

Introduction to Symplectic Field Theory

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Abstract

We sketch in this article a new theory, which we call *Symplectic Field Theory* or SFT, which provides an approach to Gromov-Witten invariants of symplectic manifolds and their Lagrangian submanifolds in the spirit of topological field theory, and at the same time serves as a rich source of new invariants of contact manifolds and their Legendrian submanifolds. Moreover, we hope that the applications of SFT go far beyond this framework.¹

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Disclosure

Despite its length, the current paper presents only a very sketchy overview of Symplectic Field Theory. It contains practically no proofs, and in a few places where the proofs are given their role is just to illustrate the involved ideas, rather than to give complete rigorous arguments.

The ideas, the algebraic formalism, and some of the applications of this new theory were presented and popularized by the authors at several conferences and seminars (e.g. [12]). As a result, currently there exists a significant mathematical community which is in some form familiar with the subject. Moreover, there are many mathematicians, including several former and current students of the authors, who are actively working on foundational aspects of the theory and its applications, and even published papers on this subject. Their results show that already the simplest versions of the theory have some remarkable corollaries (cf. [63]). We hope that the present paper will help attracting even more people to SFT.

Of course, our ideas give just a small new twist to many other active directions of research in Mathematics and Physics (Symplectic topology, Gromov-Witten invariants and quantum cohomology, Floer homology theory, String theory, just to mention few), pioneered by V.I. Arnold, C. Conley-E. Zehnder, M. Gromov, S.K. Donaldson, E. Witten, A. Floer, M. Kontsevich and others (see [2, 9, 35, 10, 18, 57, 67, 68, 46, 47]). Many people independently contributed results and ideas, which may be considered as parts of SFT. Let us just mention here the work of Yu.Chekanov [8], K.Fukaya–K.Ono–Y.-G.Oh–H.Ohta [22], A. Gathmann [24], E.Ionel–T.Parker [43, 41], Y.Ruan–A.-M.Li [58]. It also draws on other results of the current authors and their coauthors (see [15, 17, 29, 28, 27, 37, 38, 1, 39, 40]). The contact-geometric ingredient of our work is greatly motivated by two outstanding conjectures in contact geometry: Weinstein’s conjecture about periodic orbits of Reeb fields [66], and Arnold’s chord conjecture [4].

Presently, we are working on a series of papers devoted to the foundations, applications, and further development of SFT. Among the applications, some of which are mentioned in this paper, are new invariants of contact manifolds and Legendrian knots and links, new methods for computing Gromov-Witten invariants, new restrictions on the topology of Lagrangian submanifolds, new non-squeezing type theorems in contact geometry etc. We are expecting new links with the low-dimensional topology and, possibly, Physics. It seems, however, that what we see at the moment is just a tip of an iceberg. The main body of Symplectic Field Theory and its applications is yet to be discovered.

Guide for an impatient reader. The paper consists of two parts. The first part, except Section 1.9 and the end of Section 1.8, contains some necessary background symplectic-geometric and analytic information. An impatient reader can try to begin reading with Section 1.9, and use the rest of the first part for the references. The second part begins with its own introduction (Section 2.1) where we present a very rough sketch of SFT. At the end of Section 2.1 we describe the plan of the remainder of the paper.

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1 Symplectic and analytic setup

1.1 Contact preliminaries

A 1-form α on a $(2n - 1)$ -dimensional manifold V is called *contact* if the restriction of $d\alpha$ to the $(2n - 2)$ -dimensional tangent distribution $\xi = \{\alpha = 0\}$ is non-degenerate (and hence symplectic). A codimension 1 tangent distribution ξ on V is called a *contact structure* if it can be locally (and in the co-orientable case globally) defined by the Pfaffian equation $\alpha = 0$ for some choice of a contact form α . The pair (V, ξ) is called a *contact manifold*. According to Frobenius' theorem the contact condition is a condition of maximal non-integrability of the tangent hyperplane field ξ . In particular,

all integral submanifolds of ξ have dimension $\leq n - 1$. On the other hand, $(n - 1)$ -dimensional integral submanifolds, called *Legendrian*, always exist in abundance. We will be dealing in this paper only with co-orientable, and moreover co-oriented contact structures. Any non-coorientable contact structure can be canonically double-covered by a coorientable one. If a contact form α is fixed then one can associate with it the *Reeb vector field* R_α , which is transversal to the contact structure $\xi = \{\alpha = 0\}$. The field R_α is uniquely determined by the equations $R_\alpha \lrcorner d\alpha = 0$; $\alpha(R_\alpha) = 1$. The flow of R_α preserves the contact form α .

The $2n$ -dimensional manifold $M = (T(V)/\xi)^* \setminus V$, called the *symplectization* of (V, ξ) , carries a natural symplectic structure ω induced by an embedding $M \rightarrow T^*(V)$ which assigns to each linear form $T(V)/\xi \rightarrow \mathbb{R}$ the corresponding form $T(V) \rightarrow T(V)/\xi \rightarrow \mathbb{R}$. A choice of a contact form α (if ξ is co-orientable) defines a splitting $M = V \times (\mathbb{R} \setminus 0)$. As ξ is assumed to be co-oriented we can pick the positive half $V \times \mathbb{R}_+$ of M , and call it symplectization as well. The symplectic structure ω can be written in terms of this splitting as $d(\tau\alpha)$, $\tau > 0$. It will be more convenient for us, however, to use additive notation and write ω as $d(e^t\alpha)$, $t \in \mathbb{R}$, on $M = V \times \mathbb{R}$. Notice that the vector field $T = \frac{\partial}{\partial t}$ is conformally symplectic: we have $\mathcal{L}_T\omega = \omega$, as well as $\mathcal{L}_T(e^t\alpha) = e^t\alpha$, where \mathcal{L}_T denotes the Lie derivative along the vector field T . All the notions of contact geometry can be formulated as the corresponding symplectic notions, invariant or equivariant with respect to this conformal action. For instance, any contact diffeomorphism of V lifts to an equivariant symplectomorphism of M ; contact vector fields on V (i.e. vector fields preserving the contact structure) are projections of \mathbb{R} -invariant symplectic (and automatically Hamiltonian) vector fields on M ; Legendrian submanifolds in M correspond to cylindrical (i.e. invariant with respect to the \mathbb{R} -action) Lagrangian submanifolds of M .

