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Introduction to S-Duality in $N = 2$ Supersymmetric Gauge Theories

(A Pedagogical Review of the Work of Seiberg and Witten)

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Abstract

In these notes we attempt to give a pedagogical introduction to the work of Seiberg and Witten on S-duality and the exact results of $N = 2$ supersymmetric gauge theories with and without matter. The first half is devoted to a review of monopoles in gauge theories and the construction of supersymmetric gauge theories. In the second half, we describe the work of Seiberg and Witten.

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Introduction

These notes are the very late written version of a series of lectures given at the Trieste Summer School in 1995. The aim was to provide an elementary introduction to the work of Seiberg and Witten on exact results concerning $N = 2$ supersymmetric extensions of Quantum Chromodynamics. We wanted to provide, in a single place, all the background material necessary to study their work in detail. We had in mind graduate students who have already gone through their Quantum Field Theory course, but we do not expect much more background to follow these lectures. We have done our best to make the treatment pedagogical. In some sections, we have heavily drawn on previous reviews, for instance in the treatment of supersymmetry we have followed Bagger and Wess [22], and in the theory of monopoles we have used the reviews by Goddard and Olive [1] and by S. Coleman [2, 3]. These sources provide excellent presentations of these topics, and we had no compelling reason to try to make improvements on their presentation. It is also quite obvious that we have drawn heavily on the original papers of Seiberg and Witten [31, 32], but we have tried to provide the necessary tools to make their reading more accessible to interested students and/or researchers. Recently there have been other lecture notes published at a more advanced level, where one can find more details and also the connection with String Theory (see, for instance, the notes by W. Lerche [40]). We would also like to mention that we have not tried to be exhaustive in quoting all the literature on the subject. A more complete reference list can be found in [40]. The reference list is intented to provide a guidance to enter the vastly growing literature on duality in String Theory and Field Theory. We apologize to those authors who may be offended by not finding their works referenced.

These notes are divided into four sections, with each section further subdivided into several subsections. Section 1 is devoted to an introduction to monopoles in gauge theories. We start with a discussion of the Dirac monopole and the idea of charge quantization, and then describe the 't Hooft-Polyakov monopole in gauge theories with spontaneous symmetry breaking. Then we introduce the notion of Bogomol'nyi bound and the BPS states. After this, we describe the topological classification of monopoles and then describe the Montonen-Olive conjecture of electric-magnetic duality. We end this section with a description of how, in the presence of a θ-term in the Lagrangian, the electric charge of a monopole is shifted by its magnetic charge. Section 2 is devoted to an introduction to supersymmetric gauge theories. First we describe the supersymmetry algebra and its representations without and with central charges and discuss its local realizations in terms of superfields. Then we construct $N = 1$ Lagrangians and finally, $N = 2$ supersymmetric Lagrangians with both gauge multiplets and matter multiplets (hypermultiplets). At the end, we calculate the $N = 2$ central charge both in the pure gauge theory, as well as in the theory with matter and establish its relation to the BPS bound. Having built the foundations in the first two sections, in section 3 we describe the Seiberg-Witten analysis of the $N = 2$ pure gauge theory with gauge group $SU(2)$. In the first two subsections, we discuss the parametrization of the moduli space and the breaking of R-symmetries. Then we describe how the chiral $U(1)$ anomaly of the theory can be used to obtain the one-loop form of the low-energy effective action. The rest of the section is devoted to finding the exact low-energy effective action by using duality and the singularity structure on the moduli space of the theory. We express the exact solution in terms of complete elliptic integrals. In section 4, we briefly describe the Seiberg-Witten analysis of $N = 2$ $SU(2)$ gauge theory with N_f matter fields. After a discussion of the general features of these theories, we describe how the duality group is no longer pure $SL(2, Z)$. Then we describe the singularity structure on the moduli space of these theories and sketch the procedure for obtaining the exact solutions. Our aim in section 4 is to give a flavour of the analysis of theories with matter and, for a deeper understanding, the interested reader is referred to the original work of Seiberg and Witten.

1 Magnetic Monopoles in Gauge Theories

In this section, we begin by reviewing the proporties of the Dirac monopole and the idea of charge quantization. Then we describe the magnetic monopoles and dyons which arise in non-Abelian gauge theories with spontaneous symmetry breaking and discuss their general properties. We also introduce the notion of the Bogomol'nyi bound and BPS states. In the last two subsections, we describe the Montonen-Olive conjecture of electric-magnetic duality and Witten's argument about how the presence of the θ-term in the Lagrangian modifies the monopole and dyon electric charges. For a more detailed discussion of most of the material in this section, the reader is referred to the review article by Goddard and Olive [1], and to the lecture notes by Coleman [2, 3].

1.1 Conventions and Preliminaries

Let us start by stating our conventions: we always take $c = 1$, and almost always $\hbar = 1$, except when it is important to make a distinction between classical and quantum effects. For index manipulations, we use the flat Minkowsky metric η of signature $\{+, -, -, -\}$. Moreover, we choose units in which Maxwell's equations take the form:

$$\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = \rho \,, & \vec{\nabla} \times \vec{B} - \partial \vec{E}/\partial t = \vec{j} \,, \\ \vec{\nabla} \cdot \vec{B} = 0 \,, & \vec{\nabla} \times \vec{E} + \partial \vec{B}/\partial t = 0 \,. \end{array} \tag{1}$$

In these units, a factor of $(4\pi)^{-1}$ appears in Coulomb's law: For a static point-like charge q at the origin, we have $\rho = q\delta^3(\vec{r})$. Integrating the first Maxwell equation over a sphere of radius r and using spherical symmetry, we get $\int_{S^2} \vec{E} \cdot d\vec{s} = 4\pi r^2 E(r) = q$. Hence, $\vec{E} = q\vec{r}/4\pi r^3$, as stated. The electrostatic potential ϕ defined by $\vec{E} = -\vec{\nabla}\phi$ is given by $\phi = q/4\pi r$.

In relativistic notation, one introduces the four-potential $A^\mu = \{\phi, \vec{A}\}$. The electric and magnetic fields are defined as components of the corresponding field strength tensor $F_{\mu\nu}$ as follows:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \,, \\ F_{0i} &= \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 = E^i \,, \\ F_{ij} &= \partial_i A_j - \partial_j A_i = -(\partial_i A^j - \partial^j A_i) = -\epsilon_{ijk} B^k \,, \end{aligned} \tag{2}$$

so that $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk} = -\frac{1}{2}\epsilon^{oi\mu\nu}F_{\mu\nu}$. The dual field strength tensor is given by

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,,$$

with $\varepsilon^{0123} = +1$. In component notation, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_x & B_y & 0 \end{pmatrix}, \qquad \widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{3}$$

In terms of the electric four-current $j^\mu = \{\rho, \vec{j}\}$, the Maxwell's equations take the compact form

$$\partial_\nu F^{\mu\nu} = -j^\mu\,, \qquad \partial_\nu \widetilde{F}^{\mu\nu} = 0\,. \tag{4}$$

Note that when $j^\mu = 0$, the above equations are invariant under the replacement $\vec{E} \to \vec{B}, \vec{B} \to -\vec{E}$. This is referred to as the electric-magnetic duality transformation. In the presence of electric sources, this transformation is no longer a symmetry of Maxwell's equations. In order to restore the duality invariance of these equations for non-zero j^μ, Dirac [4] introduced the magnetic four-current $k^\mu = \{\sigma, \vec{k}\}$ and modified Maxwell's equations to

$$\partial_\nu F^{\mu\nu} = -j^\mu\,, \qquad \partial_\nu \widetilde{F}^{\mu\nu} = -k^\mu\,. \tag{5}$$

The above equations are now invariant under a combined duality transformation of the fields and the currents which can be written as

$$F \to \widetilde{F}\,,\; \widetilde{F} \to -F\,; \qquad j^\mu \to k^\mu\,,\; k^\mu \to -j^\mu\,. \tag{6}$$

Note that the full invariance group of the equations (5) is larger than this discrete duality. In fact, they are invariant under a continuous $SO(2)$ group which rotates the electric and magnetic quantities into each other.

For point-like electric and magnetic sources, the current densities can be written as

$$\begin{aligned} j^\mu(x) &= \sum_a q_a \int dx_a^\mu \delta^4(x - x_a)\,, \\ k^\mu(x) &= \sum_a g_a \int dx_a^\mu \delta^4(x - x_a)\,. \end{aligned}$$

A particle of electric charge q and magnetic charge g experiences a Lorentz force given by

$$m\,\frac{d^2x^\mu}{d\tau^2} = (qF^{\mu\nu} + g\,\widetilde{F}^{\mu\nu})\frac{dx_\nu}{d\tau}\,.$$

Although, at the level of quations of motion, Dirac's modification of Maxwell's theory seems trivial, it has highly non-trivial consequences in quantum theory. One way of realizing the problem is to note that the vector potential $\vec{A}$ is indispensable in the quantum formulation of the theory. On the other hand, equations $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{\nabla} \cdot \vec{B} \neq 0$ are not compatible unless the vector potential $\vec{A}$ has singularities and, hence, is not globally well-defined. It turns out that these singularities are gauge dependent and, therefore, their presence should not be experimentally detectable. In the classical theory, which can be formulated in terms of $\vec{B}$ alone, this requirement is trivially satisfied. However, in quantum theory, it leads to the important phenomenon of the quantization of electric charge. In the following, we will first give a semiclassical derivation of this effect and then describe a derivation based on the notion of the Dirac string.

1.2 A Semiclassical Derivation of Charge Quantization

Consider a non-relativistic charge q in the vicinity of a magnetic monopole of strength g, situated at the origin. The charge q experiences a force $m\ddot{\vec{r}} = q\dot{\vec{r}} \times \vec{B}$, where $\vec{B}$ is the monopole field given by $\vec{B} = g\vec{r}/4\pi r^3$. The change in the orbital angular momentum of the electric charge under the effect of this force is given by

$$\frac{d}{dt}\left(m\vec{r} \times \dot{\vec{r}}\right) = m\vec{r} \times \ddot{\vec{r}} = \frac{qg}{4\pi r^3}\,\vec{r} \times \left(\dot{\vec{r}} \times \vec{r}\right) = \frac{d}{dt}\left(\frac{qg}{4\pi}\,\frac{\vec{r}}{r}\right).$$

Hence, the total conserved angular momentum of the system is

$$\vec{J} = \vec{r} \times m\dot{\vec{r}} - \frac{qg}{4\pi}\,\frac{\vec{r}}{r}\,. \tag{7}$$

The second term on the right hand side (henceforth denoted by $\vec{J}_{em}$) is the contribution coming from the elecromagnetic field. This term can be directly computed by using the fact that the momentum density of an electromagnetic field is given by its Poynting vector, $\vec{E} \times \vec{B}$, and hence its contribution to the angular momentum is given by

$$\vec{J}_{em} = \int d^3x\,\vec{r} \times (\vec{E} \times \vec{B}) = \frac{g}{4\pi}\int d^3x\,\vec{r} \times \left(\vec{E} \times \frac{\vec{r}}{r^3}\right).$$

In components,

$$\begin{aligned} J^i_{em} &= \frac{g}{4\pi}\int d^3x E^j \left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\frac{1}{r} = \frac{g}{4\pi}\int d^3x E^j \partial_j(\hat{x}^i) \\ &= \frac{g}{4\pi}\int_{S^2} \hat{x}^i \vec{E} \cdot d\vec{s} - \frac{g}{4\pi}\int d^3x \vec{\nabla} \cdot \vec{E}\hat{x}^i\,. \end{aligned} \tag{8}$$

When the separation between the electric and magnetic charges is negligible compared to their distance from the boundary S^2, the contribution of the first integral to $\vec{J}_{em}$ vanishes by spherical symmetry. We are therefore left with

$$\vec{J}_{em} = -\frac{gq}{4\pi}\hat{r}\,. \tag{9}$$

Returning to equation (7), if we assume that orbital angular momentum is quantized. Then it follows that

$$\frac{qg}{4\pi} = \frac{1}{2} n\hbar\,, \tag{10}$$

where n is an integer. Note that in the above we have assumed the total angular momentum of the charge-monopole system to be quantized in half-integral units. This is a strange assumption considering that we did not have to treat the electrically charged particle or the monopole as fermios. both of the components are bosonic. However, it turns out that this actually is the case and that the situation does not contradict the spin-statistics theorem [5, 6]. We will not discuss this issue further but remark that the derivation of the same equation presented in the next subsection does not depend on this assumption.

Equation (10) is the Dirac charge quantization condition. It implies that if there exists a magnetic monopole of charge g somewhere in the universe, then all electric charges are quantized in units of $2\pi\hbar/g$. If we have a number of purely electric charges q_i and purely magnetic charges g_j, then any pair of them will satisfy a quantization condition:

$$\frac{q_i g_j}{4\pi\hbar} = \frac{1}{2} n_{ij} \tag{11}$$

Thus, any electric charge is an integral multiple of $2\pi\hbar/g_j$. For a given g_j, let these charges have n_{0j} as the highest common factor. Then, all the electric charges are multiples of $q_0 = n_{0j} 2\pi\hbar/g_j$. Note that q_0 itself may not exist in the spectrum. Similar considerations apply to the quantization of magnetic charge.

Till now, we have only dealt with particles that carry either an electric or a magnetic charge. Let us now consider dyons, *i.e.,* particles that carry both electric and magnetic charges. Consider two dyons of charges (q_1, g_1) and (q_2, g_2). For this system, we can repeat the calculation of $\vec{J}_{em}$ by following the steps in (8) where now the electromagnetic fields are split as $\vec{E} = \vec{E}_1 + \vec{E}_2$ and $\vec{B} = \vec{B}_1 + \vec{B}_2$. The answer is easily found to be

$$\vec{J}_{em} = -\frac{1}{4\pi} \left(q_1 g_2 - q_2 g_1 \right) \hat{r} \tag{12}$$

The charge quantization condition is thus generalized to

$$\frac{q_1 g_2 - q_2 g_1}{4\pi\hbar} = \frac{1}{2} n_{12} \tag{13}$$

This is referred to as the Dirac-Schwinger-Zwanziger condition [7, 8]. This condition is invariant under the $SO(2)$ transformation $(q + ig) \rightarrow e^{i\phi}(q + ig)$ which is also a symmetry of (5).

1.3 The Dirac String

In the following, we present a more rigorous derivation of the Dirac quantization condition which is based on the notion of a Dirac string. Let $\vec{B}_{mon}$ denote the magnetic field around a monopole. Since $\vec{\nabla}\cdot\vec{B}_{mon}\neq 0$, it is not possible to construct a well-defined $\vec{A}_{mon}$ such that $\vec{B}_{mon}=\vec{\nabla}\vec{A}_{mon}$. To overcome this problem, Dirac intruduced a semi-infinite solenoid (or string) running from $(0,0,-\infty)$ to the monopole position, $(0,0,0)$. This solenoid carries a magnetic field $\vec{B}_{sol}=g\theta(-z)\delta(x)\delta(y)\hat{z}$ which also is not divergence free. However, the total magnetic field,

$$\vec{B}=\vec{B}_{mon}+\vec{B}_{sol}=\frac{g}{4\pi r^2}+g\theta(-z)\delta(x)\delta(y)\hat{z}\,, \tag{14}$$

satisfies $\vec{\nabla}\cdot\vec{B}=g\delta(\vec{r})-g\delta(\vec{r})=0$. It is therefore possible to construct a non-singular $\vec{A}=\vec{A}_{mon}+\vec{A}_{sol}$ corresponding to the monopole-solenoid system. In fact, the singular $\vec{A}_{sol}$ associated with the Dirac string is used to cancel the singularity in $\vec{A}_{mon}$. The position of the singularity in $\vec{A}_{mon}$, and therefore the position of the Dirac string, can be shifted by singular gauge transformations. Since the Dirac string is an artificial construct, it should be unobservable and should not contribute to any physical process. However, in an Aharonov-Bohm experiment, the presence of the string can affect the wavefunction of an electric charge by contributing to its phase. This resulting Aharonov-Bohm phase along a contour Γ encircling the string and enclosing an area S can be easily computed to be

$$e\oint_\Gamma \vec{A}_{sol}\cdot d\vec{l}=e\int_S \vec{B}_{sol}\cdot d\vec{s}=eg\,.$$

Here, $e=q/\hbar$ is the electromagnetic coupling constant. This phase, and therefore the Dirac string, is unobservable provided

$$eg=2\pi n\,,$$

which is again the Dirac charge quantization condition.

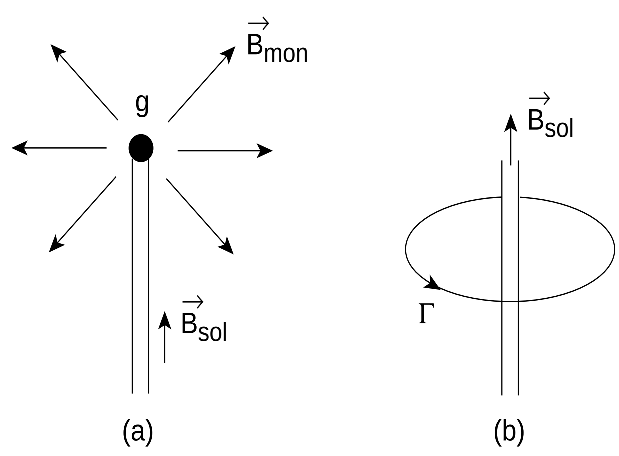

Figure 1

1.4 The Georgi-Glashow Model: A Simple Theory with Monopoles

Till now we have been working in the framework of particle mechanics where both electric and magnetic charges are point-like objects and are introduced by hand into the theory. However, in field theory, these objects can also arise as solitons which are non-trivial solutions of the field equations with localized energy density. If the gauge field configuration associated with a soliton solution has a magnetic character, the soliton can be identified as a magnetic monopole. The simplest solitons are found in the Sine-Gordon theory, which is a scalar field theory in $1+1$ dimensions. However, by Derrick's theorem, a scalar field theory in more than two dimensions cannot support static finite energy solutions. This is basically due to the fact that because of the non-trivial structure of the soliton at large distances, the total energy of the configuration diverges. This situation can be cured by the addition of gauge fields to the theory. Thus, a scalar theory with gauge interactions in four dimensions can admit static finite energy field configurations. The stability of such solitonic configurations are often related to the fact that they are characterized by conserved topological charges. For a more detailed discussion of these issues, the reader is referred to [1, 2, 3]. In the following, we describe the Georgi-Glashow model which is a simple theory in $3+1$ dimensions with monopole solutions.

The Georgi-Glashow model is a Yang-Mills-Higgs system which contains a Higgs multiplet ϕ^a $(a = 1, 2, 3)$ transforming as a vector in the adjoint representation of the gauge group $SO(3)$, and the gauge fields $W_\mu = W_\mu^a T^a$. Here, T^a are the hermitian generators of $SO(3)$ satifying $[T^a, T^b] = i f^{abc} T^c$. In the adjoint representaion, we have $(T^a)_{bc} = -i f^a_{bc}$ and, for $SO(3)$, $f^{abc} = \epsilon^{abc}$. The field strength of W_μ and the cavariant derivative on ϕ^a are defined by

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu]\,, \\ D_\mu \phi^a &= \partial_\mu \phi^a - e\epsilon^{abc} W_\mu^b \phi^c\,. \end{aligned} \tag{15}$$

The minimal Lagrangian is then given by

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{2} D^\mu \phi^a D_\mu \phi^a - V(\phi)\,, \tag{16}$$

where,

$$V(\phi) = \frac{\lambda}{4} \left(\phi^2 - a^2\right)^2\,. \tag{17}$$

The equations of motion following from this Lagrangian are

$$(D_\nu G^{\mu\nu})^a = -e\,\epsilon^{abc}\,\phi^b\,(D^\mu \phi)^c, \qquad D^\mu D_\mu \phi^a = -\lambda \phi^a (\phi^2 - a^2)\,. \tag{18}$$

The field strength also satifies the Bianchi identity

$$D_\nu\, \widetilde{G}^{\mu\nu a} = 0\,. \tag{19}$$

Let us find the vacuum configurations in this theory. Using the notations $G_a^{0i} = -\mathcal{E}_a^i$ and $G_a^{ij} = -\epsilon^{ij}{}_k \mathcal{B}_a^k$, the energy density is written as

$$\theta_{00} = \frac{1}{2}\left((\mathcal{E}_a^i)^2 + (\mathcal{B}_a^i)^2 + (D^0\phi_a)^2 + (D^i\phi_a)^2\right) + V(\phi)\,. \tag{20}$$

Note that $\theta_{00} \geq 0$, and it vanishes only if

$$G_a^{\mu\nu} = 0, \quad D_\mu \phi = 0, \quad V(\phi) = 0\,. \tag{21}$$

The first equation implies that in the vacuum, W_μ^a is pure gauge and the last two equations define the Higgs vacuum. The structure of the space of vacua is determined by $V(\phi) = 0$ which solves to $\phi^a = \phi^a_{vac}$ such that $|\phi_{vac}| = a$. The space of Higgs vacua is therefore a two-sphere (S^2) of radius a in the field space. To formulate a perturbation theory, we have to choose one of these vacua and hence, break the gauge symmetry spontaneously (this is the usual Higgs mechanism). The part of the symmetry which keeps this vacuum invariant, still survives and the corresponding unbroken generator is $\phi^c_{vac} T^c/a$. The gauge boson associated with this generator is $A_\mu = \phi^c_{vac} W^c_\mu / a$ and the electric charge operator for this surviving $U(1)$ is given by

$$Q = \hbar e \frac{\phi^c_{vac} T^c}{a} \,. \tag{22}$$

If the group is compact, this charge is quantized. The perturbative spectrum of the theory can be found by expanding ϕ^a around the chosen vacuum as

$$\phi^a = \phi^a_{vac} + \phi'^a \,.$$

A convenient choice is $\phi^c_{vac} = \delta^{c3} a$. The perturbative spectrum (which becomes manifest after choosing an appropriate gauge) consists of a massive Higgs (H), a massless photon (γ) and two charged massive bosons ($W^\pm$):

	Mass	Spin	Charge
H	$a(2\lambda)^{\frac{1}{2}}\hbar$	0	0
γ	0	$\hbar$	0
$W^\pm$	$ae\hbar = aq$	$\hbar$	$\pm q = \pm e\hbar$

In the next section, we investigate the existence of monopoles (non-perturbative states) in the Georgi-Glashow model.

1.5 The 't Hooft - Polyakov Monopole

Let us look for time-independent, finite energy solutions in the Georgi-Glashow model. Finiteness of energy requires that as $r \to \infty$, the energy density θ_{00} given by (20) must approach zero faster than $1/r^3$. This means that as $r \to \infty$, our solution must go over to a Higgs vacuum defined by (21). In the following, we will first assume that such a finite energy solution exists and show that it can have a monopole charge related to its soliton number which is, in turn, determined by the associated Higgs vacuum. This result is proven without having to deal with any particular solution explicitly. Next, we will describe the 't Hooft-Polyakov ansatz for explicitly constructing one such monopole solution. We will also comment on the existence of Dyonic solutions. For convenience, in this section we will use the vector notation for the $SO(3)$ gauge group indices and not for the spatial indices.

The Topological Nature of Magnetic Charge: Let $\vec{\phi}_{vac}$ denote the field $\vec{\phi}$ in a Higgs vacuum. It then satisfies the equations

$$\vec{\phi}_{vac} \cdot \vec{\phi}_{vac} = a^2 \,,$$
$$\partial_\mu \vec{\phi}_{vac} - e\, \vec{W}_\mu \times \vec{\phi}_{vac} = 0 \,, \tag{23}$$

which can be solved for $\vec{W}_\mu$. The most general solution is given by

$$\vec{W}_\mu = \frac{1}{ea^2}\,\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac} + \frac{1}{a}\vec{\phi}_{vac} A_\mu\,. \tag{24}$$

To see that this actually solves (23), note that $\partial_\mu \vec{\phi}_{vac} \cdot \vec{\phi}_{vac} = 0$, so that

$$\frac{1}{ea^2}(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} = \frac{1}{ea^2}\left(\partial_\mu \vec{\phi}_{vac} a^2 - \vec{\phi}_{vac}(\vec{\phi}_{vac} \cdot \partial_\mu \phi_{vac})\right) = \frac{1}{e}\partial_\mu \vec{\phi}_{vac}\,.$$

The first term on the right-hand side of Eq. (24) is the particular solution, and $\vec{\phi}_{vac} A_\mu$ is the general solution to the homogeneous equation. Using this solution, we can now compute the field strength tensor $\vec{G}_{\mu\nu}$. The field strength $F_{\mu\nu}$ corresponding to the unbroken part of the gauge group can be identified as

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{a}\vec{\phi}_{vac} \cdot \vec{G}_{\mu\nu} \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{a^3 e}\vec{\phi}_{vac} \cdot (\partial_\mu \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac})\,. \end{aligned} \tag{25}$$

Using the equations of motion in the Higgs vacuum it follows that

$$\partial_\mu F^{\mu\nu} = 0\,, \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0\,.$$

This confirms that $F_{\mu\nu}$ is a valid $U(1)$ field strength tensor. The magnetic field is given by $B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk}$. Let us now consider a static, finite energy solution and a surface Σ enclosing the core of the solution. We take Σ to be far enough so that, on it, the solution is already in the Higgs vacuum. We can now use the magnetic field in the Higgs vacuum to calculate the magnetic charge g_Σ associated with our solution:

$$g_\Sigma = \int_\Sigma B^i ds^i = -\frac{1}{2ea^3}\int_\Sigma \epsilon_{ijk}\,\vec{\phi}_{vac} \cdot \left(\partial^j \vec{\phi}_{vac} \times \partial^k \vec{\phi}_{vac}\right) ds^i\,. \tag{26}$$

It turns out that the expression on the right hand side is a topological quantity as we explain below: Since $\phi^2 = a$; the manifold of Higgs vacua ($\mathcal{M}_0$) has the topology of S^2. The field $\vec{\phi}_{vac}$ defines a map from Σ into $\mathcal{M}_0$. Since Σ is also an S^2, the map $\phi_{vac} : \Sigma \to \mathcal{M}_0$ is characterized by its homotopy group $\pi_2(S^2)$. In other words, ϕ_{vac} is characterized by an integer ν (the winding number) which counts the number of times it wraps Σ around $\mathcal{M}_0$. In terms of the map ϕ_{vac}, this integer is given by

$$\nu = \frac{1}{4\pi a^3}\int_\Sigma \frac{1}{2}\epsilon_{ijk}\vec{\phi}_{vac} \cdot \left(\partial^j \vec{\phi}_{vac} \times \partial^k \vec{\phi}_{vac}\right) ds^i\,. \tag{27}$$

Comparing this with the expression for magnetic charge, we get the important result

$$g_\Sigma = -\frac{4\pi\nu}{e}\,. \tag{28}$$

Hence, the winding number of the soliton determines its monopole charge. Note that the above equation differs from the Dirac quantization condition by a factor of 2. This is because the smallest electric charge which could exist in our model is $q_0 = e\hbar/2$ in terms of which, (28) reduces to the Dirac condition.

An Ansatz for Monopoles: Now we describe an ansatz proposed by 't Hooft [9] and Polyakov [10] for constructing a monopole solution in the Georgi-Glashow model. For a spherically symmetric, parity-invariant, static solution of finite energy, they proposed:

$$\begin{aligned} \phi^a &= \frac{x^a}{er^2} H(aer) \,, \\ W_i^a &= -\epsilon^a_{ij} \frac{x^j}{er^2} \left(1 - K(aer)\right) \,, \qquad W_0^a = 0 \,. \end{aligned} \tag{29}$$

For the non-trivial Higgs vacuum at $r \to \infty$, they chose $\phi^c_{vac} = ax^c/r = a\hat{x}^c$. Note that this maps an S^2 at spatial infinity onto the vacuum manifold with a unit winding number. The asymptotic behaviour of the functions $H(aer)$ and $K(aer)$ are determined by the Higgs vacuum as $r \to \infty$ and regularity at $r = 0$. Explicitly, defining $\xi = aer$, we have: as $\xi \to \infty$, $H \sim \xi$, $K \to 0$ and as $\xi \to 0$, $H \sim \xi$, $(K - 1) \sim \xi$. The mass of this solution can be parametrized as

$$M = \frac{4\pi a}{e} f\left(\lambda/e^2\right)$$

For this ansatz, the equations of motion reduce to two coupled equations for K and H which have been solved exactly only in certain limits. For $r \to 0$, one gets $H \to ec_1r^2$ and $K = 1 + ec_2r^2$ which shows that the fields are non-singular at $r = 0$. For $r \to \infty$, we get $H \to \xi + c_3 exp(-a\sqrt{2\lambda}r)$ and $K \to c_4\xi exp(-\xi)$ which leads to $W_i^a \approx -\epsilon^a_{ij}x^j/er^2$. Once again, defining $F_{ij} = \phi^c G^c_{ij}/a$, the magnetic field turns out to be $B^i = -x^i/er^3$. The associated monopole charge is $g = -4\pi/e$, as expected from the unit winding number of the solution. It should be mentioned that 't Hooft's definition of the Abelian field strength tensor is slightly different but, at large distances, it reduces to the form given above.

Note that in the above monopole solution, the presence of the Dirac string is not obvious. To extract the Dirac string, we have to perform a singular gauge transformation on this solution which rotates the non-trivial Higgs vacuum $\phi^c_{vac} = a\hat{x}^c$ into the trivial vacuum $\phi^c_{vac} = a\delta^{c3}$. In the process,the gauge field develops a Dirac string singularity which now serves as the source of the magnetic charge [9].

The Julia-Zee Dyons:

The 't Hooft-Polyakov monopole carries one unit of magnetic charge and no electric charge. The Georgi-Glashow model also admits solutions which carry both magnetic as well as electric charges. An ansatz for constructing such a solution was proposed by Julia and Zee [11]. In this ansatz, ϕ^a and W_i^a have exactly the same form as in the 't Hooft-Polyakov ansatz, but W_0^a is no longer zero: $W_0^a = x^a J(aer)/er^2$. This serves as the source for the electric charge of the dyon. It turns out that the dyon electric charge depends of a continuous parameter and, at the classical level, does not satisfy the quantization condition. However, semiclassical arguments [12, 13] show that, in CP invariant theories, and at the quantum level, the dyon electric charge is quantized as $q = n\hbar e$. This can be easiy understood if we recognize that a monopole is not invariant under a guage transformation which is, of course, a symmetry of the equations of motion. To treate the associated zero-mode properly, the gauge degree of freedom should be regarded as a collective coordinate. Upon quantization, this collective coordinate leads to the existence of electrically charged states for the monopole with discrete charges. In the presence of a CP violating term in the Lagrangian, the situation is more subtle as we will discuss later. In the next subsection, we describe a limit in which the equations of motion can be solved exactly for the 'tHooft-Polyakov and the Julia-Zee ansatz. This is the limit in which the soliton mass saturates the Bogomol'nyi bound.

