Introduction to Pseudodifferential and Fourier Integral Operators

Volume 1

Pseudodifferential Operators

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INTRODUCTION TO PSEUDODIFFERENTIAL AND FOURIER INTEGRAL OPERATORS
François Treves
VOLUME 1: PSEUDODIFFERENTIAL OPERATORS
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Volume 1

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Preface

I have tried in this book to describe those aspects of pseudodifferential and Fourier integral operator theory whose usefulness seems proven and which, from the viewpoint of organization and "presentability," appear to have stabilized. Since, in my opinion, the main justification for studying these operators is pragmatic, much attention has been paid to explaining their handling and to giving examples of their use. Thus the theoretical chapters usually begin with a section in which the construction of special solutions of linear partial differential equations is carried out, constructions from which the subsequent theory has emerged and which continue to motivate it: parametrices of elliptic equations in Chapter I (introducing pseudodifferential operators of type 1, 0, which here are called standard), of hypoelliptic equations in Chapter IV (devoted to pseudodifferential operators of type ρ, δ), fundamental solutions of strongly hyperbolic Cauchy problems in Chapter VI (which introduces, from a "naive" standpoint, Fourier integral operators), and of certain nonhyperbolic forward Cauchy problems in Chapter X (Fourier integral operators with complex phase).

Several chapters—II, III, IX, XI, and XII—are devoted entirely to applications.

Chapter II provides all the facts about pseudodifferential operators needed in the proof of the Atiyah–Singer index theorem, then goes on to present part of the results of A. Calderon on uniqueness in the Cauchy problem, and ends with a new proof (due to J. J. Kohn) of the celebrated sum-of-squares theorem of L. Hörmander, a proof that beautifully demonstrates the advantages of using pseudodifferential operators.

The subject of Chapter III is boundary problems for elliptic equations. It is perhaps the only place in the book where I have departed somewhat from standard procedure. The overall approach is the one made familiar by the works of A. Calderon, R. Seeley, and others: to transfer the problem from the domain where it was originally posed onto the boundary of that domain, where it becomes an "interior" but in general pseudodifferential rather than differential problem. The main difference is that regardless of the nature of the boundary conditions, I construct from the start the operator that effects the transfer to the boundary and show it to be a standard pseudodifferential operator (with respect to the tangential variables, depending smoothly on the variable normal to the boundary), a kind of exponential to which all the results of Chapter I are applicable. I then show that the testing of the essential properties (regularity of the solutions up to the boundary, Fredholm character, etc.) can be done for the interior problem on the boundary, which concerns the *Calderon operator* of the boundary problem. For instance, the Calderon operator of a boundary problem of the Lopatinski–Shapiro type, called *coercive* in this book, is elliptic. As a consequence the regularity up to the boundary of the solutions is an immediate corollary of the property that pseudodifferential operators are pseudolocal. It suffices to apply it to the "exponential" which effects the transfer to the boundary. Analogous results are discussed for problems of principal type, especially the subelliptic ones, for example certain oblique derivative problems, and for the $\bar{\partial}$-Neumann problem when the conditions (on the number of positive or negative eigenvalues of the Levi matrix) for hypoellipticity with loss of one order of differentiation are satisfied.

The text goes back to elliptic boundary problems at the end of Chapter V to discuss the question of analyticity up to the boundary, under the right circumstances, by exploiting the theory of analytic pseudodifferential operators, which makes up the contents of Chapter V.

Applications of Fourier integral operators are sprinkled throughout Volume 2. Chapter IX describes in great detail the reduction of suitable systems of pseudodifferential equations to the main "standard forms." An example is the microlocal transformation of systems of the induced $\bar{\partial}$ type to systems of Mizohata equations, under the hypothesis that the Levi matrix is nondegenerate. Chapter XI presents applications of Fourier integral operators with complex phase, in particular to operators that can be transformed microlocally into Mizohata's, and to establishing subelliptic estimates. It is shown that the latter can be used to refine the Carleman estimates that lead to uniqueness in the Cauchy problem (and thus improve the result in Chapter II).

Chapter XII presents three applications to the study of the spectrum of the Laplace–Beltrami operator $-\Delta$ on a compact Riemannian manifold: (1) the classical estimate, due to V. G. Akumovic, of the number $N(\lambda)$ of eigenvalues not exceeding $\lambda \sim +\infty$; (2) the generalization by J. Chazarain of the Poisson formula, relating the lengths of the closed geodesics to the singularities of the distribution on the real line, $\mathrm{Tr}(\exp(it\sqrt{-\Delta}))$; (3) the derivation of the existence of certain sequences of eigenvalues from that of Lagrangian submanifolds of the cotangent bundle on which the Riemannian length of covectors is constant and which satisfy Maslov's quantization condition. This last section of the book follows very closely the presentation of A. Weinstein [1].

With the exception of the elliptic boundary problems in Chapter III, the applications are never studied in their own right, with the pretense of describing them fully, but only as examples of what can be achieved by using pseudodifferential or Fourier integral operators. This is why I have refrained from embarking on the study of other major areas of application of the theory: solvability of linear PDEs, diffraction, well-posedness of the Cauchy problem. On the latter the reader is referred to the works of Ivrii [1–3], Ivrii–Petkov [1], and Hörmander [17].