Notice that the Hamiltonian vector field on $V \times \mathbb{R}$, defined by the Hamiltonian function $H = e^t$ is invariant under translations $t \mapsto t + c$, and projects to the Reeb vector field R_α under the projection $V \times \mathbb{R} \rightarrow \mathbb{R}$.

The symplectization of a contact manifold is an example of a symplectic manifold with *cylindrical* (or rather conical) ends. We mean by that a possibly non-compact symplectic manifold (W, ω) with ends of the form $E^+ = V^+ \times [0, \infty)$ and $E^- = V^- \times (-\infty, 0]$, such that V^\pm are compact manifolds, and $\omega|_{V^\pm} = d(e^t\alpha^\pm)$, where α^\pm are contact forms on V^\pm . In other words, the ends E^\pm of (W, ω) are symplectomorphic, respectively, to the positive or negative halves of the symplectizations of contact manifolds $(V^\pm, \xi^\pm = \{\alpha^\pm = 0\})$. We will consider the splitting of the ends and the contact forms α^\pm to be parts of the structure of a symplectic manifold with cylindrical ends. We will also call (W, ω) a *directed symplectic cobordism* between the contact manifolds (V^+, ξ^+) and (V^-, ξ^-) , and denote it by $\overrightarrow{V^-V^+}$.

Sometimes we will have to consider the compact part $W^0 = W \setminus (\text{Int}E^+ \cup \text{Int}E^-)$ of a directed symplectic cobordism $\overrightarrow{V^-V^+}$. If it is not clear from the context we will refer to W^0 as a *compact*, and to W as a *completed* symplectic cobordism.

Let us point out that “symplectic cobordism” is *not an equivalence relation, but rather a partial order*. Existence of a directed symplectic cobordism $\overrightarrow{V^-V^+}$ does not imply the existence of a directed symplectic cobordism $\overrightarrow{V^+V^-}$, even if one does not fix contact forms for the contact structures ξ^\pm . On the other hand, directed symplectic

cobordisms $\overrightarrow{V_0V_1}$ and $\overrightarrow{V_1V_2}$ can be glued, in an obvious way, into a directed symplectic cobordism $\overrightarrow{V_0V_2} = \overrightarrow{V_0V_1} \odot \overrightarrow{V_1V_2}$.

Contact structures have no local invariants. Moreover, any contact form is locally isomorphic to the form $\alpha_0 = dz - \sum_1^{n-1} y_i dx_i$ (Darboux' normal form). The contact structure ξ_0 on \mathbb{R}^{2n-1} given by the form α_0 is called *standard*. The standard contact structure on S^{2n-1} is formed by complex tangent hyperplanes to the unit sphere in \mathbb{C}^n . The standard contact structure on S^{2n-1} is isomorphic in the complement of a point to the standard contact structure on \mathbb{R}^{2n-1} . According to a theorem of J. Gray (see [34]) contact structures on closed manifolds have the following stability property: *Given a family ξ_t , $t \in [0, 1]$, of contact structures on a closed manifold M , there exists an isotopy $f_t : M \rightarrow M$, such that $df_t(\xi_0) = \xi_t$; $t \in [0, 1]$.* Notice that for contact forms the analogous statement is wrong. For instance, the topology of the 1-dimensional foliation determined by the Reeb vector field R_α is very sensitive to deformations of the contact form α .

The conformal class of the symplectic form $d\alpha|_\xi$ depends only on the cooriented contact structure ξ and not on the choice of the contact form α . In particular, one can associate with ξ an almost complex structure $J : \xi \rightarrow \xi$, compatible with $d\alpha$ which means that $d\alpha(X, JY)$; $X, Y \in \xi$, is an Hermitian metric on ξ . The space of almost complex structures J with this property is contractible, and hence the choice of J is homotopically canonical. Thus a co-oriented contact structure ξ defines on M a *stable almost complex structure* $\tilde{J} = \tilde{J}_\xi$, i.e. a splitting of the tangent bundle $T(V)$ into the Whitney sum of a complex bundle of (complex) dimension $(n - 1)$ and a trivial 1-dimensional real bundle. The existence of a stable almost complex structure is necessary for the existence of a contact structure on V . If V is open (see [36]) or $\dim V = 3$ (see [51, 50]) this property is also sufficient for the existence of a contact structure in the prescribed homotopy class. It is still unknown whether this condition is sufficient for the existence of a contact structure on a closed manifold of dimension > 3 . However, a positive answer to this question is extremely unlikely. The homotopy class of \tilde{J}_ξ , which we denote by $[\xi]$ and call the *formal* homotopy class of ξ , serves as an invariant of ξ . For an open V it is a complete invariant (see [36]) up to homotopy of contact structures, but not up to a contact diffeomorphism. For closed manifolds this is known to be false in many, but not all dimensions. The theory discussed in this paper serves as a rich source of contact invariants, both of closed and open contact manifolds.