1.6 The Bogomol'nyi Bound and the BPS States

In this subsection, we derive the Bogomol'nyi bound [14] on the mass of a dyon in term of its electric and magnetic charges which are the sources for $F^{\mu\nu} = \vec{\phi} \cdot \vec{G}^{\mu\nu}/a$. Using the Bianchi identity (19) and the first equation in (18), we can write the charges as

$$\begin{aligned} g &\equiv \int_{S^2_\infty} B_i dS^i = \frac{1}{a}\int \mathcal{B}^a_i \phi^a dS^i = \frac{1}{a}\int \mathcal{B}^a_i (D^i\phi)^a d^3x\,, \\ q &\equiv \int_{S^2_\infty} E_i dS^i = \frac{1}{a}\int \mathcal{E}^a_i \phi^a dS^i = \frac{1}{a}\int \mathcal{E}^a_i (D^i\phi)^a d^3x\,. \end{aligned} \tag{30}$$

Now, in the center of mass frame, the dyon mass is given by

$$M \equiv \int d^3x \theta_{00} = \int d^3x \left(\frac{1}{2}\left[(\mathcal{E}^a_k)^2 + (\mathcal{B}^a_k)^2 + (D_k\phi^a)^2 + (D_0\phi^a)^2 \right] + V(\phi) \right)\,,$$

where, $\theta_{\mu\nu}$ is the energy momentum tensor. After a little manipulation, and using the expressions for the electric and magnetic charges given in (30), this can be written as

$$\begin{aligned} M &= \int d^3x \left(\frac{1}{2}\left[(\mathcal{E}^a_k - D_k\phi^a \sin\theta)^2 + (\mathcal{B}^a_k - D_k\phi^a\cos\theta)^2 + (D_0\phi^a)^2 \right] + V(\phi) \right) \\ &+ a(q\sin\theta + g\cos\theta)\,, \end{aligned} \tag{31}$$

where θ is an arbitrary angle. Since the terms in the first line are positive, we can write $M \geq (q\sin\theta + g\cos\theta)$. This bound is maximum for $\tan\theta = q/g$. Thus we get the Bogomol'nyi bound on the dyon mass as

$$M \geq a\sqrt{g^2+q^2}\,.$$

For the 't Hooft-Polyakov solution, we have $q = 0$, and thus, $M \geq a|g|$. But $|g| = 4\pi/e$ and $M_W = ae\hbar = aq$, so that

$$M \geq a\frac{4\pi}{e} = \frac{4\pi}{e^2\hbar}M_W = \frac{4\pi\hbar}{q^2}M_W = \frac{\nu}{\alpha}M_W\,.$$

Here, α is the fine structure constant and $\nu = 1$ or $1/4$, depending on whether the electron charge is q or $q/2$. Since α is a very small number ($\sim 1/137$ for electromagnetism), the above relation implies that the monopole is much heavier than the W-bosons associated with the symmetry breaking.

From (31) it is clear that the bound is not saturated unless $\lambda \to 0$, so that $V(\phi) = 0$. This is the Bogomol'nyi-Prasad-Sommerfield (BPS) limit of the theory [14, 15]. Note that in this limit, $\phi^2_{vac} = a^2$ is no longer determined by the theory and, therefore, has to be imposed as a boundary condition on the Higgs field. Moreover, in this limit, the Higgs scalar becomes massless. Now, to saturate the bound we have to set

$$D_0\phi^a = 0\,, \qquad \mathcal{E}^a_k = (D_k\phi)^a \sin\theta\,, \qquad \mathcal{B}^a_k = (D_k\phi)^a \cos\theta\,, \tag{32}$$

where, $\tan\theta = q/g$. In the BPS limit, one can use the 't Hooft-Polyakov (or the Julia-Zee) ansatz either in (18), or in (32) to obtain the exact monopole (or dyon) solutions [14, 15]. These solutions automatically saturate the Bogomol'nyi bound and are referred to as the BPS states. Also, note that in the BPS limit, all the perturbative excitations of the theory saturate this bound and, therefore, belong to the BPS spectrum. As we will see later, the BPS bound appears in a very natural way in theories with $N = 2$ supersymmetry.

1.7 Monopoles from a Distance

Till now, we have described the monopoles arising in the Georgi-Glashow model in terms of the structure of the Higgs vacuum of the theory. In this section, we will consider monopoles in a general Yang-Mills-Higgs system and relate the Higgs vacuum description to a description in terms of the unbroken gauge fields. These are the gauge fields which remain massless and are relevant for the study of monopoles at large distances. This formulation is convenient for describing non-abelian monopoles.

Let ϕ transform as a vector in a given representation of a gauge group G. For convenience of notation, in the following we do not distinguish between the group element g and a given realization of it. Writing the gauge fields as $W_\mu = T^a W^a_\mu$, we can construct the covariant derivative of ϕ and the curvature tensor of W_μ as

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + ieW_\mu\phi\,, \\ G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu]\,. \end{aligned}$$

The Lagrangian density $\mathcal{L}$, the stress-energy tensor $\theta_{\mu\nu}$ and the gauge current j^a_μ are then given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(G^a_{\mu\nu})^2 + (D^\mu\phi)^\dagger D_\mu\phi - V(\phi)\,, \\ \theta_{\mu\nu} &= -G^a_{\mu\lambda}G^{a\lambda}_\nu + (D_\mu\phi)^\dagger(D_\nu\phi) + (D_\nu\phi)^\dagger(D_\mu\phi) - g_{\mu\nu}\mathcal{L}\,, \\ \theta_{00} &= \frac{1}{2}\mathcal{E}^{ia}\mathcal{E}^a_i + \frac{1}{2}\mathcal{B}^{ia}\mathcal{B}^a_i + D_0\phi^\dagger D_0\phi + D_i\phi^\dagger D_i\phi + V(\phi)\,, \\ j^a_\mu &= ie\phi^\dagger T^a D_\mu\phi - ie(D_\mu\phi)^\dagger T^a\phi\,. \end{aligned} \tag{33}$$

Here, the Higgs potential is gauge invariant: $V(g\phi) = V(\phi)$. The equations of motion following from the above Lagrangian are

$$D^\mu D_\mu\phi^a = -\frac{\partial V}{\partial\phi^a}\,, \qquad D^\nu G^a_{\mu\nu} = -j^a_\mu\,. \tag{34}$$

When the gauge group is $SO(3)$ spontaneously broken to $U(1)$, we can work out the Bogomol'nyi bound exactly as in the previous section and the outcome is

$$M \geq \sqrt{2}\, a\sqrt{g^2 + q^2}\,. \tag{35}$$

For a general gauge group G, the Higgs vacuum, as in the Georgi-Glashow model, is defined by

$$V(\phi) = 0\,, \qquad D_\mu \phi = 0\,.$$

The first equation defines the vacuum manifold $\mathcal{M}_0 \equiv \{\phi : V(\phi) = 0\}$, and the second equation leads to

$$[D_\mu, D_\nu]\phi = G_{\mu\nu}\phi = 0\,.$$

Thus, in the Higgs vacuum, $G_{\mu\nu}$ takes values in a subgroup of the gauge group G which keeps the Higgs vacuum invariant. We denote this unbroken subgroup of G by H. The generators of G which do not keep the Higgs vacuum invariant are of course broken and the corresponding gauge bosons become massive. If $V(\phi)$ does not have extra global symmetries and accidental minima, then it is reasonable to assume that the action of G on $\mathcal{M}_0$ is transitive. This means that any point in $\mathcal{M}_0$ is related to any other point (and, in particular, to a reference point ϕ_0) by some element of G. Therefore, the little group or the invariance group, $H \subset G$, of any point in $\mathcal{M}_0$ is isomorphic to the the little group of any other point. Hence, the structure of $\mathcal{M}_0$ is described by the right coset G/H.

For a solution to have finite energy, at sufficiently large distances from the core of the solution the field ϕ must take values in the Higgs vacuum. Let Σ be a 2-dimensional surface around the core such that, on this surface, ϕ is already in $\mathcal{M}_0$. On this surface, ϕ describes a map from Σ (with the topology of S^2) into $\mathcal{M}_0$. This map is characterised by its homotopy class which has to be an element of $\pi_2(\mathcal{M}_0) \simeq \pi_2(G/H)$. As described before, the associated topological number is the magnetic charge of the solution. As long as no monopole crosses the surface Σ, ϕ remains a continuous function of time and its homotopy class does not change. To show that the map ϕ satisfies the group properties of $\pi_2(G/H)$, one has to consider several widely separated monopoles and study how their magnetic charges combine. For a discussion of this issue, see [2].

The above discussion of the topological characterisation of the monopole is in terms of the structure of the Higgs vacuum. However, it is more natural to have a description in terms of the unbroken gauge fields. The relationship between these two descriptions is contained in the equation $D_\mu\phi = 0$ which is valid on the S^2 surface Σ. Let us parametrise S^2 by a square $\{0 \leq s, t \leq 1\}$. The map $r(s,t)$ from this square to the sphere is single valued everywhere except on the boundary of the square which is identified with a single point on the sphere:

$$r(0,t) = r(1,t) = r(s,0) = r(s,1) = r_0\,.$$

For fixed s, as t varies from 0 to 1, $r(s,t)$ describes a closed path on S^2.

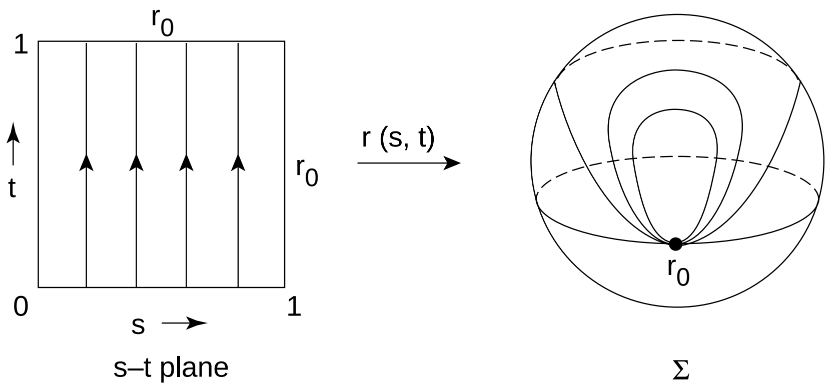

Figure 2

The covariant derivative along t becomes

$$D_t\phi(s,t) = \frac{\partial r^i}{\partial t} D_i\phi(s,t) = 0\,.$$

For constant s, this can be solved as

$$\phi(s,t) = g(s,t)\phi(s,0)\,,$$

where, $g(s,t)$ satisfies $D_t g = 0$, $g(s,0) = 1$ and is given by the path-ordered integral

$$g(s,t) = P\left(\exp\left(ie\int_0^t \vec{W}\cdot\frac{\partial\vec{r}}{\partial t}\,dt\right)\right)\,. \tag{36}$$

Clearly, $g(s,t)$ is an element of G which gauge transforms any $\phi(s,t)$ into a reference $\phi(s,0)$ and $\phi(s,0) = \phi_0$ is independent of s. Therefore, $g(s,t)$ contains the same topological information as $\phi(s,t)$ and is characterised by a homotopy class in $\pi_2(G/H)$. Since at $s = 0, 1$ we have $\partial r/\partial t = 0$, then, $g(0,t) = g(1,t) = 1$. For $t = 1$, $\phi(s,1) = g(s,1)\phi_0$. But since $\phi(s,1) = \phi_0$, we conclude that $g(s,1)$ must be an element of the unbroken gauge group H and denote it by $h(s)$:

$$h(s) \equiv g(s,1) = P\,\exp\left(ie\int_0^1 \vec{W}\cdot\frac{\partial\vec{r}}{\partial t}\,dt\right)\,. \tag{37}$$

Since $h(1) = h(0) = 1$, as we move along s, $h(s)$ describes a closed path in H and is thus, an element of $\pi_1(H)$. However, note that as we vary t from 1 to 0, $g(s,t)$ continuously interpolates between $h(s)$ and the identity. Therefore, $h(s)$ can describe only those closed paths in H which are homotopic to the trivial path when H is embedded in G. We denote the subgroup of $\pi_1(H)$ which corresponds to such paths by $\pi_1(H)_G$. If the homotopy class of the map $\phi(s,t)$ is changed in $\pi_2(G/H)$, the homotopy class of $h(s)$ changes in $\pi_1(H)_G$. In fact, there is a one-to-one correspondence between $\pi_2(G/H)$ and $\pi_1(H)_G$ and the two groups are isomorphic [2]. Thus, we can equally well characterise the monopole by its homotopy class in $\pi_1(H)_G$.

Let us discuss this in some more detail. Since any closed path in H is also a closed path in G, there is a natural homomorphism from $\pi_1(H)$ into $\pi_1(G)$. As the discussion above shows, $\pi_1(H)_G$ is in the kernel of this homomorphism. Moreover, for a compact connected group G, $\pi_2(G) = 0$, which implies that $\pi_2(G)$ can be embedded in $\pi_2(G/H)$ as its identity element. What we have done above, basically, is to construct part of the following exact sequence:

$$\pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H)$$

Note that if G is simply connected then, $\pi_1(G) = 0$ and $\pi_1(H)_G = \pi_1(H)$. So that the full $\pi_1(H)$ enters the physical description of the monopole. For non-simply connected G, this possibility can be realised in the presence of a Dirac string. Such a string appears as a singular point on the (s,t) plane, in the presence of which, it is no longer possible to continuously deform $h(s)$ to the identity map. Therefore, the homotopy class of $h(s)$ is no longer restricted to $\pi_1(H)_G$. However, for non-simply connected G, it is still possible to have a description in terms of an unrestricted π_1 group provided we embed G in its universal covering group $\tilde{G}$. Let us denote the little group of ϕ by $\hat{H}$. Since $\tilde{G}$ is necessarily connected, $\pi_1(\tilde{G}) = 0$ and $\pi_1(\hat{H}) = \pi_1(H)_G$. As an example, let us consider the Georgi-Glashow model. Here, $G = SO(3)$ (with $(Ta)_{ij} = -i\epsilon_{aij}$) and $\tilde{G} = SU(2)$ (with $T_a = \frac{1}{2}\sigma_a$). The homotopically distinct paths in H (or $\hat{H}$) are:

$$h(t) = \exp(it\,\vec{\phi}\cdot\vec{T}\,4\pi N/a), \quad \text{for } \ 0 \le t \le 1\,.$$

The integer N, which characterises the elements of π_1, is determined as follows:

$$\begin{array}{llll} \tilde{G} = SU(2), & \tilde{H} = U(1), & \vec{\phi}\cdot\vec{T} \in \frac{1}{2}\mathbf{Z}, & \Rightarrow\ N \in \mathbf{Z}\,, \\ G = SO(3), & H = SO(2), & \vec{\phi}\cdot\vec{T} \in \mathbf{Z}, & \Rightarrow\ N \in \frac{1}{2}\mathbf{Z}\,. \end{array}$$

Since $SO(3) \sim S^2/\mathbf{Z}_2$, only paths with integer N contribute in both cases. Later we show that $g = 4\pi N/e$. For $SU(2)$, $q_0 = e\hbar/2$ and $gq_0/4\pi\hbar = N/2$. For $SO(3)$, $q_0 = e\hbar$ and $gq_0/4\pi\hbar = N$.

For Glashow-Weinberg-Salam model, $G = SU(2) \times U(1)$ and $H = U(1)$ is a linear combination of $SU(2)$ and $U(1)$. Although $\pi_1(H) = \mathbf{Z}$, $\pi_1(H)_G = 0$. Therefore, in this model, any non-trivial monopole must have a Dirac string.

We set out to describe the monopole in terms of the unbroken gauge fields (the H-fields). Although, we have obtained a description in terms of $\pi_1(H)_G$, it is not manifest that $h(s)$, as given by (37), involves only the H-fields. We show this in the following: The quantity $g^{-1}(s,t)D_s g(s,t)$ is invariant under a t-dependent gauge transformation. Moreover, by construction, $D_t g(s,t) = 0$. Hence, we can write

$$\partial_t(g^{-1}D_s g) = D_t(g^{-1}D_s g) = g^{-1}[D_t, D_s]g = ieg^{-1}G_{ij}g\frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s}\,. \tag{38}$$

Let us integrate the first and the last terms above from $t = 0$ to $t = 1$. Since $g^{-1}D_s g = 0$ at $t = 0$ and $g^{-1}D_s g = h^{-1}dh/ds$ at $t = 1$, we get

$$h^{-1}\frac{dh}{ds} = ie\int_0^1 dt\, g^{-1}G_{ij}\, g\,\frac{\partial r^i}{\partial t}\,\frac{\partial r^j}{\partial s}\,. \tag{39}$$

Since G_{ij} was calculated on Σ, it involves only the H-gauge fields. The conjugation by g does not bring in a dependence on the massive gauge fields as is evident from the left-hand side of the equation. Hence the map is given entirely in terms of the H-fields without any reference to the Higgs field. As a simple application, consider the Dirac monopole. Integrating (39) from $s = 0$ to $s = 1$, we get $h(1) = exp(ie \int \vec{B} \cdot \vec{ds})$. Since $h(1) = 1$, this leads to the Dirac quantization condition $eg = 2\pi n$ or $qg/4\pi\hbar = n/2$. Another interesting consequence of (39) is a possible explanation for the fractional charges of quarks. We describe this in the next section.

1.8 The Monopole and Fractional Charges

So far, we have seen how the existence of a monopole can quantize the electric charge in integer units. In the physical world, however, we also come across fractionally charged quarks. In the following, we see how the existence of a monopole can also account for these fractional charges [16, 1].

Let us represent our adjoint Higgs by $\phi = \phi^a T^a$, where T^a are the fundamental representation matrices. Moreover, we only consider ϕ on the surface Σ as described in the previous subsection. With ϕ in the Higgs vacuum, a generator T^a belongs to the unbroken subgroup H of G provided $[T^a, \phi] = 0$. This implies that ϕ itself is a generator of H and commutes with its other generators. Thus the Lie Algebra of H is of the form $L(H) = u(1) \oplus L(K)$ and we choose $L(K)$ to be orthogonal to $u(1)$: $\mathrm{Tr}(\phi K^a) = 0$ for $K^a \in K$. Locally, H has the structure $U(1) \times K$, though, this is not necessarily the global structure. We refer to K as the colour group and identify the $U(1)$ as corresponding to electromagnetism. The gauge fields in H can be decomposed as $W^\mu = A^\mu \phi/a + X^\mu$, with $\mathrm{Tr}(\phi X^\mu) = 0$. Expanding the covariant derivative $\partial_\mu + ieW_\mu$ we can identify the electric charge operator which couples to A_μ as $Q = (e\hbar/a)\phi$.

Since $h^{-1}dh/ds$ is a generator of H, we may write

$$h^{-1}\frac{dh}{ds} = ie\alpha(s)\,\frac{\phi_0}{a} + i\beta_a K^a\,. \tag{40}$$

Using $\mathrm{Tr}(T_a T_b) = \delta_{ab}$, $\phi(r) = g\phi_0 g^{-1}$ and equation (39), we get

$$\begin{aligned}\alpha(s) &= -\frac{i}{ae}\mathrm{Tr}\left(\phi_0 h^{-1}\frac{dh}{ds}\right)\\ &= \frac{1}{a}\int_0^1 \mathrm{Tr}\left(\varphi(r)G_{ij}\right)\frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s}\,dt\,.\end{aligned}$$

Identifying the electromagnetic field strength tensor as $F^{\mu\nu} = \mathrm{Tr}\left(\phi G^{\mu\nu}/a\right)$, we get $\alpha(s) = d\Omega/ds$, where, $\Omega(s)$ is the magnetic flux in a solid angle subtended at the origin by the path $0 \le t \le 1$ at fixed s on S^2. Substituting this back in (40) and integrating from $s = 0$ to s, gives

$$h(s) = k(s)e^{iQ\Omega(s)/\hbar}\,.$$

Since $h(1) = 1$ and $\Omega(1) = g$ (where g is the total magnetic charge inside Σ), the quantization condition is replaced by

$$e^{igQ/\hbar} = k(1)^{-1} = k \in K\,.$$

The left-hand side is invariant under K, therefore, we can at most have $k \in \mathbf{Z}(K)$, the center of K. If we take $K = SU(N)$, then, $k = e^{2\pi i n/N}$ with $n = 1, 2, \ldots N$. If all values of n are allowed, then $U(1) \cap K \subset Z_N$. This corresponds to the fact that globally, H cannot be decomposed as $U(1) \times K$. Now, let $|s>$ be a colour singlet. Then $k|s>=|s>$ and $exp(igq_s/\hbar) = 1$, which is again the Dirac quantization condition. Thus, if q_0 and g_0 are the units of electric and magnetic charges for colour singlets, then $q_s = n_s q_0$ and $g = mg_0$ with $g_0 q_0 = 2\pi\hbar$. The colour non-singlet states $|c>$ can be classified according to their behaviour under the center of the colour group K:

$$k|c>= e^{2\pi i\, t(c)/N}|c>= e^{igQ/\hbar}|c>\,,$$

where, $t(c)$ is an integer mod N. For a minimal monopole g_0, we obtain $g_0 q_c/\hbar = 2\pi(m+t(c)/N)$, hence,

$$q_c = q_0(m + \frac{t(c)}{N})\,.$$

If we set $N = 3$, as for QCD, and $m = 0$, then $q_c = q_0/3, 2q_0/3, q_0$.

1.9 Non-Abelian Magnetic Charge and the Montonen-Olive Conjecture

In this subsection we first consider the generalization of charge quantisation to non-abelian monopoles [17], and then describe the electric-magnetic duality conjecture of Montonen and Olive [18].

Goddard, Nuyts and Olive [17] attempted to classify all H-monopole configurations. To describe such a monopole, we consider a static configuration and choose the gauge $\vec{r}\cdot\vec{W}^a = 0$. At large distances, it is reasonable to write the magnetic componenets of the field strength as

$$G_{ij} = \frac{1}{4\pi r^2}\,\epsilon_{ijk}\,\hat{r}^k G(r)\,,$$

where $D_\mu G(r) = 0$. Since $G(r)$ transforms in the adjoint representation, we can write $G(r) = g(s,t)G_0 g(s,t)^{-1}$. Substituting the above expression in (39), and integrating over s, we get

$$\begin{aligned} \ln\left(h(s)\right) &= \frac{ie}{4\pi r^2}G_0\int_0^s ds\int_0^1 dt\,\epsilon_{ijk}\hat{r}^k\,\frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s} \\ &= \frac{ie}{4\pi}G_0\,\Omega(s)\,. \end{aligned} \tag{41}$$

Here, $\Omega(s)$ is the solid angle subtended at the origin by the loop $0 \le t \le 1$, $s = \text{const}$ on S^2. The elements in $\pi_1(H)_G$ are, therefore, given by

$$h(s) = e^{\frac{ie}{4\pi}G_0\Omega(s)}\,.$$

Since, $h(1) = 1$, the above equation implies that

$$e^{ieG_0} = 1\,, \tag{42}$$

which is the generalized charge quantization condition. Clearly, G_0 is arbitrary upto a conjugation in the gauge group. This freedom can be used to solve the charge quantization condition as follows: Assume that H is compact and connected and let T denote an Abelian subgroup of H generated by its Cartan subalgebra. Then, any element of H is conjugate to at least one element of T. Thus, it is always possible to find a frame in which $G_0 = \beta^a T^a$. The coefficients β^a are still not unique as they transform under the Weyl group of H which keeps this parametrization of G_0 unchanged. Therefore, the equivalence classes of β^a, related by the action of the Weyl group, are the gauge invariant objects which characterise the non-abelian magnetic charges. The β^a are determined by the quantization condition $exp(ie\beta^a T^a) = 1$. To solve this, let ω^a denote a weight vector of H in the given representation, and let $\Lambda(H)$ denote the weight lattice. Then the quantization condition implies that

$$e\beta^a\omega^a \in 2\pi\mathbf{Z}\,, \qquad \text{for all } \omega \in \Lambda(H)\,.$$

Note that the factors of e and 2π are convention dependent. Thus, $e\beta$ lies on a lattice dual to $\Lambda(H)$: $e\beta \in \Lambda^*(H)$. This dual lattice can by itself be regarded as the weight lattice of a dual group H^v which has $e\beta$'s as its weight vectors (For details, see [17]). Moreover, $(H^v)^v = H$. H is referred to as the electric group and H^v as the magnetic group. The magnetic charges are related to H^v in the same way that electric charges are related to H. A simple example of a dual pair of groups is provided by $SO(3)$ and $SU(2)$. In this case, $G_0 = \beta T^3$. T^3 has integral eigenvalues for $SO(3)$ and half-integral eigenvalues for $SU(2)$. The quantization condition $exp(ie\beta T^3) = 1$ gives:

$$\begin{aligned}
\text{for } H = SO(3): \quad e\beta &= 4\pi\frac{n}{2} = 4\pi \times (\text{weight of } SU(2))\,, \\
\text{for } H = SU(2): \quad e\beta &= 4\pi n = 4\pi \times (\text{weight of } SO(3))\,.
\end{aligned}$$

For a general $SU(N)$ group, the dual relation is given by $(SU(NM)/\mathbf{Z}_N)^v = (SU(NM)/\mathbf{Z}_M)$ and, in particular, $SU(N)^v = SU(N)/Z_N$.

Now, we will briefly describe the Montonen-Olive conjecture [18] which is based on the above results. This conjecture states that a gauge theory is characterized by $H \times H^v$, and that we have two equivalent descriptions of the theory: One in terms of H-gauge fields with normal charged particles in the perturbative spectrum and another, in terms of H^v-gauge fields with monopoles in the perturbative spectrum. The Noether currents (associated with electric charges) also get interchanged with topological currents (associated with magnetic charges). Hence, the coupling constant $q/\hbar$ of the H-theory is replaced by $g/\hbar$ in the in the H^v-theory. Since, $g \sim 1/e$, this conjecture relates a strongly coupled theory to a weekly coupled one, and vice-versa. As a result of this, it is not easy to either prove or disprove this conjecture. Montonen and Olive provided some semiclassical evidence in favour of this conjecture in the BPS limit of the Georgi-Glashow model. This model contains a Higgs boson, a photon and two massive charged vector bosons in its perturbative spectrum, and magnetic monopoles as solitonic classical solutions. The unbroken gauge group is self dual, $H = U(1) = H^v$, therefore, the dual theory has the same form as the original one with the monoples as elementary states and the massive gauge bosons as solitonic solutions. The gauge boson mass in the dual theory (where it appears as a soliton) can be computed using the BPS formula and turns out to have the right value. Moreover, the long-range force between two monopoles as obtained by Manton [19], can be obtained by calculating the potential between W-bosons in the dual theory and turns out to be the same.

Since the Montonen-Olive duality is non-perturbative in nature, it cannot be verified in a perturbative framework unless we have some kind of control over the perturbative and non-perturbative aspects of the theory. Such a control is provided by superysymmetry. In fact, in the $N = 4$ super Yang-Mills theory, some very non-trivial predictions of this duality were verified in [20]. In later parts, we will consider in detail the analogue of the Montonen-Olive duality in $N = 2$ supersymmetric gauge theories. A prerequisite for this, however, is the introduction of the θ-term in the Yang-Mills action which affects the electric charges of dyons.

1.10 The θ-Parameter and the Monopole Charge

In this section we will show, following Witten [21], that in the presence of a θ-term in the Lagrangian, the magnetic charge of a particle always contributes to its electric charge.

As shown by Schwinger and Zwanziger, for two dyons of charges (q_1, g_1) and (q_2, g_2), the quantization condition takes the form

$$q_1 g_2 - q_2 g_1 = 2\pi n\hbar \tag{43}$$

For an electric charge q_0 and a dyon (q_n, g_n), this gives $q_0 g_n = 2\pi n\hbar$. Thus, the smallest magnetic charge the dyon can have is $g_0 = 2\pi\hbar/q_0$. For two dyons of the same magnetic charge g_0 and electric charges q_1 and q_2, the quantization condition implies $q_1 - q_2 = nq_0$. Therefore, although the difference of electric charges is quantized, the individual charges are still arbitrary. This arbitrariness in the electric charge of dyons can be fixed if the theory is CP invariant as follows: Under a CP transformation $(q, g) \to (-q, g)$. If the theory is CP invariant, the existence of a state (q, g_0) necessarily leads to the existence of $(-q, g_0)$. Applying the quantization condition to this pair, we get $2q = q_0 \times integer$. This implies that $q = nq_0$ or $q = (n + \frac{1}{2})q_0$, though at a time we can either have dyons of integral or half-odd integral charge, and not both together.

In the above argument, it was essential to assume CP invariance to obtain integral or half-odd inegral values for the electric charges of dyons. However, in the real world, CP invariance is violated and there is no reason to expect that the electric charge should be quantized as above. To study the effect of CP violation, we consider the Georgi-Glashow model with an additional θ-term which is the source of CP violation:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}(D_\mu \vec{\phi})^2 - \lambda(\phi^2 - a^2)^2 + \frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} . \tag{44}$$

Here, $\tilde{F}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$ and the vector notation is used to represent indices in the gauge space. The presence of the θ-term does not affect the equations of motion but changes the physics since the theory is no longer CP invariant. We want to construct the electric charge operator in this theory. The theory has an $SO(3)$ gauge symmetry but the electric charge is associated with an unbroken $U(1)$ which keeps the Higgs vacuum invariant. Hence, we define an operator N which implements a gauge rotation around the $\hat{\phi}$ direction with gauge parameter $\Lambda^a = \phi^a/a$. These transformations correspond to the electric charge. Under N, a vector $\vec{v}$ and the gauge fields $\vec{A}_\mu$ transform as

$$\delta\vec{v} = \frac{1}{a}\,\vec{\phi}\times\vec{v}\,, \quad \delta\vec{A}_\mu = \frac{1}{ea}\,D_\mu\vec{\phi}\,.$$

Clearly, $\vec{\phi}$ is kept invariant. At large distances where $|\phi| = a$, the operator $e^{2\pi i N}$ is a 2π-rotation about $\hat{\phi}$ and therefore $\exp(2\pi i N) = 1$. Elsewhere, the rotation angle is $2\pi|\phi|/a$. However, by Gauss' law, if the gauge transformation is 1 at ∞, it leaves the physical states invariant. Thus, it is only the large distance behaviour of the transformation which matters and the eigenvalues of N are quantized in integer units. Now, we use Noether's formula to compute N:

$$N = \int d^3x \left(\frac{\delta\mathcal{L}}{\delta\partial_0 A_i^a}\,\delta A_i^a + \frac{\delta\mathcal{L}}{\delta\partial_0\phi^a}\,\delta\phi^a\right)\,.$$

Since $\delta\vec{\phi} = 0$, only the gauge part (which also includes the θ-term) contributes:

$$\begin{aligned}
\frac{\delta}{\delta\partial_0 A_i^a}\left(F^a_{\mu\nu}F^{a\mu\nu}\right) &= 4F^{aoi} = -4\mathcal{E}^{ai}\,,\\
\frac{\delta}{\delta\partial_0 A_i^a}\left(\tilde{F}^a_{\mu\nu}F^{a\mu\nu}\right) &= 2\epsilon^{ijk}F^a_{jk} = -4\mathcal{B}^{ai}\,.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
N &= \frac{1}{ae}\int d^3x D_i\vec{\phi}\cdot\vec{\mathcal{E}}^i - \frac{\theta e}{8\pi^2 a}\int d^3x\, D_i\vec{\phi}\cdot\vec{\mathcal{B}}^i\\
&= \frac{1}{e}Q - \frac{\theta e}{8\pi^2}M\,,
\end{aligned}$$

where, we have used equations (30). Here, Q and M are the electric and magnetic charge operators with eigenvalues q and g, respectivly, and N is quantized in integer units. This leads to the following formula for the electric charge

$$q = ne + \frac{\theta e^2}{8\pi^2}g\,.$$

For the 't Hooft-Polyakov monopole, $n = 1$, $g = -4\pi/e$, and therefore, $q = e(1-\theta/2\pi)$. For a general dyonic solution we get

$$g = \frac{4\pi}{e}m, \qquad q = ne + \frac{\theta e}{2\pi}m\,. \tag{45}$$

Thus, in the presence of a θ-term, a magnetic monopole always carries an electric charge which is not an integral multiple of some basic unit.