To complete this brief outline of the contents of the book: the global theory of Fourier integral operators is described in Chapter VIII, following the laying out of the symplectic geometry background in Chapter VII. Clean phases, rather than nondegenerate ones, are used in the microlocal representations of the operators. This simplifies composition in Section 6 of Chapter VIII, and pays off nicely in the applications to Riemannian geometry in Chapter XII.

There are important aspects of pseudodifferential and Fourier integral operator theory that this book does not discuss. First, this book is totally L^2 oriented. Not a word is said about pseudodifferential action on L^p spaces for $p \neq 2$. I felt I was not qualified to go into this area. Besides, there is great advantage in restricting one's outlook to L^2, for one thus can exploit Fourier transforms to the full. This is also why the book does not deal in any depth with the kernels $K(x, y)$ associated with the operators. After all, perhaps the main thrust of pseudodifferential operators is to substitute, as often as possible, the calculus of symbols for that of kernels. Symbolic calculus has been traditionally based on Fourier (or Laplace) transforms, whose natural framework is L^2 or the Schwartz space $\mathscr{S}'$ of tempered distributions. This is of course not to deny that certain applications, such as continuity between L^p spaces and even between spaces of Hölder continuous functions, require less coarse treatment. But such questions and many others are beyond the scope of this book.

Closer to its contents are the classes of pseudodifferential operators introduced in the last few years by various authors, most notably by Beals and Fefferman [1], Boutet de Monvel [3], Hörmander [19], and Unterberger [1]. For a systematic study, see Beals [1]. In this connection my feeling has been that this is more advanced mathematics, which the reader should not have too much difficulty in learning once he has digested some of the material in this book. The same applies to the global theory of pseudodifferential operators in Euclidean space (see Kumano-Go [1]) and to various extensions of Fourier integral operators, such as the one in Guillemin [2] or those based on the *Airy function*, which turn up naturally in the study of certain problems where the characteristics are double (as occurs, for example, in geometrical optics; see, for instance, Taylor [2] and Egorov [2]). One important item related to Fourier integral operators, and which is missing from this book but undoubtedly should have been in it, is the *metaplectic representation.* On this subject I must content myself with referring the reader to other texts, for instance Leray [1] and Weinstein [1].

The prerequisites for a serious study of the material in the book vary from chapter to chapter. Most of the time they are the standard requirements in real and complex analysis and in functional analysis, with a smattering of distribution theory, whose essential concepts and notation are recalled in the section on notation and background. Manifolds, their tangent and cotangent bundles, and more general vector bundles are defined in Chapter I. Complements of "basic" differential geometry are provided in Chapter VII, following a section devoted to symplectic linear algebra and preceding one devoted to symplectic differential geometry. I hope that some chapters will be useful to anybody eager to learn the fundamental aspects of pseudodifferential and Fourier integral operator theory, or willing to teach it for the first time—I am thinking mainly of Chapters I–III and VI–VIII. Other chapters (Chapters V and IX–XI) are intended more for reference or specialized study and use. Still other chapters fall in between these two categories.

The book is rather informally written—to some this will seem an understatement—due mainly to my inclinations, lack of time, and a certain sense of urgency, the sense that a book with more or less these contents is overdue. I have not hesitated to borrow from the available literature, especially from the original article of Hörmander [11], from the lecture notes of Duistermaat [1], and from the article by Melin and Sjöstrand [1].

In matters of terminology I have tried to be as much of a conformist as I could. But one notation I could not resign myself to adopting is $L^m(\quad)$ for the spaces of pseudodifferential operators. L is overused in mathematics: Lebesgue spaces, sets of linear transformations, linear partial differential operators, Lagrangian manifolds are all called L this or that. On the other hand, capital psi, Ψ, is underused, and very naturally associated with pseudo, so I write $\Psi^m(\quad)$ in the place of $L^m(\quad)$. Perhaps the only other novelty is the term *microdistribution*, which seems to me the natural analogue, in the context of distributions, of the name *microfunction* introduced by M. Sato in hyperfunction theory (see Sato [1]).

On the other hand, I have stuck to the name Fourier integral operator, although I tend to agree that it is not the most felicitous and that it may have been more equitable to use Maslov operator instead, as many Russian authors do. But Fourier integral operator is the term that people everywhere outside the Soviet Union use and are used to, and it might be too late to reverse the trend. I do not quite understand J. Dieudonné when he contends in his beautiful treatise [1] on calculus on manifolds that the name distorts the purpose of Fourier integral operators, which have succeeded, according to Dieudonné, in "eliminating" the Fourier transform. I believe rather that their purpose, and their effect, is to extend the applications of the Fourier transform from Euclidean spaces to manifolds.

François Treves

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Fourier Integral Operators

Notation and Background

1. Euclidean Spaces

$\mathbb{R}^n$: n-dimensional (real) Euclidean space
$\mathbb{R}_n$: dual of $\mathbb{R}^n$
$x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n)$, also $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$: variables and coordinates in $\mathbb{R}^n$
$\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$: variables and coordinates in $\mathbb{R}_n$
$\mathbb{C}^n$, $\mathbb{C}_n$: n-dimensional complex space and its dual
$z = (z^1, \ldots, z^n)$, also $z = (z_1, \ldots, z_n)$: variables and coordinates in $\mathbb{C}^n$
$x \cdot \xi = x^1\xi_1 + \cdots + x^n\xi_n$: scalar product between $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}_n$
$|x| = [(x^1)^2 + \cdots + (x^n)^2]^{1/2}$, $|\xi| = [\xi_1^2 + \cdots + \xi_n^2]^{1/2}$: Euclidean norms in $\mathbb{R}^n$ and in $\mathbb{R}_n$
$\bar{z} = (\bar{z}^1, \ldots, \bar{z}^n)$: the complex conjugate of z
$z \cdot \bar{z}' = z^1\bar{z}'^1 + \cdots + z^n\bar{z}'^n$: the *hermitian product* in $\mathbb{C}^n$
$|z| = [|z^1|^2 + \cdots + |z^n|^2]^{1/2} = [z \cdot \bar{z}]^{1/2}$: Euclidean norm in $\mathbb{C}^n$
Ω: *open* subset of a Euclidean space
$\Omega \backslash S$: *complement* of a subset S in Ω
$S \subset\subset \Omega$: means that the *closure* of S is a *compact* subset of Ω (then S is said to be *relatively compact* in Ω)