1.2 Dynamics of Reeb vector fields

Let (V, ξ) be a $(2n - 1)$ -dimensional manifold with a co-orientable contact structure with a fixed contact form α . For a generic choice of α there are only countably many periodic trajectories of the vector field R_α . Moreover, these trajectories can be assumed *non-degenerate* in the sense that the linearized Poincaré return map A_γ along any closed trajectory γ , including multiples, has no eigenvalues equal to 1. Let us denote by $\mathcal{P} = \mathcal{P}_\alpha$ the set of all periodic trajectories of R_α , including multiples. ²

²As it is explained below in Section 1.8 the orientation issues require us to exclude certain multiple periodic orbits out of consideration. Namely, let us recall that real eigenvalues of symplectic matrices

The reason for a such choice is discussed in Section 1.8 below. We will also fix a point m_γ on each *simple* orbit from \mathcal{P} . Non-degenerate trajectories can be divided into *odd* and *even* depending on the sign of the Lefschetz number $\det(I - A_\gamma)$. Namely, we call γ odd if $\det(I - A_\gamma) < 0$, and even otherwise. The parity of a periodic orbit γ agrees with the parity of a certain integer grading which is defined if certain additional choices are made, as it is described below.

If $H_1(V) = 0$ then for each $\gamma \in \mathcal{P}$ we can choose and fix a surface F_γ spanning the trajectory γ in V . We will allow the case $H_1(V) \neq 0$, but will require in most of the paper that the torsion part is trivial³. In this case we choose a basis of $H_1(V)$, represent it by oriented curves C_1, \dots, C_K , and choose a symplectic trivialization of the bundle $\xi|_{C_i}$ for each chosen curve. We recall that the bundle ξ is endowed with the symplectic form $d\alpha$ whose conformal class depends only on ξ . For any periodic orbit $\gamma \in \mathcal{P}$ let us choose a surface F_γ with $[\partial F_\gamma] = [\gamma] - \sum n_i [C_i]$. The coefficients n_i are uniquely defined because of our assumption that $H_1(V)$ is torsion-free.

The above choices enable us to define the *Conley-Zehnder index* $\text{CZ}(\gamma)$ of γ as follows. Choose a homotopically unique trivialization of the symplectic vector bundle $(\xi, d\alpha)$ over each trajectory $\gamma \in \mathcal{P}$ which extends to $\xi|_{F_\gamma}$ (and coincides with a chosen trivialization of $\xi|_{C_i}$ if C_i is not homologically trivial). The linearized flow of R_α along γ defines then a path in the group $Sp(2n - 2, \mathbb{R})$ of symplectic matrices, which begins at the unit matrix and ends at a matrix with all eigenvalues different from 1. The Maslov index of this path (see [3, 56]) is, by the definition, the Conley-Zehnder index $\text{CZ}(\gamma)$ of the trajectory γ . See also [39], Section 3, for an axiomatic description of the Conley-Zehnder index using our normalization conventions.

Notice that by changing the spanning surfaces for the trajectories from \mathcal{P} one can change Conley-Zehnder indices by the value of the cohomology class $2c_1(\xi)$, where $c_1(\xi)$ is the first Chern class of the contact bundle ξ . In particular, mod 2 indices can be defined independently of any spanning surfaces, and even in the case when $H_1(V) \neq 0$. In fact,

$$(-1)^{\text{CZ}(\gamma)} = (-1)^{n-1} \text{sign}(\det(I - A_\gamma)).$$

1.3 Splitting of a symplectic manifold along a contact submanifold

Let V be a hypersurface of contact type, or in a different terminology, a symplectically convex hypersurface in a symplectic manifold (W, ω) . This means that ω is exact, $\omega = d\beta$, near V , and the restriction $\alpha = \beta|_V$ is a contact form on V . Equivalently, one can say that the conformally symplectic vector field X , ω -dual to β , is transversal to V . Let us assume that V divides W , $W = W_+ \cup W_-$, where the notation of the parts are chosen in such a way that X serves as an inward transversal for W_+ , and an outward transversal for W_- . The manifolds W_\pm can be viewed as compact directed symplectic cobordisms such that W_- has only positive contact boundary (V, α) , while the same contact manifold serves as a negative boundary of W_+ .

different from ± 1 come in pairs λ, λ^{-1} . Let $\gamma \in \mathcal{P}$ be a simple periodic orbit and A_γ its linearized Poincaré return map. If the total multiplicity of eigenvalues of A_γ from the interval $(-1, 0)$ is odd, then we exclude from \mathcal{P} all even multiples of γ .

³ The case when $H_1(V)$ has torsion elements is discussed in Section 2.9.1 below.

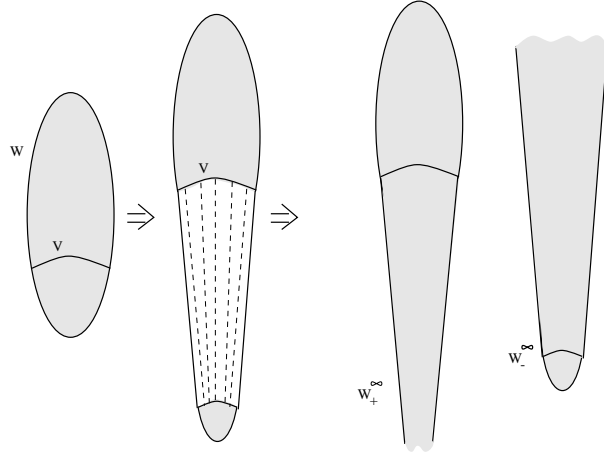


Figure 1: Splitting of a closed symplectic manifold W into two completed symplectic cobordisms W_-^∞ and W_+^∞

