
Introduction to Symmetry Analysis

BRIAN J. CANTWELL
Stanford University



CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

© Cambridge University Press 2002

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

This book was set using LaTeX and Mathematica[™]

Typeface Times Roman 10/13 pt. System L^AT_EX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Cantwell, Brian.

Introduction to symmetry analysis / Brian J. Cantwell.

p. cm. – (Cambridge texts in applied mathematics)

Includes bibliographical references and index.

ISBN 0-521-77183-8 – ISBN 0-521-77740-2

1. Differential equations – Numerical solutions 2. Symmetry (Physics)
I. Title. II. Series.

QA371 .C195 2002

515'.35 – dc21

2001037835

ISBN 0 521 77183 8 hardback
ISBN 0 521 77740 2 paperback

Contents

Author's Preface	<i>page</i> xvii
Historical Preface	xxi
Rise of the Academies	xxi
Abel and Galois	xxiv
Lie and Klein	xxvi
1870	xxviii
Lie's Arrest	xxix
Gauss, Riemann, and the New Geometry	xxx
The Erlangen Program	xxxiv
Lie's Career at Leipzig	xxxvi
A Falling Out	xxxvi
Lie's Final Return to Norway	xxxvii
After 1900	xxxviii
The Ariadne Thread	xxxix
Suggested Reading	xl
1 Introduction to Symmetry	1
1.1 Symmetry in Nature	1
1.2 Some Background	3
1.3 The Discrete Symmetries of Objects	4
1.3.1 The Twelfefold Discrete Symmetry Group of a Snowflake	4
1.4 The Principle of Covariance	10
1.5 Continuous Symmetries of Functions and Differential Equations	12
1.5.1 One-Parameter Lie Groups in the Plane	13

1.5.2	Invariance of Functions, ODEs, and PDEs under Lie Groups	14
1.6	Some Notation Conventions	23
1.7	Concluding Remarks	25
1.8	Exercises	29
	References	31
2	Dimensional Analysis	33
2.1	Introduction	33
2.2	The Two-Body Problem in a Gravitational Field	34
2.3	The Drag on a Sphere	38
	2.3.1 Some Further Physical Considerations	43
2.4	The Drag on a Sphere in High-Speed Gas Flow	44
2.5	Buckingham's Pi Theorem – The Dimensional-Analysis Algorithm	47
2.6	Concluding Remarks	50
2.7	Exercises	50
	References	54
3	Systems of ODEs and First-Order PDEs; State-Space Analysis	55
3.1	Autonomous Systems of ODEs in the Plane	55
3.2	Characteristics	56
3.3	First-Order Ordinary Differential Equations	57
	3.3.1 Perfect Differentials	57
	3.3.2 The Integrating Factor; Pfaff's Theorem	59
	3.3.3 Nonsolvability of the Integrating Factor	60
	3.3.4 Examples of Integrating Factors	62
3.4	Thermodynamics; The Legendre Transformation	65
3.5	Incompressible Flow in Two Dimensions	68
3.6	Fluid Flow in Three Dimensions – The Dual Stream Function	69
	3.6.1 The Method of Lagrange	70
	3.6.2 The Integrating Factor in Three and Higher Dimensions	72
	3.6.3 Incompressible Flow in Three Dimensions	73
3.7	Nonlinear First-Order PDEs – The Method of Lagrange and Charpit	74
	3.7.1 The General and Singular Solutions	77
3.8	Characteristics in n Dimensions	79
	3.8.1 Nonlinear First-Order PDEs in n Dimensions	82
3.9	State-Space Analysis in Two and Three Dimensions	83
	3.9.1 Critical Points	84

3.9.2	Matrix Invariants	84
3.9.3	Linear Flows in Two Dimensions	85
3.9.4	Linear Flows in Three Dimensions	88
3.10	Concluding Remarks	90
3.11	Exercises	91
	References	95
4	Classical Dynamics	96
4.1	Introduction	96
4.2	Hamilton's Principle	98
4.3	Hamilton's Equations	101
4.3.1	Poisson Brackets	103
4.4	The Hamilton–Jacobi Equation	105
4.5	Examples	108
4.6	Concluding Remarks	119
4.7	Exercises	119
	References	120
5	Introduction to One-Parameter Lie Groups	121
5.1	The Symmetry of Functions	121
5.2	An Example and a Counterexample	122
5.2.1	Translation along Horizontal Lines	122
5.2.2	A Reflection and a Translation	123
5.3	One-Parameter Lie Groups	123
5.4	Invariant Functions	125
5.5	Infinitesimal Form of a Lie Group	127
5.6	Lie Series, the Group Operator, and the Infinitesimal Invariance Condition for Functions	128
5.6.1	Group Operators and Vector Fields	130
5.7	Solving the Characteristic Equation $X\Psi[x] = 0$	130
5.7.1	Invariant Points	132
5.8	Reconstruction of a Group from Its Infinitesimals	132
5.9	Multiparameter Groups	134
5.9.1	The Commutator	135
5.10	Lie Algebras	136
5.10.1	The Commutator Table	137
5.10.2	Lie Subalgebras	138
5.10.3	Abelian Lie Algebras	138
5.10.4	Ideal Lie Subalgebras	139
5.11	Solvable Lie Algebras	139
5.12	Some Remarks on Lie Algebras and Vector Spaces	142