2. The Multi-Index Notation

$\mathbb{Z}$: set of integers >0 or ≤ 0
$\mathbb{Z}_+$: set of integers ≥ 0
$\mathbb{Z}_+^n$: set of *n-tuples* $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{Z}_+$ for each $j = 1, \ldots, n$
$\beta \leq \alpha$: means $\beta_j \leq \alpha_j$ for every $j = 1, \ldots, n$ $(\alpha, \beta \in \mathbb{Z}_+^n)$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$: *length* of $\alpha \in \mathbb{Z}_+^n$

$\alpha! = \alpha_1! \cdots \alpha_n!, \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$ if $\alpha, \beta \in \mathbb{Z}_+^n$ and $\beta \leq \alpha$

$x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$ if $x \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$

$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ if $\xi \in \mathbb{R}_n$

$\partial_x^\alpha = (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n}$ (also denoted by ∂^α)

$\partial_\xi^\alpha = (\partial/\partial \xi_1)^{\alpha_1} \cdots (\partial/\partial \xi_n)^{\alpha_n}$

$D_x^\alpha = \left(\dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x^n}\right)^{\alpha_n}$ (also denoted by D^α)

$D_\xi^\alpha = \left(\dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial \xi_n}\right)^{\alpha_n}$

Also, if u is a C^∞ function of x:

$u^{(\alpha)} = \partial_x^\alpha u$

$\partial_x u = u_x = \operatorname{grad} u = (\partial u/\partial x^1, \ldots, \partial u/\partial x^n)$

$D_j u = \dfrac{1}{\sqrt{-1}}\dfrac{\partial u}{\partial x^j}, j = 1, \ldots, n$

Taylor Expansion

(0.1) $$u(x) = \sum_{\alpha \in \mathbb{Z}_+^n} (1/\alpha!)(x-y)^\alpha\, \partial^\alpha u(y)$$

Leibniz Formula

(0.2) $$\partial^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} u\, \partial^\beta v$$

Transposed Leibniz Formula

(0.3) $$v\, \partial^\alpha u = \sum (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^{\alpha-\beta}[u\, \partial^\beta v]$$

[To prove (0.3) multiply the left-hand side by a C^∞ function w vanishing outside a compact set and integrate by parts $\int wv\, \partial^\alpha u\, dx = (-1)^{|\alpha|} \int \partial^\alpha(wv) u\, dx$, apply (0.2) and integrate by parts "back".]

Differential Operators in Ω

Linear partial differential operators are polynomials in $D = (D_1, \ldots, D_n)$ with *coefficients* belonging to $C^\infty(\Omega)$, such as

$$P(x, D) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha.$$

If, for some α of length m, c_α does not vanish identically in Ω, m is called the *order* of $P(x, D)$. When the coefficients c_α are constant, we write $P(D)$.

${}^tP(x, D)$: *transpose* of $P(x, D)$, defined by

$${}^tP(x, D)u(x) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}D^\alpha[c_\alpha(x)u(x)]$$

$P(x, D)^*$: *adjoint* of $P(x, D)$, $P(x, D)^* = \overline{{}^tP(x, D)}$. The bar means that the coefficients have been replaced by their complex conjugates.
$\Delta = (\partial/\partial x^1)^2 + \cdots + (\partial/\partial x^n)^2$: the *Laplace operator* in $\mathbb{R}^n$;
$(\partial/\partial \bar{z}^j) = \frac{1}{2}(\partial/\partial x^j + \sqrt{-1}\,\partial/\partial y^j)$, $j = 1, \ldots, n$: the *Cauchy–Riemann operators* in $\mathbb{C}^n$.

3. Functions and Function Spaces

supp f: the *support* of the function f, i.e., the closure of the set of points at which f does *not* vanish
$C^m(\Omega)$: space of m times continuously differentiable *complex-valued* functions in Ω ($m \in \mathbb{Z}_+$ or $m = +\infty$)
$C_c^\infty(\Omega)$: space of C^∞ complex functions in Ω having compact support; the elements of $C_c^\infty(\Omega)$ are often called *test functions* in Ω
$C_c^\infty(K)$: space of C^∞ complex functions in $\mathbb{R}^n$ which vanish identically outside the compact set K

The C^∞ Topology

Let K be any compact subset of Ω, m any integer ≥ 0. For any $\phi \in C^\infty(\Omega)$, set

$$p_{m,K}(\phi) = \operatorname*{Max}_{x\in K} \sum_{|\alpha|\leq m} |\partial^\alpha\phi(x)|.$$