It is very useful to represent the charged states as points on the complex plane, with electric charges along the real axis and magnetic charges along the imaginary axis. A state can thus be represented as

$$q + ig = e(n + m\tau)\,, \tag{46}$$

where,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{47}$$

In this parametrisation, the Bogomol'nyi bound (35) takes the form

$$M \geq \sqrt{2}|ae(n + m\tau)|\,. \tag{48}$$

Note that (46) implies that all states lie on a two-dimensional lattice with lattice parameter τ and (48) implies that the BPS bound for a state is proportional to the distance of its lattice point from the origin. These equations play a very important role in the subsequent discussions.

2 Supersymmetric Gauge Theories

In this section we will explain some aspects of supersymmetry and supersymmetric field theories which are relevant to the work of Witten and Seiberg. We start by explaining our conventions and then briefly describe the representations of supersymmetry algebra with and without central charges. We then discuss the representations of $N = 1$ supersymmetry in terms of quantum fields and construct Lagrangians with $N = 1$ and $N = 2$ supersymmetry. Most of the material in this section is by now standard and can be found in [22, 23, 24]. Towards the end of this section, we will explicitly calculate the central charges in $N = 2$ theories with and without matter.

2.1 Conventions

We start by descrbing our conventions. We use the flat metric $\eta_{ab} = \text{diag}\ (1, -1, -1, -1)$. The spinors of the Lorentz group $SL(2, C) \sim SU(2)_L \times SU(2)_R$ are written with dotted and undotted components and, under $SL(2, C)$, transform as

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta\,, \qquad \bar{\psi}'_{\dot{\alpha}} = M^*_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}\,.$$

Spinor indices are raised or lowered with the ϵ-tensor,

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (i\sigma_2)\,.$$

By definition, this tensor is invariant under a $SL(2, C)$ transformation: $\epsilon^{\alpha\beta} = M^\alpha_{\ \gamma} \epsilon^{\gamma\delta} M_\delta{}^\beta$. This can be written as $M^T \sigma_2 M = \sigma_2$ which implies $\sigma_2 M = (M^T)^{-1} \sigma_2$ Using this, we can write the transformations of the spinors with raised indices as

$$\psi'^\alpha = \psi^\beta (M^{-1})_\beta^{\ \alpha}\,, \qquad \bar{\psi}'^{\dot{\alpha}} = \bar{\psi}^{\dot{\beta}} (M^*)^{-1\ \dot{\alpha}}_{\dot{\beta}}\,.$$

Now, let us define

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \equiv (1, \vec{\sigma})\,,$$

then,

$$\sigma^\mu P_\mu = \begin{pmatrix} P^0 - P^3 & -P^1 + iP^2 \\ -P^1 - iP^2 & P^0 + P^3 \end{pmatrix} ,$$

and $\det(\sigma^\mu P_\mu) = P_\mu P^\mu$. We can raise the indices on σ^μ using the ϵ-tensor and define $\bar{\sigma}$ as

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = -(\sigma^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} .$$

Numerically, this gives,

$$(\bar{\sigma}^\mu) = (i\sigma_2)(\sigma^\mu)^T(i\sigma_2)^T = \sigma_2(\sigma^\mu)^T\sigma_2 = (1, -\vec{\sigma}) .$$

With these conventions, Lorentz transformations are generated by

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha^{\ \beta} &= \frac{1}{4}[\sigma^\mu_{\alpha\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\beta} - (\mu \leftrightarrow \nu)] , \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{1}{4}[\bar{\sigma}^{\mu\dot{\alpha}\beta}\sigma^\nu_{\beta\dot{\beta}} - (\mu \leftrightarrow \nu)] . \end{aligned}$$

For the scalar product of spinors, we use the following conventions

$$\begin{aligned} \psi\chi &= \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi , \\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} , \\ (\psi\chi)^\dagger &= \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi} . \end{aligned}$$

We list some more spinor identities

$$\begin{aligned} \chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi , \\ (\chi\sigma^\mu\bar{\psi})^\dagger &= \psi\sigma^\mu\bar{\chi} , \\ \chi\sigma^\mu\bar{\sigma}^\nu\psi &= \psi\sigma^\nu\bar{\sigma}^\mu\chi , \\ (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger &= \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi} . \end{aligned}$$

In the above basis, the Dirac matrices and Dirac and Majorana spinors are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \qquad \psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} , \qquad \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} . \tag{49}$$

As usual, one defines $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Consider a massless fermion moving in the z-direction. Then, $P^\mu = E(1, 0, 0, 1)$, and the Dirac equation gives $(\gamma^0 - \gamma^3)\psi = 0$. Since the helicity operator is now $J_3 = \frac{i}{2}\gamma^1\gamma^2$, one gets, $J_3\psi = \frac{i}{2}(\gamma^0)^2\gamma^1\gamma^2\psi = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^3\psi = -\frac{1}{2}\gamma_5\psi$. Hence,

$$\gamma_5 = +1 \Rightarrow -ve \text{ helicity} , \gamma_5 = -1 \Rightarrow +ve \text{ helicity} .$$

2.2 Supersymmetry Algebra without Central Charges

In the absence of central charges, the supersymmetry algebra is written as

$$\begin{aligned} &\{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu\delta^I_J , \\ &\{Q^I_\alpha, Q^J_\beta\} = 0 , \ \{\bar{Q}_{\dot{\alpha}I} , \bar{Q}_{\dot{\beta}J}\} = 0 . \end{aligned} \tag{50}$$

Here, Q and $\bar{Q}$ are the supersymmetry generators and transform as spin-half operators under the angular momentum algebra. The indices I, J run from 1 to N, where N is the total number of supersymmetries. Moreover, the supersymmetry generators commute with the momentum operator P_μ and hence, with P^2. Therefore, all states in a given representation of the algebra have the same mass. For a theory to be supersymmetric, it is necessary that its particle content form a representation of the above algebra. The supersymmetry algebra can be embedded in the super-Poincaré algebra and its representations can be obtained systematically using Wigner's method. In the following, we will give a brief description of the representations of supersymmetry algebra.

Massless Irreducible Representations: For massless states, we can always go to a frame where $P^\mu = M(1,0,0,1)$. Then the supersymmetry algebra becomes

$$\{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4M \end{pmatrix} \delta^I_J \,.$$

Now, in a unitary theory the norm of a state is always positive definite. Since Q_α and $\bar{Q}_{\dot{\alpha}}$ are conjugate to each other, and $\{Q_1, \bar{Q}_{\dot{1}}\} = 0$, it follows that $Q_1|phys >= \bar{Q}_{\dot{1}}|phys >= 0$. As for the other generators, it is convenient to rescale them as

$$a^I = \frac{1}{2\sqrt{M}} Q^I_2 \,, \qquad (a^I)^\dagger = \frac{1}{2\sqrt{M}} \bar{Q}^I_{\dot{2}} \,.$$

Then, the supersymmetry algebra takes the form

$$\{a^I, (a^J)^\dagger\} = \delta^{IJ} \,, \quad \{a^I, a^J\} = 0 \,, \quad \{(a^I)^\dagger, (a^J)^\dagger\} = 0 \,.$$

This is a Clifford algebra with $2N$ generators and has a 2^N-dimensional representation. From the point of view of the angular momentum algebra, a^I is a rising operator and $(a^I)^\dagger$ is a lowering operator for the helicity of massless states. We choose the vacuum such that $J_3|\Omega_\lambda >= \lambda|\Omega_\lambda >$ and $a^I|\Omega_\lambda >= 0$ for all I. Other states are generated by the action of $(a^I)^\dagger$'s on the vacuum state. From anti-symmetry it follows that a state with m $(a^I)^\dagger$'s, and hence with helicity $\lambda - m/2$, will have a degeneracy of ${}^N\mathbf{C}_m$. The helicity of all states so constructed will span the range λ to $\lambda - N/2$. Some examples are:

$$\begin{array}{lll} N=1: & |\lambda>, \quad |\lambda-1/2> \\ N=2: & |\lambda>, \;\; 2\,|\lambda-1/2>, \quad |\lambda-1> \\ N=4: & |\lambda>, \;\; 4\,|\lambda-1/2>, \;\; 6\,|\lambda-1>, \;\; 4\,|\lambda-3/2>, \;\; |\lambda-2> \end{array}$$

The irreducible representations are not necessarily CPT invariant. Therefore, if we want to assign physical states to these representations, we have to suplement them with their CPT conjugates. If a representation is CPT self-conjugate, it is left unchanged. Below, we list the representations after the addition of the CPT conjugates and indicate the particle spectra which can be assiged to them:

$$\begin{array}{rrllllll} N=1, & \lambda=1/2: & |1/2>, & |0>, & |-1/2>, & |0> & & \\ & \lambda=1: & |1>, & |1/2>, & |-1>, & |-1/2> & & \\ N=2, & \lambda=1/2: & |1/2>, & 2|0>, & |-1/2>, & |-1/2>, & 2|0>, & |1/2> \\ & \lambda=1: & |1>, & 2|1/2>, & |0>, & |-1>, & 2|-1/2>, & |0> \\ N=4, & \lambda=1: & |1>, & 4|1/2>, & 6|0>, & 4|-1/2>, & |-1> & \end{array}$$

Thus, for $N = 1$, the representation contains a Majorana spinor and a complex scalar if $\lambda = 1/2$ (scalar multiplet), or a massless vector and a Majorana spinor if $\lambda = 1$ (vector multiplet). For $N = 2$ and $\lambda = 1/2$, we have two Majorana spinors (or one Dirac spinor) with two complex scalars. This representation has the same particle content as two copies of the $N = 1$, $\lambda = 1/2$ multiplet. For $N = 2$ and $\lambda = 1$, we have a massless vector, two Majorana spinors and a complex scalar. Note that this multiplet has the same particle content as the two $N = 1$ multiplets for $\lambda = 1/2$ and $\lambda = 1$ put together. For $N = 4$, the representation is self-conjugate and accommodates a massless vector, two Dirac fermions and three complex scalars.

Massive Irreducible Representations: For massive states, we can always go to the rest frame where $P_\mu = (M, 0, 0, 0)$ and define

$$a^I_\alpha = Q^I_\alpha/\sqrt{2M}\,, \quad (a^I_\alpha)^\dagger = \bar{Q}_{\dot{\alpha}I}/\sqrt{2M}\,.$$

Then the supersymmetry algebra reduces to

$$\{a^I_1, (a^J_1)^\dagger\} = \delta^{IJ}\,, \qquad \{a^I_2, (a^J_2)^\dagger\} = \delta^{IJ}\,,$$

with all other anti-commutators vanishing. The Clifford vacuum is defined by $a^I_\alpha|\Omega >= 0$ and the representation is constructed by applying $(a^I_\alpha)^\dagger$'s on Ω. Let $|\Omega >$ be a spin singlet. Then there are ${}^{2N}\mathrm{C}_m$ states at level m and the dimension of the the representation is given by $\sum_{m=0}^{2N} {}^{2N}\mathbf{C}_m = 2^{2N}$. The maximum spin which can be reached is $N/2$ and not N as one might naively expect. This is because $(a^I_1)^\dagger(a^I_2)^\dagger = \frac{1}{2}\epsilon^{\alpha\beta}(a^I_\alpha)^\dagger(a^I_\beta)^\dagger$ is a scalar. Thus the state with $m = 2N$ has spin zero, as the vacuum. The degeneracy of states with a given spin is labelled by the irreducible representations of the group $USp(2N)$ which we will not discuss here. Instead, let us consider the simplest example. For $N = 1$, the massive representation contains $2^2 = 4$ states,

$$|\Omega >, \quad a^\dagger_\alpha|\Omega >, \quad \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta}a^\dagger_\alpha a^\dagger_\beta|\Omega >\,,$$

with spin content $(0) \oplus (1/2) \oplus (0)$. Here, (j) denotes a state of total spin j and degeneracy $2j + 1$. Thus, in the above example, we have a Weyl (or Majorana) spinor and a complex scalar (λ, ϕ). For $N = 2$, the representation contains $2^4 = 16$ states which, under the $SU(2)$ of angular momentum, decompose as $5(0) \oplus 4(1/2) \oplus 1(1)$. The $N = 4$ massive multiplet has $2^8 = 256$ states and inclueds a spin 2 state.

Till now we have considered representations based on a singlet vacuum. Let us consider a vacuum $|\Omega_j >$ of spin j which is $2j + 1$-fold degenerate. The representation now contains $(2j + 1)2^{2N}$ states. The spectrum is worked out by combining the $j = 0$ representation of the Clifford algebra and a spin j, using the angular momentum addition rules. For example, to obtain the $N = 1$ representation based on $|\Omega_j >$, we combine $(0) \oplus (1/2) \oplus (0)$ with (j) to obtain $(j) \oplus (j + 1/2) \oplus (j - 1/2) \oplus (j)$. For $j = 1/2$, we get $(1/2) \oplus (1) \oplus (0) \oplus (1/2)$ which corresponds to a gauge field, a Dirac fermion and a scalar field, all of the same mass. Note that in all cases we get the same number of bosonic and fermionic degrees of freedom.

2.3 Supersymmetry Algebra with Central Charges

As shown by Haag, Lapuszanski and Sohnius [25], the supersymmetry algebra (50) admits a central extension and can be generalised to

$$\begin{aligned}
\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J \,, \\
\{Q^I_\alpha, Q^J_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ} \,, \\
\{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ} \,,
\end{aligned} \tag{51}$$

where, Z and Z^* are the central charge metrices which are antisymmetric in I and J. Let us focus on the case of even N. Using a unitary transformation, we can skew-diagonalize Z: $\tilde{Z}^{IJ} = U^I_A U^J_B Z^{AB}$, so that it takes the form $Z = \epsilon \otimes D$, where D is an $N/2$-dimensional diagonal matrix. Thus, the index I which counts the number of supersymmetries can be decomposed into (a, m), with $a = 1, 2$ coming from the antisymmetric tensor ϵ, and $m = 1, ..., N/2$ coming from the diagonal matrix D. By a further chiral rotation, we may choose the eigenvalues of D to be real. Once we have skew-diagonalized, it is sufficient to consider just the $N = 2$ supersymmetry, for which the algebra takes the form

$$\begin{aligned}
\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^a_b \,, \\
\{Q^a_\alpha, Q^b_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z \,, \\
\{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab} Z \,.
\end{aligned} \tag{52}$$

Since Z commutes with all the generators, we can fix it to be the eigenvalue for the given representation. Now, let us define:

$$a_\alpha = \frac{1}{2}\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\} \,, \qquad b_\alpha = \frac{1}{2}\{Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\} \,.$$

Then, the algebra (51) reduces to

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z) \,, \qquad \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2}Z) \,, \tag{53}$$

with all other anticommutators vanishing. Since all physical states have positive definite norm, it follows that for massless states, the central charge is trivially realised (*i.e.*, $Z = 0$). For massive states, this leads to a bound on the mass $M \geq \sqrt{2}|Z|$. When $M = \sqrt{2}|Z|$, one set of operators in (53) is trivially realized and the algebra resembles the massless case and the dimension of representation is greatly reduced. For example, a reduced massive $N = 2$ multiplet has the same number of states as a massless $N = 2$ multiplet. Thus the representations of the $N = 2$ algebra with a central charge can be classified as either long multiplets (when $M > \sqrt{2}|Z|$) or short multiplets (when $M = \sqrt{2}|Z|$).

The mass bound $M \geq \sqrt{2}|Z|$ is reminiscent of the Bogomol'nyi bound in the Georgi-Glashow model. In fact, it turns out that in the supersymmetric version of the Georgi-Glashow model (which is based on the algebra without central charges) the solitonic solutions do give rise to a central extension term in the supersymmetry algebra, thus realizing (51)[26]. The origin of the central charge is easy to understand: The supersymmetry charges Q and $\bar{Q}$ are space integrals of local expressions in the fields (the time component of the super-currents). In calculating their anticommutators, one encounters surface terms which are normally neglected. However, in the presence of electric and magnetic charges, these surface terms are non-zero and give rise to a central charge. As we will explicitly show towards the end of this section, it is found that

$$Z = a(q + ig) = ae(n + m\tau)\,, \tag{54}$$

so that $M \geq \sqrt{2}|Z|$ coincides with the Bogomol'nyi bound (35). From (53) it is clear that the BPS states (which saturate the bound) are annihilated by half of the supersymmetry generators and thus belong to reduced representations of (51). An important consequence of this is that, for BPS states, the relationship between their charges and masses is dictated by supersymmetry and does not receive perturbative or non-perturbative corrections in quantum theory. This is so because a modification of this relation implies that the states no longer belong to a short multiplet. On the other hand, quantum correction are not expected to generate the extra degrees of freedom needed to convert a short multiplet into a long multiplet. Since there is no other possibility, we conclude that for short multiplets the relation $M = \sqrt{2}|Z|$ is not modified either perturbatively or non-perturbatively.

2.4 Local Representations of N=1 Supersymmetry

In this subsection we describe the action of supersymmetry on the local fields in a quantum field theory. It is well known that the Poincaré group naturally acts on the space-time coordinates. All other objects transform as components of tensors or spinors defined on the space-time manifold. Similarly, the supersymmetry transformations naturally act on an extension of the space-time, called the "superspace". The quantum fields then transform as components of a "superfield" defined on the superspace. In the following, we first describe these notions and then introduce the chiral and vector superfields.

Superspace : The superspace is obtained by adding four spinor degrees of freedom $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ to the space-time coordinates x^μ. The spinor index is raised and lowered with the ϵ-tensor and $\theta\theta = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2$. Similarly, $\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = 2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}}$. We also have

$$\theta^\alpha\theta^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\,, \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\,, \qquad \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}\,.$$

These formulae are the basis for Fierz rearrangements.

Under the supersymmetry transformations (50) with $N = 1$ and transformation parameters ξ and $\bar{\xi}$, the superspace coordinates are taken to transform as

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}\,, \\ \theta &\rightarrow \theta' = \theta + \xi\,, \\ \bar{\theta} &\rightarrow \bar{\theta}' = \bar{\theta} + \bar{\xi}\,. \end{aligned} \tag{55}$$

Since these transformations are implemented by the operator $\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}$, we can easily obtain the representation of the supercharges acting on the superspace as

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\partial_\mu\,, \qquad \bar{Q}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\,. \tag{56}$$

These satisfy $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$. Moreover, using the chain rule, it is easy to see that $\partial/\partial x^\mu$ is invariant under (55) but not $\partial/\partial\theta$ and $\partial/\partial\bar\theta$. Therefore, we introduce the super-covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\,, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\,\partial_\mu\,. \tag{57}$$

They satisfy $\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$ and commute with Q and $\bar{Q}$.

Superfields: A superfield is a function on the superspace, say, $F(x,\theta,\bar\theta)$. Since the θ-coordinates are anti-commuting, the most general $N=1$ superfield can always be expanded as

$$\begin{aligned} F(x,\theta,\bar\theta) &= f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ &+ \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x)\,. \end{aligned} \tag{58}$$

Clearly, any function of superfields is, by itself, a superfield. Under supersymmetry, the superfield transforms as $\delta F = (\xi Q + \bar\xi\bar Q)F$, from which, the transformation of the component fields can be obtained. Note that since $d(x)$ is the component of highest dimension in the multiplet, its variation under supersymmetry is always a total derivative of other components. Thus, ignoring surface terms, the space-time integral of this component is invariant under supersymmetry. This tells us that a supersymmetric Lagrangian density may be constructed as the highest dimension component of an appropriate superfield. To describe physical systems, we do not need all components of the superfield. The relevant components are selected by imposing appropriate constraints on the superfield.

Chiral Multiplets: The $N=1$ scalar multiplet is represented by a superfield with one constraint:

$$\bar{D}_{\dot\alpha}\Phi = 0\,.$$

This is referred to as the chiral superfield. Note that for $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, we have

$$\bar{D}_{\dot\alpha}y^\mu = 0, \quad \bar{D}_{\dot\alpha}\theta^\beta = 0\,.$$

Therefore, any function of (y,θ) is a chiral superfield. It can be shown that this also is a necessary condition. Hence, any chiral superfield can be expanded as

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,. \tag{59}$$

Here, A and ψ are the fermionic and scalar components respectively and F is an auxiliary field required for the off-shell closure of the algebra. Similarly, an anti-chiral superfield is defined by $D_\alpha\Phi^\dagger = 0$ and can be expanded as

$$\Phi^\dagger(y^\dagger,\bar\theta) = A^\dagger(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta\bar\theta F^\dagger(y^\dagger)\,, \tag{60}$$

where, $y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\bar\theta$. The product of chiral superfields is a chiral superfield. In general, any arbitrary function of chiral superfields is a chiral superfield:

$$\begin{aligned}\mathcal{W}(\Phi_i) &= \mathcal{W}(A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \\ &= \mathcal{W}(A_i) + \frac{\partial\mathcal{W}}{\partial A_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial\mathcal{W}}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i A_j}\psi_i\psi_j\right). \end{aligned} \tag{61}$$

$\mathcal{W}$ is referred to as the superpotential. In terms of the original variables, Φ and $\Phi^\dagger$ take the form

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= A(x) + i\theta\sigma^\mu\bar\theta\partial_\mu A - \frac{1}{4}\theta^2\bar\theta^2\Box A \\ &\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta\theta F(x)\,, \end{aligned} \tag{62}$$

$$\begin{aligned}\Phi^\dagger(x,\theta,\bar\theta) &= A^\dagger(x) - i\theta\sigma^\mu\bar\theta\partial_\mu A^\dagger - \frac{1}{4}\theta^2\bar\theta^2\Box A^\dagger \\ &\quad + \sqrt{2}\bar\theta\bar\psi(x) + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\,\theta\sigma^\mu\,\partial_\mu\bar\psi + \bar\theta\bar\theta F^\dagger(x)\,. \end{aligned} \tag{63}$$

Vector Multiplet: This multiplet is represented by a real superfield satifying $V = V^\dagger$. In components, it takes the form

$$\begin{aligned}V(x,\theta,\bar\theta) = \; & C + i\theta\chi - i\bar\theta\bar\chi + \tfrac{i}{2}\theta^2(M+iN) - \tfrac{i}{2}\bar\theta^2(M-iN) \\ & -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta(\bar\lambda + \tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi) \\ & -i\bar\theta^2\theta(\lambda + \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi) + \tfrac{1}{2}\theta^2\bar\theta^2(D - \tfrac{1}{2}\Box C)\,. \end{aligned}$$

Many of these components can be gauged away using the abelian gauge transformation $V \to V + \Lambda + \Lambda^\dagger$, where Λ ($\Lambda^\dagger$) are chiral (antichiral) superfields. In the so called Wess-Zumino gauge, we set $C = M = N = \chi = 0$, so that

$$V = -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D\,.$$

In this gauge, $V^2 = \frac{1}{2}A_\mu A^\mu\theta^2\bar\theta^2$ and $V^3 = 0$. The Wess-Zumino gauge breaks supersymmetry, but not the gauge symmetry of the abelian gauge field A_μ. The Abelian field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V\,, \qquad \bar W_{\dot\alpha} = -\frac{1}{4}D^2\bar D_{\dot\alpha}V\,.$$

W_α is a chiral superfield. Since it is gauge invariant, it can be computed in the Wess-Zumino gauge and takes the form

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\,, \tag{64}$$

where, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the familiar abelian field strength tensor.

In the non-Abelian case, V belongs to the adjoint representation of the gauge group: $V = V_A T^A$, where, $T^{A\dagger} = T^A$. The gauge transformations are now implemented by

$$e^{-2V} \to e^{-i\Lambda^\dagger} e^{-2V} e^{i\Lambda} \qquad \text{where, } \Lambda = \Lambda_A T^A$$

The non-Abelian gauge field strength is defined by

$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V}$$

and transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda} .$$

In components, it takes the form

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + \theta^2 \sigma^\mu D_\mu \bar{\lambda}^a \right) , \tag{65}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu , \qquad D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + f^{abc} A^b_\mu \bar{\lambda}^c .$$

In the next section, we will construct supersymmetric Lagrangians in terms of superfields.

2.5 Construction of N=1 Lagrangians

In this section we will construct the $N = 1$ Lagrangians for the scalar and the vector multiplets. These serve as the building blocks for the $N = 2$ Lagrangian which is our real interest. As stated before, a supersymmetric Lagrangian can be constructed as the highest component of a superfield. Thus the problem reduces to that of finding appropriate superfields.

Lagrangian for the Scalar Multiplet: Let us first consider the product of a chiral and an anti-chiral superfield $\Phi^\dagger_i \Phi_j$. This is a general superfield and its highest component can be computed using (62) and (63) as

$$\begin{aligned} \Phi^\dagger_i \Phi_j \mid_{\theta^2\bar{\theta}^2} \; = \; & - \frac{1}{4} A^\dagger_i \Box A_j - \frac{1}{4} \Box A^\dagger_i A_j + F^\dagger_i F_j + \frac{1}{2} \partial_\mu A^\dagger_i \partial^\mu A_j \\ & - \frac{i}{2} \psi_j \sigma^\mu \partial_\mu \bar{\psi}_i + \frac{i}{2} \partial_\mu \psi_j \sigma^\mu \bar{\psi}_i . \end{aligned}$$

Dropping some total derivatives and summing over $i = j$, we get the free field Lagrangian

$$\mathcal{L} = \Phi^\dagger_i \Phi_i \mid_{\theta^2\bar{\theta}^2} = \partial_\mu A^\dagger_i \partial^\mu A_i + F^\dagger_i F_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i .$$

This is the free Lagrangian for a massless scalar and a massless fermion with an auxiliary field which can be eliminated by its equation of motion. Supersymmetric interaction terms can be constructed in terms of the superpotential (61) and its conjugate, which are holomorphic functions of Φ and $\Phi^\dagger$, respectively. Moreover, note that the space of the fields Φ may have a non-trivial metric g^{ij} in which case the scalar kinetic term, for example, takes the form $g^{ij} \partial_\mu A^\dagger_i \partial^\mu A_j$, with appropriate modifications for other terms. In such cases, the free field Lagrangian above has to be replaced by a non-linear σ-model. Thus, the most general $N = 1$ supersymmetric Lagrangian for the scalar multiplet (including the interaction terms) is given by

$$\mathcal{L} = \int d^4\theta \, K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\Phi^\dagger) .$$

Note that the θ-integrals pick up the highest component of the superfield and in our conventions, $\int d^2\theta\theta^2 = 1$ and $\int d^2\bar\theta\bar\theta^2 = 1$. In terms of the non-holomorphic function $K(A, A^\dagger)$, the metric on the field space is given by $g^{ij} = \partial^2 K/\partial A_i \partial A_j^\dagger$. For this reason, the function $K(\Phi, \Phi^\dagger)$ is referred to as the Kähler potential.

For a renormalizable theory, the forms of K and $\mathcal{W}$ are not arbitrary and are constrained by R-symmetry. This symmetry acts on the chiral superfields as follows

$$\begin{aligned} R\,\Phi(x,\theta) = \Phi'(x,\theta) &= e^{2in\alpha}\Phi(x, e^{-i\alpha}\theta)\,, \\ R\,\Phi^\dagger(x,\bar\theta) = \Phi'^\dagger(x,\theta) &= e^{-2in\alpha}\Phi^\dagger(x, e^{i\alpha}\bar\theta)\,. \end{aligned}$$

Under this, the component fields transform as

$$\begin{aligned} A &\to e^{2in\alpha} A\,, \\ \psi &\to e^{2i(n-1/2)\alpha}\psi\,, \\ F &\to e^{2i(n-1)\alpha} F\,. \end{aligned}$$

We refer to n as the R-character. Since $\theta \to e^{+i\alpha}\theta$, or $d^2\theta \to e^{-2i\alpha}d^2\theta$, The R-character of the superfields in each term of $\mathcal{W}$ must add up to one. Similarly, K should be R-neutral. The vector multiplet is real and it has no natural R-symmetry. This symmetry plays an important role in the study of supersymmetric gauge theories and we will come back to it in the next section.

<u>Lagrangians for the Vector Multiplet:</u> As mentioned in the previous section, the Abelian field strength W, given by (64), is a chiral superfield. Using the expansion there, one can easily compute that

$$W^\alpha W_\alpha\,|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,.$$

Hence, the usual abelian supersymmetric Lagrangian (which does not contain the $F\widetilde{F}$ term) is given by

$$\mathcal{L} = \frac{1}{4g^2}\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha}\right)\,.$$

Similarly, in the non-Abelian case, using the normalization $\mathrm{Tr}T^aT^b = \delta^{ab}$, we have

$$\mathrm{Tr}(W^\alpha W_\alpha\,|_{\theta\theta}) = -2i\lambda^a\sigma^\mu D_\mu\bar\lambda^a + D^aD^a - \frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\,, \qquad (66)$$

and, hence, the usual non-Abelian supersymmetric Lagrangian (without the $F\widetilde{F}$-term) is given by

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{Tr}\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha}\right)\,.$$

However, we are interesed in the supersymmetric analogue of the Lagrangian (44) which also contains a θ-term. From (66), it is obvious that the super Yang-Mills Lagrangian with a θ-term can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\left(\tau\,\mathrm{Tr}\int d^2\theta\, W^\alpha W_\alpha\right) \\ &= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde F^{a\mu\nu} + \frac{1}{g^2}(\frac{1}{2}D^aD^a - i\lambda^a\sigma^\mu D_\mu\bar\lambda^a)\,, \qquad (67) \end{aligned}$$

where, $\tau = \theta/2\pi + 4\pi i/g^2$. Note that τ can be regarded as a constant chiral superfield.