5.13	Concluding Remarks	145
5.14	Exercises	145
	References	148
6	First-Order Ordinary Differential Equations	149
6.1	Invariant Families	149
6.2	Invariance Condition for a Family	152
6.3	First-Order ODEs; The Integrating Factor	155
6.4	Using Groups to Integrate First-Order ODEs	157
6.5	Canonical Coordinates	162
6.6	Invariant Solutions	166
6.7	Elliptic Curves	172
6.8	Criterion for a First-Order ODE to Admit a Given Group	174
6.9	Concluding Remarks	176
6.10	Exercises	176
	References	177
7	Differential Functions and Notation	178
7.1	Introduction	179
7.1.1	Superscript Notation for Dependent and Independent Variables	181
7.1.2	Subscript Notation for Derivatives	181
7.1.3	Curly-Brace Subscript Notation for Functions That Transform Derivatives	183
7.1.4	The Total Differentiation Operator	184
7.1.5	Definition of a Differential Function	185
7.1.6	Total Differentiation of Differential Functions	186
7.2	Contact Conditions	187
7.2.1	One Dependent and One Independent Variable	187
7.2.2	Several Dependent and Independent Variables	188
7.3	Concluding Remarks	189
7.4	Exercises	190
	References	190
8	Ordinary Differential Equations	191
8.1	Extension of Lie Groups in the Plane	191
8.1.1	Finite Transformation of First Derivatives	191
8.1.2	The Extended Transformation Is a Group	193
8.1.3	Finite Transformation of the Second Derivative	195
8.1.4	Finite Transformation of Higher Derivatives	195

8.1.5	Infinitesimal Transformation of the First Derivative	197
8.1.6	Infinitesimal Transformation of the Second Derivative	198
8.1.7	Infinitesimal Transformation of Higher-Order Derivatives	199
8.1.8	Invariance of the Contact Conditions	200
8.2	Expansion of an ODE in a Lie Series; The Invariance Condition for ODEs	201
8.2.1	What Does It Take to Transform a Derivative?	202
8.3	Group Analysis of Ordinary Differential Equations	203
8.4	Failure to Solve for the Infinitesimals That Leave a First-Order ODE Invariant	203
8.5	Construction of the General First-Order ODE That Admits a Given Group – The Riccati Equation	204
8.6	Second-Order ODEs and the Determining Equations of the Group	207
8.6.1	Projective Group of the Simplest Second-Order ODE	208
8.6.2	Construction of the General Second-Order ODE that Admits a Given Group	213
8.7	Higher-Order ODEs	213
8.7.1	Construction of the General p th-Order ODE That Admits a Given Group	214
8.8	Reduction of Order by the Method of Canonical Coordinates	215
8.9	Reduction of Order by the Method of Differential Invariants	216
8.10	Successive Reduction of Order; Invariance under a Multiparameter Group with a Solvable Lie Algebra	217
8.10.1	Two-Parameter Group of the Blasius Equation	219
8.11	Group Interpretation of the Method of Variation of Parameters	228
8.11.1	Reduction to Quadrature	230
8.11.2	Solution of the Homogeneous Problem	231
8.12	Concluding Remarks	233
8.13	Exercises	233
	References	236
9	Partial Differential Equations	237
9.1	Finite Transformation of Partial Derivatives	238
9.1.1	Finite Transformation of the First Partial Derivative	238
9.1.2	Finite Transformation of Second and Higher Partial Derivatives	239
9.1.3	Variable Count	241
9.1.4	Infinitesimal Transformation of First Partial Derivatives	242