Then, as K and m vary in all possible manners, the $p_{m,K}$ form a basis of continuous seminorms on $C^\infty(\Omega)$. Actually, it suffices to let K range over an *exhausting sequence of compact subsets* of Ω, $\{K_\nu\}_{\nu=0,1,\ldots}$; this means that K_ν is contained in the interior of $K_{\nu+1}$ and that every compact subset of Ω is contained in some K_ν. Set $p_m = p_{m,K_m}$; the seminorms p_m define the topology of $C^\infty(\Omega)$. Every neighborhood of a "point" ϕ_o of $C^\infty(\Omega)$ contains a neighborhood

$$V_{m,\varepsilon} = \{\phi \in C^\infty(\Omega); p_m(\phi - \phi_o) \leq \varepsilon\}$$

for a suitable choice of $m \in \mathbb{Z}_+$ and $\varepsilon > 0$. A sequence of C^∞ functions in Ω, ϕ_j ($j = 1, 2, \ldots$), converges to ϕ_o in $C^\infty(\Omega)$ if and only if, for every $\alpha \in \mathbb{Z}^n_+$, $\partial^\alpha \phi_j$ converges to $\partial^\alpha \phi_o$ uniformly on every compact subset of Ω.

The topology of $C^\infty(\Omega)$ can be defined by a metric such as

$$\operatorname{dist}(\phi, \psi) = \sum_{m=0}^{\infty} 2^{-m} \inf(1, p_m(\phi - \psi)).$$

All such metrics are equivalent, and turn $C^\infty(\Omega)$ into a complete metric space. Equipped with its natural (i.e., the C^∞) topology $C^\infty(\Omega)$ is a Fréchet space, i.e., a locally convex topological vector space that is metrizable and complete. In the C^∞ topology every *bounded and closed* set is *compact.* (A subset of $C^\infty(\Omega)$ is bounded if every seminorm $p_{m,K}$ is bounded on it.) This property follows easily from the Ascoli–Arzela theorem.

The Natural Topology of $C_c^\infty(\Omega)$

For any compact subset K of Ω, $C_c^\infty(K)$ is a *closed* linear subspace of $C^\infty(\Omega)$ and is equipped with the induced (or relative) topology. Set-theoretically,

$$C_c^\infty(\Omega) = \bigcup_{K \subset\subset \Omega} C_c^\infty(K). \tag{0.4}$$

Then a *convex* subset of $C_c^\infty(\Omega)$ is open if and only if its intersection with every subspace $C_c^\infty(K)$ is open in the latter.

The topology of $C_c^\infty(\Omega)$ is used only through the following properties:

(i) A *sequence* converges in $C_c^\infty(\Omega)$ if and only if it is contained in $C_c^\infty(K)$ for some compact subset K of Ω and converges in $C_c^\infty(K)$.
(ii) A subset B of $C_c^\infty(\Omega)$ is *bounded* if and only if it is contained and bounded in *some* $C_c^\infty(K)$.
(iii) A linear map of $C_c^\infty(\Omega)$ into an arbitrary locally convex space E is *continuous* if and only if its restriction to every subspace $C_c^\infty(K)$, $K \subset\subset \Omega$, is *continuous* (i.e., if the image of every convergent sequence is a convergent sequence).

From (ii) it follows that *every bounded and closed subset of $C_c^\infty(\Omega)$ is compact.*

$L^p(\Omega)$: Lebesgue space of (equivalence classes) of pth power integrable functions in Ω ($1 \le p < +\infty$)
$\|u\|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p \, dx)^{1/p}$, norm in $L^p(\Omega)$
$L^2(0, T; E)$: space of L^2 functions in the interval $[0, T]$ valued in the Hilbert space E

$L^\infty(\Omega)$: Lebesgue space of (equivalence classes) of essentially bounded functions in Ω

$\|u\|_{L^\infty(\Omega)}$: the norm in $L^\infty(\Omega)$

$L^p_{\text{loc}}(\Omega)$: space of *locally* L^p-functions in Ω, i.e., of the functions f such that given any test function ϕ in Ω, $\phi f \in L^p(\Omega)$

$L^p_c(\Omega)$: subspace of $L^p(\Omega)$ consisting of the functions $f \in L^p(\Omega)$ such that $\operatorname{supp} f \subset\subset \Omega$

$(u, v)_{L^2(\Omega)} = \int u(x)\overline{v(x)}\, dx$: the scalar or hermitian product in $L^2(\Omega)$, often also denoted by $(u, v)_0$

C^∞, C^∞_c, L^p, L^p_{loc}, L^p_c: the spaces when $\Omega = \mathbb{R}^n$

$\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$: the Schwartz space of C^∞ functions ϕ in $\mathbb{R}^n$ *rapidly decaying at infinity*, which means that, given any pair of integers $m, M \geq 0$,

$$q_{m,M}(\phi) = \sup_{x\in\mathbb{R}^n}\left[(1+|x|)^M \sum_{|\alpha|\leq m} |\partial^\alpha \phi(x)|\right] < +\infty$$

Topology of $\mathcal{S}$

It is defined by the seminorms $q_{m,M}$ as (m, M) ranges over $\mathbb{Z}^2_+$; $\mathcal{S}$ is a Fréchet space and every bounded and closed subset of $\mathcal{S}$ is compact (a subset of $\mathcal{S}$ is bounded if all the seminorms $q_{m,M}$ are bounded on it).