Interaction Terms and the General $N = 1$ Lagrangian: Let the chiral superfields Φ_i belong to a given representation of the gauge group in which the generators are the matrices T^a_{ij}. The kinetic energy term $\Phi_i^\dagger \Phi_i$ is invariant under global gauge transformations $\Phi' = e^{-i\Lambda}\Phi$. In the local case, to insure that Φ' remains a chiral superfield, Λ has to be a chiral superfield. The supersymmetric gauge invariant kinetic energy term is then given by $\Phi^\dagger e^{-2V}\Phi$. We are now in a position to write down the full N=1 supersymmetric Lagrangian as

$$\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\left(\tau \mathrm{Tr}\int d^\theta\, W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar\theta\, \Phi^\dagger e^{-2V}\Phi + \int d^2\theta\, \mathcal{W} + \int d^2\bar\theta\, \bar{\mathcal{W}}\,. \tag{68}$$

Note that since each term is separately invariant, the relative normalisation between the scalar part and the Yang-Mills part is not fixed by $N = 1$ supersymmetry. In the above, we have set the normalization of the scalar part to one, but later, we will change this by rescaling the scalar multiplet Φ. In terms of component fields, the above Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} \;=\; & - \; \frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - \frac{i}{g^2}\,\lambda^a\sigma^\mu D_\mu\bar\lambda^a + \frac{1}{2g^2}D^a D^a \\
& + \; (\partial_\mu A - iA^a_\mu T^a A)^\dagger(\partial^\mu A - iA^{a\mu}T^a A) - i\,\bar\psi\bar\sigma^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\
& - \; D^a A^\dagger T^a A - i\sqrt{2}\,A^\dagger T^a\lambda^a\psi + i\sqrt{2}\,\bar\psi T^a A\bar\lambda^a + F_i^\dagger F_i \\
& + \; \frac{\partial\mathcal{W}}{\partial A_i}F_i + \frac{\partial\bar{\mathcal{W}}}{\partial A_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i\partial A_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\bar{\mathcal{W}}}{\partial A_i^\dagger\partial A_j^\dagger}\bar\psi_i\bar\psi_j\,.
\end{aligned} \tag{69}$$

Here, $\mathcal{W}$ denotes the scalar component of the superpotential. The auxiliary fields F and D^a can be eliminated by using their equations of motion. The terms involving these fields, thus, give rise to the scalar potential

$$V = \sum_i \left|\frac{\partial\mathcal{W}}{\partial A_i}\right|^2 - \frac{1}{2}g^2(A^\dagger T^a A)^2\,. \tag{70}$$

2.6 The $N = 2$ Supersymmetric Lagrangian for Gauge Fields

The on-shell $N = 1$ scalar multiplet (A, ψ) and vector multiplet (A_μ, λ), put together, have the same field content as the on-shell $N = 2$ vector multiplet $(A, \psi, \lambda, A_\mu)$. The Lagrangian (69) contains all these fields but as such is not $N = 2$ supersymmetric. We now assume that $(A, \psi, \lambda, A_\mu)$ form an $N = 2$ vector multiplet and discuss the restrictions which this assumtion imposes on the $N = 1$ Lagrangian in (69). First, since A^a_μ and λ^a belong to the adjoint representation of the gauge group, A_i and ψ_i should also belong to the same representation if they are to be part of the same multiplet. Hence, $T^a_{ij} = -if^a_{ij}$ and the sets of indices $\{i\}$ and $\{a\}$ coincide. Second, since the two supersymmetry generators in the $N = 2$ algebra appear on the same footing, the same must be the case with the fermions ψ^a and λ^a in (69). To satisfy this condition, we set the superpotential $\mathcal{W}$ to zero since it couples only to ψ^a. This condition also fixes the arbitrary relative normalization between the Yang-Mills part and the scalar part of the Lagrangian since it requires that the kinetic energy terms for both fermions should have the same normalization. This is achieved by scaling $\Phi \to \Phi/g$ in (69). It turns out that if the Lagrangian (69) satisfies these conditions, then it has $N = 2$ supersymmetry. The terms containing the auxiliary fields now take the form

$$\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}DD + D\,[A^\dagger\,,\,A] + F^\dagger F\right)\,,$$

where, we have used the notation $\Phi = \Phi^a T^a$, with T^a in the fundamental representation. On eliminating D and F, we get the scalar potential

$$V = -\frac{1}{2g^2}\mathrm{Tr}\left([A^\dagger\,,\,A]^2\right)\,. \tag{71}$$

The full Lagrangian with $N = 2$ supersymmetry can now be written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\,\mathrm{Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\,\Phi^\dagger e^{-2V}\Phi\right)\right] \\ &= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} + (D_\mu A)^\dagger D^\mu A - \frac{1}{2}[A^\dagger\,,\,A]^2 \\ &\qquad - i\,\lambda\sigma^\mu D_\mu\bar\lambda - i\,\bar\psi\bar\sigma^\mu D_\mu\psi - i\sqrt{2}\,[\lambda,\psi]\,A^\dagger - i\sqrt{2}\,[\bar\lambda,\bar\psi]\,A\Big)\,,\end{aligned} \tag{72}$$

where, in the component expansion, the auxiliary fields have been eliminated. Note that (72) is the supersymmetric generalization of the Yang-Mills-Higgs theories described in section 1, with the Higgs potential V given by (71), and with a θ-term. Therefore, the discussion in section 1 also applies to our $N = 2$ supersymmetric theory. In particular, the Higgs vacuum is defined by $D_\mu A = 0, V = 0$, and the potential vanishes for non-zero field configurations provided A commutes with $A^\dagger$. Thus the model admits monopole and dyonic solutions and contains massive gauge bosons. For example, if the gauge group is $SU(2)$ or $SO(3)$, it is broken down to $U(1)$ and two of the gauge bosons become massive.

Now, suppose that we are interested in the behaviour of this model at energies lower than some cutoff Λ which is smaller than the mass of the lightest massive state in the theory. At such energies, we will not encounter any on-shell massive states and the physics can be described by the Wilsonian low-energy effective action. This effective action is obtained by completely integrating out all massive states as well as integrating out all massless excitations above the scale Λ [36, 37]. This, in general, is a complicated procedure and cannot be carried out explicitly. Fortunately, in our model, the general form of the Wilsonian effective action is severely constrained by $N = 2$ supersymmetry. This is easiest to see when the theory (72) is formulated in term of $N = 2$ superfields as described below.

The $N = 2$ Superspace Formulation: The $N = 2$ superspace is obtained by adding four more fermionic degrees of freedom, say, $\tilde\theta$ and $\bar{\tilde\theta}$, to the $N = 1$ superspace. Thus, a generic $N = 2$ superfield can be written as $F(x, \theta, \bar\theta, \tilde\theta, \bar{\tilde\theta})$. We need a superfield which has the same components as the $N = 2$ vector multiplet. This is obtained by imposing the constraints of chirality and reality on a general $N = 2$ superfield [27]. We briefly describe this following the approach of

[28]. An $N=2$ chiral superfield Ψ is defined by the constraints $\bar{D}_{\dot\alpha}\Psi = 0$ and $\bar{\tilde{D}}_{\dot\alpha}\Psi = 0$, where, the supercovariant derivative $\bar{\tilde{D}}_{\dot\alpha}$ is defined in the same way as $\bar{D}_{\dot\alpha}$ with θ replaced by $\tilde\theta$. The expansion of Ψ can be arranged in powers of $\tilde\theta$ and can be written as

$$\Psi = \Psi^{(1)}(\tilde y, \theta) + \sqrt{2}\tilde\theta^\alpha \Psi^{(2)}_\alpha(\tilde y, \theta) + \tilde\theta^\alpha\tilde\theta_\alpha \Psi^{(3)}(\tilde y, \theta)\,,$$

where, $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\tilde\theta\sigma^\mu\bar{\tilde\theta}$. This expansion helps us relate the $N=2$ formalism to the $N=1$ language we have been using so far. Clearly, the component $\Psi^{(1)}$ has the same form as the $N=1$ chiral superfield Φ. The remaining two components are constrained by a reality condition. The outcome is that $\Psi^{(2)}_\alpha = W_\alpha(\tilde y, \theta)$ as given in (65), and $\Psi^{(3)}$ is given by

$$\Psi^{(3)}(\tilde y, \theta) = \Phi^\dagger(\tilde y - i\theta\sigma\bar\theta, \theta, \bar\theta) exp\left[2gV(\tilde y - i\theta\sigma\bar\theta, \theta, \bar\theta\right]\,|_{\bar\theta\bar\theta}\;.$$

Here, $\Phi(\tilde y - i\theta\sigma\bar\theta, \theta, \bar\theta)$ is to be understood as the expansion in (62). Clearly, Ψ has the same field content as the $N=2$ vector multiplet. One can verify that in terms of the $N=2$ superfield Ψ, the $N=2$ Lagrangian (72) is given by the compact expression

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im\,Tr}\int d^2\theta d^2\tilde\theta\,\frac{1}{2}\tau\Psi^2\,. \tag{73}$$

Using the $N=2$ chiral superfield Ψ (also referred to as the $N=2$ vector superfield), we can now construct the most general $N=2$ Lagrangian for the gauge fields: Corresponding to any function $\mathcal{F}(\Psi)$, we can construct a Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{4\pi}\mathrm{Im\,Tr}\int d^2\theta d^2\tilde\theta\,\mathcal{F}(\Psi) \\ &= \frac{1}{8\pi}\mathrm{Im}\left(\int d^2\theta\,\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha + 2\int d^2\theta d^2\bar\theta\,(\Phi^\dagger e^{2gV})^a\mathcal{F}_a(\Phi)\right)\,. \end{aligned}\tag{74}$$

Here, $\mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$, $\mathcal{F}_{ab}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ and $\mathcal{F}$ is referred to as the $N=2$ prepotential. From the above, we can easily read off the Kähler potential as $\mathrm{Im}(\Phi^{\dagger a}\mathcal{F}_a(\Phi)$. This gives rise to a metric $g_{ab} = \mathrm{Im}(\partial_a\partial_b\mathcal{F})$ on the space of fields. A metric of this form is called a special Kähler metric. If we demand renormalisability, then $\mathcal{F}$ has to be quadratic in Ψ as in (73). However, if we want to write a low-energy effective action, then renormalisability is not a criterion and $\mathcal{F}$ can have a more complicated form. In particular, we can start from the microscopic theory (72), corresponding to a quadratic prepotential, and try to construct the modified $\mathcal{F}$ for the low-energy Wilsonian effective action. The exact determination of this function is the subject of the work of Seiberg and Witten.

2.7 The $N=2$ Supersymmetric Lagrangian for Matter Fields

Since matter fields and gauge fields transform under different representations of the gauge group, they cannot be part of the same multiplet. The $N=2$ matter supermultiplet is called the hypermultiplet and contains one pair of complex scalars and one pair of two-component spinors, all transforming under the same representation of the gauge group. From our discussion of the representations of the supersymmetry algebra it follows that if a hypermultiplet is massive, then its mass should appear as a central extension in the supersymmetry algera. This is so because otherwise $N = 2$ supersymmetry requires a larger number of components than is contained in a hypermultiplet. In $N = 1$ notation, a hypermultiplet contains a chiral superfield Q and an anti-chiral superfield $\widetilde{Q}^\dagger$, both transforming under the same representatin N_c of the gauge group $SU(N_c)$ We denote the components of Q and $\widetilde{Q}$ by (q, ψ_q, F_q) and $(\widetilde{q}, \psi_{\widetilde{q}}, F_{\widetilde{q}})$ respectively. The form of the Lagrangian for N_f hypermultiplets (labelled by an index i), interacting with a $N = 2$ vector multiplet, can be partly inferred from the $N = 1$ theories and is given by

$$\mathcal{L} = \int d\theta^4 \left(Q_i^\dagger e^{-2V} Q_i + \widetilde{Q}_i e^{2V} \widetilde{Q}_i^\dagger\right) + \int d\theta^2 \left(\sqrt{2}\widetilde{Q}_i \Phi Q_i + m_i \widetilde{Q}_i Q_i\right) + \, h.c. + \cdots \tag{75}$$

Here, the dots represent the Lagrangian for the pure $N = 2$ vector multiplet and we have supressed the gauge group indices. The term $\sqrt{2}\widetilde{Q}_i\Phi Q_i$ is related, by $N = 2$ supersymmetry, to the coupling of the hypermultiplet with the $N = 1$ vector multiplet V. The presence of this term is not required in an $N = 1$ Lagrangian. As in normal QCD, when all masses m_i are equal, the theory is invariant under the global flavour group $SU(N_f)$.

Eliminating the auxiliary fields F_q and $F_{\widetilde{q}}$, which appear in the hypermultiplet, results in a contribution to the scalar potential given by (compare with (70))

$$V = \frac{1}{2} g^2 \sum_a D_a D^a \,,$$

with

$$D^a = \sum_{i=1}^{N_f} \left(q_i^\dagger \lambda^a q_i - \widetilde{q}_i \lambda^a \widetilde{q}_i^\dagger\right) . \tag{76}$$

Here, λ^a are the gauge group generators in the fundamental representaion. This term is referred to as the D-term. It must be mentioned that the $N = 2$ algebra has a global $SU(2)_R$ symmetry which should also be a symmetry of the Lagrangian. However, in our decomposition of the hypermultiplet in terms of Q and $\widetilde{Q}^\dagger$, this symmetry is not manifest since it rotates the scalar components q and $\widetilde{q}^\dagger$ as a doublet. It is not difficult to write an $N = 2$ Lagrangian with manifest $SU(2)_R$ symmetry [23].

<u>Minimization of the D-term</u>: To find the vacuum of the theory, now we also have to minimize the D-term contribution and the hypermultiplet mass term contribution to the scalar potential [29, 30]. For non-zero quark masses, the only solution is $q = \widetilde{q} = 0$. Thus, only the scalar A can have a non-zero vacuum expectation value. But when $m_i = 0$, then the D-term can be minimized for a set of non-vanishing q and $\widetilde{q}$ (the potential has flat directions). However, the $\widetilde{Q}\Phi Q$ term in the Lagrangian now requires $A_{vac} = 0$. In the following, we determine these flat directions.

The minimum of the D-term corresponds to $D^a = 0$. To solve this, note that $q_\alpha^{(i)}$ (where α denotes a colour index) can be regarded as a set of N_c vectors in $\mathbf{C}^{N_f}$. Thus, we can construct the matrix of scalar products,

$$\sum_i q_\alpha^{(i)} q_\beta^{\dagger(i)} = q_\alpha \cdot q_\beta^\dagger = (qq^\dagger)_{\alpha\beta}\,.$$

In terms of this,

$$\sum_i q_i^\dagger \lambda^a q_i = \mathrm{Tr}\,\left(qq^\dagger \lambda^a\right)\,.$$

Similarly, we construct the matrix $\widetilde{q}_\alpha^\dagger \cdot \widetilde{q}_\beta = (\widetilde{q}^\dagger \widetilde{q})_{\alpha\beta}$. Hence, $D^a = 0$ can be written as

$$D^a = \mathrm{Tr}\left[\left(qq^\dagger - \widetilde{q}^\dagger\widetilde{q}\right)\,\lambda^a\right] = 0\,.$$

Since λ^a are in an irreducible representation of $SU(N_c)$, the above condition is solved by

$$(qq^\dagger) - (\widetilde{q}^\dagger\widetilde{q}) = c^2 \mathbf{1}_{N_c}\,. \tag{77}$$

Here, we can distinguish two cases: $N_f < N_c$ and $N_f \geq N_c$.

1) $N_f < N_c$: In this case, the matrix $(qq^\dagger)$ has rank N_f and thus it has $N - N_f$ zero eigenvalues. The same applies to the matrix $(\widetilde{q}^\dagger\widetilde{q})$. By using $SU(N_c) \times SU(N_f) \times U(1)_R$ rotations, $(qq^\dagger)$ can be diagonalized and condition (77) then implies that $(\widetilde{q}^\dagger\widetilde{q})$ must also be diagonal. Hence, in this basis, we can solve for q and $\widetilde{q}$ as

$$q = \begin{pmatrix} v_1^{(1)} & 0 & \cdots & 0 \\ 0 & v_2^{(2)} & & \\ \vdots & & \ddots & \\ & & & v_{N_f}^{(N_f)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix}, \quad \widetilde{q} = \begin{pmatrix} \tilde{v}_1^{(1)} & 0 & \cdots & & 0 & \cdots \\ 0 & \tilde{v}_2^{(2)} & & & 0 & \cdots \\ \vdots & & \ddots & & & \\ 0 & & & \tilde{v}_{N_f}^{(N_f)} & 0 & \cdots \end{pmatrix}.$$

The presence of zero eigenvalues imply that $c = 0$ and therefore, $v_i^{(i)} = \tilde{v}_i^{(i)}$. In this case, the gauge symmetry is broken down to $SU(N_c - N_f)$ except for $N_c = N_f - 1$, where it is totally broken. $2N_f N_c - N_f^2$ quark superfields become heavy and the remaining N_f^2 quark superfields remain massless and correspond to the Goldstone bosons of broken global symmetries.

2) $N_f \leq N$: In this case $(qq^\dagger)$ has rank N_c and, generically, its eigenvalues are not zero. As a result, $c \neq 0$. Arguing as above, in this case the vacuum values of q and $\widetilde{q}$ can be written as

$$q = \begin{pmatrix} v_1^{(1)} & 0 & \cdots & & 0 & \cdots \\ 0 & v_2^{(2)} & & & 0 & \cdots \\ \vdots & & \ddots & & & \\ 0 & & & v_{N_c}^{(N_c)} & 0 & \cdots \end{pmatrix}, \quad \widetilde{q} = \begin{pmatrix} \tilde{v}_1^{(1)} & 0 & \cdots & 0 \\ 0 & \tilde{v}_2^{(2)} & & \\ \vdots & & \ddots & \\ & & & \tilde{v}_{N_c}^{(N_c)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

and

$$\tilde{v}_i^{(i)} = \sqrt{|v_i^{(i)}|^2 - c^2}\,.$$

In this case, the gauge group is completely broken, and depending on the values of $v_i^{(i)}$, the pattern of chiral symmetry breaking quite complicated with many possibilities. For $N_f > N$, there are surviving R-symmetries.

2.8 Central Charges in the $N = 2$ Pure Gauge Theory

We have seen that the $N = 2$ supersymmetry algebra with central charge Z implies the bound $M \geq \sqrt{2}|Z|$ on the particle masses. It was also stated that this bound is the same as the BPS bound on the masses which is determined in terms of the electric and magnetic charges. In this section, we prove the above statement by explicitly calculating Z following [26].

In the supersymmetry algebra, The central charge Z appears in the commutator of the supercharges Q^I_α which, in turn, are space integrals of S^{I0}_α (Here, $S^{I\mu}_\alpha$ denotes the supercurrent). Thus, we first have to compute S^{I0}'s in terms of the basic fields, and then evaluate their commutators. The central charge is related to a surface term in the space integral of this commutator which is non-zero if the field configuration corresponds to electric and magnetic charges.

As a warm up exercise, we start with the theory for $N = 1$ chiral superfields. The Lagrangian is given by

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger \Phi + \int d^2\theta\, \mathcal{W}(\Phi) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\Phi^\dagger)\,.$$

Defining $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, the superfield can be expanded as $\Phi = A(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y)$. In this basis, $Q_\alpha = \partial/\partial\theta^\alpha$, $\bar{Q}_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\,\partial/\partial y^\mu$, and the supersymmetry variations, $\delta_\epsilon = \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$, of the fields are given by

$$\begin{array}{ll}
\delta A = \sqrt{2}\epsilon\psi\,, & \delta\bar{A} = \sqrt{2}\bar{\epsilon}\bar{\psi}\,, \\
\delta\psi = \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu A\,, & \delta\bar{\psi} = -i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar{A} + \sqrt{2}\bar{F}\bar{\epsilon}\,, \\
\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi\,, & \delta\bar{F} = i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar{\psi}\,.
\end{array}$$

Using these transformations, we can compute the variation of the terms involving the superpotential $\mathcal{W}$ as

$$\begin{aligned}
\delta(\mathcal{W}'F - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A^2}\psi\psi) &= \mathcal{W}''\delta AF + \mathcal{W}'\delta F - \frac{1}{2}\mathcal{W}'''\delta A\psi\psi + \mathcal{W}''\psi\delta\psi \\
&= \mathcal{W}''\sqrt{2}\epsilon\psi F + i\sqrt{2}\mathcal{W}'\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi + \sqrt{2}\mathcal{W}''\psi(i\sigma^\mu\bar{\epsilon}\partial_\mu A + \epsilon F) \\
&= \partial_\mu\left(i\sqrt{2}\frac{\partial\mathcal{W}}{\partial A}\bar{\epsilon}\bar{\sigma}^\mu\psi\right)\,.
\end{aligned}$$

Note that we have used $\mathcal{W}'''\delta A\psi\psi = \mathcal{W}'''\sqrt{2}(\epsilon\psi)\psi\psi = 0$. Since $\mathcal{W}'''$ is totally symmetric, this statement is also true in the presence of many fields ψ^i. Thus, we have,

$$\delta(\int d^2\theta\mathcal{W} + \int d^2\bar{\theta}\bar{\mathcal{W}}) = i\sqrt{2}\partial_\mu\left(\frac{\partial\mathcal{W}}{\partial A_j}\bar{\epsilon}\bar{\sigma}^\mu\psi^j - \frac{\partial\bar{\mathcal{W}}}{\partial\bar{A}_j}\bar{\psi}_j\bar{\sigma}^\mu\epsilon\right)\,.$$

Next we calculate the variation of the kinetic terms,

$$\mathcal{L}_D = \partial_\mu A^\dagger \partial^\mu A + F^\dagger F - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + \frac{i}{2}\partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi \,.$$

A slightly lengthy computation yields:

$$\begin{aligned} \delta\mathcal{L}_D = & - \frac{i}{\sqrt{2}}\partial_\mu (F\bar{\psi}\bar{\sigma}^\mu \epsilon) + \frac{i}{\sqrt{2}}\partial_\mu (\epsilon\psi\partial^\mu A^\dagger - \epsilon\sigma^{\nu\mu}\psi\partial_\nu A^\dagger) \\ & + \frac{i}{\sqrt{2}}\partial_\mu (F^\dagger \bar{\epsilon}\bar{\sigma}^\mu \psi) + \frac{i}{\sqrt{2}}\partial_\mu (\bar{\epsilon}\bar{\psi}\partial^\mu A - \bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\epsilon}\partial_\nu A) \,. \end{aligned}$$

Adding the variations of the superpotential and the kinetic terms, and using the definitions

$$\sigma^{\mu\nu} = \frac{1}{2}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \,, \quad \bar{\sigma}^{\mu\nu} = \frac{1}{2}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \,,$$

we get the supercurrent as

$$S^\rho_{matter} = \sqrt{2}\epsilon\sigma^\nu \bar{\sigma}^\rho \psi \partial_\nu A^\dagger + i\sqrt{2}\frac{\partial\bar{\mathcal{W}}}{\partial A^\dagger}\bar{\psi}\bar{\sigma}^\rho \epsilon + \sqrt{2}\bar{\psi}\bar{\sigma}^\rho \sigma^\nu \bar{\epsilon}\partial_\nu A - i\sqrt{2}\frac{\partial\mathcal{W}}{\partial A}\bar{\epsilon}\bar{\sigma}^\rho \psi \,. \tag{78}$$

Note that for convenience, we have included the supersymmetry transformation parameters $\epsilon, \bar{\epsilon}$ in the definition of the supercurrent. So, expanding in components, we actually have:

$$S^\rho = \epsilon^\alpha S^\rho_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{S}^{\rho\dot{\alpha}} \,. \tag{79}$$

Next we consider the inclusion of gauge fields which are described by a vector multiplet. The supersymmetry variations of the fields in this multiplet are given by

$$\begin{aligned} \delta A^a_\mu &= -i\bar{\epsilon}\bar{\sigma}_\mu \lambda^a + i\bar{\lambda}^a \bar{\sigma}_\mu \epsilon \,, \\ \delta D^a &= \bar{\epsilon}\bar{\sigma}^\mu D_\mu \lambda^a + D_\mu \lambda^a \bar{\sigma}^\mu \epsilon \,, \\ \delta\lambda^a &= \tfrac{1}{2}\sigma^{\mu\nu}\epsilon F^a_{\mu\nu} + i\epsilon D^a \,, \\ \delta\bar{\lambda}^a &= \tfrac{1}{2}\bar{\epsilon}\bar{\sigma}^{\nu\mu} F^a_{\mu\nu} - i\bar{\epsilon} D^a \,. \end{aligned} \tag{80}$$

Furthermore, in the presence of the gauge interactions, the matter field transformations need some modification:

$$\begin{aligned} \delta A &= \sqrt{2}\epsilon\psi \,, \\ \delta\psi &= \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu \bar{\epsilon} D_\mu A \,, \\ \delta\bar{\psi} &= \sqrt{2}\bar{\epsilon}F^\dagger - i\sqrt{2}\epsilon\sigma^\mu D_\mu A^\dagger \,, \\ \delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu D_\mu \psi - 2iT^a A\bar{\epsilon}\bar{\lambda}^a \,. \end{aligned} \tag{81}$$

The last term in δF is needed to cancel part of the variation of ψ in the term $A^\dagger T^a \lambda^a \psi$ in the Lagrangian (69). The part of the Lagrangian describing the pure vector superfieled can be written as

$$\mathcal{L} = \frac{1}{g^2}\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{i}{2}\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{i}{2}D_\mu \bar{\lambda}^a \bar{\sigma}^\mu \lambda^a + \frac{1}{2}D^2\right) + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \,.$$

If we ignore the $\theta F\tilde{F}$ term, then the supercurrent obtained from this part of the Lagrangian takes the form

$$S^\rho_{gauge} = -\frac{i}{2g^2}\left(\bar{\lambda}^a\bar{\sigma}^\rho\sigma^{\mu\nu}\epsilon F^a_{\mu\nu} + \bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\lambda^a F^a_{\mu\nu}\right) .$$

For the theory coupled to matter, the supercurrent is not just the sum of the above, plus the gauge-covariantized version of S^ρ_{matter}. We also expect some contribution from the terms $D^a A^\dagger T^a A$ and $iA^\dagger T^a\lambda^a\psi + h.c.$. Taking this into account, the supercurrent for the interacting $N=1$ Lagrangian (69) is given by

$$\begin{aligned} S^\rho = & - \frac{i}{2g^2}\left(\bar{\lambda}^a\bar{\sigma}^\rho\sigma^{\mu\nu}\epsilon + \bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\lambda^a\right) F^a_{\mu\nu} - \left(\bar{\epsilon}\bar{\sigma}^\rho\lambda^a + \bar{\lambda}^a\bar{\sigma}^\rho\epsilon\right) A^\dagger T^a A \\ & + \sqrt{2}\epsilon\sigma^\mu\bar{\sigma}^\rho\psi D_\mu A^\dagger + \sqrt{2}\bar{\psi}\bar{\sigma}^\rho\sigma^\mu\bar{\epsilon}D_\mu A + i\sqrt{2}\frac{\partial\bar{\mathcal{W}}}{\partial A^\dagger}\bar{\psi}\bar{\sigma}^\rho\epsilon - i\sqrt{2}\frac{\partial\mathcal{W}}{\partial A}\bar{\epsilon}\bar{\sigma}^\rho\psi . \end{aligned} \tag{82}$$

Now, let us turn our attention to the pure $N=2$ gauge theory. As discussed in section 1, this is obtained from the $N=1$ theory (69) by setting $\mathcal{W}=0$ and by scaling the chiral superfield Φ to Φ/g. Furthermore, Φ is now a vector in the adjoint representation of the gauge group. Thus, relabelling (λ,ψ) as (λ_1,λ_2), we can easily write down one of the $N=2$ supercurrents as

$$\begin{aligned} g^2 S^\rho_{(1)} = & - \frac{i}{2}\left(\bar{\lambda}^a_1\bar{\sigma}^\rho\sigma^{\mu\nu}\epsilon + \bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\lambda^a_1\right) F^a_{\mu\nu} - \left(\bar{\epsilon}\bar{\sigma}^\rho\lambda^a_1 + \bar{\lambda}^a_1\bar{\sigma}^\rho\epsilon\right) A^\dagger T^a A \\ & + \sqrt{2}\,\epsilon\sigma^\mu\bar{\sigma}^\rho\lambda^a_2\, D_\mu A^{a\dagger} + \sqrt{2}\,\bar{\lambda}^a_2\bar{\sigma}^\rho\sigma^\mu\bar{\epsilon}\, D_\mu A^a . \end{aligned}$$

The $N=2$ theory is also invariant under a second set of supersymmetry transformations (with parameter ϵ') which is obtained from (80) and (81) by the replacement $\lambda\to\psi$, $\psi\to-\lambda$. This corresponds to a transformation of the (λ,ψ) doublet by an element of $SU(2)_R$, which is a symmetry group of the $N=2$ algebra. The associated conserved current is then obtained from $S^\rho_{(1)}$ by the replacement $\lambda_1\to\lambda_2$, $\lambda_2\to-\lambda_1$:

$$\begin{aligned} g^2 S^\rho_{(2)} = & - \frac{i}{2}\left(\bar{\lambda}^a_2\bar{\sigma}^\rho\sigma^{\mu\nu}\epsilon' + \bar{\epsilon}'\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\lambda^a_2\right) F^a_{\mu\nu} - \left(\bar{\epsilon}'\bar{\sigma}^\rho\lambda^a_2 + \bar{\lambda}^a_2\bar{\sigma}^\rho\epsilon'\right) A^\dagger T^a A \\ & - \sqrt{2}\,\epsilon'\sigma^\mu\bar{\sigma}^\rho\lambda^a_1\, D_\mu A^{a\dagger} - \sqrt{2}\,\bar{\lambda}^a_1\bar{\sigma}^\rho\sigma^\mu\bar{\epsilon}'\, D_\mu A^a . \end{aligned}$$

Let us first concentrate on $S^\mu_{(1)}$. Using the identities

$$\begin{aligned} \sigma^a\bar{\sigma}^b\sigma^c &= \eta^{ab}\sigma^c - \eta^{ac}\sigma^b + \eta^{bc}\sigma^a + i\epsilon^{abcd}\sigma_d , \\ \bar{\sigma}^a\sigma^b\bar{\sigma}^c &= \eta^{ab}\bar{\sigma}^c - \eta^{ac}\bar{\sigma}^b + \eta^{bc}\bar{\sigma}^a - i\epsilon^{abcd}\bar{\sigma}_d , \end{aligned}$$

along with $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi$, this supercurrent can be rewritten as

$$\begin{aligned} g^2 S^\mu_{(1)} = & - \epsilon\sigma_\nu\bar{\lambda}^a_1(iF^{a\mu\nu} + \widetilde{F}^{a\mu\nu}) + \sqrt{2}\,\epsilon\sigma^\nu\bar{\sigma}^\mu\lambda^a_2\, D_\nu A^{\dagger a} + \epsilon\sigma^\mu\bar{\lambda}^a_1 A^\dagger T^a A \\ & + (\bar{\epsilon}\ \textit{dependent terms}) . \end{aligned}$$

From the above, we can easily read off the components $S^\mu_{(1)\alpha}$ (see eq. (79)) as

$$g^2 S^\mu_{(1)\alpha} = \sigma_{\nu\alpha\dot{\alpha}}\bar{\lambda}^{a\dot{\alpha}}_1(iF^{a\mu\nu} + \widetilde{F}^{a\mu\nu}) + \sqrt{2}(\sigma^\nu\bar{\sigma}^\mu\lambda^a_2)_\alpha D_\nu A^{\dagger a} + \sigma^\mu_{\alpha\dot{\alpha}}\bar{\lambda}^{a\dot{\alpha}}_1 A^\dagger T^a A .$$

After lowering the spinor index using

$$\bar\lambda_1^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\lambda_{1\dot\beta} = i(\sigma_y\lambda_1^\dagger)^{\dot\alpha}\,, \qquad \text{(Note that } \bar\lambda_{\dot\beta} = \lambda_\beta^\dagger)$$

and using vector notation for spatial components of vectors, The $\mu = 0$ component of the current takes the form

$$\begin{aligned} g^2 S^0_{(1)\alpha} &= - \;\; i(\vec\sigma\sigma_y\lambda_1^{\dagger a})_\alpha \cdot (i\vec F^a + \vec{\widetilde F})^a \\ &+ \;\; \sqrt{2}\lambda^a_{2\alpha} D_0 A^{\dagger a} + \sqrt{2}(\vec\sigma\cdot\vec D A^{\dagger a}\lambda^a_2)_\alpha + i(\sigma_y\lambda_1^{\dagger a})_\alpha A^\dagger T^a A\,, \end{aligned}$$

where,

$$\vec F^a = F^{a0i}\,, \quad \vec{\widetilde F}^a = \widetilde F^{a0i}\,.$$

As before, the expression for $S^0_{(2)\alpha}$ is obtained from the above by the replacements $\lambda_1 \to \lambda_2$, $\lambda_2 \to -\lambda_1$.