9.1.5	Infinitesimal Transformation of Second and Higher Partial Derivatives	243
9.1.6	Invariance of the Contact Conditions	245
9.2	Expansion of a PDE in a Lie Series – Invariance Condition for PDEs	247
9.2.1	Isolating the Determining Equations of the Group – The Lie Algorithm	247
9.2.2	The Classical Point Group of the Heat Equation	248
9.3	Invariant Solutions and the Characteristic Function	255
9.4	Impulsive Source Solutions of the Heat Equation	257
9.5	A Modified Problem of an Instantaneous Heat Source	263
9.6	Nonclassical Symmetries	269
9.6.1	A Non-classical Point Group of the Heat Equation	270
9.7	Concluding Remarks	272
9.8	Exercises	273
	References	275
10	Laminar Boundary Layers	277
10.1	Background	277
10.2	The Boundary-Layer Formulation	279
10.3	The Blasius Boundary Layer	281
10.3.1	Similarity Variables	282
10.3.2	Reduction of Order; The Phase Plane	284
10.3.3	Numerical Solution of the Blasius Equation as a Cauchy Initial-Value Problem	291
10.4	Temperature Gradient Shocks in Nonlinear Diffusion	293
10.4.1	First Try: Solution of a Cauchy Initial-Value Problem – Uniqueness	294
10.4.2	Second Try: Solution Using Group Theory	295
10.4.3	The Solution	298
10.4.4	Exact Thermal Analogy of the Blasius Boundary Layer	299
10.5	Boundary Layers with Pressure Gradient	301
10.6	The Falkner–Skan Boundary Layers	305
10.6.1	Falkner–Skan Sink Flow	310
10.7	Concluding Remarks	313
10.8	Exercises	313
	References	317
11	Incompressible Flow	318
11.1	Invariance Group of the Navier–Stokes Equations	318

11.2	Frames of Reference	321
11.3	Two-Dimensional Viscous Flow	323
11.4	Viscous Flow in a Diverging Channel	325
11.5	Transition in Unsteady Jets	329
	11.5.1 The Impulse Integral	320
	11.5.2 Starting-Vortex Formation in an Impulsively Started Jet	325
11.6	Elliptic Curves and Three-Dimensional Flow Patterns	353
	11.6.1 Acceleration Field in the Round Jet	353
11.7	Classification of Falkner–Skan Boundary Layers	357
11.8	Concluding Remarks	360
11.9	Exercises	361
	References	362
12	Compressible Flow	364
12.1	Invariance Group of the Compressible Euler Equations	365
12.2	Isentropic Flow	369
12.3	Sudden Expansion of a Gas Cloud into a Vacuum	370
	12.3.1 The Gasdynamic–Shallow-Water Analogy	371
	12.3.2 Solutions	372
12.4	Propagation of a Strong Spherical Blast Wave	375
	12.4.1 Effect of the Ratio of Specific Heats	382
12.5	Compressible Flow Past a Thin Airfoil	384
	12.5.1 Subsonic Flow, $M_\infty < 1$	386
	12.5.2 Supersonic Similarity, $M_\infty > 1$	389
	12.5.3 Transonic Similarity, $M_\infty \approx 1$	390
12.6	Concluding Remarks	391
12.7	Exercises	392
	References	393
13	Similarity Rules for Turbulent Shear Flows	395
13.1	Introduction	395
13.2	Reynolds-Number Invariance	397
13.3	Group Interpretation of Reynolds-Number Invariance	401
	13.3.1 One-Parameter Flows	401
	13.3.2 Temporal Similarity Rules	403
	13.3.3 Frames of Reference	404
	13.3.4 Spatial Similarity Rules	405
	13.3.5 Reynolds Number	407
13.4	Fine-Scale Motions	408
	13.4.1 The Inertial Subrange	410