The following inclusions are all continuous and have dense image:

(0.5) $$C^\infty_c(\Omega) \begin{array}{c}\nearrow L^p(\Omega) \searrow \\ \searrow C^\infty(\Omega) \nearrow\end{array} L^p_{\text{loc}}(\Omega) \qquad (1 \leq p < +\infty);$$

(0.6) $$C^\infty_c \hookrightarrow \mathcal{S} \hookrightarrow L^p \qquad (1 \leq p < +\infty).$$

Let E be a Banach space, or more generally a locally convex space. We denote by $C^m(\Omega; E)$, $C^\infty_c(\Omega; E)$, $C^\infty_c(K; E)$, $L^p(\Omega; E)$, $L^p_{\text{loc}}(\Omega; E)$, $L^p_c(\Omega; E)$, $\mathcal{S}(\mathbb{R}^n; E)$ the analogues of the preceding spaces but relative to functions valued in E. The definitions are the same except that the absolute value in $\mathbb{C}$ (where the functions were valued) must be replaced by the norm, or the continuous seminorms, in the space E (see Treves [3], Section 39).

4. Distributions and Distribution Spaces

$\mathcal{D}'(\Omega)$: the space of distributions in Ω, i.e., of the continuous linear maps $C^\infty_c(\Omega) \to \mathbb{C}$, i.e., the *dual* of $C^\infty_c(\Omega)$

$\mathcal{E}'(\Omega)$: the space of *compactly supported* distributions in Ω, by definition the dual of $C^\infty(\Omega)$

supp T: the *support* of the distribution T, i.e., the intersection of all closed subsets in whose complement T vanishes identically

$\langle T, \phi\rangle = T(\phi) = \int T(x)\phi(x)\,dx$: the *duality bracket* between test functions and distributions. Thus $T \in \mathscr{D}'(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, but we can also take $T \in \mathscr{E}'(\Omega)$ and $\phi \in C^\infty(\Omega)$.

Convergence of Distributions

It is the uniform convergence on the bounded subsets of $C_c^\infty(\Omega)$. For *sequences* it is the same as the weak convergence: $T_j \to T_o$ $(j = 1, 2, \ldots)$ if and only if $\langle T_j, \phi\rangle \to \langle T_o, \phi\rangle$ for each test function ϕ.

Bounded Sets of Distributions

A set B of distributions is bounded if and only if for *each* $\phi \in C_c^\infty(\Omega)$,

$$\sup_{T\in B} |\langle T, \phi\rangle| < +\infty.$$

Sets that are bounded and closed in $\mathscr{D}'(\Omega)$ (or in $\mathscr{E}'(\Omega)$) are compact.

Differential Operators Acting on Distributions

If $P(x, D)$ is a differential operator in Ω its action on $T \in \mathscr{D}'(\Omega)$ is defined by the integration-by-parts formula

$$\langle P(x,D)T, \phi\rangle = \langle T, {}^tP(x,D)\phi\rangle, \qquad \phi \in C_c^\infty(\Omega). \tag{0.7}$$

It is clear that $P(x, D)$ defines a continuous linear map of $\mathscr{D}'(\Omega)$ (resp., of $\mathscr{E}'(\Omega)$) into itself. A particular case is that of a differential operator of order *zero*, that is, *multiplication* by a C^∞ function ψ: $\langle \psi T, \phi\rangle = \langle T, \phi\psi\rangle$, which defines a continuous endomorphism of $\mathscr{D}'(\Omega)$ (resp., $\mathscr{E}'(\Omega)$). Note that supp $P(x, D)T \subset$ supp T: *differential operators decrease the support.*

Local Structure of a Distribution

Given any $T \in \mathscr{D}'(\Omega)$ and any open set $\Omega' \subset\subset \Omega$, there is a finite set of continuous functions f_j and of differential operators P_j in Ω $(j = 1, \ldots, N)$ such that

$$T = \sum_{j=1}^{N} P_j f_j \qquad \textit{in } \Omega', \tag{0.8}$$

that is,

$$\langle T, \phi \rangle = \sum_{j=1}^{N} \int f_j(x)\, {}^t P_j \phi(x)\, dx, \qquad \phi \in C_c^\infty(\Omega').$$

If T has compact support, the finite-sum representation (0.8) may be taken to be valid in Ω itself, and the continuous functions f_j can be taken to vanish outside an arbitrary neighborhood of supp T.

Distributions That Are Functions

A distribution T in Ω is said to be a function if there is $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\text{(0.9)} \qquad \langle T, \phi \rangle = \int f(x)\phi(x)\, dx, \qquad \phi \in C_c^\infty(\Omega).$$

For now write T_f instead of T if (0.9) holds. Then $f \mapsto T_f$ is a continuous (linear) injection of $L^1_{\text{loc}}(\Omega)$ into $\mathscr{D}'(\Omega)$. In turn it defines the continuous injections into $\mathscr{D}'(\Omega)$ of

$$C^m(\Omega) \quad (0 \le m \le +\infty), \qquad C_c^\infty(\Omega), \qquad L^p_{\text{loc}}(\Omega) \quad (1 \le p \le +\infty).$$

We also have the continuous injections into $\mathscr{E}'(\Omega)$ of

$$C_c^m(\Omega) \quad (0 \le m \le +\infty), \quad L_c^p(\Omega) \quad (1 \le p \le +\infty).$$