To evaluate the central charge, we are interested in the anti-commutator

$$\{Q_{(1)\alpha}\,,\,Q_{(2)\beta}\} = \left\{\int d^3x\, S^0_{(1)\alpha}(\vec x,0)\,,\,\int d^3y\, S^0_{(2)\beta}(\vec y,0)\right\}\,.$$

As noticed by Olive and Witten [26], a non-zero contribution to this comes from certain boundary terms which measure electric and magnetic charges. To check this result, we have to look, in the anti-commutator, for terms of the form $\int d^3x\partial_i(A^{\dagger a}F^{aoi} + A^{\dagger a}\widetilde F^{aoi})$. For this, we need only retain the relevant terms in the supercurrents:

$$\begin{aligned} g^2 S^0_{(1)\alpha} &= -i(\vec\sigma\sigma_y\lambda_1^{\dagger a})_\alpha\cdot(i\vec F^a + \vec{\widetilde F})^a + \sqrt{2}\,(\vec\sigma\cdot\vec D A^{\dagger a}\lambda^a_2)_\alpha + \cdots\,, \\ g^2 S^0_{(2)\alpha} &= -i(\vec\sigma\sigma_y\lambda_2^{\dagger a})_\alpha\cdot(i\vec F^a + \vec{\widetilde F})^a - \sqrt{2}\,(\vec\sigma\cdot\vec D A^{\dagger a}\lambda^a_1)_\alpha + \cdots\,. \end{aligned} \tag{83}$$

Using this, we get

$$\begin{aligned} \{Q_{(1)\alpha}\,,\,Q_{(2)\beta}\} &= \frac{1}{g^4}\int d^3x\int d^3y\,[i\sqrt{2}(\sigma^i\sigma_y)_{\alpha\gamma}\sigma^j_{\beta\lambda}\{\lambda^{\dagger a}_{1\gamma},\lambda^b_{1\lambda}\}(iF^a_{0i} + \widetilde F^a_{0i})D_j A^{\dagger b} \\ &\qquad -i\sqrt{2}(\sigma^j)_{\alpha\gamma}(\sigma^i\sigma_y)_{\beta\lambda}\{\lambda^{\dagger a}_{2\gamma},\lambda^b_{2\lambda}\}(iF^a_{0i} + \widetilde F^a_{0i})D_j A^{\dagger b}] \\ &= \frac{1}{g^2}\int d^3x\, i\sqrt{2}\left[(\sigma^i\sigma_y\sigma^{j\,T})_{\alpha\beta} - (\sigma^i\sigma_y\sigma^{j\,T})_{\beta\alpha}\right](iF^a_{0i} + \widetilde F^a_{0i})D_j A^{\dagger a}\,. \end{aligned}$$

The term within the square brackets, involving the σ matrices, can be simplified if we use $\sigma^i\sigma_y = -\sigma_y\sigma^{i\,T}$, so that

$$(\sigma^i\sigma_y\sigma^{j\,T})_{\alpha\beta} = -(\sigma_y\sigma^{i\,T}\sigma^{j\,T})_{\alpha\beta} = -[\sigma_y(\delta^{ij} - i\epsilon^{ijk}\sigma^T_k)]_{\alpha\beta}\,.$$

Subtracting from this a similar equation with α and β interchanged, we get the term within the square brackets as equal to $-2(\sigma_y)_{\alpha\beta}\delta^{ij} = 2i\epsilon_{\alpha\beta}\delta^{ij}$. Thus the commutator takes the form

$$\{Q_{(1)\alpha}, Q_{(2)\beta}\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta}\int d^3x(iF^{a0i} + \widetilde F^{a0i})D_i A^{\dagger a}\,.$$

Using the Bianchi identity for the magnetic part, and the equation of motion for the electric part, one can easily show that this is the same as

$$\{Q_{(1)\alpha}, Q_{(2)\beta}\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta}\int d^3x\,\partial_i\left[(iF^{a0i} + \widetilde{F}^{a0i})A^{\dagger a}\right] .$$

Similarly,

$$\{\bar{Q}_{(1)\dot{\alpha}}, \bar{Q}_{(2)\dot{\beta}}\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\dot{\alpha}\dot{\beta}}\int d^3x\,\partial_i\left[(-iF^{a0i} + \widetilde{F}^{a0i})A^{a}\right] .$$

Thus, in the commutator of the supercurrents, we have recovered the total derivatives which are nothing but the electric and magnetic charge densities:

$$Q_{ele} = -\frac{1}{ag}\int d^3x\partial_i(F^{a0i}A^a) = gn_e\,, \qquad Q_{mag} = -\frac{1}{ag}\int d^3x\partial_i(\widetilde{F}^{a0i}A^a) = \frac{4\pi}{g}n_m\,.$$

Here, a is the value of A in the Higgs vacuum. The charge quantization condition used above corresponds to integral fundamantal charges as in $SU(2)$ breaking to $U(1)$ with fileds in the adjoint representation. Now, comparing with (52), we can easily read off the $N=2$ central charge as $Z = -ia(n_e + (4\pi i/g^2)n_m)$. Recall that the phase of Z is convention dependent and we are mainly interested in its magnitude.

To find the effect of the θ-parameter, we can either modify the calculation by adding the contribution from $\theta F\widetilde{F}$, or simply use the Witten effect described in section (1.10). As we learnt there, the effect of the θ parameter is to shift the electric charge to $Q_{ele} = gn_e + (\theta g^2/8\pi^2)Q_{mag}$. Using this (and ignoring an overall factor of $-i$), we get the central charge as

$$Z = a\,(n_e + \tau_{cl}n_m)\,, \quad \text{where,} \quad \tau_{\rm cl} = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{\mathrm{g}^2}\,.$$

Using supersymmetry algebra this implies a mass bound

$$M \geq \sqrt{2}\,|Z| = \sqrt{2}\,|\,a\,(n_e + \tau_{cl}n_m)\,|\,,$$

which is the BPS bound. So far, we have considered the microscopic $N=2$ action (72). At the level of the effective action (74) specified by a prepotential $\mathcal{F}$, the central charge takes the form

$$Z = an_e + a_D n_m\,,$$

where $a_D = \partial\mathcal{F}/\partial a$. This formula can be motivated as follows: As we will see in the next section, the theory (74) has a dual description in which the magnetic monopoles, and not the electric charges, appear as fundamental objects. In this description, n_e and n_m are interchanged and also a is replaced by a_D. Thus, in the dual theory, one can easily infer the contribution of the monopole to the BPS bound to be $a_D n_m$. The full duality group, as we will see in the next section, is $SL(2,Z)$ under which a and a_D transform as a doublet. Combining these facts leads to the central extension formula given above. Similar constructions apply to arbitrary groups $SU(N)$ with adjoint matter.

2.9 Central Charge in N=2 Gauge Theory with Matter

When matter in the fundamental representation is added to the $N = 2$ pure gauge theory described above, there is an important change: the central charge also receives contributions from the masses of the matter fields. In $N = 2$ supersymmetry, matter fields are part of hypermultiplets. A hypermultiplet can be most easily described in the $N = 1$ notation as consisting of a chiral superfield Q and an anti-chiral superfield $\widetilde{Q}^\dagger$. Q ($\widetilde{Q}$) contains components q, ψ_q ($\widetilde{q}, \psi_{\widetilde{q}}$) and transforms in the N ($\bar{N}$) representation of the gauge group $SU(N)$. Under the $SU(2)_R$ symmetry of the $N = 2$ algebra, $q, \widetilde{q}^\dagger$ form a doublet while ψ_q and $\psi^\dagger_{\widetilde{q}}$ are singlets. Since the fields in a hypermultiplet have spin $\leq 1/2$, they belong to a short representation of the $N = 2$ algebra and, therefore, must satify the relation $M = \sqrt{2}Z$. The fact that Z, as derived in the previous section, does not satisfy this relation in the presence of bare hypermultiplet masses, indicates that the central charge must receive further contributions from the hypermultiplets. We will calculate this contribution below.

Beside the standard kinetic terms and gauge couplings for the chiral fields Q and $\widetilde{Q}$, the $N = 2$ Lagrangian in the presence of matter also contains the $N = 1$ superpotential given by

$$\sum_{i=1}^{N_f} \sqrt{2}\widetilde{Q}_i \Phi Q_i + \sum_{i=1}^{N_f} m_i \widetilde{Q}_i Q_i + h.c.$$

Here, the index i runs over the N_f quark flavours and Φ is the chiral superfield of the vector multiplet in the adjoint representation. The first term is related to the gauge coupling of the matter fields by $N = 2$ and the second term is an $N = 2$ invariant mass term. When all masses are equal, the theory also has a $SU(N_f)$ flavour symmetry which is broken down to smaller subgroups when the masses are not equal. For all masses unequal, it breaks down to $U(1)^{N_f}$.

The contribution of Q and $\widetilde{Q}$ to the supercurrents can be easily calculated using (78), with the superpotential $\mathcal{W}$ as given above. Thus, as the contribution to $S^\mu_{(1)\alpha}$, we get (supressing both colour and flavour indices):

$$\begin{aligned}
\epsilon^\alpha S^\mu_{(1)\alpha} = \cdots &+ \sqrt{2}\,\epsilon\sigma^\nu\bar{\sigma}^\mu\psi_q D_\nu q^\dagger + i\sqrt{2}\, m\, \widetilde{q}^\dagger \bar{\psi}_q \bar{\sigma}^\mu \epsilon \\
&+ \sqrt{2}\,\epsilon\sigma^\nu\bar{\sigma}^\mu\psi_{\widetilde{q}} D_\nu \widetilde{q}^\dagger + i\sqrt{2}\, m\, q^\dagger \bar{\psi}_{\widetilde{q}} \bar{\sigma}^\mu \epsilon + \cdots .
\end{aligned}$$

To obtain the corresponding terms in the second supercurrent, we have to make the replacements $q \to \widetilde{q}^\dagger$, $\widetilde{q}^\dagger \to -q$,

$$\begin{aligned}
\epsilon^\alpha S^\mu_{(2)\alpha} = \cdots &+ \sqrt{2}\,\epsilon\sigma^\nu\bar{\sigma}^\mu\psi_q D_\nu \widetilde{q} - i\sqrt{2}\, m\, q \bar{\psi}_q \bar{\sigma}^\mu \epsilon \\
&- \sqrt{2}\,\epsilon\sigma^\nu\bar{\sigma}^\mu\psi_{\widetilde{q}} D_\nu q + i\sqrt{2}\, m\, \widetilde{q} \bar{\psi}_{\widetilde{q}} \bar{\sigma}^\mu \epsilon + \cdots .
\end{aligned}$$

Using the standard canonical commutation relations and following a procedure similar to the previous section, we can calculate the contribution of these extra terms to the anti-commutator of the supercharges. The additional term is of the form $2i\epsilon_{\alpha\beta} \sum_i m_i S_i$, with

$$S_i = \int d^3x \left(D_0 q_i^\dagger q_i + q_i D_0 q_i^\dagger - \frac{i}{2}\psi^\dagger_{qi}\psi_{qi} + \frac{i}{2}\psi_{qi}\psi^\dagger_{qi} - (q \to \widetilde{q}, \psi_q \to \psi_{\widetilde{q}}) \right) ,$$

where, the index i is not summed over. Clearly, this is a conserved charge associated with a global $U(1)$ symmetry under which Q_i and $\widetilde{Q}_i$ carry charges $+1$ and -1,respectively. These are the $U(1)$ factors of the broken flavour group. Taking this extra term into account, the formula for the central charge takes the form

$$Z = n_e a + n_m a_D + \sum_i \frac{1}{\sqrt{2}}\, m_i S_i \,. \tag{84}$$

This formula is crucial in the Seiberg-Witten analysis of $N = 2$ $SU(2)$ gauge theory with quark hypermultiplets.

3 The Seiberg-Witten Analysis of $N = 2$ Supersymmetric Yang-Mills Theory

In this section, we analyze the $N = 2$ pure $SU(2)$ theory following Seiberg and Witten [31]. $N = 2$ has powerful Ward identities which, together with some physical input, lead to interesting conclusions like monopole condensation and - after $N = 2$ is softly broken to $N = 1$ - to confinement due to condensation of the massless monopoles. The chain of reasoning is long and elaborate. For the theories under consideration, when the θ-angle is set to zero, the BPS bound is given by

$$\begin{aligned} M &\geq \sqrt{2}\,|Z|\,, \\ Z &= a\,(n_e + \frac{i}{\alpha} n_m)\,, \quad \text{with} \quad \alpha = \frac{g^2}{4\pi}\,. \end{aligned}$$

The form of Z is invariant under $n_e \leftrightarrow n_m$ accompanied by $\alpha \leftrightarrow 1/\alpha$ and $a \leftrightarrow a/\alpha$. This may be regarded as some evidence for the Montonen-Olive conjecture of electromagnetic duality, although we have seen that Z gets an interesting renormalization in $N = 2$ theories. While the conjecture in its original form makes sense in $N = 4$ theories, for $N = 2$, we can still have an $SL(2, Z)$ transformation acting on the parameter $\tau = \theta/2\pi + i4\pi/g^2$ which embodies a weaker form of Montonen-Olive conjecture. By combining these transformations with global symmetries and the requirement of positivity of kinetic energy, Seiberg and Witten were able to formulate a procedure for determining the exact form of the low-energy theory as will be described below.

3.1 Parametrization of the Moduli Space

The classical action for the $N = 2$ supersymmetric Yang-Mills theory (72) containts the scalar potential

$$V = \frac{1}{2g^2} \mathrm{Tr}\left([\phi^\dagger, \phi]^2 \right) \,.$$

Therefore, the Higgs vacuum is defined by $[\phi, \phi^\dagger] = 0$, which implies that ϕ takes values in the Cartan subalgebra of the gauge group: $\phi = \phi_i H^i$. Thus, generically, the gauge group G is broken to the subgroup H which is generated by elements from the Cartan subalgebra. Elements in G/H do not leave the Higgs vacuum invariant, but being gauge transformations, they relate physically equivalent vacua. On the other hand, once a given basis for the Cartan subalgebra is chosen, then different vacuum values of ϕ, within this subalgebra, correspond to different physical theories. Thus these degrees of freedom in ϕ (*i.e.*, the ϕ_i's) parametrize the space of physically inequivalent vacua, or the moduli space of the theory. The dimension of this moduli space is equal to the rank r of the gauge group G. However, this parametrization of the moduli space is not the desired one as there is still some residual gauge invariance: The coset G/H contains elements which, while not leaving the vacuum invariant, do not take ϕ out of the Cartan subalgebra. These transformations are precisely the Weyl reflections. Therefore, the correct parametrization of the moduli space is given not by ϕ, but by Weyl invariant functions constructed out of it.

The Weyl invariants are obtained from the characteristic equation,

$$\det(\lambda - \phi) = 0\,.$$

Since Weyl reflections act on ϕ by conjugation, $\det(\lambda-\phi)$ is invariant. Hence, if we expand this quantity as a polynomial in powers of λ, then the coefficients are Weyl invariant quantities. In the following, we express these coefficients as simple functions of ϕ. Let $a_1, \cdots, a_N$ denote the roots of the characteristic equation, or the eigenvalues of ϕ. If the gauge group is $SU(N)$ (or $SO(N)$), we have $\mathrm{Tr}\phi = \sum a_i = 0$. For generic values of a_i, the gauge group is broken to $U(1)^r$. When some of the eigenvalues coincide, the unbroken group jumps from $U(1)^r$ to something at least as big as $U(1)^{r-1} \times SU(2)$. In terms of a_i, the characteristic polynomial takes the form:

$$\lambda^N + \lambda^{N-2} \sum_{i<j} a_i a_j - \lambda^{N-3} \sum_{i<j<k} a_i a_j a_k + \cdots + (-1)^N \prod_{i=1}^{N} a_i = 0\,.$$

For $SU(2)$, $\phi = \frac{1}{2} a \sigma_3$ and it can be easily checked that the desired Weyl invariant is $u = \mathrm{Tr}(\phi^2) = \frac{1}{2} a^2$. For $SU(3)$, the coefficients in the characteristic polynomial are $a_1 a_2 + a_1 a_3 + a_2 a_3$ and $a_1 a_2 a_3$, where a_1, a_2, a_3 are the eigenvalues of ϕ. Using $(\mathrm{Tr}\phi)^2 = 0$, and $(\mathrm{Tr}\phi)^3 = 0$, one can easily write the Weyl invariants as

$$\begin{aligned} u &= \tfrac{1}{2}\mathrm{Tr}(\phi^2) = -(a_1 a_2 + a_1 a_3 + a_2 a_3)\,, \\ v &= -\tfrac{1}{3}\mathrm{Tr}(\phi^3) = a_1 a_2 a_3\,. \end{aligned}$$

In general, for $SU(N)$ similar formulae can be worked out. The coefficients of the characteristic polynomial are the "Chern" classes of ϕ,

$$\det(\lambda - \phi) = \lambda^N - \lambda^{N-1} c_1(\phi) + \lambda^{N-2} c_2(\phi) + \cdots + (-1)^j \lambda^{N-j} c_j(\phi) + \cdots + (-1)^N c_N(\phi) = 0\,.$$

The coefficients $c_j(\phi)$ can be easily determined from the following formal expansion,

$$\begin{aligned} \det(\lambda - \phi) &= \lambda^N \det(1 - \phi/\lambda) = \lambda^N e^{\mathrm{Tr}\ln(1-\phi/\lambda)} = \lambda^N \exp\left(-\sum_{n=1}^{\infty} \frac{\mathrm{Tr}\,(\phi^n)}{n\lambda^n} \right) \\ &= \lambda^N - \frac{1}{2}\mathrm{Tr}(\phi^2)\lambda^{N-2} - \frac{1}{3}\mathrm{Tr}(\phi^3)\lambda^{N-3} + \cdots\,. \end{aligned}$$

Note that the series expansion for $\ln(1-\phi/\lambda)$ makes sense only for $\lambda >> \phi$. Therefore, in the above expansion only terms with positive powers of λ are relevant and they provide all the Weyl invariant quantities we need for a gauge invariant parametrization of the moduli space.

In the above, we have treated ϕ classically. In quantum field theory, we parametrize the moduli space by the vacuum expectation values of the corresponding classical Weyl invariants. For example, the moduli space of the $SU(2)$ $N=2$ supersymmetric Yang-Mills theory is parametrized by $u = <\mathrm{Tr}(\phi^2)>$, which at the classical limit, reduces to $a^2/2$.

3.2 R-Symmetry and its Breaking

The N-extended supersymmetry algebra (50) is invariant under global $U(N)$ rotations of the N supercharges. Therefore, a supersymmetric theory should also exhibit such a global symmetry, usually referred to as R-symmetry. The action of this global symmetry on the supercharges can be easily translated into a transformation of the superspace variables. For example, for $N=1$, the $U(1)$ R-symmetry acts on the supercharge as $Q \rightarrow e^{-i\alpha}Q$. From this we can obtain its action on the superspace coordinates as $\theta \rightarrow e^{i\alpha}\theta$ and $\bar{\theta} \rightarrow e^{-i\alpha}\bar{\theta}$ (see (56)). For $N=2$, we have a $U(1)$ which acts as above on the $N=2$ superspace coordinates $\theta^I, \bar{\theta}_I$, along with an $SU(2)$ R-symmetry which rotates the index I of the supercharges. In order to keep the supersymmetric Lagrangian invariant under these transformations, we have to assign appropriate transformation properties to various superfields. From this, we can easily obtain the behaviour of the component fields under R-symmetry. Below, we will describe this in a little more detail.

Action on the $N=2$ Vector Multiplet: As discussed in the previous section, the $N=2$ vector multiplet contains a vector field A_μ, two Weyl spinors λ, ψ and a scalar ϕ, all transforming in the adjoint representation of the gauge group G. These components can be arranged as

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array} .$$

The $SU(2)_R$ transformation acts on the rows in the above diagram and rotates the fermions (λ, ψ) into each other while keeping A_μ and ϕ invariant. In the $N=1$ formalism, this multiplet decomposes into a vector superfield $V(A_\mu, \lambda)$, and a chiral superfield $\Phi(\phi, \psi)$. Therefore, the only part of $SU(2)_R$ which remains manifest in the $N=1$ language is a $U(1)_J$ subgroup which does not mix λ and ψ. This subgroup is generated by σ_3 and acts as $(\lambda, \psi) \rightarrow (e^{i\alpha}\lambda, e^{-i\alpha}\psi)$. The action of the $U(1)_{\mathcal{R}}$ on the $N=1$ superfields was discussed in the previous section. Below, we summarize these transformations:

$$\begin{aligned} U(1)_{\mathcal{R}} \quad &: \Phi \rightarrow e^{2i\alpha}\Phi(e^{-i\alpha}\theta), \quad V \rightarrow V(e^{-i\alpha}\theta)\,, \\ U(1)_J \quad &: \Phi \rightarrow \Phi(e^{-i\alpha}\theta), \qquad\quad V \rightarrow V(e^{-i\alpha}\theta)\,. \end{aligned} \tag{85}$$

Action on the $N=2$ Hypermultiplet: In $N=2$, the matter fields appear in hypermultiplets, each containing two complex scalars $(q, \tilde{q}^\dagger)$ and two Weyl fermions $(\psi_q, \tilde{\psi}_q^\dagger)$. All these components transform in the same (usually the fundamental) representation of the gauge group. The components of a hypermultiplet can be arranged as

$$\begin{array}{ccc} & \psi_q & \\ q & & \widetilde{q}^{\dagger} \\ & \widetilde{\psi}_q^{\dagger} & \end{array} .$$

Again, $SU(2)_R$ acts on the rows and thus rotates $q, \widetilde{q}^{\dagger}$. In the $N=1$ language, the hypermultiplet is decomposed in terms of chiral multiplets $Q(q, \psi_q)$, and $\widetilde{Q}(\widetilde{q}, \widetilde{\psi}_q)$, which carry dual gauge quantum numbers. In this decomposition, only the $U(1)_J$ subgroup of $SU(2)_R$ is manifest. Moreover, in the $N=1$ decomposition, a hypermultiplet interacting with a vector multiplet gives rise to a superpotential term

$$\mathcal{W} = \sqrt{2}\widetilde{Q}\Phi Q.$$

Since $\mathcal{W}$ should carry two units of $U(1)_{\mathcal{R}}$ charge, Q and $\widetilde{Q}$ are neutral. We summarize these transformations below:

$$\begin{aligned} U(1)_{\mathcal{R}}:&\quad Q \to Q(e^{-i\alpha}\theta)\,, \qquad \widetilde{Q} \to \widetilde{Q}(e^{-i\alpha}\theta)\,, \\ U(1)_J:&\quad Q \to e^{i\alpha}Q(e^{-i\alpha}\theta)\,, \quad \widetilde{Q} \to e^{i\alpha}\widetilde{Q}(e^{-i\alpha}\theta)\,. \end{aligned} \tag{86}$$

For later convenience, we list below the transformations of all component fields under $U(1)_{\mathcal{R}}$ and $U(1)_J$:

$$\begin{array}{lccc} U(1)_{\mathcal{R}}: & \phi & \to & e^{2i\alpha}\phi\,, \\ & (\psi, \lambda) & \to & e^{i\alpha}(\psi, \lambda)\,, \\ & (\psi_q, \widetilde{\psi}_q) & \to & e^{-i\alpha}(\psi_q, \widetilde{\psi}_q)\,, \\ & (A_\mu, q, \widetilde{q}) & \to & (A_\mu, q, \widetilde{q})\,. \\ & & & \\ U(1)_J: & (\lambda, q) & \to & e^{i\alpha}(\lambda, q)\,, \\ & (\psi, \widetilde{q}^{\dagger}) & \to & e^{-i\alpha}(\psi, \widetilde{q}^{\dagger})\,, \\ & (\phi, A_\mu, \psi_q, \widetilde{\psi}_q^{\dagger}) & \to & (\phi, A_\mu, \psi_q, \widetilde{\psi}_q^{\dagger})\,. \end{array}$$

Note that we can combine the two-component spinors λ and $\bar{\psi}$ into a four-component Dirac spinor ψ_D (see (49)). The spinor ψ_D transforms as $e^{i\alpha}\psi_D$ under $U(1)_J$ and as $e^{i\alpha\gamma_5}\psi_D$ under $U(1)_{\mathcal{R}}$. Similarly, $U(1)_{\mathcal{R}}$ acts as a chiral $U(1)$ on the Dirac spinor constructed out of ψ_q and $\widetilde{\psi}_q^{\dagger}$, though now with the opposite charge. Thus $U(1)_{\mathcal{R}}$ is a chiral symmetry and is, therefore, broken by a chiral anomaly as will be discussed below.

Breaking of R-Symmetries: Classically, our theory has the full global $SU(2)_R \times U(1)_{\mathcal{R}}$ as a symmetry group. However, quantum mechanically, $U(1)_{\mathcal{R}}$ is broken to a discrete subgroup due to anomalies. This subgroup can be easily determined from elementary instanton considerations (the more direct method will be described in the next subsection). To compute the anomaly for the gauge group $SU(N_c)$, note that by the index theorem, in the presence of an instanton, there is one zero-mode for each left moving fermion in the fundamental or antifundamental representaion and $2N_c$ zero-modes for each left-handed fermion in the adjoint representation. A correlation function in this theory involves integrations over the fermionic collective coordinates corresponding to these zero-modes. For a correlator to be non-zero, it should contain enough fermion insertions to soak the zero-modes. Hence, in the presence of N_f flavours, the first non-vanishing correlator is

$$G = \langle \lambda(x_1) \cdots \lambda(x_{2N_c}) \psi(y_1) \cdots \psi(y_{2N_c}) \psi_q(z_1) \cdots \psi_q(z_{N_f}) \widetilde{\psi}_q(u_1) \cdots \widetilde{\psi}_q(u_{N_f}) \rangle \, . \tag{87}$$

Under the $U(1)_{\mathcal{R}}$, G transforms as

$$G \to e^{i\alpha(4N_c - 2N_f)} \, G \, .$$

Hence $U(1)_{\mathcal{R}}$ is broken to the discrete group $\mathbf{Z}_{4N_c - 2N_f}$. In the following, we focus on the pure Yang-Mills theory so that $N_f = 0$. In this case, $U(1)_{\mathcal{R}} \to \mathbf{Z}_{4N_c}$ and is represented by $e^{2\pi i\alpha}$, where $\alpha = n/4N_c \, , n = 1, \cdots 4N_c$. Thus the global symmetry group is $SU(2)_R \times \mathbf{Z}_{4N_c}$. However, note that the centre of $SU(2)_R$, which acts as $(\lambda, \psi) \to e^{i\pi}(\lambda, \psi)$, is also contained in $\mathbf{Z}_{4N_c}$ (corresponding to $n = 2N_c$). Hence, the true symmetry group is

$$\left(SU(2)_R \times \mathbf{Z}_{4N_c} \right) / \mathbf{Z}_2 \, .$$

This surviving R-symmetry is broken further by the Higgs vacuum. The field ϕ^2 has charge 4 under $\mathbf{Z}_{4N_c}$ and transforms to $e^{2\pi i n/N_c} \phi^2$, which is invariant only for $n = N_c, 2N_c, 3N_c, 4N_c$. Therefore, if the vacuum is characterized by non-zero ϕ^2, then $\mathbf{Z}_{4N_c}$ is broken down to $\mathbf{Z}_4$. This is the situation for the $SU(2)$ gauge theory ($N_c = 2$). In this case, all elements which do not keep ϕ^2 invariant, act as a $\mathbf{Z_2} : \phi^2 \to -\phi^2$. Therefore, the final R-symmetry group for the $SU(2)$ gauge theory is $(SU(2)_R \times \mathbf{Z_4})/\mathbf{Z_2}$.

For the gauge group $SU(3)$, we have $\mathbf{Z_{4N_c}} = \mathbf{Z_{12}}$. The vacuum is parametrized by ϕ^2 and ϕ^3, which, under $\mathbf{Z}_{4N_c}$, transform to $e^{2\pi i n/N_c} \phi^2$ and $e^{3\pi i n/N_c} \phi^3$ respectively. Invariance of ϕ^2 requires $n = N_c, 2N_c, 3N_c, 4N_c$, breaking $\mathbf{Z_{12}}$ to $\mathbf{Z_4}$. Invariance of ϕ^3 picks $n = 2N_c, 4N_c$. Thus, for the $SU(3)$ gauge theory, $\mathbf{Z}_{12}$ is broken to $\mathbf{Z}_2$. The broken $\mathbf{Z}_6$ subgroup acts non-tivially on the moduli space. For $SU(4)$ and higher gauge groups, no subgroup of $\mathbf{Z}_{4N_c}$ survives.

3.3 Low-Energy Effective Action for $N = 2$ Gauge Theory

Let us consider the $N = 2$ supersymmetric Yang Mills theory based on a group G of rank r which is spontaneously broken by a non-zero $\langle \phi \rangle$. Far from the points where two eigenvalues of $\langle \phi \rangle$ coincide, the only massless fields are the vector supermultiplets associated with the unbroken subgroup $U(1)^r$ of G. At sufficiently low energies, non of the massive states will appear as physical states and an effective description of the theory can be given in terms of the massless fields alone. In principle, this low-energy theory can be obtained by integrating out all the massive modes as well as massless modes above a low-energy cutoff. In practice, this procedure is not easy to implement. Seiberg and Witten formulated an indirect procedure for determining the exact low-energy theory as we will in the remaining part of this section.

As discussed in the previous section, the effective action (the part with at most two derivatives) is fully determined by a prepotential $\mathcal{F}$ which is only a function of r massless vector supermultiplets. In the $N = 1$ language, the corresponding Lagrangian takes the form (with A_i denoting chiral superfields):

$$\frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\,\frac{\partial\mathcal{F}}{\partial A^i}\bar{A}^i+\int d^2\theta\,\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A_i\partial A_j}W^{\alpha i}W^j_\alpha\right].$$

In terms of $\mathcal{F}(A)$, the Kähler potential is given by $K = \mathrm{Im}\left(\bar{A}_i\partial\mathcal{F}(A)/\partial A_i\right)$. If a_i denotes the scalar component of the chiral superfield A_i, then the metric on the space of fields, and therefore, the one on the space of Higgs vacuua, is given by

$$ds^2 = g_{i\bar{j}}\,da^i d\bar{a}^j = \mathrm{Im}\,\frac{d^2\mathcal{F}}{\partial a_i\partial a_j}\,da^i d\bar{a}^j\,.$$

Note that, by virtue of $N = 2$ supersymmetry, this metric is the same as the generalized gauge coupling which appears in front of the $F^i_{\mu\nu}F^{\mu\nu i}$ term in the above Lagrangian. This relationship is not modified in the quantum theory.

In the following, we will be mainly interested in the $SU(2)$ gauge group spontaneously broken to $U(1)$. In this case the effective low-energy Lagrangian takes the form

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\,\frac{\partial\mathcal{F}(A)}{\partial A}\bar{A}+\int d^2\theta\,\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A^2}W^\alpha W_\alpha\right], \tag{88}$$

and the metric on the field space is given by

$$ds^2 = \mathrm{Im}(\tau)\,da d\bar{a}\,, \quad \text{where,} \quad \tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2}. \tag{89}$$

Our aim in this subsection is to determine the perturbative form of $\mathcal{F}$ following [33]. The determination of the exact form of the prepotential, including non-perturbative corrections, is the subject of the later subsections.

To determine the one-loop perturbative correction to $\mathcal{F}$, we start from the exact microscopic theory (72) with gauge group $SU(2)$. This theory is asymptotically free and therefore at high energies one can perform reliable perturbative calculations. As indicated in the previous subsection, the $U(1)_{\mathcal{R}}$ symmetry of this theory is broken by the standard chiral anomaly. Thus we have

$$\partial_\mu J^\mu_5 = -\frac{N_c}{8\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\,.$$

This implies that, to one-loop, under a $U(1)_{\mathcal{R}}$ transformation, the effective Lagrangian changes by

$$\delta\mathcal{L}_{eff} = -\frac{\alpha N_c}{8\pi^2}F\widetilde{F}\,. \tag{90}$$

If the theory also contains N_f fermions in the fundamental representation, then in the above expression, N_c has to be replaced by $N_c - N_f/2$. Note that since $(32\pi^2)^{-1}\int F\widetilde{F}$ is an integer, the anomaly breaks $U(1)_{\mathcal{R}}$ to $\mathbf{Z}_{4N_c}$ (or to $\mathbf{Z}_{4N_c-2N_f}$ in the presence of matter). The same result was obtained in the previous subsection from a consideration of fermion zero modes in an instanton background. Moreover, the anomaly can be regarded as causing a shift in the θ-angle. In the following, we use this freedom to set θ to zero by an appropriate chiral rotation of the fermions.