Let

$$(W_-^\infty, \omega_-^\infty) = (W_-, \omega) \cup (V \times [0, \infty), d(e^t \alpha))$$

and

$$(W_+^\infty, \omega_+^\infty) = (V \times (-\infty, 0], d(e^t \alpha)) \cup (W_+, \omega)$$

be the completions, and

$$(W_-^\tau, \omega_-^\tau) = (W_-, \omega) \cup (V \times [0, \tau], d(e^t \alpha))$$

and

$$(W_+^\tau, \omega_+^\tau) = (V \times [-\tau, 0], d(e^t \alpha)) \cup (W_+^0, \omega)$$

partial completions of W_\pm . Let us observe that the symplectic manifolds

$$(W_-, e^{-\tau} \omega_-^\tau), \quad (V \times [-\tau, \tau], d(e^t \alpha)) \quad \text{and} \quad (W_+, e^\tau \omega_+^\tau)$$

fit together into a symplectic manifold (W^τ, ω^τ) , so that $W^0 = W$. Hence when $\tau \rightarrow \infty$ the deformation (W^τ, ω^τ) can be viewed as a decomposition of the symplectic manifold $W = W^0$ into the union of two completed symplectic cobordisms W_+^∞ and W_-^∞ . We will write $W = W_- \odot W_+$ and also $W^\infty = W_-^\infty \odot W_+^\infty$.

Let us give here two important examples of the above splitting construction.

Example 1.3.1 Suppose $L \subset W$ is a Lagrangian submanifold. Its neighborhood is symplectomorphic to a neighborhood of the 0-section in the cotangent bundle $T^*(L)$. The boundary V of an appropriately chosen neighborhood has contact type, and thus we can apply along V the above splitting construction. As the result we split W into W_+^∞ symplectomorphic to $W \setminus L$, and W_-^∞ symplectomorphic to $T^*(L)$.

Example 1.3.2 Let M be a hyperplane section of a Kähler manifold W , or more generally a symplectic hyperplane section of a symplectic manifold W , in the sense of Donaldson (see [11]). Then M has a neighborhood with a contact boundary V . The affine part $W \setminus M$ is a Stein manifold in the Kählerian case, and in any case has a structure of a symplectic Weinstein manifold \widetilde{W} (notice that the symplectic structure of \widetilde{W} does not coincide with the induced symplectic structure on $W \setminus M$ but contains $W \setminus M$ as an open symplectic submanifold). The Weinstein manifold $W \setminus M$ contains an isotropic deformation retract Δ . The splitting of W along V produces W^∞ symplectomorphic to \widetilde{W} , and W_+^∞ symplectomorphic to $W \setminus \Delta$. If Δ is a smooth Lagrangian submanifold, then we could get the same decomposition by splitting along the boundary of a tubular neighborhood of L , as in Example 1.3.1.

1.4 Compatible almost complex structures

According to M. Gromov (see [35]) an almost complex structure J is called *tamed* by a symplectic form ω if ω is positive on complex lines. If, in addition, one adds the calibrating condition that ω is J -invariant, then J is said *compatible* with ω . For symplectic manifolds with cylindrical ends one needs further compatibility conditions at infinity, as it is described below.

At each positive, or negative end $(V \times \mathbb{R}_\pm, d(e^t \alpha))$ we require J to be invariant with respect to translations $t \mapsto t \pm c$, $c > 0$ at least for sufficiently large t . We also require the contact structure $\xi^\pm|_{V \times t}$ to be invariant under J , and define $J \frac{\partial}{\partial t} = R_\alpha$, where R_α is the Reeb vector field (see 1.2 above) of the contact form α . In the case when $W = V \times \mathbb{R}$ is the symplectization of a manifold V , i.e. W is a cylindrical manifold, we additionally require J to be globally invariant under all translations along the second factor.

To define a compatible almost complex structure J in the above Examples 1.3.1 and 1.3.2 one needs to specify a contact form α on the contact manifold V . In the case of the boundary of a tubular neighborhood of a Lagrangian submanifold L a natural choice of a contact form is provided by a Riemannian metric on L . The Reeb vector field for such a form α generates on V the geodesic flow of the metric.

When V is the boundary of a neighborhood of a hyperplane section M then there exists another good choice of a contact form. It is a S^1 -invariant connection form α on the principal S^1 -bundle $V \rightarrow M$, whose curvature equals the symplectic form $\omega|_M$. The contact manifold $(V, \xi = \{\alpha = 0\})$ is called the *pre-quantization* of the symplectic manifold (M, ω) . Orbits of the Reeb field R_α are all closed and coincide with the fibers of the fibration, or their multiples. Notice that though the Reeb flow in this case looks extremely nice and simple, all its periodic orbits are highly degenerate, see Section 2.9.2 below. Notice that the symplectization W of V can be viewed as the total space of a complex line bundle L associated with the S^1 -fibration $V \rightarrow M$ with the zero-section removed. It is possible and convenient to choose J compatible with the structure of this bundle, and in such a way that the projection $W \rightarrow M$ becomes holomorphic with respect to a certain almost complex structure J_M on M compatible with ω .

Let us describe now what the symplectic splitting construction from Section 1.3 looks like from the point of view of a compatible almost complex structure.