These injections all have a *dense* image. This can most quickly be seen as follows: since all bounded subsets of $C_c^\infty(\Omega)$, or of $C^\infty(\Omega)$, have compact closure, these spaces are reflexive: $C_c^\infty(\Omega)$ is the dual of $\mathscr{D}'(\Omega)$, $C^\infty(\Omega)$ that of $\mathscr{E}'(\Omega)$. To prove that a subspace M of $\mathscr{D}'(\Omega)$ (resp., $\mathscr{E}'(\Omega)$) is dense, it suffices to show that any function $\phi \in C_c^\infty(\Omega)$ (resp., $C^\infty(\Omega)$) such that $\langle T, \phi \rangle = 0$ for all $T \in M$ must be identically zero. Take $M = C_c^\infty(\Omega)$. Let $\psi \in C^\infty(\Omega)$ be such that $\int \psi\phi\, dx = 0$ for all $\phi \in M$. Choose $\phi = \chi\bar{\psi}$, with $\chi \in C_c^\infty(\Omega)$, $\chi \ge 0$ arbitrary. We must have $\int |\psi|^2 \chi\, dx = 0$ for all such χ, hence $\psi \equiv 0$. □

If $\mathcal{O}$ is any open subset of Ω, one can say that T is a (locally L^1) function in $\mathcal{O}$, if this is true of the restriction of T to $\mathcal{O}$ (i.e., to $C_c^\infty(\mathcal{O})$). Then one can further specify the kind of function that T is, for instance a C^∞ function.

sing supp T: the *singular support* of T, i.e., the smallest closed set in the complement of which T is a C^∞ function

If $P(x, D)$ is a differential operator in Ω and T is a C^∞ function in $\mathcal{O}$, so is $P(x, D)T$: *differential operators decrease the singular support.*

δ: *Dirac measure* at the origin in $\mathbb{R}^n$; this is the distribution $\phi \mapsto \phi(0)$ $(\phi \in C_c^\infty(\mathbb{R}^n))$
δ_{x_o}, or $\delta(x - x_o)$: Dirac measure at the point x_o
$\delta^{(\alpha)} = \partial_x^\alpha \delta$: αth derivative of the Dirac measure.

These distributions are *not* functions, for $\alpha \neq 0$ they are not even *Radon measures,* that is, continuous linear functionals on the space C^0 of continuous functions in $\mathbb{R}^n$ (equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}^n$).

$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$, the dual of $\mathcal{S}$: $\mathcal{S}'$ is the space of tempered (or slowly growing at infinity) distributions in $\mathbb{R}^n$.

Structure of a Tempered Distribution

Given any $T \in \mathcal{S}'$ there is a continuous function f in $\mathbb{R}^n$ such that, for suitable integers M, $m \geq 0$,

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{-M} |f(x)| < +\infty,$$

and

$$T = (1 - \Delta)^m f. \tag{0.10}$$

In (0.10), Δ is the Laplace operator.

The dual of $\mathcal{S}'$ is $\mathcal{S}$; $\mathcal{S}$ is continuously embedded and dense in $\mathcal{S}'$ (see the preceding argument), and thus this is also true of C_c^∞.

We often write $\mathcal{D}'$, $\mathcal{E}'$ instead of $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$ respectively.

If E is a Banach space or, more generally, a locally convex topological vector space, we use the following notation:

$\mathcal{D}'(\Omega; E)$: the space of continuous linear maps $C_c^\infty(\Omega) \to E$, equipped with the topology of uniform convergence on the bounded subsets of $C_c^\infty(\Omega)$
$\mathcal{E}'(\Omega; E)$: the subspace of $\mathcal{D}'(\Omega; E)$ consisting of the compactly supported E-valued distributions in Ω

5. Convolution and Fourier Transform of Distributions

Until otherwise specified all functions and distributions in this subsection are defined in the whole of $\mathbb{R}^n$.

Convolution of Functions

$f * g$: the *convolution* of two functions f, g:

(0.11) $$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

One may assume that $f \in L^1_{\text{loc}}$, $g \in L^1_c$, or that both f, g belong to L^1, or one may make other assumptions such as

$$f \in L^p, \quad g \in L^q, \qquad 1 \le p, q \le +\infty, \qquad 1/p + 1/q - 1 \ge 0.$$

Then $f * g \in L^r$ with $1/r = 1/p + 1/q - 1$, and we have the *Hölder inequalities*

(0.12) $$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}.$$

In particular we may take $p = 1$, $1 \le q \le +\infty$ to be arbitrary, and we get

(0.13) $$\|f * g\|_{L^q} \le \|f\|_{L^1}\|g\|_{L^q}.$$

Thus L^q is a convolution L^1-module, and L^1 is a Banach algebra.

Convolution of Distributions with Functions

$\check{\phi}$: if ϕ is a function in $\mathbb{R}^n$, $\check{\phi}(x) = \phi(-x)$.
Let $\phi \in C_c^\infty$, $T \in \mathscr{D}'$ or, alternatively, $\phi \in C^\infty$, $T \in \mathscr{E}'$.
$T * \phi$: convolution of T with ϕ, written also $\phi * T$:

(0.14) $$\langle T * \phi, \psi \rangle = \langle T, \check{\phi} * \psi \rangle, \qquad \psi \in C_c^\infty.$$

Observe that $\psi \mapsto \check{\phi} * \psi$ is a continuous endomorphism (i.e., linear map into itself) of C_c^∞, or of C^∞. Then

$$(\phi, T) \mapsto T * \phi$$

is a separately continuous bilinear map of $C_c^\infty \times \mathscr{D}'$, or of $C^\infty \times \mathscr{E}'$, into C^∞.