The one-loop form of $\mathcal{F}$ can be determined from the requirement that under a $U(1)_{\mathcal{R}}$ transformation, the low-energy effective action (88) change as in (90). The variation $\delta\mathcal{L}_{eff}$ could come only from terms in $\mathcal{L}_{eff}$ which are quadratic in $F_{\mu\nu}$. Writing only the relevent terms from the Lagrangian (88), this means

$$\frac{1}{16\pi}\mathrm{Im}\left[\mathcal{F}''(e^{2i\alpha}A)(-FF+iF\widetilde{F})\right]=\frac{1}{16\pi}\mathrm{Im}\left[\mathcal{F}''(A)(-FF+iF\widetilde{F})\right]-\frac{\alpha N_c}{8\pi^2}F\widetilde{F}\,.$$

The form of the prepotential is therefore restricted by

$$\mathcal{F}''(e^{2i\alpha}A)=\mathcal{F}''(A)-\frac{2\alpha N_c}{\pi}\,,$$

or, for infinitesimal α, by

$$\frac{\partial^3\mathcal{F}}{\partial A^3}=\frac{N_c}{\pi}\frac{i}{A}\,.$$

The above equation can be easily integrated to give the one-loop expression for the prepotential as

$$\mathcal{F}_{1-loop}(A)=\frac{i}{2\pi}A^2\ln\frac{A^2}{\Lambda^2}\,. \tag{91}$$

Here, Λ is a fixed dynamically generated scale like Λ_{QCD}. It is known that, due to $N=2$ supersymmetry, the above one-loop expression for the prepotential does not receive higher order perturbative corrections. This is related to the fact that in this theory the perturbative β-function is only a one-loop effect. Thus, in this theory, the one-loop nature of the perturbative β-function is consistent with the well known one-loop nature of the anomaly. This is not the case in $N=1$ theories where the anomaly is, of course, still a one-loop effect but not the β-function (see, for example, [36, 37]).

Although, the prepotential, as given in (91), is exact in perturbation theory, it does receive non-perturbative corrections due to instanton effects as argued in [33]. The general form of these corrections can be fixed as follows: First, it is clear that a correction to $\mathcal{F}$ coming from a configuration of instanton number k should be proportional to the k-instanton factor $exp(-8\pi^2k/g^2)$ (since the prepotential is a holomorphic function, it cannot receive corrections from anti-instanton configurations). Using the one-loop β-function of the theory given by $\beta(g)=-g^3/4\pi^2$, the k-instanton factor can be written as

$$e^{-8\pi^2k/g^2}=\left(\frac{\Lambda}{a}\right)^{4k}\,.$$

Furthermore, following the approach of Seiberg in [38, 39], we note that the broken $U(1)_{\mathcal{R}}$ symmetry is restored if we assign a charge of 2 to Λ. With this modification, the prepotential should transform under $U(1)_{\mathcal{R}}$ as a field of charge 4, without a non-homogeneous term. This implies that the k-instanton correction should also be proportional to A^2. Putting these together, the prepotential including generic non-perturbative corrections can be written as

$$\mathcal{F}=\frac{i}{2\pi}A^2\ln\frac{A^2}{\Lambda^2}+\sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{A}\right)^{4k}A^2\,.$$

The coefficients $\mathcal{F}_k$ are not field dependent due to the fact that in a supersymmetric theory, instantons contribute to the path integral only through zero-modes[34]. The coefficient $\mathcal{F}_1$ was calculated in [33] and found to be non-zero. The determination of the exact form of $\mathcal{F}$ is the subject of the work of Seiberg and Witten.

The one-loop expression for $\mathcal{F}$ can also be obtained from integrating the expression for the β-function, choosing A as one of the integration limits: In the classical low-energy theory, which is obtained by simply dropping the massive fields from the microscopic Lagrangian (and not integrating over them), the prepotential is given by $\mathcal{F}(A) = \frac{1}{2}\tau_{cl}A^2$, where, $\tau_{cl} = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. In the quantum theory, for large a (which is the *vev* of the scalar somponent of A), asymptotic freedom takes over and the theory is weakly coupled. Therefore, in this limit, a good approximation to the quantum behaviour of the theory can be obtained by replacing g by the corresponding running couplings $g(a)$ (we have set θ to zero by a chiral redefinition of the fermion fields). In the limit of large a, $g(a)$ and hence $\mathcal{F}(a)$ can be obtained in perturbation theory by integrating the β-function.

For other gauge groups we note that for every root α, $(\alpha > 0)$, we have massive fields $W^{\alpha}, W^{-\alpha}$. For simply-laced groups, $\alpha^2 = 2$, and

$$\mathcal{F} = \frac{i}{4\pi} \sum_{\alpha>0} (\vec{\alpha}\cdot\vec{a})^2 \ln \frac{(\vec{\alpha}\cdot\vec{a})^2}{\Lambda^2} \, .$$

The coefficient of the logarithm follows from the one-loop β-function and it also leads to the unbroken global $U(1)_R$-symmetry.

The one-loop expression for $\mathcal{F}$, coupled with the fact that a well defined theory should have a positive kinetic energy term, leads to very interesting consequences. For large $|a|$, using (91), we can calculate $\tau(a) = \frac{i}{\pi}(\ln\frac{a^2}{\Lambda^2} + 3)$. This is a multi-valued function, though the metric on the field space given by $\mathrm{Im}\tau$ is single valued. However, since $\mathrm{Im}\tau(a)$ is a harmonic function, it cannot have a global minimum. Thus, if it is globally defined, it cannot be positive everywhere. Therefore, the positivity of the kinetic energy requires that $\tau(a)$ is defined only locally. This means that in the region of the complex plane where $\tau(a)$ becomes negative, one needs a different description of the theory. In the next subsection we describe how these equivalent descriptions could be obtained.

3.4 Duality

To find the duality transformations which relate different descriptions of the same theory, we consider the gauge field terms in the bosonic part of the action. Working in the Minkowski space with conventions $(F_{\mu\nu})^2 = -(\widetilde{F}_{\mu\nu})^2$ and $\widetilde{\widetilde{F}} = -F$, these terms can be written as

$$\frac{1}{32\pi}\,\mathrm{Im}\int \tau(a)(F + i\widetilde{F})^2 = \frac{1}{16\pi}\,\mathrm{Im}\int \tau(a)(F^2 + i\widetilde{F}F)\, .$$

Now we regard F as an independent field and implement the Bianchi identity $dF = 0$ by introducing a Lagrange multiplier vector field V_D. To fix the Lagrange multiplier term, $U(1) \subset$

$SU(2)$ is normalized such that all $SO(3)$ fields have integer charges. Then, as discussed in section 1, all matter fields in the fundamental representation of $SU(2)$ will have half-integer charges. With this convention, a magnetic monopole satisfies $\epsilon^{0\mu\nu\rho}\partial_\mu F_{\nu\rho} = 8\pi\delta^{(3)}(x)$. The Lagrange multiplier term can now be constructed by coupling V_D to a monopole:

$$\frac{1}{8\pi}\int V_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \frac{1}{8\pi}\int \widetilde{F}_D F = \frac{1}{16\pi}\,\mathrm{Re}\int(\widetilde{F}_D - iF_D)(F + i\widetilde{F})\,,$$

where, $F_{D\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$. Adding this to the gauge field action and integrating over F, gives the dual theory

$$\frac{1}{32\pi}\,\mathrm{Im}\int\left(-\frac{1}{\tau}\right)(F_D + i\widetilde{F}_D)^2 = \frac{1}{16\pi}\,\mathrm{Im}\int\left(-\frac{1}{\tau}\right)(F_D^2 + i\widetilde{F}_D F_D)\,.$$

The dualization can also be performed in an $N=1$ supersymmetric language. In this case, the relevant term in the action is

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta\tau(A)W^2\,.$$

The Bianchi identity is replaced by $\mathrm{Im}\mathcal{D}W = 0$. This can be implemented by introducing a vector superfield V_D and the corresponding Lagrange multiplier term becomes

$$\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^4\theta\, V_D\mathcal{D}W = \frac{1}{4\pi}\,\mathrm{Re}\int d^4x d^4\theta\, i\mathcal{D}V_D W = -\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^2\theta\, W_D W\,.$$

Adding this to the action and integrating out W, gives the dual action

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta\left(-\frac{1}{\tau(A)}W_D^2\right)\,.$$

Thus, the effect of the duality transformation is to replace a gauge field which couples to electric charges by a dual gauge field which couples to magnetic charges, and at the same time, transform the gauge coupling as

$$\tau \longrightarrow \tau_D = -\frac{1}{\tau}\,. \tag{92}$$

We recognize this as the electric-magnetic duality of section 1. Also, note that the action is invariant under the replacement $\tau \longrightarrow \tau + 1$. This transformation, along with the one in (92), generates the $SL(2,\mathrm{Z})$ group which, therefore, is the full duality group of our theory. This group acts on τ as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \tag{93}$$

where, $ad - bc = 1$ and $a, b, c, d \in \mathrm{Z}$.

$N=2$ supersymmetry relates the dual description of the gauge fields to a dual description for the scalar fields. To see this, let us introduce $h(A) = \partial\mathcal{F}/\partial A$. In terms of this, $\tau(A) = \partial h(A)/\partial A$ and the scalar kinetic energy term becomes $\mathrm{Im}\int d^4\theta\, h(A)\bar{A}$. For the dual theory corresponding to (92), let us introduce the corresponding dual variables $A_D, \mathcal{F}_D, h_D(A_D)$ and τ_D. Then, equation (92) implies that $A_D = h = \partial\mathcal{F}/\partial A$ and $h_D = -A$. Under this transformation, the scalar kinetic energy term retains its form,

$$\mathrm{Im}\int d^4\theta\, h(A)\bar{A} = \mathrm{Im}\int d^4\theta\, h_D(A_D)\bar{A}_D\,.$$

In the following, we use the notations A_D and $h(A)$ interchangeably.

We now consider the effect of the full $SL(2,Z)$ group on A and $\mathcal{F}$, or equivalently on A and $A_D = \partial\mathcal{F}/\partial A$. The transformation (93) implies that

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix} . \tag{94}$$

Note that, in general, we could also add a constant column matrix C to the right-hand side of the above equation. However, for non-zero C, the BPS mass formula for the theory in the absence of matter is not invariant under the above transformation. Thus in this case we should set $C = 0$. However, in the presence of matter fields, the BPS mass formula is modified and a non-zero C is allowed. This case will be discussed in the last section. On the space of the scalar fields, the transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ are implemented by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{95}$$

These matrices generate the $SL(2,\mathrm{Z})$ group.

The transformation of $\mathcal{F}$ can be easily obtained from (94), or equivanlently from

$$\begin{aligned} A'_D &= aA_D + bA\,, \\ A' &= cA_D + dA\,. \end{aligned}$$

The first equation can be integrated with respect to A' by using the second equation and the result is

$$\mathcal{F}' = \frac{1}{2}\beta\delta A^2 + \frac{1}{2}\alpha\gamma A_D^2 + \beta\gamma A A_D + \mathcal{F}\,.$$

Let us now come back to the metric on the moduli space as given by (89). In terms of the variable a_D, this takes the form

$$ds^2 = \mathrm{Im}\, da_D d\bar{a} = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a}_D)\,,$$

which is invariant under the $SL(2,\mathrm{Z})$ transformations. To make the description more precise, note that the moduli space $\mathcal{M}$ is a complex one dimensional manifold and let u be a holomorphic coordinate on this manifold. Finally, we will identify u as $\langle \mathrm{Tr}\phi^2\rangle$. Let a and a_D be the coordinates on a space $X \cong \mathrm{C}^2$ on which we can choose a symplectic form $\omega = \mathrm{Im}\, da_D \wedge d\bar{a}$. The functions $(a_D(u), a(u))$ give a map f from $\mathcal{M}$ to X. In other words, they determine a section of X regarded as an $SL(2,\mathrm{Z})$ bundle over $\mathcal{M}$. In terms of the coordinate u, the metric on the moduli space takes the form

$$ds^2 = \mathrm{Im}\, \frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}}\, du d\bar{u} = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}}\frac{da}{du}\right) du d\bar{u}\,.$$

This is the pull-back of the Kähler metric associated with the symplectic form ω and is, therefore, manifestly $SL(2,\mathrm{Z})$ invariant. Choosing $u = a$, we get back to the original formula. Note that for arbitrary $(a_D(u), a(u))$, the metric is not positive. However, the solution we will obtain later determines these functions such that the metric is always positive.

Higher Dimensions: For a group of rank r, the metric takes the form

$$ds^2 = \mathrm{Im}\,\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}\, da_i d\bar{a}_j \,.$$

Introduce a space $X \simeq \mathbf{C}^{2r}$ with coordinates $a^i, a_{Dj} = \partial\mathcal{F}/\partial a^j$. Endow X with the symplectic form $\omega = \frac{i}{2}\sum_i (da^i \wedge d\bar{a}_{Di} - da_{Di} \wedge d\bar{a}^i)$ and the holomorphic 2-form $\omega_h = \sum_i da^i \wedge da_{Di}$. Choose u^s $(s = 1, ..., r)$ as coordinates on the moduli space $\mathcal{M}$. Then, $a^i(u), a_{Dj}(u)$ give a map f from $\mathcal{M}$ to X such that $f^*(\omega_h) = 0$ (this condition ensures that $a_{Dj} = \partial\mathcal{F}/\partial a_j$). Thus, the metric takes the form

$$ds^2 = \mathrm{Im}\,\frac{\partial a_{Di}}{\partial u^r}\,\frac{\partial \bar{a}^i}{\partial \bar{u}^s}\, du^r d\bar{u}^s \,.$$

This is the metric associated with $f^*(\omega)$ and is therefore invariant under $Sp(2r, \mathrm{R})$: If we write $v^T = (a_D, a)$, then the metric is invariant under $v \to Mv + c$, where, $M \in Sp(2r, \mathbf{R})$. In the absence of matter, $c = 0$ and moreover, only the $Sp(2r, \mathbf{Z})$ part survives.

If the moduli space $\mathcal{M}$ has a non-trivial structure, then, on being taken around a close loop, the vector $v^T = (a_D, a)$ will get transformed by an element of the monodromy group. As will be shown later, the monodromy group is a subgroup of $SL(2, \mathrm{Z})$, and its determination is essential for solving the problem.

3.5 Dyon Masses

As described in sections 1 and 2, for the microspcopic $SU(2)$ theory, the BPS bound is given by $M \geq \sqrt{2}|Z|$, where,

$$Z = a(n_e + \tau_{cl}\, n_m)\,,$$

and $\tau_{cl} = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$. All states that saturate this bound fall in a short multiplet of the $N = 2$ algebra. In a more general $N = 2$ theory, like the low-energy effective action we have been studying, this formula is slightly modified. Suppose, the theory contains matter fields in hypermultiplets. When $a \neq 0$, these fields, which in the $N = 1$ formalism are described by chiral fields $M, \widetilde{M}$, become massive. If a hypermultiplet carries charge n_e, then its coupling to the chiral field A is uniquely fixed by $N = 2$ supersymmetry as

$$\sqrt{2}\, n_e\, AM\widetilde{M}\,.$$

From this we can easily see that for such a state, $Z = a\, n_e$. On the other hand, if we consider a magnetic monopole carrying n_m units of magnetic charge, then a manipulation very similar to the one in subsection 1.6 leads to the BPS bound $\sqrt{2}|n_m\, a_D|$ or, equivalently, $Z = n_m\, a_D$. In general, one can compute this formula for dyons and obtain

$$Z = an_e + a_D n_m\,. \tag{96}$$

If the gauge group is of rank r, then we have

$$Z = a^i n_{e,i} + h_i(a) n_m^i = a \cdot n_e + a_D \cdot n_m \,.$$

Since masses are physically observable, the mass formula should be invariant under the monodromies on the moduli space. Therefore, if $v = (a_D, a)^T$ transforms to Mv (with $M \in Sp(2r, \mathrm{Z})$), then the vector $w = (n_m, n_e)$ should transform to wM^{-1}. Note that this once again requires M to be an integral matrix.

Under the action of the monodromy, the coupling matrix $\tau_{ij} = \partial a_D i/\partial a_j$ transforms to $(A\tau + B)(c\tau + D)^{-1}$, where, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. This is very similar to the transformation of the period matrix of a genus r Riemann surface under the action of the monodromy group on the moduli space of genus r surfaces. This makes it reasonably appealing to identify our r-dimensional moduli space of vacua $\mathcal{M}$ with the moduli space of genus r Riemann surfaces. The variables a_i, a_{Dj} are then related to the periods of these surfaces and can be calculated. Moreover, $\mathrm{Im}\,\tau$ is always positive definite, which resolves the problem of geting a negative kinetic term. To check this hypothesis for our $SU(2)$ theory, we first have to determine the monodromy structure on the moduli space $\mathcal{M}$. This is the subject of the next subsection.

3.6 Monodromies on the Moduli Space of the $SU(2)$ Theory:

We have discussed the possibility of the existence of monodromies on the moduli space and pointed out that understanding the monodromy structure may help us solve the theory, *i.e.*, to determine the exact non-perturbative prepotential for the effective low-energy theory. This is equivalent to determining the functions $a_D(u)$ and $a(u)$. In this subsection, we set out to identify the singularities on the moduli space $\mathcal{M}$ and calculate the associalted monodromies.

Monodromy at large u: At large $|a|$, the theory is asymptotically free and $u = \frac{1}{2}a^2$. To a good approximation, the prepotential is given by the one-loop formula $\mathcal{F}(a) = (i/2\pi)a^2 \ln(a^2/\Lambda^2)$, from which we obtain

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln\left(\frac{a}{\Lambda}\right) + \frac{ia}{\pi}\,.$$

If we make a close loop on the u-plane around $u = 0$, we get $\ln u \to \ln u + 2\pi i$, and $\ln a \to \ln a + i\pi$, hence,

$$\begin{aligned} a_D &\to -a_D + 2a\,, \\ a &\to -a\,. \end{aligned}$$

This is implemented by the monodromy matrix

$$M_\infty = PT^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{97}$$

acting on $(a_D, a)^T$. Here, P is the negative of the identity element in $SL(2, \mathrm{Z})$ and T is as defined in (95). Under the action of this monodromy, the magnetic and electric quantum numbers of BPS states change as $(n_m, n_e) \to (-n_m, -n_e - 2n_m)$ so that the mass formula is unchanged.

Monodromies at finite u: The monodromy at $u = \infty$ implies that there exist other monodromies somewhere else on the u-plane. If these monodromies commute with M_∞, then a^2 is a good global coordinate. However, this cannot be the case as then the positivity of the kinetic energy is violated. To get a non-abelian monodromy group, we need at least two singularities on the u-plane, and at finite u, with non-trivial monodromies around them. These singularities will be related by the broken discrete symmetry $u \to -u$. We make this minimal assumption on the number of extra singularities. Then a loop enclosing both these singularities should reproduce the monodromy M_∞

What is the origin of these singularities on the moduli space? To obtain the Wilsonian low-energy effective action, we have integrated out all massive states in the theory. The massive particle loops then give rise to a non-trivial metric on the moduli space. The values of the masses, however, depend on the modulus u, and it may so happen that for certain values of u, some of the masses become zero. Then, at these points, we end up integrating out massless states and thus create singularities on the moduli space. The nature of a singularity, *i.e.,* the monodromy associated with it, depends on the properties of the particle which becomes massless at the singularity. Since, for finite u (the non-perturbative regime), the mass spectrum as a function of u is not known, the singularities cannot be found in a straightforward way. The way to proceed is to assume that some generic states become massless at certain values of u (say $u = 1$ and $u = -1$) and find the corresponding monodromies (say M_1 and M_{-1}). The massless states are then specified by the condition $M_1 M_{-1} = M_\infty$.

Naively, one may expect that the massive gauge boson multiplets contribute to the singularity when they become massless. Classically, this happens at the point $u = 0$, which, in quantum theory, may get shifted to a non-zero u. However, as argued in [31], a spin-1 multiplet becoming massless does not lead to a consistent picture. We will not repeat this argument here.

The only other massive states in the theory with spin $\leq 1/2$ are monopoles and dyons, which, due to the spin condition, belong to short $N = 2$ multiplets and, therefore, are BPS states. These states are described by hypermultiplets. Seiberg and Witten suggested that the singularities arise when some of these non-perturbative states become massless. The problem now is to calculate the associated monodromies. Note that the hypermultiplets for monopoles and dyons cannot be coupled to the fundamental fields of our theory in a local way. However, in the subsection on duality it was seen that it is possible to go to a dual description of the theory in which some dual gauge fields couple to the monopoles or dyons in exactly the same way that the usual gauge fields couple to a particle of unit electric charge. Thus we only have to calculate the monodromy when a massive electrically charged hypermultiplet becomes massless and then, using the duality transformation, find the monodromy for a generic monopole or dyon. Let us consider a dual description of the theory in which a certain monopole or dyon appears as an elementary state, and let us label this description by a letter, say q. Near the point where this state is massless, all massive fields can be integrated out and the theory is essentially a $U(1)$ theory coupled to a hypermultiplet. If we denote the *vev* of the scalar field in this description of the theory by $a(q)$, then the mass of a BPS state of unit electric charge goes to zero when $a(q) = 0$ at some $u = u_q$. Thus, near this point, $a(q)$ is a good local coordinate and can be expanded as $a(q) \approx c_q\,(u - u_q)$. Moreover, near this point, the one-loop $U(1)$ β-function implies (see the next subsection):

$$\tau(a(q)) = -\frac{i}{\pi}\ln\frac{a(q)}{\Lambda},$$

from which we obtain

$$a_D(q) = -\frac{i}{\pi}\,a(q)\ln\frac{a(q)}{\Lambda} + \frac{i}{\pi}.$$

Moving on a closed loop around u_q so that $(u-u_q)\to e^{2\pi i}(u-u_q)$, we get the monodromy

$$\begin{array}{rcl} a_D(q) & \to & a_D(q) + 2a(q)\,, \\ a(q) & \to & a(q)\,. \end{array} \tag{98}$$

Let us now calculate the monodromy when a (n_m, n_e) dyon becomes massless (the dyon is stable or marginally stable if n_m and n_e are coprime). The first step is to find a dual description of the theory in which this dyon appears as an elementary stat of charge $(0,1)$. Under a generic $SL(2,\mathrm{Z})$ transformation we get a $(n_m(q), n_e(q))$ dyon with

$$\left(\begin{array}{c} a_D(q) \\ a(q) \end{array}\right) = \left(\begin{array}{c} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{array}\right), \qquad \left(\begin{array}{c} n_m(q) \\ n_e(q) \end{array}\right) = \left(\begin{array}{c} n_m\delta - n_e\gamma \\ -n_m\beta + n_e\alpha \end{array}\right), \tag{99}$$

so that $Z = n_m a_D + n_e a$ is invariant. In the above, $\alpha\delta - \beta\gamma = 1$. Now we choose the parameters such that $n_m(q)=0, n_e(q)=1$. With this choice, $(a_D(q), a(q))$ become the variables in terms of which the dyon couples to the $SL(2,\mathrm{Z})$ transformed gauge field in the same way that the unit electric charge couples to usual gauge fields. In particular, when the dyon becomes massless at some point on the moduli space, the associated monodromy, in this description, is given by (98). Inverting the first equation in (99), we get $a_D = -\beta a(q) + n_e a_D(q)$ and $a = \alpha a(q) - n_m a_D(q)$. Thus we can easily find the action of the monodromy on the original variables as

$$\left(\begin{array}{c} a_D \\ a \end{array}\right) \to \left(\begin{array}{cc} 1+2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1-2n_e n_m \end{array}\right)\left(\begin{array}{c} a_D \\ a \end{array}\right). \tag{100}$$

Denoting the monodromy matrix as $M(n_m, n_e)$, we note that $\mathrm{Tr}\,M(n_m,n_e) = 2$. Thus the monodromy always belongs to the parabolic subgroup of $SL(2,\mathrm{Z})$.

Now we calculate the monodromies at $u=\pm 1$. Let us assume that a (m,n) dyon becomes massless at $u=1$ and a (m',n') dyon becomes massless at $u=-1$. The the associated monodromies should satisfy

$$M_1(m,n)\,M_{-1}(m',n') = M(\infty)\,. \tag{101}$$

Using (97) and (100), this can be written as

$$\left(\begin{array}{cc} 1+2mn & 2n^2 \\ -2m^2 & 1-2mn \end{array}\right) = \left(\begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array}\right)\left(\begin{array}{cc} 1-2m'n' & -2n'^2 \\ 2m'^2 & 1+2m'n' \end{array}\right),$$

leading to the equations

$$\begin{array}{rcl} 1+mn & = & m'n' + 2m'^2\,, \\ m^2 & = & m'^2\,, \\ n^2 & = & n'^2 + 1 + 2m'n'\,, \\ 1-mn & = & -m'n'\,. \end{array}$$

These imply that $m = \pm 1$ and $m' = \pm 1$. For each combination of (m, m'), we can easily determine n' in terms of n and get the following possible sets of solutions

$$\begin{array}{lccccccc} (m,n): & (1,n) & , & (-1,n) & , & (-1,n) & , & (1,n)\,, \\ (m',n'): & (1,n-1) & , & (1,-n-1) & , & (-1,n+1) & , & (-1,-n+1)\,. \end{array}$$

There do not seem to be any solutions where M_∞ is factorized into a product of more than two such parabolic $M(m,n)$. Moreover, note that the solution allows only dyons of unit magnetic charge to contribute to the monodromy. This is consistent with the result that, semiclassically, only these dyons are stable.

Let us consider another consistency check: In general, under the action of the monodromy, the quantum numbers of dyons will change. However, we expect that the particular dyon which becomes massless and is the source of the singularity should remain invariant under the monodromy (as it is the properties of this dyon which determine the monodromy matrix). This can be easily checked. The eigenvalue equation

$$(q_m, q_e) \begin{pmatrix} 1+2mn & 2n^2 \\ -2m^2 & 1-2mn \end{pmatrix} = (q_m, q_e)\,, \tag{102}$$

leads to $nq_m - mq_e = 0$, provided m and n do not vanish simultaneously. Clearly, $q_m = m, q_n = n$ is a solution of this equation. If we restrict ourselves to stable dyons, then this is the only solution. Thus, knowing the monodromy, we can find the dyon which gives rise to it.

The simplest solution to the equation (101) corresponds to $m = m' = 1, n = 0, n' = -1$. The monodromy matrices are then given by

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{103}$$

A comparison with (102) implies that M_1 arises due to a monopole becoming massless at $u = 1$ and M_{-1} arises when a $(1, -1)$ dyon becomes massless at $u = -1$. At the point where the monopoles become massless, we have $a_D = 0$ and where the $(1, -1)$ dyon becomes massless, we have $a - a_D = 0$ as is evident from $Z = n_m a_D + n_e a$. Note that the monodromy at infinity, M_∞, shifts the electric charge by 2 units. Hence, at the points where condensation takes place, the electric charge is really defined modulo 2. We can conjugate the representation of the fundamental group by M_∞^n.

3.7 $U(1)$ β-function

To construct the monodromies in the previous subsection, we have used the β-function for a $U(1)$ theory interacting with a hypermultiplet. Let us look at this in some more detail. If we have Weyl fermions with charge Q_f and complex scalars with charge Q_s, then their contribution to the $U(1)\beta$-functions is

$$\beta(g) \equiv \mu \frac{d}{d\mu} g = \frac{g^3}{16\pi^2} \left(\sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right).$$

If we denote the coefficient of g^3 by b and define $\alpha = g^2/4\pi$, then the above equation can be rewritten as

$$\mu \frac{d}{d\mu}\left(\frac{1}{\alpha}\right) = -8\pi b\,.$$

Consider hypermultiplet which is a reduced multiplet of $N = 2$ with spin $\leq 1/2$. In $N = 1$ language, this is described by chiral superfields $M, \widetilde{M}$ and contains two Weyl fermions and two complex scalars, all with the same charge Q. Hence we get

$$b = \frac{1}{16\pi^2}\, Q^2 \left(2 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3}\right) = \frac{1}{8\pi^2}\, Q^2\,.$$

Now remember that, using the anomaly, we have set the θ-parameter to zero by a chiral rotation of the fermions. As a result we have $\tau = i/\alpha$, so that,

$$\mu \frac{d\tau}{d\mu} = -\frac{i}{\pi} Q^2\,.$$

Identifying μ with the natural scale of the theory which is a and setting $Q = 1$, we obtain

$$\tau \simeq -\frac{i}{\pi} \ln \frac{a}{\Lambda}\,.$$

This is the expression which was used in the determination of the monodromies at finite u. If we are interested in the contribution from a monopole multiplet, then we can perform the above calculation in terms of the dual variables. The answer then becomes $\tau_D \simeq -(i/\pi) \ln (a_D/\Lambda)$.

3.8 Monopole Condensation and Confinement

In this subsection we describe how confinement in $N = 1$ gauge theory can be understood in terms our macroscopic $N = 2$ theory. The $N = 2$ vector superfield $\mathcal{A}$ can be decomposed, in the $N = 1$ formalism, into a vector superfield W_α and a chiral superfield Φ. To break $N = 2$ down to $N = 1$, one can add a superpotential $W = m\ \mathrm{Tr}\Phi^2$ to the action. This gives a bare mass to the Φ multiplet and, therefore, the low-energy theory is a pure Abelian gauge theroy. This low-energy theory has $\mathbf{Z}_4$ chiral symmetry and is believed to have a mass gap with confinement and spontaneous breaking of $\mathbf{Z}_4$ to $\mathbf{Z}_2$. Seiberg and Witten gave a macroscopic description of this phenomenon based on the $N = 2$ picture as will be described below.

In $N = 2$ theory, the massless spectrum in the semiclassical limit contains only the Abelian vector multiplet $\mathcal{A}$. Let us analyze the effect of turning on a mass m for the scalar multiplet Φ. In the low-energy theory, $\mathrm{Tr}\,\Phi^2$ is represented by a chiral superfield U. Its scalar component is $u = \langle \mathrm{Tr}\phi^2 \rangle$, which is a holomorphic function on the moduli space. For small m, we can simply add $W_{eff} = mU$ to the low-energy Lagrangian. This presumably removes the vacuum degeneracy and gives a mass to the scalar multiplet. To make the Abelian gauge field also massive (so that the theory has a mass gap), we need either (i) extra light gauge fields giving rise to a strongly coupled non-Abelian theory, or (ii) light charged fields giving rise to a Higgs mechanism. Thus, in either case, somewhere on $\mathcal{M}$ extra massless states must appear. As indicated before, one cannot get extra light gauge fields. Therefore, we consider option (ii) with the light charged fields being monopoles or dyons. Near the point with massless monopoles, we go to a dual description of the theory and use the $N = 1$ chiral superfields $M, \widetilde{M}$ to describe the monopole hypermultiplet. In this dual description, the full $N = 1$ superpotential then becomes

$$\hat{W} = \sqrt{2} A_D M \widetilde{M} + m\, U(A_D)\,.$$

The low-energy vacuum structure is easy to analyze. Vacua correspond to solutions of

$$d\hat{W} = 0\,,$$

satisfying $|M| = |\widetilde{M}|$, so that the D-term vanishes. For $m = 0$, $M = \widetilde{M} = 0$ and a_D is arbitrary. Thus we recover the $N = 2$ moduli space. If $m \neq 0$, then

$$\begin{aligned} \sqrt{2} M \widetilde{M} + m\, \frac{du}{dA_D} &= 0 \quad , \\ a_D M = a_D \widetilde{M} &= 0 \quad . \end{aligned}$$

Assuming that $du/da_D \neq 0$, we get $M, \widetilde{M} \neq 0$, so that $a_D = 0$ and $M = \widetilde{M} = (-mu'(0)/\sqrt{2})^{1/2}$. Since M is charged, its vacuum expectation value generates a mass for the gauge field through the Higgs mechanism. Thus, by a simple analysis, we reproduce the expected mass gap of the microscopic theory. To understand charge confinement, note that, since the hypermultiplet $M\widetilde{M}$ describes monopoles, we have a magnetic Higgs mechanism. Thus, $M \neq 0$ means that massless magnetic monopoles condense in vacuum. This gives rise to the confinement of electric charges by the dual (*i.e.,* magnetic) Meissner effect.

