way.

Let $\left\{x_{s} \mid s \in S\right\}$ and $\left\{y_{t} \mid t \in T\right\}$ be elements in the unit ball of a Hilbert space $K$. Then the mapping $u: B\left(\ell_{2}(T), \ell_{2}(S)\right) \rightarrow B\left(\ell_{2}(T), \ell_{2}(S)\right)$ that takes $(a(s, t))_{(s, t) \in S \times T}$ to $\left(a(s, t)\left\langle x_{s}, y_{t}\right\rangle\right)_{(s, t) \in S \times T}$ is a complete contraction. In addition, if $S=T$ and $x_{t}=y_{t}$ for all $t$, then $u$ is completely positive.