13.5	Application: Experiment to Measure Small Scales in a Turbulent Vortex Ring	413
13.5.1	Similarity Rules for the Turbulent Vortex Ring	415
13.5.2	Particle Paths in the Turbulent Vortex Ring	417
13.5.3	Estimates of Microscales	419
13.5.4	Vortex-Ring Formation	420
13.5.5	Apparatus Design	422
13.6	The Geometry of Dissipating Fine-Scale Motion	424
13.6.1	Transport Equation for the Velocity Gradient Tensor	425
13.7	Concluding Remarks	437
13.8	Exercises	438
	References	444
14	Lie–Bäcklund Transformations	446
14.1	Lie–Bäcklund Transformations – Infinite Order Structure	447
14.1.1	Infinitesimal Lie–Bäcklund Transformation	449
14.1.2	Reconstruction of the Finite Lie–Bäcklund Transformation	452
14.2	Lie Contact Transformations	453
14.2.1	Contact Transformations and the Hamilton–Jacobi Equation	457
14.3	Equivalence Classes of Transformations	457
14.3.1	Every Lie Point Operator Has an Equivalent Lie–Bäcklund Operator	459
14.3.2	Equivalence of Lie–Bäcklund Transformations	459
14.3.3	Equivalence of Lie–Bäcklund and Lie Contact Operators	461
14.3.4	The Extended Infinitesimal Lie–Bäcklund Group	461
14.3.5	Proper Lie–Bäcklund Transformations	462
14.3.6	Lie Series Expansion of Differential Functions and the Invariance Condition	462
14.4	Applications of Lie–Bäcklund Transformations	464
14.4.1	Third-Order ODE Governing a Family of Parabolas	466
14.4.2	The Blasius Equation $y_{xxx} + yy_{xx} = 0$	471
14.4.3	A Particle Moving Under the Influence of a Spherically Symmetric Inverse-Square Body Force	474
14.5	Recursion Operators	478
14.5.1	Linear Equations	479
14.5.2	Nonlinear Equations	481

14.6	Concluding Remarks	496
14.7	Exercises	496
	References	497
15	Variational Symmetries and Conservation Laws	498
15.1	Introduction	498
	15.1.1 Transformation of Integrals by Lie–Bäcklund Groups	499
	15.1.2 Transformation of the Differential Volume	499
	15.1.3 Invariance Condition for Integrals	500
15.2	Examples	504
15.3	Concluding Remarks	511
15.4	Exercises	512
	References	513
16	Bäcklund Transformations and Nonlocal Groups	515
16.1	Two Classical Examples	517
	16.1.1 The Liouville Equation	517
	16.1.2 The Sine–Gordon Equation	519
16.2	Symmetries Derived from a Potential Equation; Nonlocal Symmetries	521
	16.2.1 The General Solution of the Burgers Equation	522
	16.2.2 Solitary-Wave Solutions of the Korteweg–de Vries Equation	533
16.3	Concluding Remarks	547
16.4	Exercises	548
	References	550
<i>Appendix 1</i>	Review of Calculus and the Theory of Contact	552
A1.1	Differentials and the Chain Rule	552
	A1.1.1 A Problem with Notation	553
	A1.1.2 The Total Differentiation Operator	554
	A1.1.3 The Inverse Total Differentiation Operator	555
A1.2	The Theory of Contact	555
	A1.2.1 Finite-Order Contact between a Curve and a Surface	555
<i>Appendix 2</i>	Invariance of the Contact Conditions under Lie Point Transformation Groups	558
A2.1	Preservation of Contact Conditions – One Dependent and One Independent Variable	558
	A2.1.1 Invariance of the First-Order Contact Condition	558

A2.1.2	Invariance of the Second-Order Contact Condition	559
A2.1.3	Invariance of Higher-Order Contact Conditions	560
A2.2	Preservation of the Contact Conditions – Several Dependent and Independent Variables	562
A2.2.1	Invariance of the First-Order Contact Condition	562
A2.2.2	Invariance of the Second-Order Contact Conditions	564
A2.2.3	Invariance of Higher-Order Contact Conditions	566
<i>Appendix 3</i>	Infinite-Order Structure of Lie–Bäcklund Transformations	569
A3.1	Lie Point Groups	569
A3.2	Lie–Bäcklund Groups	570
A3.3	Lie Contact Transformations	570
A3.3.1	The Case $m > 1$	573
A3.3.2	The Case $m = 1$	573
A3.4	Higher-Order Tangent Transformation Groups	574
A3.5	One Dependent Variable and One Independent Variable	580
A3.6	Infinite-Order Structure	581
A3.6.1	Infinitesimal Transformation	582
Reference		583
<i>Appendix 4</i>	Symmetry Analysis Software	584
A4.1	Summary of the Theory	586
A4.2	The Program	588
A4.2.1	Getting Started	589
A4.2.2	Using the Program	591
A4.2.3	Solving the Determining Equations and Viewing the Results	593
A4.3	Timing, Memory and Saving Intermediate Data	594
A4.3.1	Why Give the Output in the Form of Strings?	597
A4.3.2	Summary of Program Functions	597
References		600
Author Index		601
Subject Index		604

Introduction to Symmetry

1.1 Symmetry in Nature

Symmetry is universal, fascinating, and of immense practical importance. As human beings we have evolved a perception of symmetry that lies at the core of our conscious life. Symmetries provide cues that help us relate to our environment and guide our movements through the world. Everyone has a taste for things that are in some way symmetrical or possess a pleasing deviation from perfect symmetry. A highly paid supermodel will often have rather symmetrical facial features. But a perfectly symmetrical face has an unnatural, androgynous look, and rarely is this associated with great beauty or a memorable persona. Perhaps the most perfect object we can imagine is a circle, yet dividing the circumference by the diameter produces the irrational number π that we can only symbolize. Perfect, unequivocal, symmetry, like perfect theory, eludes us always.