3.9 The Solution of the Model

In this subsection, we will identify the moduli space of the $N = 2$ theories with the moduli space of genus 1 Riemann surfaces and then use this identification to calculate $a_D(u)$ and $a(u)$.

Let us first summarize what we have learnt about the structure of the moduli space: The moduli space $\mathcal{M}$ is the u-plane with singularities at $1, -1, \infty$ and a $\mathbf{Z}_2$ symmetry acting as $u \to -u$ (Fig. 3). Over this punctured plane there is a flat $SL(2, \mathbf{Z})$ bundle V, which has $(a_D, a)^T$ as a section. This bundle has monodromies

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{104}$$

around the singularities.

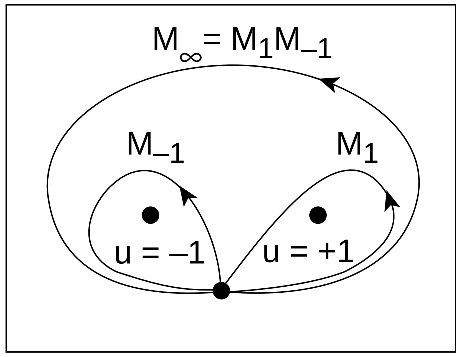

Figure 3

To be more precise, the quantities $(a_D(u), a(u))$ form a holomorphic section of the bundle $V \otimes \mathbf{C}$, and have the following asymptotic behaviour

$$
u \approx \infty : \qquad \begin{cases} a \cong \sqrt{2u} \\ a_D \approx i\, \frac{\sqrt{2u}}{\pi} \ln u \end{cases} ,
$$

$$
u = 1 : \qquad \begin{cases} a_D \approx c_0(u-1) \\ a \approx a_0 + \frac{i}{\pi}\, a_D \ln a_D \end{cases} .
$$

where, a_0 and c_0 are constants. For $u = -1$ we get a behaviour similar to $u = 1$ but with a_D replaced by $a - a_D$. The metric on the moduli space is $ds^2 = \mathrm{Im}(\tau)\, |da|^2$ with

$$
\tau = \frac{da_D/du}{da/du} . \tag{105}
$$

To insure positivity of kinetic energy, $\mathrm{Im}(\tau)$ should be positive definite. The monodromies generate a subgroup $\Gamma(2)$ of $SL(2, \mathrm{Z})$ and, in fact, the u-plane with its singularities is the quotient of the upper half plane H by $\Gamma(2)$. This quotient has three cusps corresponding to the three singularites.

The space $H/\Gamma(2)$, which is the u-plane, also parametrizes the family of curves E_u described by the equation

$$
y^2 = (x-1)(x+1)(x-u) , \tag{106}
$$

where x is a complex variable. First, note that this equation is invariant under the transformations

$$
w : \{u \to -u\, , x \to -x\, , y \to \pm iy\} ,
$$

that generate a $\mathbf{Z}_4$ symmetry, out of which only a $\mathbf{Z}_2$ subgroup acts on u. This is the same as the symmetry structure on $\mathcal{M}$. Now, let us describe the curve which the above equation represents. The curve basically is the x-space the topology of which is determined by the requirement that y is a single valued function. Sine the equation is quadratic in y, if we move along a close loop on the x-space around anyone of the three zeros of y, then we get $y \to -y$. The same is also true for a loop which contains all the zeros, or equivalently, a loop around the point $x = \infty$ (this is because there is an odd number of zeros). Therefore, if y is to be a single valued function, then the x-space should be a double cover of the complex plane $\mathbf{C}$ with the point at infinity added to it. Furthermore, this space should have four branch points at $x = -1, 1, u, \infty$ which are joined pairwise by two cuts (Fig. 4). To fix attention, consider one branch cut from -1 to 1, and another from u to ∞. The two sheets are joined along these cuts so that on crossing a cut, we move from one sheet to the other. It is on this space that y is single valued.

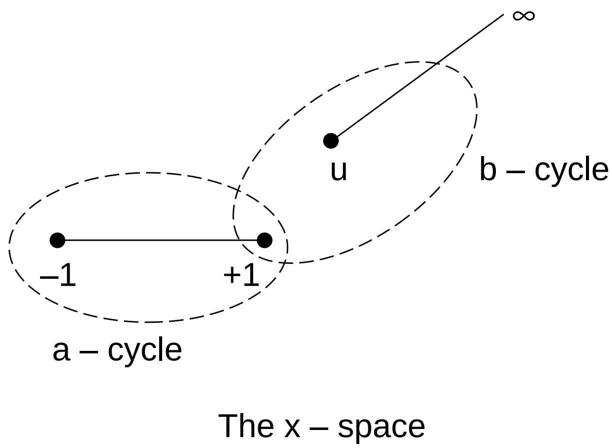

Figure 4

The x-space so obtained is nothing but a genus one Riemann surface. This can be easily visualized as follows: On a torus, draw a circle c_1 along the a-cycle and translate this along the torus to get a circle c_2 (Fig. 5(a)). Now, squash the circles c_1 and c_2 into line segments l_1 and l_2 (Fig. 5(b)). This divides the torus into two halves joined along these segments. If we now cut open both the two halves, the surface we get is the same as the x-space described above with l_1 and l_2 as the two branch cuts and with the point at infinity mapped to a point at finite x (Fig. 5(c)).

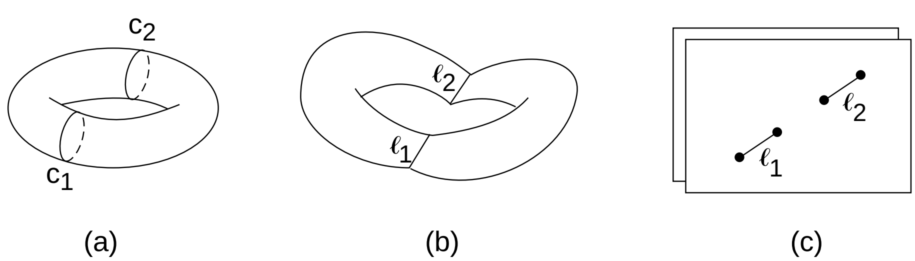

Figure 5

From this, it is clear that a loop on the x-plane that goes around one of the two cuts corresponds to the a-cycle of the torus. On the other hand, a loop which intersects both cuts (in our case, goes around 1 and u), corresponds to the b-cycle on the torus. Note that when two of the branch points on the x-space coincide, a cycle on the curve E_u shrinks to zero size and the curve becomes singular. For example, in our case, if $u = \infty$, the a-cycle shrinks to zero size and when $u = 1$, this happens to the b-cycle. Thus the singularities on the u-plane are at the points where a curve in the family E_u develops a vanishing cycle.

To identify a_D and a, on the genus one Riemann surface E_u, we pick up two independent one cycles γ_1 and γ_2, normalized such that their intersection number is one:

$$\gamma_1 \cdot \gamma_2 = 1 .$$

These cycles, which continuously vary with u, form a local basis for the first homology group $V_u = H^1(E_u, \mathrm{C})$ of E_u. A cycle can be paired with elements λ from the first cohomology group

$$\gamma \to \oint_\gamma \lambda .$$

λ can be thought of as a meromorphic $(1,0)$-form on E_u with vanishing residue, modulo exact forms. The vanishing of the residue insures that the pairing is invariant under continuous deformations of γ even across poles of λ. By virtue of this pairing, we can also regrad λ as an element of V_u. For the one-forms on E_u, we can choose a basis

$$\lambda_1 = \frac{dx}{y} , \qquad \lambda_2 = \frac{xdx}{y} .$$

Upto scalar multiplication, λ_1 is the unique holomorphic differential on E_u and if we define

$$b_i = \oint_{\gamma_i} \lambda_1 , \qquad \text{for} \quad i = 1, 2 ,$$

then the torus is characterized by a parameter

$$\tau_u = b_1/b_2 , \qquad \text{with} \quad \mathrm{Im}(\tau_u) > 0 .$$

Let us consider an arbitrary section

$$\lambda = a_1(u)\lambda_1 + a_2(u)\lambda_2 ,$$

of V_u and, for the moment, make the identification

$$a_D = \oint_{\gamma_1} \lambda , \qquad a = \oint_{\gamma_2} \lambda .$$

If λ is a form with vanishing residue then, on circling a singularity, a_D and a transform in the right way, simply according to how γ_1 and γ_2 transform under a subgroup of $SL(2, Z)$ (this is further explained in subsection 4.6 below). On the other hand, if λ has a pole with a non-vanishing residue, then it is possible that the integration path may move across this pole and, as a result, a_D and a may no longer transform under a pure $SL(2,Z)$. This second possibility is of course not consistent with the symmetries of the BPS mass formula in the absence of fermions with non-zero bare masses, as we already discussed in a previous subsection. Therefore, λ should not have a pole with a non-vanishing residue (In the presence of fermions with non-zero bare masses, the situation is different and will be discussed at the end of the next section). The above identification of a_D and a implies that

$$\frac{da_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du}\,, \qquad \frac{da}{du} = \oint_{\gamma_2} \frac{d\lambda}{du}\,.$$

To fix the arbitrariness in λ, we use the condition $\mathrm{Im}\tau > 0$ for the metric on $\mathcal{M}$ as defined in (105). First, suppose that

$$\frac{d\lambda}{du} = f(u)\lambda_1 = f(u)\frac{dx}{y}\,.$$

Then,

$$\frac{da_D}{du} = f(u)\, b_1\,, \qquad \frac{da}{du} = f(u)\, b_2\,,$$

so that

$$\tau = \frac{b_1}{b_2} = \tau_u\,.$$

Since $\mathrm{Im}\,\tau_u > 0$, we get $\mathrm{Im}\,\tau > 0$. As argued by Seiberg and Witten, the converse is also true, so $d\lambda/du$ does not depend on λ_2. The function $f(u)$ is fixed by the asymptotic behaviour of the theory near the singularities on the u plane and is given by $f(u) = -\sqrt{2}/4\pi$. With this, we can obtain λ as

$$\lambda = \frac{\sqrt{2}}{2\pi}\,\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{2\pi}\,\frac{dx y}{x^2-1} = \frac{\sqrt{2}}{2\pi}\,\frac{dx}{y}(x-u)\,.$$

To calculate a and a_D, we have to choose a specific basis of one-cycles on E_u. We identify γ_2 with the a-cycle on the torus, or equivalently, with a curve which loops around the points -1 and 1 on the x plane. We can deform this curve so that it lies entirely along the cut from -1 to 1 and back. Thus, $a(u)$ is given by

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{107}$$

For γ_1, we choose the curve which loops around the points 1 and u and get

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{108}$$

It can be checked that with this choice of the one-cycles, a and a_D have the desired behaviour near the singularities [31].

Explicit Formulae for $a(u)$ and $a_D(u)$: One can easily find explicit formulae for the a and a_D in terms complete elliptic integrals $E(k)$ and $K(k)$ by using the integral representation of hypergeometric functions $F(\alpha,\beta,\gamma;z)$ [35]. The hypergeometric functions are given by

$$\begin{aligned} F(\alpha,\beta,\gamma;z) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dx\, x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n\geq 0}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!}\,. \end{aligned}$$

Comparing this with the integral representation of a in (107), we can easily see that

$$a(u) = \sqrt{2(1+u)}\, F\left(-1/2, 1/2, 1; 2/(1+u)\right)\,.$$

As for a_D, let us first make the substitution $x = (u-1)t + 1$ in (108). This gives

$$a_D(u) = \frac{i}{\pi}(u-1)\int_0^1 dt\, t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}(1-\frac{1-u}{2}t)^{-\frac{1}{2}}\,.$$

Comparing this with the expression for the hypergeometirc functions, we get

$$a_D(u) = \frac{i}{2}(u-1)F\left(1/2, 1/2, 2; (1-u)/2\right)\,.$$

In terms of the hypergeometric functions, the complete elliptic integrals are given by

$$\begin{aligned} K(k) &= \frac{\pi}{2}F\left(1/2, 1/2, 1; k^2\right)\,, \\ E(k) &= \frac{\pi}{2}F\left(-1/2, 1/2, 1; k^2\right)\,. \end{aligned}$$

Further, we define $k'^2 = 1 - k^2$ and set

$$E'(k) \equiv E(k')\,, \qquad K'(k) \equiv K(k')\,.$$

Now, using the identity

$$c(1-z)F(a,b,c;z) - cF(a,b-1,c;z) + (c-a)zF(a,b,c+1;z) = 0\,,$$

with $c = 1, a = b = 1/2$ and $z = (1-u)/2$, in the expression for a_D, we get

$$\begin{aligned} a(u) &= \frac{4}{\pi k}E(k)\,, \\ a_D(u) &= \frac{4}{i\pi}E(i\frac{k'}{k}) + \frac{4i}{\pi k^2}K(i\frac{k'}{k})\,. \end{aligned}$$

Here, $k^2 = 2/(1+u)$ and $(1-u)/2 = -k'^2/k^2$. These expressions can be further simplified if we note that

$$F(a,b,c;z) = (1-z)^{-a}F(a,c-b,c;z/(z-1)) = (1-z)^{-b}F(b,c-a,c;z/(z-1))\,.$$

This implies that $K(ik'/k) = kK(k') = kK'(k)$ and $E(ik'/k) = k^{-1}E(k') = k^{-1}E'(k)$, so that

$$\begin{aligned} a(u) &= \frac{4}{\pi k}\, E(k)\,, \\ a_D(u) &= \frac{4}{\pi i}\, \frac{E' - K'}{k}\,. \end{aligned}$$

The generalized coupling τ can be calculated by using

$$\begin{aligned} \frac{dE}{dk} &= \frac{E-K}{k}\,, \quad \frac{dK}{dk} = \frac{1}{kk'^2}\,(E - k'^2 K)\,, \\ \frac{dE'}{dk} &= -\frac{k}{k'^2}\,(E' - K')\,, \quad \frac{dK'}{dk} = -\frac{1}{kk'^2}\,(E' - k^2 K')\,, \end{aligned}$$

and is given by

$$\tau = \frac{\partial a_D}{\partial a} = \frac{da_D/dk}{da/dk} = \frac{iK'}{K}\,.$$

4 The Seiberg-Witten Analysis of $N = 2$ Gauge Theory With Matter

In this section, we consider the $N = 2$ supersymmetric gauge theory, with gauge group $SU(2)$, coupled to N_f matter multiplets. The analysis involves many technical issues and the properties of the theory depend on the value of N_f. As in usual QCD, matter fields (fermions) and gauge bosons contribute to the β-function with opposite signs. Thus, to insure asymptotic freedom, we require $N_f \leq 4$. The theory with $N_f = 4$ is finite to all orders in perturbation theory and, when the quarks are massless, it is very likely conformally invariant non-perturbatively. Seiberg and Witten provided evidence that this theory is an example of an $SL(2, Z)$ invariant theory. The only other $SL(2, Z)$ invariant theory known in four dimensions is the $N = 4$ supersymmetric gauge theory which also has a vanishing β-function. However, in the following, we will mainly concentrate on $N_f \leq 3$. For the $N_f = 4$ case, the reader is referred to original work of Seiberg and Witten [32].

In the presence of matter, it is convenient to choose a charge normalization different from the previous sections. In $N = 2$ pure gauge theory all fields transform in the adjoint representation and the charges of particles and monopoles are integers. Hence, in the formula $Z = an_e + a_D n_m$, n_e and n_m are integers. With this normalization, when quarks are present, n_e could be half-integral. In this section we choose a slightly different convention: To ensure that n_e is always integral, we multiply it by 2 and divide a by 2 to keep the mass formula unchanged. Since a_D is kept unchanged, it is now given by $2a_D = \partial\mathcal{F}/\partial a$. In terms of the gauge invariant quantity $u = \mathrm{Tr}\phi^2$, as $|u| \to \infty$, the asymptotic behaviour of a and a_D with this normalization becomes

$$a \cong \frac{1}{2}\sqrt{2u}\,, \quad a_D \simeq i\frac{4}{\pi} a \log a\,.$$

Furthermore, the effective coupling is also rescaled to $\tau = \partial a_D/\partial a = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. This rescaling of the charge affects the form of the curve which determines the solution of the model, keeping the physics unchanged.

In this section we are going to study the theory with gauge group $SU(2)$. In this case, the most interesting dynamics appear in the Coulomb branch of the moduli space, where $SU(2)$ breaks down to $U(1)$. It is, therefore, instructive to first study $N = 2$ supersymmetric QED.

4.1 $N = 2$ Supersymmetric QED

The $N = 2$ supersymmetric QED contains the photon vector multiplet and k quark hypermultiplets. The photon multiplet consists of the $N = 1$ chiral superfields (W_α, A) and each quark hypermultiplet consists of two $N = 1$ chiral superfields M^i and $\widetilde{M}_i$, with $U(1)$ charges $+1$ and -1, respectively. The only renormalizable $N = 2$ compatible superpotential is:

$$\mathcal{W} = \sqrt{2} A M^i \widetilde{M}_i + \sum_i m_i M^i \widetilde{M}_i \,.$$

When $m_i = 0$, the theory has a global symmetry group $SU(k) \times SU(2)_R \times U(1)_{\mathcal{R}}$. $SU(k)$ is a flavour symmetry group acting on the k hypermultiplets, while $SU(2)_R$ rotates the two supersymmetries and $U(1)_{\mathcal{R}}$ is the usual R-symmetry which is afflicted by an anomaly. As in the case without matter, the strategy is to first determine the classical moduli space and then to see whether the quantum moduli space is qualitatively different. The symmetries of the theory are very useful in determining the structure of these moduli spaces.

In general, the classical moduli space has a Higgs branch and a Coulomb branch. The Higgs branch is defined by some of the M^i's acquiring vacuum expectation values. In this case, both in QED and in QCD with gauge group $SU(2)$, the gauge group is completely broken. For $k = 1$, there is no Higgs phase. When $k \geq 2$, and $m_i = 0$, then, using the global symmetries together with the the vanishing condition for the D-term, it is possible to rotate M and $\widetilde{M}$ to the form $M = (B, 0, 0, ...)$ and $\widetilde{M} = (0, B, 0, ...)$. From this one can read off the different patterns of symmetry breaking as a function of the number of flavours k. A theory on the Higgs branch does not contain monopoles or dyons and hence, the dynamical possibilities are not as rich as on the Coulomb branch. The classical moduli space is a hyperKähler manifold and the symmetries of the theory lead to a unique hyperKähler metric on it. As a result of this uniqueness, the metric does not receive quantum mechanical corrections. Therefore, on the Higgs branch, the classical and the quantum moduli spaces are the same.

In the case at hand, if all $m_i = 0$, then the Coulomb branch is defined by $< A > \neq 0$ which, in turn, implies that $< M^i > = < \widetilde{M}_i > = 0$. The $U(1)$ gauge group remains unbroken while all the M's become massive. At a generic point on the moduli space, the effective low-energy threory (which involves only the massless modes) is a pure $N = 2$ gauge theory and the Kähler potential is of the special geometry type : $K = \mathrm{Im} a_D(a)\bar{a}$. This is related by $N = 2$ supersymmetry to the the gauge kinetic term,

$$\int d^2\theta \, \frac{\partial a_D}{\partial a} \, W^\alpha W_\alpha \,.$$

The Kähler potential (and thus the metric) receive quantum corrections, but the one-loop approximation to K is exact and also there are no non-perturbative corrections since this theory does not contain instantons. Using the one-loop β-function for QED with k hypermultiplets, we obtain

$$a_D = -\frac{ik}{2\pi}\, a \ln(a/\Lambda)\,. \qquad (109)$$

To reproduce this formula one should be careful about the extra factors of 2 in the supersymmetric way of defining g and the normalization convention mentioned earlier. The metric $\mathrm{Im}(\tau)$ obtained from this is zero at $|a| = \Lambda/e$ and thus the effective coupling constant is singular. Because of this Landau pole singularity, the theory does not make sense in the ultra-violet region unless embedded in a larger theory which is asymptotically free. If fermion mass terms are added, the singularities on the Coulomb branch can move. Since, when $a = -\frac{1}{\sqrt{2}}m_i$ one electron becomes massless, we have

$$a_D = -\frac{i}{2\pi}\sum_i (a + m_i/\sqrt{2})\ \ln\frac{(a + m_i/\sqrt{2})}{\Lambda}\,.$$

To each bare mass is now associated a singularity on the moduli space where the particle becomes massless. Depending on the possible equality of two or more masses, one can have Higgs and Coulomb branches touching, leading to an intricate structure.

Here, we also see a manifestation of the modified form of the central charge of the $N = 2$ algebra discussed in section 2. For pure $N = 2$, we have $Z = n_e a + n_m a_D$. However, now the electron masses are not just $\sqrt{2}|a|$, rather the i-th multiplet has mass $|\sqrt{2}a + m_i|$. As shown in section 2, this is consistent with the fact that the $U(1)$ charges S_i of the massive hypermultiplets appears in the formula for the central charge Z as

$$Z = n_e a + n_m a_D + \sum_i S_i m_i/\sqrt{2}\,.$$

4.2 $N = 2$ Supersymmetric QCD with Matter

The $N = 2$ supersymmetric QCD coupled to N_f matter hypermultiplets contains the $N = 1$ superpotential

$$W = \sum_{i=1}^{N_f}(\sqrt{2}\widetilde{Q}_i\Phi Q^i + m_i\tilde{Q}_i Q^i)\,.$$

In general, when $m_i = 0$, this theory has a global symmetry group $SU(N_f)\times SU(2)_R\times U(1)_{\mathcal{R}}$. In the special case, when the gauge group is $SU(2)$, the flavour group is enlarged to $O(2N_f)$. This is due to the fact that for $SU(2)$ the fundamental representation and its conjugate are isomorphic and, therefore, Q^i and $\widetilde{Q}_i$ can be combined into a $2N_f$-dimensional vector transforming under $O(2N_f)$. Thus, for the special case of the $SU(2)$ gauge group, the theory also has a parity symmetry group $\mathbf{Z}_2 \subset O(2N_f)$ acting as

$$\rho : Q_1 \leftrightarrow \widetilde{Q}_1\,, \qquad (110)$$

with all other fields remaining unchanged. This parity plays an important role in the analysis of the theory. The global symmetry group of the theory is actually a quotient of $O(2N_f) \times SU(2)_R \times U(1)_{\mathcal{R}}$. The quotient is to be taken because a $Z_2 \subset U(1)_{\mathcal{R}}$ is the same as $(-1)^F$ contained in the Lorentz group. This, combined with the center of $SU(2)_R$, is the same as the $\mathbf{Z}_2$ in the center of $O(2N_f)$.

For $N_f = 0, 1$, the theory has only a Coulomb branch with ($< \phi > \neq 0$). On this branch $SU(2)$ is broken to $U(1)$ and all quarks are massive. Moreover, $U(1)_{\mathcal{R}}$ is spontaneously broken because Φ has $U(1)_{\mathcal{R}}$ charge 2. For $N_f \geq 2$ we can either have a Coulomb branch or Higgs branches. On a Higgs branch, the gauge symmetry is fully broken and the dynamics is not very rich. We will not analyse this branch in detail, but will only mention that, as in the QED case, the metric on the moduli space is uniquely determined by the symmetries and does not receive quantum correction.

Some Properties of the Quantum Theory: The perturbative β-function of our theory (which, due to supersymmetry, is only a one-loop effect) is given by

$$\beta(g) = -\frac{4 - N_f}{16\pi^2} g^3. \tag{111}$$

Therefore, to insure asymptotic freedom, we consider only $N_f = 0, 1, 2, 3, 4$.

In the previous section, from the counting of fermion zero-modes in an instanton background, we found that $U(1)_{\mathcal{R}}$ is broken to a discrete subgroup $\mathbf{Z}_{4N_c - 2N_f}$. This was obtained by requiring the invariance of the correlation function G given by (87) under $U(1)_{\mathcal{R}}$. For $N_c = 2$, we have an extra discrete symmetry group (110) which permutes the fermion zero-modes ψ_{q_1} and $\widetilde{\psi}_{q_1}$. This group is anomalous as it changes G to $-G$. Therefore, now we can also allow $U(1)_{\mathcal{R}}$ transformations which do not keep G invariant, but change it by a sign. This sign can be compensated for by an anomalous $\mathbf{Z}_2$ transformation. Thus, for $N_c = 2$, $U(1)_{\mathcal{R}}$ is broken to the discrete subgroup $\mathbf{Z}_{2(4N_c - 2N_f)} = \mathbf{Z}_{4(4-N_f)}$. This can be combined with the anomalous $\mathbf{Z}_2$ to get a discrete $\mathbf{Z}_{4(4-N_f)}$ anomaly-free subgroup with the action (see (85) and (86)):

$$\begin{array}{ll} W_\alpha \to \omega W_\alpha(\omega^{-1}\theta) & \\ \Phi \to \omega^2 \Phi(\omega^{-1}\theta) & \\ Q^1 \to \widetilde{Q}_1(\omega^{-1}\theta) & \\ \widetilde{Q}_1 \to Q^1(\omega^{-1}\theta) & \\ \left.\begin{array}{l} Q^i \to Q^i(\omega^{-1}\theta) \\ \widetilde{Q}_i \to \widetilde{Q}_i(\omega^{-1}\theta) \end{array}\right\} & \qquad i \neq 1\,, \end{array}$$

where, $\omega = \exp(2i\pi/4(4 - N_f))$. For $N_f = 0$ we do not have the quarks to cancel the anomaly and only the square of the above transformations is anomaly free. This case was discussed in the previous section. Furthermore, it can be seen that a subgroup $\mathbf{Z}_2 \subset \mathbf{Z}_{4(4-N_f)}$ is the same as $(-1)^F$ in the Lorentz group. Combining the above transformations with the $U(1)_J$ subgroup of $SU(2)_R$ (see (85) and (86)), we find a $\mathbf{Z}_{4(4-N_f)}$ symmetry which commutes $N = 1$

supersymmetry

$$\begin{array}{ll} \Phi \to \omega^2\Phi(\theta) & \\ Q^1 \to \omega^{-1}\widetilde{Q}_1(\theta) & \\ \widetilde{Q}_1 \to \omega^{-1}Q^1(\theta) & \\ \left.\begin{array}{l} Q^i \to \omega^{-1}Q^i(\theta) \\ \widetilde{Q}_1 \to \omega^{-1}\widetilde{Q}_i(\theta) \end{array}\right\} & i \neq 1\,. \end{array} \tag{112}$$

Under this transformation, the gauge invariant order parameter $u = \mathrm{Tr}\phi^2$ has charge 4 and transforms as $u \to \exp 2\pi i/(4-N_f)u$. This further breaks $\mathbf{Z}_{4(4-N_f)}$ down to $\mathbf{Z}_4$. The remaining $\mathbf{Z}_{4-N_f}$ acts non-trivially on the u-plane. Note that for $N_f = 0$, the subgroup which does not keep u invariant is only a $\mathbf{Z}_2$.

As in the $N_f = 0$ case, the large u-behaviour of $a_D(u)$ is determined by the one-loop β-function (111):

$$\begin{aligned} a &\cong \frac{1}{2}\sqrt{2u} + \cdots\,, \\ a_D &\simeq i\,\frac{4-N_f}{2\pi}\,a(u)\ln\frac{u}{\Lambda_{N_f}^2} + \cdots\,. \end{aligned} \tag{113}$$

Here, the dots represent non-perturbative instanton corrections. The generic form of these corrections can be obtained by arguments similar to the ones used for the $N_f = 0$ case in the previous section. First, a correction coming from a k-instanton configuration is proportional to the k-instanton factor, which, using the β-function (111), can be written as

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda_{N_f}}{a}\right)^{k(4-N_f)}\,. \tag{114}$$

Following Seiberg [38, 39], we can restore the broken part of the $U(1)_{\mathcal{R}} \times \mathbf{Z}_2(\rho)$ symmetry by assigning appropriate charges to u and Λ_{N_f}. Thus we assign charge 4 and even ρ-parity to u and charge $2(4-N_f)$ and odd ρ-parity to $\Lambda_{N_f}^{4-N_f}$. Note that with this assignment, the one-instanton factor $\exp(-8\pi^2/g^2)$ will have an odd ρ-parity which compensates for the odd parity under $\mathbf{Z}_2(\rho)$ of the correlation function G in (87), thus keeping it invariant . For the special case of $N_f = 4$, $U(1)_{\mathcal{R}}$ is non-anomalous and u has charge 4. $Z_2(\rho)$ in $O(8)$ is still anomalous, but again, it can be treated as unbroken by assigning odd parity to the instanton factor $\exp(-8\pi^2/g^2)$. Invariance under $U(1)_{\mathcal{R}}$ with the above charge assignments implies that each correction term should contain a factor of $\sqrt{u}$. Moreover, since the metric on the u-plane is invariant under the ρ-parity, configurations with odd instanton numbers cannot contribute to a and a_D. Putting these facts together, we can write the generic form of the corrected a and a_D as

$$\begin{aligned} a &= \frac{1}{2}\sqrt{2u}\left(1 + \sum_{n=1}^{\infty} a_n(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\right)\,, \\ a_D &= i\frac{4-N_f}{2\pi}a(u)\ln\frac{u}{\Lambda_{N_f}^2} + \sqrt{u}\sum_{n=0}^{\infty} a_{Dn}(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\,. \end{aligned}$$

The difficult part is to compute the coefficients a_n and a_{Dn}.

Since the gauge group $SU(2)$ breaks to $U(1)$, the theory will have massive charged states and we want to know how the unbroken global symmetry acts on them. The unbroken part of the global symmetry, which is the part that keeps u invariant, is obtained by raising the transformations (112) to the power $4 - N_f$. This unbroken transformation changes the sign of ϕ and therefore acts as charge conjugation on the charged fields. For $N_f = 1, 3$; $4 - N_f$ is odd and the unbroken transformation contains odd powers of ρ. It, therefore, acts as the parity in $O(2N_f)$. Hence, in this case, the parity (110) is realized on the spectrum but it reverses the signs of electric and magnetic charges. States of given charge belong to $SO(2)$ (for $N_f = 1$) or $SO(6)$ (for $N_f = 3$) multiplets. For $N_f = 2, 4$; $4 - N_f$ is even and the unbroken transformation contains only even powers of ρ. All the odd powers of ρ, which amount to the parity transformation, are part of the broken transformations. Thus, in this case, the parity symmetry is broken and the states are only in $SO(4)$ (for $N_f = 2$) or $SO(8)$ (for $N_f = 4$) representations.

4.3 BPS Saturated States

On the Higgs branch $SU(2)$, is completely broken and there are no electric or magnetic charges. Thus the central charge of the $N = 2$ algebra only contains contributions from the $U(1)$ charges of the hypermultiplets. We will not discuss this in any detail.