Convolution among Distributions

$\check{T}$: the distribution in $\mathbb{R}^n$ defined by

(0.15) $$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle, \qquad \phi \in C_c^\infty.$$

$S * T$, also denoted $\int S(y)T(x-y)\,dy$ or $\int S(x-y)T(y)\,dy$: convolution of $S \in \mathscr{E}'$, $T \in \mathscr{D}'$

(0.16) $$\langle S * T, \phi \rangle = \langle S, \check{T} * \phi \rangle, \qquad \phi \in C_c^\infty.$$

By what precedes, $\phi \mapsto \check{T} * \phi$ is a continuous linear map of C_c^∞ into C^∞, and therefore (0.16) is a good definition: $S * T \in \mathscr{D}'$, and

$$(S, T) \mapsto S * T$$

is a separately continuous bilinear map of $\mathscr{E}' \times \mathscr{D}'$ into $\mathscr{D}'$. We have

$$\text{(0.17)} \qquad \operatorname{supp}(S * T) \subset \operatorname{supp} S + \operatorname{supp} T,$$

where the right-hand side is the set of the *vector sums* $x + y$ of any element x of supp S with any element $y \in \operatorname{supp} T$.

By virtue of (0.17), $\mathscr{E}'$ is a convolution algebra. In $\mathscr{E}'$ convolution is commutative; the Dirac measure δ at the origin is the identity.

Convolution of $m + 1$ distributions, of which m have compact support, makes sense, and it is associative. If $P(D)$ is a differential operator in $\mathbb{R}^n$, then for any $S \in \mathscr{E}'$, $T \in \mathscr{D}'$, we have

$$\text{(0.18)} \qquad P(D)(S * T) = [P(D)S] * T = S * [P(D)T].$$

Since, for all $T \in \mathscr{D}'$,

$$\text{(0.19)} \qquad \delta * T = T,$$

we also have

$$\text{(0.20)} \qquad P(D)T = [P(D)\delta] * T.$$

Fourier Transforms of Functions

If $u \in \mathscr{S}$, its Fourier transform is

$$\text{(0.21)} \qquad \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

The Fourier transformation $u \mapsto \hat{u}$ *defines an isomorphism of* $\mathscr{S}(\mathbb{R}^n)$ *onto* $\mathscr{S}(\mathbb{R}_n)$. The inverse of this isomorphism is given by the *Fourier inversion formula*:

$$\text{(0.22)} \qquad u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi.$$

(Other authors follow slightly different conventions.) The Fourier transformation extends as an isomorphism of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}_n)$, and we have the *Plancherel–Parseval formulas*:

$$\text{(0.23)} \qquad \int |u|^2\, dx = (2\pi)^{-n} \int |\hat{u}|^2\, d\xi,$$

$$\int u\bar{v}\,dx = (2\pi)^{-n}\int \hat{u}\bar{\hat{v}}\,d\xi \qquad (u, v \in L^2(\mathbb{R}^n)). \tag{0.24}$$

Also worth mentioning is the *Lebesgue theorem*, which states that *the Fourier transform of a function* $f \in L^1(\mathbb{R}^n)$ *is a continuous function in* $\mathbb{R}_n$ *converging to zero at infinity.*

Fourier Transforms of Distributions

$\hat{T}$: Fourier transform of the *tempered* distribution T, defined as follows:

$$(2\pi)^{-n}\int \hat{T}(\xi)\overline{\hat{\phi}(\xi)}\,d\xi = \int T(x)\overline{\phi(x)}\,dx, \qquad \phi \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier transformation $T \mapsto \hat{T}$ *is an isomorphism of* $\mathcal{S}'(\mathbb{R}^n)$ *onto* $\mathcal{S}'(\mathbb{R}_n)$. It extends the Fourier transformation on $L^2(\mathbb{R}^n)$.

Theorems of Paley–Wiener (–Schwartz)

In order for a tempered distribution u on $\mathbb{R}_n$ to be the Fourier transform of a compactly supported distribution (resp., C^∞ function), it is necessary and sufficient for u to be a C^∞ function slowly growing at infinity (resp., rapidly decaying at infinity, i.e., belonging to $\mathcal{S}$) extendable to $\mathbb{C}_n$ as an *entire function* $u(z)$ *of exponential type*, that is, satisfying everywhere the *Cauchy–Riemann equations*,

$$\partial u/\partial \bar{z}_j = 0, \qquad j = 1, \ldots, n, \tag{0.25}$$

and such that, for suitable constants A, $B > 0$,

$$|u(z)| \leq A\, e^{B|z|}, \qquad z \in \mathbb{C}_n. \tag{0.26}$$

When $T \in \mathcal{E}'$ we have

$$\hat{T}(\xi) = \int e^{-ix\cdot\xi}T(x)\,d\xi. \tag{0.27}$$

In particular, the Fourier transform of the Dirac measure is the constant function 1,

$$\hat{\delta} = 1, \tag{0.28}$$

and if $P(D) = \sum_{|\alpha|\leq m} c_\alpha D^\alpha$ is any differential operator with constant coefficients in $\mathbb{R}^n$, then

$$\widehat{P(D)\delta} = P(\xi) \qquad \Big(= \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha\Big). \tag{0.29}$$

Fourier Transform of a Convolution

If $S \in \mathscr{E}'$, $T \in \mathscr{S}'$, we can form $S * T$ and compute its Fourier transform. We have

$$\widehat{S * T} = \hat{S}\hat{T}. \tag{0.30}$$

Since $\hat{S} \in C^\infty$, the right-hand side is well defined; since all derivatives of $\hat{S}$ are slowly growing at infinity, it is a tempered distribution (as expected).