Objects of the natural world universally exhibit some form of symmetry. Despite an astonishing variety of shapes, all members of the animal kingdom possess body architectures that can be sorted into only about 37 basic types. Almost all animals possess bilateral symmetry; they must eat, and to eat efficiently two hands, grasping symmetrically, are better than one. Animals must move, and to move efficiently it is essential to be balanced about the center of mass. When asymmetric development does occur, it is invariably associated with some unusual, very specific adaptation, as in the case of the bottom-dwelling flounder with both eyes on the same side of its head. The whorls and spirals of plant organs produced by the response of an expanding growth surface to surrounding mechanical constraints [1.1] have been the subject of scientific inquiry for centuries. The nearly perfect spheres that fill the universe – stars, planets, moons, and the like – are shaped primarily by gravitational forces, which act in a three-dimensional universe where no one direction or position is distinguished from another. Free space is homogeneous and isotropic. We marvel at the incredible

variety of delicate geometrical forms associated with the six-sided symmetry of snowflakes or the regular crystalline structure of gems formed over millennia by heat, pressure, and water, their shape a consequence of the forces that act on an atomic scale according to the symmetries of the electronic outer shells that participate in bonding. Anyone who studies fluid mechanics is struck by the aesthetic symmetry of shock wave patterns or bubbly flows or any of the myriad spiral patterns that mark the vortical world that flows over, around, and through us.

There have been many attempts to quantify the relationship between symmetry and beauty. A fine example of this can be found in the fascinating work of George David Birkhoff (1884–1944) [1.2], who was one of the preeminent American mathematicians of the early 20th century and is generally credited with developing the ergodic theorem in the kinetic theory of gases. Birkhoff was originally motivated by the desire to identify what it was that made one musical piece beautiful and another not. He felt that beauty had a universal character and therefore it should be possible to quantify it mathematically, and so he developed what he called the “aesthetic measure.” Ultimately he applied this measure to a wide variety of objects – everything from musical pieces to vases to floor tilings. Today such an attempt to categorize music seems naive in view of the vast range of musical technique – everything from guitar “resonant buzz” invented accidentally by country singer Marty Robbins (but claimed by “Spirit in the Sky” Norman Greenbaum) to the patriotic screechings of Jimi Hendrix to the asynchronous beat of Dave Brubeck. No simple measure can cover it all.

Although the use of symmetries to categorize objects is interesting in its own right, that is not the purpose of this text. Our main interest is in the symmetries inherent in the physical laws that govern the natural world. Knowledge of these symmetries will be used to enhance our understanding of complex physical phenomena, to simplify and solve problems, and, ultimately, to deepen our understanding of nature. The primary goal of this text is to develop the methods of symmetry analysis based on Lie groups for the uninitiated reader and to use these methods to find and express the symmetry properties of ordinary differential equations, partial differential equations, integrals, and the solution functions that they govern. The text is directed primarily at first- and second-year graduate students in science and engineering, but it may also be useful to advanced researchers who would like to gain some familiarity with symmetry methods. The student is expected to be familiar with classical approaches to the solution of differential equations, although the early chapters provide much of the required background in terms that should be understandable to an upper-level undergraduate.

1.2 Some Background

My first encounter with Lie groups came while browsing in the GALCIT aeronautics library at Caltech in 1975. I ran across the book by Abraham Cohen [1.3], first published in 1911. The first few chapters of this book give a very lucid description of the concept of a Lie group and the idea of invariance under a group. Cohen's book makes interesting reading when one realizes that at the time it was written, Sophus Lie's ideas were still a brand-new development, yet they were seen as important enough to warrant a full-blown textbook treatment. In his 1906 treatise on *The Theory of Differential Equations* Andrew Forsyth devotes several chapters to Lie groups and Bäcklund transformations. It is a fact, however, that shortly thereafter, Lie's ideas fell into obscurity and remained so until soon after World War II. As researchers began to turn more and more often to nonlinear problems and as the inherent importance of symmetries began to be recognized, Lie's ideas gained renewed interest.