On the Coulomb branch, the simplest BPS saturated states are the quarks with zero bare mass and which acquire masses $M = \sqrt{2}|a|$ after the spontaneous breaking of the gauge symmetry. These form a set of BPS states which transform as a vector of $SO(2N_f)$. Besides these, there are BPS states which transform as a spinor of $SO(2N_f)$ arising as follows: Since the gauge symmetry is broken to $U(1)$, the theory contains monopoles. In the presence of a monopole each $SU(2)$ doublet of fermions has one zero-mode. Since there are N_f hypermultiplets, there are $2N_f$ fermion doublets and, therefore, $2N_f$ fermion zero-modes. After quantization, these zero-modes give rise to a $2N_f$-dimensional Dirac algebra which provides a spinor representation of $SO(2N_f)$. Thus the fermion zero-modes turn the monopole into a spinor of $SO(2N_f)$. This is very similar to the quantization of the Ramond sector of the Superstring theory. The presence of spinors indicate that, at the quantum level, the symmetry group is a universal cover of $SO(2N_f)$, or $Spin(2N_f)$.

One of the monopole's collective coordinates is a charge rotation. Upon quantization this leads to a spectrum of electrically charged states for the monopole. A 2π rotation by the electric charge operator, however, is not the identity. Using the Witten effect, such a rotation gives a topologically non-trivial gauge transformation with eigenvalue $e^{i\theta}(-1)^H$ for a monopole with $n_m = 1$. Here, $(-1)^H$ is the centre of $SU(2)$ which is odd for states in the hypermultiplet and even for states in the vector multipler. In this section we normalize the charge operator so that elementary quarks have charges ± 1 and massive gauge bosons have charges ± 2. With this normalization, the above mentioned gauge rotation can be written as an operator statement

$$e^{i\pi Q} = e^{i\theta n_m}(-1)^H.$$

where, $Q = n_e + n_m\theta/\pi$. Since, $n_e \in \mathbf{Z}$, this relation implies that states of even n_e have $(-1)^H = 1$ and states of odd n_e have $(-1)^H = -1$. The factor $(-1)^H$ can be identified with the chirality operator for the spinor representation of $SO(2N_f)$. For $N_f = 1, 3$, the $SO(2N_f)$ parity transformation acts on the spectrum and it guarantees that a dyon transforming as a positive chirality spinor of $SO(2N_f)$ is degenerate with a particle of opposite electric and magnetic charges and opposite $SO(2N_f)$ chirality. No such relation exists for $N_f = 2, 4$.

For $N_f > 0$ we will see that the spectrum contains states with $n_m > 1$. A convenient way of labelling states is in terms of their behaviour under the centre of $Spin(2N_f)$.

- $N_f = 2$: $Spin(4) = SU(2) \times SU(2)$ with centre $\mathbf{Z}_2 \times \mathbf{Z}_2$. The representations of the center can be labeled by (ϵ, ϵ'), where $\epsilon = 0$ for vector-like irreducible representation and $\epsilon = 1$ for spinor-like irreducible representations. An elementary quark transforms as $(\epsilon, \epsilon') = (1, 1)$. Multiple quark states then transform as $(n_e \; mod \; 2, n_e \; mod \; 2)$. Since monopoles behave like spinors, we have

$$(\epsilon, \epsilon') = ((n_e + n_m) \; mod \; 2, n_e \; mod \; 2) \qquad \text{for } N_f = 2\,.$$

- $N_f = 3$: $Spin(6) = SU(4)$ with centre $\mathbf{Z}_4$. The elementary quarks are in the 6 of $Spin(6)$, thus they have charge 2 with respect to the centre. Since monopoles behave like spinors, we conclude that $\mathbf{Z}_4$ acts as:

$$\exp \frac{i\pi}{4}(n_m + 2n_e) \qquad \text{for } N_f = 3\,.$$

- $N_f = 4$: $Spin(8)$ has centre $\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, using similar arguments as above, we get, (ϵ, ϵ')=$(n_m \; mod \; 2, n_e \; mod \; 2)$. The four representations of the center are labeled by the representations of of $Spin(8)$ which realize them: $(0, 1) \equiv v$, $(1, 0) \equiv s$, $(1, 1) \equiv c$ and $(0, 0) \equiv o$. Here, v refers to the vector representation, s and c to two spinor representatios and o to the trivial representation. $Spin(8)$ has a triality group of outer automorphisms which is isomorphic to the permutation group $\mathbf{S}_3$ acting on v, s and c.

If an $N = 2$ invariant mass is turned on, say $m_{N_f} \neq 0$, then $SO(2N_f)$ explicitly breaks to $SO(2N_f - 2) \times SO(2)$, and the global Abelian charge associated with $SO(2)$ appears in the central charge

$$Z = n_e a + n_m a_D + S_{N_f} \frac{m_{N_f}}{\sqrt{2}}, \qquad M = \sqrt{2}|Z|\,.$$

Hence for $a = \pm m_{N_f}/\sqrt{2}$ one of the elementary quarks becomes massless. This will be crucial later in determining the global structure of the quantum moduli space on the Coulomb branch.

4.4 Duality

As in the $N_f = 0$ case, there is an $SL(2, Z)$ duality group which acts on the fields and the couplings. In the presence of matter, some new issues appear which will be discussed below.

As before, we can compute the monodromy matrices for (a_D, a) around the singular points on the moduli space. The simplest case corresponds to having a single quark with non-zero bare mass, and investigating what happens as $a \approx a_0 \equiv m_{N_f}/\sqrt{2}$. From the QED analysis we know that there is a logarithmic singularity at a_0 where this quark becomes massless. Near the singularity we have

$$\begin{aligned} a &\approx a_0\,, \\ a_D &\approx -\tfrac{i}{2\pi}(a-a_0)\ln(a-a_0)+c\,. \end{aligned}$$

The monodromy can now be easily computed to be

$$\begin{aligned} a &\to a\,, \\ a_D &\to a_D + a - \frac{m_{N_f}}{\sqrt{2}}\,. \end{aligned}$$

Unlike the $N_f = 0$ case, now we have an inhomogeneous transformation as the column vector $(a_D, a)^T$ picks up a shift under the monodromy (besides the usual $SL(2, Z)$ transformation). The possibility of such a shift was also noticed for the pure gauge theory case in the previous section. There, however, the shift was not part of the monodromy group and, furthermore, it was not compatible with the symmetries of the BPS mass formula. In the presence of matter, this shift is allowed since the BPS mass formula is modified. Moreover, now the shift naturally appears as a part of the monodromy group. To write a monodromy matrix, we construct a column vector $(m/\sqrt{2}, a_D, a)^T$. The monodromy can now be written as

$$\begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix} \to \mathcal{M} \begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix},$$

with the monodromy matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

If we arrange the charges as a row vector $W = (S, n_m, n_e)$, then invariance of Z (or of M) implies that, under the monodromy, $W \to W\mathcal{M}^{-1}$. In general the matrix $\mathcal{M}$ can be of the form

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ r & k & l \\ q & n & p \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ lq - pr & p & -l \\ nr - kq & -n & k \end{pmatrix},$$

with $\det \mathcal{M} = 1$.

Note that under the monodromy, the electric and magnetic charges can mix with each other but will not pick up contributions proportional to S which is a global symmetry charge. On the other hand, S can pick up contributions proportional to n_e and n_m which are related to the local gauge symmetry. For the example considered above,

$$(S, n_m, n_e) \to (S + n_m, n_m, -n_m + n_e)\,.$$

This leads once again to issues of marginal stability: For large values of m_{N_f}, the singularity is in the weak-coupling region of large u where a semi-classical treatment of monopole is reliable. In the semi-classical quantization, the global $U(1)$ charge is carried only by fermionic zero-modes. Since there is only a finite number of these modes, one cannot construct states with arbitrarily large values of S. This means that, although, by acting with $\mathcal{M}^{-1}$ we can increase the value of S at will, what might have started as a one-particle state comes back as a multiparticle state. This is possible if at some point, when going around the singularity, the single-particle state becomes unstable and decays into a multi-particle state. Thus, although the formalism is $SL(2,Z)$ covariant, the spectrum is not.

Seiberg and Witten provide many compelling arguments to suggest that the theory with $N_f = 4$, like the theory with $N = 4$, is $SL(2,Z)$ invariant, though the details are quite different. In the $N_f = 4$ theory, the global symmetry group is $Spin(8)$ which is the universal cover of $SO(8)$. The states $(n_m, n_e) = (0,1)$ are the elementary hypermultiplets in the vector representation v of $Spin(8)$. The states $(n_m, n_e) = (1,0)$ are in a spinor representaion s, and $(n_m, n_e) = (1,1)$ are in a spinor representation c of $Spin(8)$. While $SL(2,Z)$ alone cannot keep this spectrum unchanged, a combination of $SL(2,Z)$ and the $Spin(8)$ triality group (which permutes the representations v, s and c) could do so. This is provided one is willing to assume that there are monopole bound states for every pair of relatively prime integers (p,q), analoguous to the situation discussed by Sen for the $N = 4$ theory [20]. For each such pair, there should exist eight states transforming in a representation 8_v, 8_s or 8_c of $Spin(8)$, depending on the *mod* 2 reduction of (p,q). $SL(2,Z)$ mixed with triality will then keep the spectrum invariant. The solution Seiberg and Witten provided for this model gives strong support to this possibility. The global symmetry group is then a semi-direct product of $Spin(8)$ and $SL(2,Z)$.

4.5 A First Look at Singularities

As in the pure $N = 2$, we first try to locate the singularities on the moduli space before computing their monodromies. We recall that, with the standard normalization of the gauge coupling constant g, the one-loop β-function which is given by (111), can be integrated to give

$$\frac{1}{\alpha_{N_f}(\mu)} = \frac{4 - N_f}{2\pi} \ln \frac{\mu}{\Lambda_{N_f}} ,$$

where, $\alpha = 4\pi/g^2$. Now, if some (say $N_f - N_f'$) of the quarks have masses $m_i = m$, and we are looking at the theory at some scale $\mu < m$, then the low energy theory contains only N_f' hypermultiplets as the degrees of freedom. The coupling $\alpha(N_f')$ is then given by an expression similar to the above one with N_f and Λ_{N_f} replaced by N_f' and $\Lambda_{N_f'}$. The scales of the theories can be related by the matching condition $\alpha_{N_f}(m) = \alpha_{N_f'}(m)$, which implies

$$\Lambda_{N_f'}^{4-N_f'} = m^{N_f - N_f'} \, \Lambda_{N_f}^{4-N_f} .$$

For instance,

$$\begin{aligned} N_f = 3 \, , N_f' = 0 \quad &\Rightarrow \quad \Lambda_0^4 = m^3 \Lambda_3 \, , \\ N_f = 1 \, , N_f' = 0 \quad &\Rightarrow \quad \Lambda_0^4 = m \Lambda_1^3 \, . \end{aligned}$$

To determine the singularity structure on the moduli space, we first consider theories with $N_f \leq 3$, and with hypermultiplet bare masses very large as compared to Λ. The singularities which arise from hypermultiplets becoming massless are now in the semi-classical (large u) region of the moduli space and can be easily identified. In the small u region, one is effectively left with an $N_f = 0$ theory with two singularities corresponding to massless monopoles and dyons. Then we slowly decrease the bare masses to zero and follow the movement of the sigularities on the moduli space.

$\underline{N_f = 3}$: Let us start with equal masses $m_i = m >> \Lambda$. In this case, the global symmetry $Spin(6) \approx SU(4)$ of the massless theory is broken to $SU(3) \times U(1)$. Classically, there is a singularity at $a = m/\sqrt{2}$ where the three quarks become massless. These electrically charged massless fields form a $\underline{3}$ representation of $SU(3)$. For $m >> \Lambda$ the singularity is in the semi-classical region, $u \sim 2a^2 >> \Lambda^2$. For $u << m^2$ the three massive quarks can be integrated out giving a $N_f = 0$ theory with scale $\Lambda_0^4 = m^3\Lambda_3$. Hence, for small u the moduli space is that of a pure $N = 2$ theory with scale Λ_0 which has two singularities with $(n_m, n_e) = (1, 0)$ and $(1, 1)$. These are the points where monopoles and dyons become massless. Clearly the massless states at these singularities are $SU(3)$ invariant.

As m is decreased, the singularity at large u moves, and although the charges of the associated states can change (through monodromy matrices), their non-Abelian charges cannot change. Hence, for any m, the massless fields at the various singularities transform as $\underline{3}$, $\underline{1}$ and $\underline{1}$ of $SU(3)$. In the $m = 0$ limit, the original global symmetry is restored and the massless states must combine into representations of $SU(4)$. The only possibility is to have two singularities combining into a $\underline{4}$ of $SU(4)$ and the other sigularity moves somewhere else remaining a singlet. Thus, we conclude that the massless $N_f = 3$ theory has two singularities with massless particles in the $\underline{4}$ and $\underline{1}$ of $SU(4)$. From our study of how different states transform under the centre, the smallest charges for the massless particles at the singularities are $(n_m, n_e) = (1, 0)$ for the $\underline{4}$, and $(n_m, n_e) = (2, 1)$ for the $\underline{1}$. Semiclassically the first state exists, but it is not known whether the monopole-monopole bound state implied by the second also exists or not.

$\underline{N_f = 2}$: In this case, with two equal masses $m_i = m >> \Lambda$, the $Spin(4)$ global symmetry of the massless theory is broken to $SO(2) \times SO(2)$. There is a singularity at $a = m/\sqrt{2}$ and the massless states there transform under one or the other of the two $SO(2)$'s. In the region $u << m^2$, we can again integrate out the quarks, leading to a $N_f = 0$ theory with $\Lambda_0^4 = m^2\Lambda_2^2$, and two singularities with $(n_m, n_e) = (1, 0), (1, 1)$. The massless fields at these singularities are singlets under $SO(2) \times SO(2)$. In all, we have four massless states associated with three singularities. As $m \to 0$, we recover the full global symmetry group of the massless theory with two flavours which is $Spin(4) \approx SU(2) \times SU(2)$. The singularities have to combine in such a way that the massless states form a representation of this unbroken group. Since each $SO(2)$ is contained in a different $SU(2)$, the only way to combine the massless states into representations of $SU(2) \times SU(2)$ is as $(\underline{2}, \underline{1})$ and $(\underline{1}, \underline{2})$. Hence, for $m = 0$, there are two singularities, and the simplest charge assignments are $(n_m, n_e) = (1, 0)$ in one spinor of $SO(4)$, and $(n_m, n_e) = (1, 1)$ in the other spinor. Recall that the transformation under the centre is $(\epsilon, \epsilon') = ((n_m + n_e) mod\, 2, n_e\ mod\, 2)$.

$\underline{N_f = 1}$: Now the massive theory has the same $SO(2)$ symmetry as the massless theory, however, the same arguments as before imply the presence of three singularities for large m. This number does not change as $m \to 0$ due to the $\mathbf{Z}_3$ symmetry acting on the u-plane. We recall that this symmetry is the subgroup $\mathbf{Z}_{4-N_f} = \mathbf{Z}_3$ of $\mathbf{Z}_{4(4-N_f)}$ in (112) which is broken by a non-zero u. From

$$a_D = i\frac{4-N_f}{2\pi}a(u)\ln\frac{u}{\Lambda_{N_f}^2} + \ldots\,,$$

we see that as a transforms homogeneously under (112), a_D is shifted by a, *i.e.*, $a \to \omega^2 a$, $a_D \to \omega^2(a_D + a)$ with $\omega = e^{i\pi/6}$. Therefore, if one of the singularities in the $m = 0$ limit corresponds to massless states with $(n_m, n_e) = (1, 0)$, then the $\mathbf{Z}_3$ symmetry implies the existance of two other singularities characterized by $(n_m, n_e) = (1, 1)$ and $(1, 2)$. Hence, even for $m = 0$, the moduli space of the $N_f = 1$ theory has three singularities corresponding to massless states $(n_m, n_e) = (1, 0), (1, 1)$ and $(1, 2)$.

4.6 Monodromies and the Determination of the Metric

As in the case of $N = 2$ theory without matter, a solution to the theory can be found by regarding the u-plane as the moduli space of a family E_u of elliptic curves parametrized by u. The quantities $a_D(u)$ and $a(u)$ can then be related to the periods of these curves. A curve in E_u becomes singular when one of the cycles on it shrinks to zero size. This corresponds to a singularity on the u-plane with a non-trivial monodromy around it. If we know the singularities and the associated monodromies on the u-plane (which are associated with the appearence of massless particles in the spectrum), then we can work backwords and determine the families of the elliptic curves from which the periods a_D and a could be computed. In this subsection, we sketch the physical arguments used by Seiberg and Witten in [32] to find the curves for $N_f = 1, 2, 3$ theories exploiting the general features of the curve for the $N_f = 0$ theory (This reference also contains a more rigorous treatment of this problem which we will not reproduce here).

For $N_f = 0$, the solution was given by a family of elliptic curves described by the equation

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u)\,,$$

with the monodromy in $\Gamma(2)$. In the present section we have changed our conventions so that (n_m, n_e) are both integers even in the presence of matter fields. This was achieved by multiplying n_e by 2 and dividing a by 2 so that $Z = a_D n_m + a n_e$ is unchanged. This change of convention can be implemented as a transformation

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix},$$

which also changes the monodromy matrix as

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} m & 2n \\ p/2 & q \end{pmatrix}.$$

Since p and n are integers modulo 2, the new monodromy matrix contains the entry $2n$ which is an integer modulo 4, while all other entries are integers. These martices form the subgroup $\Gamma_0(4)$ of $SL(2,Z)$. In the new convention, the monodromies in (104) take the form

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \tag{115}$$

and the solution is given by a family of curves described by

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x\,. \tag{116}$$

As discussed in the previous section, any genus-one curve can be represented by a cubic

$$y^2 = F(x) = (x - e_1)(x - e_2)(x - e_3)\,.$$

This describes a space x as a double cover of the complex plane branched over e_1, e_2, e_3 and ∞. The curve becomes singular when two branch points coincide (*e.g.*, $e_1 = e_2 \neq e_3$ or $e_3 \to \infty$). In this case the singularity is called stable. If more than two branch points coincide, then the singularity is not stable, but a u-dependent reparametrization of x and y can always convert this into a stable singularity. For the curve (116) the branch points are at

$$x = 0,\ \frac{1}{2}(u \pm \sqrt{u^2 - \Lambda^4}),\ \infty\,.$$

For $u = \pm\Lambda^2$ we have two stable singularities, but the $u \to \infty$ singularity is not stable.

To understand the properties of this curve better, let us first consider a generic situation: For a stable singularity at, for instance, $u = 0$, the family of curves near $u = 0$ can be written in the form

$$y^2 = (x - 1)(x^2 - u^n)\,,$$

for some integer n. The monodromy around $u = 0$ is then conjugate to T^n where the matrix T is defined in (95). This can be understood as follows: Consider the holomorphic Abelian differential $\omega = dx/y$ on a curve $y^2 = (x - 1)(x^2 - \lambda)$, where $\lambda = u^n$. The periods can be written as

$$\omega_1 = \int_{\Gamma_1} \frac{dx}{y}\,, \qquad \omega_2 = \int_{\Gamma_2} \frac{dx}{y}\,,$$

where Γ_1 is a path from $u = -\lambda^{1/2}$ to $u = \lambda^{1/2}$, and Γ_2 is a path from $u = \lambda^{1/2}$ to $u = 1$ (Fig. 6(a)). As $\lambda \to e^{2\pi i}\lambda$, $\lambda^{1/2} \to -\lambda^{1/2}$ and the cut Γ_1 moves as in Fig. 6(b). This simply exchanges the branches of the integrand and therefore $\omega_1 \to \omega_1$.

As for ω_2, the path Γ_2 is transformed as in Fig. 7.

Thus, $\omega_2 \to \omega_2 + \omega_1$. Since $\lambda = u^n$, when $u \to e^{2\pi i}u$, λ makes n turns and we obtain the n-th power of the monodromy. Thus, in terms of λ, the monodromy is conjugate to T while in terms of u, it is conjugate to T^n as we wanted to show.

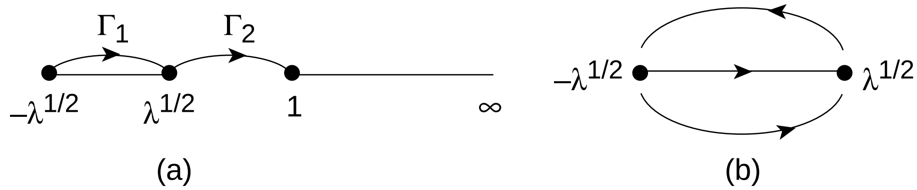

Figure 6

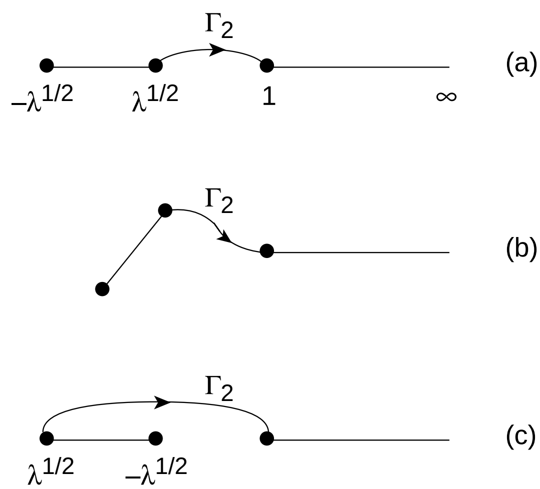

Figure 7

Given any polynomial, a useful quantity is its discriminant defined as

$$\Delta = \prod_{i<j} (e_i - e_j)^2,$$

where the e_i's are the roots. Δ can be expressed in terms of the coefficients of the polynomial. Clearly, at a singularity when two branch points coincide, $\Delta = 0$ (except for a singularity at ∞). For the example $y^2 = (x-1)(x^2 - u^n)$, $\Delta \sim u^n$ near $u = 0$ where the monodromy is conjugate to T^n. Hence, in the stable case, the exponent of the monodromy is the order of zero of Δ. For instance, for the family of curves described by (116), Δ has first-order zeros at $u = \pm\Lambda^2$, and the monodromies around these points are conjugate to T. Let us now look at $u \to \infty$. For large u the branch points are approximately at $x = 0$, $\Lambda^4/4u$, u, ∞. Thus the singularity at $u = \infty$ is not stable as more than two branch points coincide in this limit. This is cured by a change of variables

$$x = x'u\,, \qquad y = y'u^{3/2}\,,$$

which shifts the branch points to 0, $\Lambda^4/4u^2$, 1, ∞. The discriminant now behaves like $\Delta \sim u^{-4}$ corresponding to a monodromy conjugate to T^{-4}. Due to the presence of $\sqrt{u}$ in $y \to y'$, the monodromy in terms of the original varialbles (x, y) has an extra minus sign and is conjugate to PT^{-4}. Thus we see that the curve (116) produces the correct monodromies of the $N_f = 0$ theory.

To obtain the curves for theories with non-zero N_f, we should know the monodromies which arise in these theories. This can be easily worked out (as in the $N_f = 0$ case) since we already know the charge spectrum of the particles which become massless at the singularities on the u-plane. First note that, as in the $N_f = 0$ theory, the monodromy at $u = \infty$ can be obtained form the perturbative β-function (111), or equivalently (113), and is given by

$$\mathcal{M}_\infty = PT^{N_f-4}.$$

The singularities at finite points in the u-plane, in general, correspond to massless magnetic monopoles with one unit of magnetic charge and n_e units of magnetic charge. To calculate the corresponding monodromy, we have to go to a dual description of the theory in which the monopole couples to the dual gauge field the way an electron couples to the usual gauge field. In this frame, the monodromy is determined by the one-loop QED β-function (109) with k hypermultiplets and is given by T^k. This has to be conjugated with $T^{n_e}S$ which converts a hypermultiplet of charge $(0,1)$ into a monopole of charge $(1,n_e)$. Hence the monodromy around a point with k magnetic monopoles with $(n_m = 1, n_e)$ is $(T^{n_e}S)T^k(T^{n_e}S)^{-1}$. Similar arguments can be applied to calculate the monodromy for the $(2,1)$ state in the $N_f = 3$ theory. In the following we list all the monodromies around the singularities described in the previous subsection:

$$\begin{aligned}
N_f = 0: &\quad STS^{-1},\ (T^2S)T(T^2S)^{-1} \to M_\infty = PT^{-4}\,, \\
N_f = 1: &\quad STS^{-1},\ (TS)T(TS)^{-1},\ (T^2S)T(T^2S)^{-1} \to M_\infty = PT^{-3}\,, \\
N_f = 2: &\quad ST^2S^{-1},\ (TS)T^2(TS)^{-1} \to M_\infty = PT^{-2}\,, \\
N_f = 3: &\quad (ST^2S)T(ST^2S)^{-1},\ ST^4S^{-1} \to M_\infty = PT^{-1}\,.
\end{aligned}$$

Using $S^2 = -1$ and $(ST)^3 = 1$, it is easy to check that, for each N_f, the product of the monodromy matrices at finite u yields M_∞. Note that in the first three cases ($N_f = 0, 1, 2$) the electric charges assigned to the singularities differ in sign from those determined by our previous arguments. This is consistent with the inherent ambiguities in relating the universal charges with the transformation properties under the centre of the group. Based on the knowledge obtained so far, in the following we sketch the arguments of Seiberg and Witten to obtain the explicit form of the curves for theories with non-zero N_f.

Properties of the $N_f = 0$ Curve: Before proceeding further, it is very useful to enumerate some properties of the $N_f = 0$ curve which are expected to remain valid even for non-zero N_f:

1. The equation describing the family of curves is of the generic form $y^2 = F(x, u, \Lambda)$, where F is a polynomial at most cubic in x and u.

2. The part of F cubic in x and u is $F_0 = x^2(x - u)$.

3. If we assign $U(1)_{\mathcal{R}}$ charge 4 to u and x and charge 2 to Λ, then F has charge 12. If y is assigned charge 6, then the curve is invariant under $U(1)_{\mathcal{R}}$ transformations.

4. F can be written as $F = F_0 + \Lambda^4 F_1$ where $F_1 = x/4$.

The property (1) remains an ansatz for $N_f > 0$. It will lead to the correct monodromy matrices, and it can also be justified in part when considering the $N_f = 4$ theory. Property (2) is a consequence of the fact that as $u \to \infty$, we must obtain the monodromy at infinity associated with a logarithm in a_D coming from the one-loop β-function. This means that one of the branch points at finite x should move to infinity as $u \to \infty$. For large u, the cubic part can be written as $(x - e_1 u)(x - e_2 u)(x - e_3 u)$. Then the desired behaviour can be obtained if two e_i's coincide and the other is different, say $e_1 = e_2 \neq e_3$. In this case, by a redefinition of x, we can bring F_0 to the form $x^2(x - u)$. Property (4) in the $N_f = 0$ theory is a consequence of the $U(1)_R$ charge assignments. Note that F has only a classical contribution plus a one-instanton term : $\Lambda^4 x/4$. Now we use these properties to determine the curves for $N_f = 1, 2, 3$.

The Curves for Massless $N_f = 1$ Theory: In this theory, from equation (114), the instanton amplitude is proportional to Λ_1^3. However, for $N_f \geq 1$, the instanton factor is odd under the ρ parity and, therefore, only even powers of it can appear in F. Since Λ_1 has $U(1)_R$ charge 2 and F has charge 12, the only possibility is

$$y^2 = x^2(x - u) + t\Lambda_1^6 \,.$$

The constant t can be absorbed in a redefinition of Λ_1: $t\Lambda_1^6 = \tilde{\Lambda}^6$. The discriminant of this family of curves is

$$\Delta = \tilde{\Lambda}_1^6(4u^3 - 27\tilde{\Lambda}_1^6) \,.$$

This has three zeros which are interchanged under the $\mathbf{Z}_3$ transformation acting on the u-plane and the associated monodromies are conjugate to T. Similarly, the monodromy at large u can be worked out to be PT^{-3}. This is consistent with what we should have for the $N_f = 1$ theory.

The Curves for Massless $N_f = 2$ Theory: In this case the instanton factor is Λ_2^2 and, again, in the absence of bare masses, only even powers of it can appear in F. Since Λ_2 has $U(1)_R$ charge 2, the general form of the curve is

$$y^2 = x^2(x - u) + (ax + bu)\Lambda_2^4 \,.$$

From our discussion of the $N_f = 2$ theory, we expect two singularities, each with two massless monopoles. Hence the monodromy at each singularity is conjugate to T^2 which means that the discriminant at each finite singularity should have a double zero. This condition determines a and b. After a rescaling of Λ_2, the family of $N_f = 2$ curves can be written as

$$y^2 = (x^2 - \tilde{\Lambda}_2^4)(x - u) \,,$$

which has the expected $\mathbf{Z}_2$ symmetry.

The curves for Massless $N_f = 3$ Theory: In this case, there are two singularities on the u-plane with monodromies conjugate to T^4 and T respectively, and there is no symmetry acting on the u-plane. Take the T^4 singularity to be at $u = 0$. The discriminant then should have a fourth-order zero at $u = 0$. This, together with the usual $U(1)_{\mathcal{R}}$ charge and ρ parity assignments, leads to

$$F = a\Lambda_3^2 x^2 + bu^2 x + cux^2 + x^3 \,.$$

Here, $b \neq 0$ since otherwise the curve is singular for all u. Requiring that the cubic part of F have the expected classical behaviour, and after some rescaling, one gets

$$y^2 = x^2(x-u) + \tilde{\Lambda}_3^2(x-u)^2 \,.$$

Note that in the above we have only considered theories without a bare mass term. Since a mass term has odd ρ-parity, in a theory with non-zero bare masses, odd powers of the instanton factor can also contribute to the equation for the curve. We will not discuss these cases here, but for the sake of completeness, will simply reproduce the final results:

$$\begin{aligned}
N_f = 1: \qquad y^2 &= x^2(x-u) + \frac{1}{4} m\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6 \,, \\
N_f = 2: \qquad y^2 &= (x^2 - \frac{1}{64}\Lambda_2^4)(x-u) + m_1 m_2 \Lambda_2^2 x - \frac{1}{64}(m_1^2 + m_2^2)\Lambda_2^4 \,, \\
N_f = 3: \qquad y^2 &= x^2(x-u) - \frac{1}{64}\Lambda_3^2(x-u)^2 - \frac{1}{64}(m_1^2 + m_2^2 + m_3^2)\Lambda_3^2(x-u) \\
&\quad + \frac{1}{4} m_1 m_2 m_3 \Lambda_3 x - \frac{1}{64}(m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2)\Lambda_3^2 \,.
\end{aligned}$$

For more details and discussions (including the $N_f = 4$ case), as well as a more rigorous method of obtaining the equations for the curves, the reader is referred to the original work of Seiberg and Witten [32].

Once the curves are determined, the rest of the procedure is very similar to the $N_f = 0$ case. We define the quantities a and a_D by the contour integrals

$$a = \int_{\gamma_1} \lambda \,, \quad a_D = \int_{\gamma_2} \lambda \,,$$

and the metric is given by $\mathrm{Im}(\tau)$ with $\tau = \frac{da_D}{du} / \frac{da}{du}$. Here λ is a holomorphic one-form such that

$$\frac{d\lambda}{du} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y} \,.$$

The difference with the $N_f = 0$ case is that now λ can have poles with non-zero residue, however, the residues should be u independent. If this is the case, then τ still undergoes an $SL(2,Z)$ transformation while (a_D, a) undergoes an $SL(2,Z)$ transformation plus a shift. As we saw in the case of QED with quarks of non-zero bare masses, this shift is actually needed and is consistent with the modified form of the BPS mass formula (84). From this it follows that the residues of λ, which are responsible for the shifts, should be proportional to the bare quark masses. This was checked in [32] for the $N_f = 2$ curve and found to be the case. In fact, the existence of residues is very restrictive and this information was used by Seiberg and Witten to drive the curve for the $N_f = 4$ theory in a rigorous way. The curves for other theories can then be determined by renormalization group flow and the results agree with what is listed above.

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