First, we assume that the original almost complex structure J on W is chosen in such a way that the contact structure $\xi = \{\alpha = 0\}$ on V consists of complex tangencies to V , and that $JX = R_\alpha$, where X is a conformally symplectic vector field, ω -dual to α , and R_α is the Reeb vector field of α . Next we define an almost complex structure J^τ on $W^\tau = W_- \cup V \times [-\tau, \tau] \cup W_+$ by setting $J^\tau|_{W_\pm} = J$ and requiring J^τ to be independent of $t \in [-\tau, \tau]$ on $V \times [-\tau, \tau]$. When $\tau \rightarrow \infty$ the almost complex structure J^τ on $W^\tau = W_- \cup V \times [-\tau, \tau]$ converges to an almost complex structure J_-^∞ on W_-^∞ compatible with ω_-^∞ , and J_+^τ on $W_+^\tau = V \times [-\tau, \tau] \cup W_+$ converges to an almost complex structure J_+^∞ on W_+^∞ compatible with ω_+^∞ .

1.5 Holomorphic curves in symplectic cobordisms

Let (V, α) be a contact manifold with a fixed contact form and $(W = V \times \mathbb{R}, \omega = d(e^t \alpha))$ its symplectization. Let us denote by $\pi_{\mathbb{R}}$ and π_V the projections $W \rightarrow \mathbb{R}$ and $W \rightarrow V$, respectively. For a map $f : X \rightarrow W$ we write $f_{\mathbb{R}}$ and f_V instead of $\pi_{\mathbb{R}} \circ f$ and $\pi_V \circ f$.

Notice that given a trajectory γ of the Reeb field R_α , the cylinder $\mathbb{R} \times \gamma \subset W$ is a J -holomorphic curve. Let us also observe that

Proposition 1.5.1 *For a J -holomorphic curve $C \subset W$ the restriction $d\alpha|_C$ is non-negative, and if $d\alpha|_C \equiv 0$ then C is a (part of a) cylinder $\mathbb{R} \times \gamma$ over a trajectory γ of the Reeb field R_α .*

Given a J -holomorphic map f of a punctured disk $D^2 \setminus 0 \rightarrow W$ we say that the map f is *asymptotically cylindrical* over a periodic orbit γ of the Reeb field R_α at $+\infty$ (resp. at $-\infty$) if $\lim_{r \rightarrow 0} f_{\mathbb{R}}(re^{i\theta}) = +\infty$ (resp. $= -\infty$), and $\lim_{r \rightarrow 0} f_V(re^{i\theta}) = \bar{f}(\theta)$, where the map $\bar{f} : [0, 2\pi] \rightarrow V$ parameterizes the trajectory γ .

The almost complex manifold (W, J) is bad from the point of view of the theory of holomorphic curves: it has a *pseudo-concave* end $V \times (-\infty, 0)$, or using Gromov's terminology its geometry at this end is not bounded. However, it was shown in [37] that Gromov compactness theorem can be modified to accommodate this situation, see Theorems 1.6.2 and 1.6.3 below. We will mention in this section only the following fact related to compactness, which motivates the usage of holomorphic curves asymptotically cylindrical over orbits from \mathcal{P}_α .

Proposition 1.5.2 *Suppose that all periodic orbits of the Reeb field R_α are non-degenerate. Let C be a non-compact Riemann surface without boundary and $f : C \rightarrow W$ a proper J -holomorphic curve. Suppose that there exists a constant $K > 0$ such that $\int_C f^* d\alpha < K$. Then C is conformally equivalent to a compact Riemann surface S_g of genus g with $s^+ + s^-$ punctures*

$$x_1^+, \dots, x_{s^+}^+, x_1^-, \dots, x_{s^-}^- \in S_g,$$

such that near the punctures $\mathbf{x}^+ = (x_1^+, \dots, x_{s^+}^+)$ the map f is asymptotically cylindrical over periodic orbits $\Gamma^+ = \{\gamma_1^+, \dots, \gamma_{s^+}^+\}$ at $+\infty$, and near the punctures $\mathbf{x}^- = (x_1^-, \dots, x_{s^-}^-)$ the map f is asymptotically cylindrical over periodic orbits $\Gamma^- = \{\gamma_1^-, \dots, \gamma_{s^-}^-\}$ at $-\infty$.

Thus holomorphic maps of punctured Riemann surfaces, asymptotically cylindrical over periodic orbits of the Reeb vector field R_α , form a natural class of holomorphic curves to consider in symplectizations as well as more general symplectic manifolds with cylindrical ends. We will define now moduli spaces of such curves.

Let $W = \overline{V^- V^+}$ be a (completed) directed cobordism, α^\pm corresponding contact forms on V^- and V^+ , \mathcal{P}^\pm the sets of all periodic orbits (including multiple ones) of the Reeb vector fields R_{α^\pm} . We assume that α^\pm satisfies the genericity assumptions from Section 1.2. Choose a compatible almost complex structure J on W . Let Γ^\pm be ordered sets of trajectories from \mathcal{P}^\pm of cardinality s^\pm . We also assume that every *simple* periodic orbit γ from \mathcal{P}^\pm comes with a fixed marker $m_\gamma \in \gamma$.