The Lie algorithm used to analyze the symmetry of mathematical expressions was developed to an advanced state through the pioneering efforts of Ovsianikov [1.5] and his students in the Soviet Union. In the United States, Garrett Birkhoff [1.6] at Harvard the son of George Birkhoff played a key role in bringing attention to Lie's ideas by clarifying the relationship between group invariance and dimensional analysis as applied to problems in fluid mechanics. Fluid mechanics, governed as it is by nonlinear equations from which a rich variety of simplified nonlinear and linear approximations can be derived, is an especially fertile source of examples and applications of group theory.

During the same period, new ideas about the role of similarity solutions as approximations to realistic complex physical problems were being developed by Barenblatt and Zel'dovich [1.7] in the Soviet Union. By the late 1960s and early 1970s the whole field was active again, and new applications of group theory were being developed by a number of researchers, including Ibragimov in the Soviet Union [1.8], Bluman and Cole at Caltech [1.9], Anderson, Kumei, and Wulfman at the University of the Pacific [1.10], Chester at Bristol [1.11], Harrison and Estabrook at the Jet Propulsion Laboratory [1.12], and many others. Today group analysis, in one form or another, is the central topic of a number of excellent textbooks, including Hansen [1.13], Ames [1.14], Olver [1.15], Bluman and Kumei [1.16], Rogers and Ames [1.17], Stephani [1.18], and most recently Ibragimov [1.19], Andreev et al. [1.20], Hydon [1.21] and Baumann [1.22]. The valuable collection of results by workers around the world contained in the CRC series edited by Ibragimov [1.23] gives testimony to the achievements of the last half century or so. Today, symmetry analysis constitutes the most important (indeed one might say the only) widely applicable method

for finding analytical solutions of nonlinear problems. The Lie algorithm can be applied to virtually any system of ODEs and PDEs. Moreover the procedure is highly systematic and amenable to programming with symbol manipulation software. As a result, sophisticated software tools are now available for analyzing the symmetries of differential equations (References [1.24], [1.25], [1.26]; see also the review of symbolic software for group analysis by Hydon [1.21] and Hereman [1.27]).

1.3 The Discrete Symmetries of Objects

For more background on the importance of symmetry, particularly in the early development of modern physics, I would recommend the works of the German–American mathematical physicist Hermann Weyl (1885–1955), who formulated the group-theoretic basis of quantum mechanics. In his monograph [1.28] Weyl writes of the role of symmetry in science and art. Weyl was a student of David Hilbert and a member of the famous group of German mathematicians at the University of Göttingen, which broke up during the Nazi era prior to the start of World War II and later re-formed as the nucleus of the Courant Institute in New York. Finally, one of my favorite readings is Feynman’s discussion of the role of symmetry in modern physics, which can be found in Chapter 52 of Volume I of the *Feynman Lectures on Physics* [1.29].

Let’s begin with a widely accepted general definition of symmetry usually attributed to Weyl.

Definition 1.1. *An object is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation. The object is then said to be invariant with respect to the given operation.*

The symmetry properties of an object can usually be expressed in terms of a set of matrices each of which, when used to transform the various points composing the object, leave it unchanged in appearance. To clarify the notion of symmetry and its mathematical description, let’s examine the rotational and reflectional symmetry of a snowflake.

1.3.1 The Twelffold Discrete Symmetry Group of a Snowflake

Transparent ice crystals form around dust particles in the atmosphere when water vapor condenses at temperatures below the freezing point. The water molecule is an isosceles triangle composed of two hydrogen atoms bonded to an oxygen atom at its apex with an angle of 104.5° between the bonds. The attraction between the hydrogen atoms of each molecule and the oxygen atoms of other molecules overcomes thermal motions, leading to the formation of

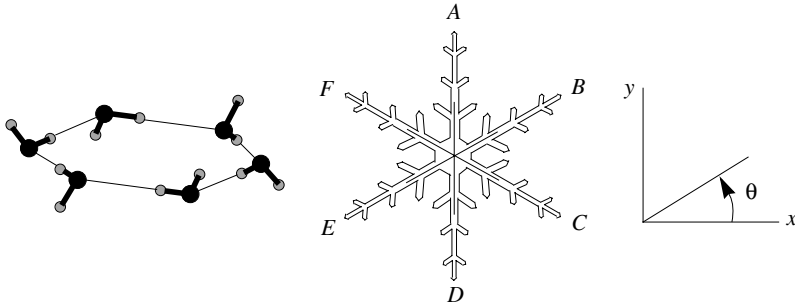


Fig. 1.1. Hexagonal structure of ice crystals and snowflakes.