Combining (0.20), (0.29), and (0.30) gives

$$\widehat{P(D)T} = P(\xi)\hat{T}; \tag{0.31}$$

in particular,

$$\widehat{D^\alpha T} = \xi^\alpha \hat{T}, \tag{0.32}$$

$$\widehat{\Delta T} = -|\xi|^2 \hat{T}. \tag{0.33}$$

Sobolev Spaces

In the following definitions s denotes an arbitrary *real* number.

$H^s = H^s(\mathbb{R}^n)$: the space of tempered distributions u in $\mathbb{R}^n$ whose Fourier transform $\hat{u}$ is a square-integrable function in $\mathbb{R}_n$ for the measure $(1 + |\xi|^2)^s \, d\xi$

$(u, v)_s$: the *inner product* in H^s,

$$(u, v)_s = (2\pi)^{-n} \int \hat{u}(\xi)\overline{\hat{v}(\xi)}(1 + |\xi|^2)^s \, d\xi$$

$\|u\|_s = [(u, u)_s]^{1/2}$: the *norm* in H^s, which is a Hilbert space when equipped with the inner product $(\; , \;)_s$

$H_c^s(K)$: the subspace of H^s consisting of the distributions having their support in the compact set K; $H_c^s(K)$ is a *closed* linear subspace of H^s

$H_c^s(\Omega)$: the union of the spaces $H_c^s(K)$ for K ranging over the collection of all compact subsets of Ω

$H_{\text{loc}}^s(\Omega)$: the space of distributions u in Ω such that $\phi u \in H^s$ for any $\phi \in C_c^\infty(\Omega)$.

The topology of $H_{\text{loc}}^s(\Omega)$ is that defined by the seminorms $u \mapsto \|\phi u\|_s$, $\phi \in C_c^\infty(\Omega)$. It suffices to take ϕ ranging over a sequence $\{\phi_\nu\}$ such that the compact sets $K_\nu = \{x \in \Omega; \phi_\nu(x) = 1\}$ exhaust Ω (see definition of the C^∞ topology), and $\phi_\nu \le \phi_{\nu+1}$, $\nu = 0, 1, \ldots$. Thus $H_{\text{loc}}^s(\Omega)$ is easily seen to be a (reflexive) Fréchet space.

The topology of $H_c^s(\Omega)$ is defined as follows: for each $K \subset\subset \Omega$, $H_c^s(K)$ is equipped with the Hilbert space structure induced by H^s. Then a *convex* set in $H_c^s(\Omega)$ is open if and only if its intersection with every $H_c^s(K)$ is open.

We have the following continuous linear injections with dense images

$$\mathscr{S} \hookrightarrow H^s \hookrightarrow H^{s'} \hookrightarrow \mathscr{S}' \qquad (s' \leq s),$$

$$C_c^\infty(\Omega) \hookrightarrow H_c^s(\Omega) \hookrightarrow H_{\text{loc}}^s(\Omega) \hookrightarrow \mathscr{D}'(\Omega).$$

We have the *set-theoretical* equalities (the first one is topological);

$$(0.34) \qquad C^\infty(\Omega) = \bigcap_s H_{\text{loc}}^s(\Omega), \qquad C_c^\infty(\Omega) = \bigcap_s H_c^s(\Omega),$$

$$(0.35) \qquad \mathscr{E}'(\Omega) = \bigcup_s H_c^s(\Omega), \qquad \mathscr{D}'^F(\Omega) = \bigcup_s H_{\text{loc}}^s(\Omega),$$

where $\mathscr{D}'^F(\Omega)$ stands for the space of *distributions of finite order* in Ω (i.e., distributions having a finite-sum representation (0.8) in the whole of Ω, not just in sets $\Omega' \subset\subset \Omega$).

$(1-\Delta)^s$: the *convolution operator*

$$u(x) \mapsto (2\pi)^{-n} \int e^{ix\cdot\xi}(1+|\xi|^2)^s \hat{u}(\xi)\, d\xi.$$

As s varies over $\mathbb{R}$, $(1-\Delta)^s$ forms a group of automorphisms of $\mathscr{S}(\mathbb{R}^n)$, or of $\mathscr{S}'(\mathbb{R}^n)$. Given any $t \in \mathbb{R}$, $(1-\Delta)^s$ is an *isometry* of H^t onto H^{t-2s}, in particular, of H^s onto H^{-s}.

We have $H^0 = L^2$; the equality applies also to the Hilbert space structures. The dense image injection $\mathscr{S} \hookrightarrow H^s$ transposes into the injection $(H^s)' \hookrightarrow \mathscr{S}'$ whose image is equal to H^{-s}. Thus the following pairs of spaces can be naturally regarded as dual pairs:

$$H^s \text{ and } H^{-s}, \qquad H_{\text{loc}}^s(\Omega) \text{ and } H_c^{-s}(\Omega),$$

and $(1-\Delta)^s$ as the natural *linear* isometry of H^s onto its *antidual*, H^{-s} (*antiduality* is defined by the bracket $\langle u, \bar{v}\rangle$, whereas duality is defined by $\langle u, v\rangle$).