Let $S = S_g$ be a compact Riemann surface of genus g with a conformal structure j , with s^+ punctures $\mathbf{x}^+ = \{x_1^+, \dots, x_{s^+}^+\}$, called positive, s^- punctures $\mathbf{x}^- = \{x_1^-, \dots, x_{s^-}^-\}$, called negative, and r marked points $\mathbf{y} = \{y_1, \dots, y_r\}$. We will also fix an *asymptotic marker* at each puncture. We mean by that a ray originating at each puncture. Alternatively, if one takes the cylinder $S^1 \times [0, \infty)$ as a conformal model of the punctured disk $D^2 \setminus 0$ then an asymptotic marker can be viewed as a point on the circle at infinity. If a holomorphic map $f : D^2 \setminus 0 \rightarrow V^\pm \times \mathbb{R}_\pm$ is asymptotically cylindrical over a periodic orbit γ , we say that a marker $\mu = \{\theta = \theta_0\}$ is mapped by f to the marker $m_\gamma \in \overline{\gamma}$, where $\overline{\gamma}$ is the simple orbit which underlines γ , if $\lim_{r \rightarrow 0} f_{V^\pm}(re^{i\theta_0}) = m_\gamma$. Let us recall (see Section 1.2 above) that we provided each periodic orbit from \mathcal{P}^\pm with a ‘‘capping’’ surface. This surface bounds $\gamma \in \mathcal{P}^\pm$ in V_\pm if γ is homologically trivial, or realizes a homology between γ and the corresponding linear combination of basic curves C_i^\pm . We will continue to rule out torsion elements in the first homology (see the discussion of torsion in Section 2.9.1 below) and choose curves $C_k \subset W$ which represent a basis of the image $H_1(V^- \cup V^+) \rightarrow H(W)$ and for each curve C_i^\pm fix a surface G_i^\pm which realizes a homology in W between C_i^\pm and the corresponding linear combination of curves C_k . All the choices enable us to associate with a relative homology class $A' \in H_2(W, \Gamma^- \cup \Gamma^+)$, $\Gamma^\pm \subset \mathcal{P}^\pm$, an absolute integral class $A \in H_2(W)$, which we will view as an element of $H_2(W; \mathbb{C})$.

Let us denote by $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ the moduli space of (j, J) -holomorphic curves $S_g \setminus (\mathbf{x}^- \cup \mathbf{x}^+) \rightarrow W$ with r marked points, which are asymptotically cylindrical over the periodic orbit γ_i^+ from Γ^+ at the positive end at the puncture x_i^+ , and asymptotically cylindrical over the periodic orbit γ_i^- from Γ^- at the negative end at the puncture x_i^- , and which send asymptotic markers to the markers on the corresponding periodic orbits. The curves from $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ are additionally required to satisfy a stability condition, discussed in the next section. We write $\mathcal{M}_g^A(\Gamma^-, \Gamma^+; W, J)$ instead of $\mathcal{M}_{g,0}^A(\Gamma^-, \Gamma^+; W, J)$, and $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$ instead of $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ if it is clear from the context which target almost complex manifold (W, J) is considered.

Notice, that we are not fixing j , and the configurations of punctures, marked points or asymptotic markers. Two maps are called equivalent if they differ by a conformal map $S_g \rightarrow S_g$ which preserves all punctures, marked points and asymptotic markers. When the manifold $W = V \times \mathbb{R}$ is cylindrical, and hence the almost complex structure J is invariant under translations along the second factor, then all the moduli spaces $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ inherit the \mathbb{R} -action. We will denote the quotient moduli space by $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)/\mathbb{R}$, and by $\mathcal{M}_{g,r,s^-,s^+}^A(W, J)$ the union $\bigcup \mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$

taken over all sets of periodic orbits $\Gamma^\pm \subset \mathcal{P}^\pm$ with the prescribed numbers s^\pm of elements. We will also need to consider the moduli space of disconnected curves of Euler characteristic $2 - 2g$, denoted by $\widetilde{\mathcal{M}}_{g,r}^A(\Gamma^-, \Gamma^+)$.

1.6 Compactification of the moduli spaces $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$

To describe the compactification we need an appropriate notion of a *stable holomorphic curve*.

Given a completed symplectic cobordism $W = \overline{V^-V^+}$ we first define a *stable curve of height 1*, or a *1-story stable curve* as a “usual” stable curve in a sense of M. Kontsevich (see [46]), i.e. a collection of holomorphic curves $h_i : S_i \rightarrow W$ from moduli spaces $\mathcal{M}_{g_i,r}^A(\Gamma_i^-, \Gamma_i^+)$ for various genera g_i which realize homology classes A_i , and collections of periodic orbits Γ_i^\pm , such that certain pairs of marked points (called special) on these curves are required to be mapped to one point in W . The stability condition means the absence of infinitesimal symmetries of the moduli space. Let us point out, however, that in the case when W is a cylindrical cobordism, and in particular the almost complex structure J is translationally invariant, we would need to consider along with the above moduli space its quotient under the \mathbb{R} -action. The stability for this new moduli space still means an absence of infinitesimal deformations, but it translates into an additional restriction on holomorphic curves. Namely, in the first case the stability condition means that each constant curve has, after removal of the marked points, a negative Euler characteristic. In the second case it additionally requires that when *all* connected components of the curve are straight cylinders $\gamma \times \mathbb{R}$, $\gamma \in \mathcal{P}$ then at least one of these cylinders should have a marked point.

One can define an arithmetic genus g of the resulting curve, the total sets \mathbf{x}^\pm and \mathbf{y} (equal to the union of sets \mathbf{x}_i^\pm and \mathbf{y}_i for the individual curves of the collection), and the total absolute homology class $A \in H_2(W)$ (see the discussion in Section 1.5 above), realized by the union of all curves of the collection.

Moduli of stable curves of height 1, denoted by ${}_1\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$, form a part of the compactification of the moduli space $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$. However, unlike the case of closed symplectic manifolds, the stable curves of height 1 are not sufficient to describe the compactification of the moduli space $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$.

A finite sequence (W_1, \dots, W_k) of symplectic manifolds with cylindrical ends is called a *chain* if the positive end of W_i matches with the negative end of W_{i+1} , $i = 1, \dots, k - 1$. This means that all data, assigned to an end, i.e. a contact form, marking of periodic orbits, and an almost complex structure, are the same for the matching ends.