hydrogen bonds, which link molecules together. The symmetry properties of the water molecule are such that if the formation temperature is below -14°C , each molecule bonds to four neighboring molecules in a repeating tetrahedral arrangement with the oxygen atoms at the corners of the tetrahedron. The tetrahedral structure gives rise to hexagonal rings of water molecules as shown in Figure 1.1. These hexagons on the molecular scale are responsible for the hexagonal symmetry of the ice crystal at macroscopic scales.

The exact structure of the ice crystal depends on its temperature history during formation. Thus, because of the infinite variability of atmospheric conditions, the shape of each snowflake is unique.

One final point before we begin: A snowflake is a three-dimensional object with a front and back. Here we wish to study only the planar symmetry of a face-on view, and so we consider the snowflake to be flat, existing entirely in a two-dimensional world. By the way, the tendency for snowflakes to be nearly flat is also explained by the crystal structure at the molecular level, which tends to be composed of relatively weakly bound planar sheets.

Figure 1.1 is my best attempt to sketch a typical snowflake. Overall it looks like a fairly symmetrical six-sided object. However, close inspection reveals a lot of detailed imperfections in my drawing. In order to have a useful discussion of the symmetry properties of the snowflake, we simply must accept the fact that we can't look at it too closely. We have to be willing to gloss over the imperfections and agree that the six corners of the snowflake are indistinguishable. The labels *A*, *B*, *C*, *D*, *E*, *F* are applied to the corners for reference purposes, but with the convention that the labels do not compromise the property that the corners themselves are indistinguishable.

This seemingly minor point is actually crucial and all-encompassing. It is central to the methods used to test for symmetry. In principle, any real object in all of its detail is completely devoid of symmetry. Therefore it is important to

recognize that the symmetries that accrue to an object apply, not to the object itself, but to its abstract representation. The moon is a sphere only when viewed from a perspective that flattens all mountain ranges, mare, rocks, pebbles, etc. Often it is the degree and manner in which a symmetry is broken that is of paramount importance. Galileo's great discovery in the seventeenth century was that the moon is not a smooth sphere but is covered with craters whose dimensions rival the largest geological features found on earth.

So it is the case today that the most important scientific questions are often associated with peeling away symmetries or searching for new symmetries of complex systems in order to reach a deeper understanding of the underlying physics. One often asks: Which parameters in a physical problem are important? Which ones are not? Occasionally, new physics is discovered when the means is found to "fix" a broken symmetry. In the modern era, the most spectacular example of this is the failure of Maxwell's equations to preserve Galilean invariance while preserving invariance under the puzzling Lorentz transformation. This led directly to Einstein's theory of special relativity, the recognition that time and space are connected, and the discovery that the speed of light is a universal invariant for all observers. A more recent example that shook the foundations of particle physics is the famous 1956 discovery by Lee and Yang [1.30], [1.31] that parity is not conserved in beta decay.

1.3.1.1 Symmetry Operations

Now, let's begin our study of the symmetries of a snowflake.

Suppose we rotate the snowflake by 30° (Figure 1.2). If we close our eyes before the rotation, then open them afterwards, we can see that an operation has been applied to the snowflake. The object is not left invariant, and the 30° rotation does not qualify as a symmetry operation. There are in fact just six rotation angles that leave the snowflake invariant: 60° , 120° , 180° , 240° , 300° , and 360° .

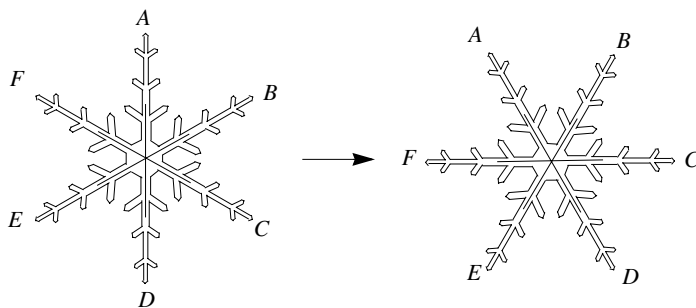
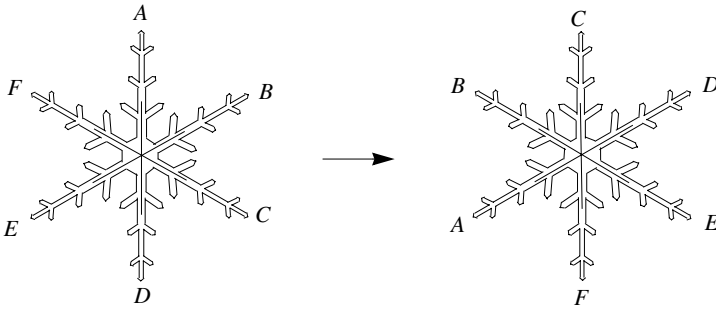


Fig. 1.2. Counterclockwise rotation by 30° .

Fig. 1.3. Counterclockwise rotation by 120° .

Now apply a rotation of 120° (Figure 1.3). In this case, there is no way we can tell that the operation has taken place (remember that the labels are not part of the object and tiny details are ignored). The snowflake is invariant, and the rotation by 120° is a symmetry operation. We can express the rotational symmetry of the snowflake mathematically as a transformation

$$\begin{aligned}\bar{x} &= x \cos \theta - y \sin \theta, \\ \bar{y} &= x \sin \theta + y \cos \theta.\end{aligned}\tag{1.1}$$

where the (x, y) coordinates are oriented as shown in Figure 1.1 and the parameter of the transformation, θ , can only take on the six discrete values given above. It is convenient (though not necessary) to think of (1.1) as a mapping of points in a given space whose coordinate axes remain fixed, rather than the usual interpretation as a rotation of the coordinate axes themselves. The object moves under the action of the transformation while the reference axes stay fixed. The six rotations are as follows:

$$\begin{aligned}C_6^1 &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, & C_6^2 &= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, & C_6^3 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ C_6^4 &= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, & C_6^5 &= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, & E &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}\tag{1.2}$$

The matrices C_6^1 , C_6^2 , C_6^3 , C_6^4 , C_6^5 , E express the rotational symmetry of *any* hexagonal object with indistinguishable sides and corners.

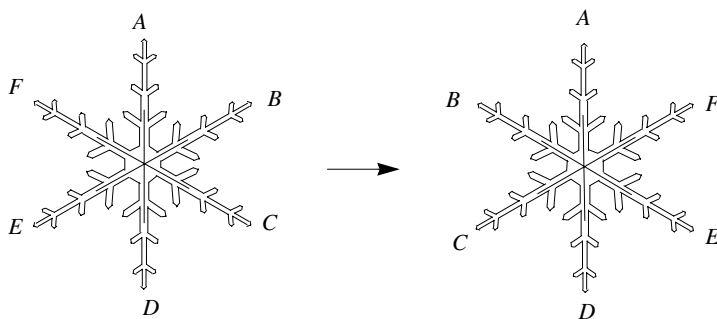


Fig. 1.4. Reflection through a vertical axis.

What about reflections? Reflection through an axis passing through $A-D$ leaves the snowflake invariant (Figure 1.4). Recall that we are considering a flat snowflake and so all operations are in the plane of the paper. If we wanted to consider the three-dimensional symmetries of a finite-thickness snowflake, then we would have to include transformations in the z -direction, either reflecting points between the front and back or rotating the object out of the plane of the paper.

The reflection through $A-D$ can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}. \quad (1.3)$$

Another reflectional symmetry is through axis $a-d$, which splits the angle between $A-D$ and $B-E$ as shown in Figure 1.5. Four other symmetry operations are: reflection through axis $B-E$, reflection through $C-F$ and reflections

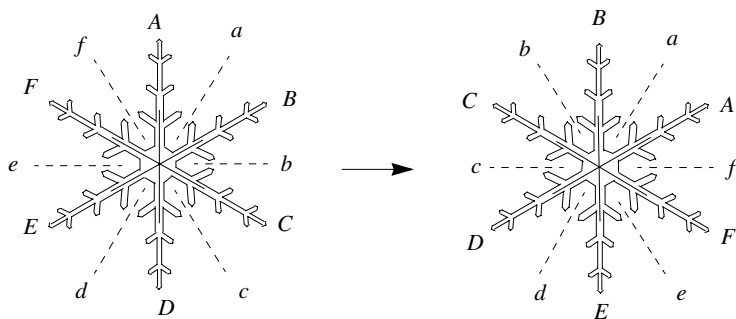


Fig. 1.5. Reflection axes of a snowflake.