Let us first suppose that none of the cobordisms which form a chain (W_1, \dots, W_k) is cylindrical. Then a *stable curve of height k* , or a *k -story stable curve* in the chain (W_1, \dots, W_k) is a k -tuple $f = (f_1, \dots, f_k)$, where $f_i \in {}_1\widetilde{\mathcal{M}}_{g_i,r_i}^{A_i}(\Gamma_i^-, \Gamma_i^+; W_i, J_i)$, such that the boundary data of the curve f_i at the positive end match the boundary data of f_{i+1} on the negative one. One also needs to impose the following additional equivalence relation regarding the asymptotic markers on multiple orbits. Suppose that γ is a k -multiple periodic orbit, so that the holomorphic curve f_i is asymptotically cylindrical over γ at the positive end at a puncture x^+ , and f_{i+1} is asymptotically cylindrical over γ at the negative end at a puncture x^- . There are k possible positions

μ_1^+, \dots, μ_k^+ and μ_1^-, \dots, μ_k^- of asymptotic markers at each of the punctures x^\pm . We assume here that the markers are numbered cyclically with respect to the orientation defined by the Reeb vector field at each of the punctures, and that the markers μ_1^+ and μ_1^- are chosen for the curves f_i and f_{i+1} . Then we identify $f = \{\dots, f_i, f_{i+1}, \dots\}$ with $(k-1)$ other stable curves of height k obtained by simultaneous cyclic shift of the asymptotic markers at the punctures x^+ and x^- .

The curves f_i , which form a k -story stable curve $f = (f_1, \dots, f_k)$ are called *floors*, or *levels* of f .

If some of the cobordisms which form the chain (W_1, \dots, W_k) , say W_{i_1}, \dots, W_{i_l} , are cylindrical then we will assume that the corresponding floors of a k -story curve in $W = (W_1, \dots, W_k)$ are defined *only up to translation*. In other words, if W_i is cylindrical for some $i = 1, \dots, k$ (i.e. $W_i = V_i \times \mathbb{R}$ and J_i is translationally invariant) then f_i should be viewed as an element of ${}_1\widetilde{\mathcal{M}}_{g_i, r_i}^{A_i}(\Gamma_i^-, \Gamma_i^+; W_i, J_i)/\mathbb{R}$, rather than ${}_1\widetilde{\mathcal{M}}_{g_i, r_i}^{A_i}(\Gamma_i^-, \Gamma_i^+; W_i, J_i)$. It will be convenient for us, however, to introduce the following convention. When speaking about stable holomorphic curves in chains which contain cylindrical cobordisms, we will treat the corresponding floors as curves representing their equivalence classes from ${}_1\widetilde{\mathcal{M}}_{g_i, r_i}^{A_i}(\Gamma_i^-, \Gamma_i^+; W_i, J_i)/\mathbb{R}$. Any statement about such curves should be understood in the sense, that *there exist* representatives for which the statement is true.

Let us define now the meaning of convergence of a sequence of holomorphic curves to a stable curve of height l . For $l = 1$ this is Gromov's standard definition (see [35]). Namely, with each stable curve $h = \{S_i, h_i\} \in {}_1\mathcal{M}_{g, r}^A(\Gamma^-, \Gamma^+)$ of height 1 we associate a nodal surface \widehat{S} obtained by identifying special pairs of marked points on $\coprod S_i$. The maps h_i fits together to a continuous map $\widehat{S} \rightarrow W$ for which we will keep the notation h . Let us consider also a smooth surface S obtained by smoothing the nodes of \widehat{S} . There exist a partitioning of S by circles into open parts diffeomorphic to surfaces S_i with removed special points, and a map $g : S \rightarrow \widehat{S}$ which is a diffeomorphism from the complement \widetilde{S} of the dividing circles in S to the complement of the double points in \widehat{S} , and which collapses the partitioning circles to double points. A sequence of holomorphic $\varphi_l : (S, j_l) \rightarrow (W, J)$ is said to converge to a stable curve h if the sequence $\varphi_l|_{\widetilde{S}}$ converges to $h_i \circ g|_{\widetilde{S}}$, and j_l converges to $g^*(j)$ uniformly on compact sets, where j is the conformal structure on the stable curve. Of course, we also require convergence of marked points and asymptotic markers. A sequence of stable curves $h^j = \{S_i^j, h_i^j\}_{i=1, \dots, k} \in {}_1\mathcal{M}_{g, r}^A(\Gamma^-, \Gamma^+)$, $j = 1, \dots$, is said to converge to a stable curve h , if h can be presented as a collection of stable curves h_i , $i = 1, \dots, k$, such that h_i^j converges to h_i in the above sense for each $i = 1, \dots, k$.

The convergence of a sequence of smooth curves to a stable curve of height $l > 1$ is understood in a similar sense. Let us assume here for simplicity that $l = 2$ and that the floors $f_1 : S_1 \rightarrow W_1$ and $f_2 : S_2 \rightarrow W_2$ of a stable curve f in a chain (W_1, W_2) are smooth, i.e. have no special marked points. As in the height 1 case let us consider

- the smooth surface S partitioned according to the combinatorics of our stable curve by circles into two open (possibly disconnected) parts U_1 and U_2 diffeomorphic to the punctured surfaces S_1 and S_2 ,

- the surface \hat{S} with double points obtained by collapsing these circles to points, and
- the projection $g : S \rightarrow \hat{S}$.

Let $(W, J) = (W_1, J_1) \odot (W_2, J_2)$ be the composition of (completed) directed symplectic cobordisms W_1 and W_2 with compatible almost complex structures J_1 and J_2 . This means that

- there exists a contact hypersurface $V \subset W$ which splits W into two cobordisms W_1^0 and W_2^0 ;
- $W_1 = W_1^0 \cup V \times [0, \infty)$, $W_2 = V \times (-\infty, 0] \cup W_2^0$;
- $J|_{W_j^0} = J_j|_{W_j^0}$, $j = 1, 2$;
- J_1 and J_2 are translationally invariant at the ends $V \times [0, \infty)$ and $(-\infty, 0] \times V$.

We denote by W^k , $k = 1, \dots$, the quotient space of the disjoint union

$$W_1^0 \amalg V \times [-k, k] \amalg W_2^0$$

obtained by identifying $V = \partial W_1^0$ with $V \times (-k)$ and $V = \partial W_2^0$ with $V \times k$, and extend the almost complex structures $J_1|_{W_1^0}$ and $J_2|_{W_2^0}$ to the unique almost complex structure J^k on W^k which is translationally invariant on $V \times [-k, k]$. We also consider W_1^k obtained by gluing W_1^0 and $V \times [0, k]$ along $V = \partial W_1^0 = V \times 0$, and W_2^k glued in a similar way from $V \times [-k, 0]$ and W_2^0 . We have $W_j = \bigcup_{k=0}^{\infty} W_j^k$, $j = 1, 2$. On the other hand, W_1^k and W_2^k can be viewed as submanifolds of W^k .

Definition 1.6.1 Suppose that we are given a sequence j^k of conformal structures on the surface S and a sequence of 1-story (j^k, J^k) -holomorphic curves $f^k : S \rightarrow W^k$. We say that this sequence converges to a stable curve $f = (f_1, f_2)$ of height 2 in (W_1, W_2) if there exist two sequences of domains $U_1^1 \subset \dots \subset U_1^i \subset \dots \subset U_1$ and $U_2^1 \subset \dots \subset U_2^i \subset \dots \subset U_2$, such that

$$\bigcup_{k=1}^{\infty} U_i^k = U_i, \quad i = 1, 2;$$

$$f^k(U_i^k) \subset W_i^k \quad \text{for } i = 1, 2, \quad k = 1, \dots;$$

for $i = 1, 2$ the holomorphic curves $f^k|_{U_i^k}$ converges to $f_i \circ g : U_i \rightarrow W_i$, and the conformal structures $j^k|_{U_i^k}$ converge to g^*j_i when $k \rightarrow \infty$ uniformly on compact sets. As in the case of stable curves of level 1 we also require convergence of marked points and asymptotic markers.

Let us emphasize that when some of the cobordisms are cylindrical then according to the convention which we introduced above one is allowed to compose the corresponding curves with translations to satisfy the above definition.

Notice that if the cobordism W_2 is cylindrical, i.e. $W_2 = V \times \mathbb{R}$ and J_2 is translationally invariant, then $W_1 \odot W_2$ can be identified with W_1 , and thus one can

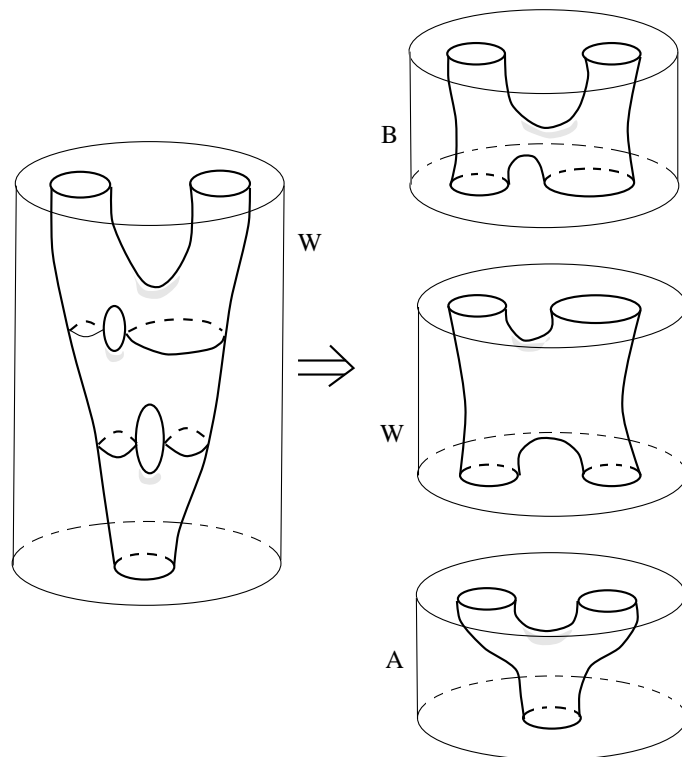


Figure 2: A possible splitting of a sequence of holomorphic curves in a completed symplectic cobordism

talk about convergence of a sequence of curves $f^k \in {}_1\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J_1)$ (where the almost complex structure J_1 is fixed!) to a 2-story curve (f_1, f_2) , where $f_1 \in {}_1\widetilde{\mathcal{M}}_{g_1, r_1}^{A_1}(\Gamma^-, \Gamma; W_1, J_1)$, $f_2 \in {}_1\widetilde{\mathcal{M}}_{g_2, r_2}^{A_2}(\Gamma, \widetilde{\Gamma}^+; V \times \mathbb{R}, J_2)/\mathbb{R}$, $g = g_1 + g_2$, $r = r_1 + r_2$, $A = A_1 + A_2$, and J_2 is translationally invariant. It is important to stress the point that the curve f_2 is defined only up to translation.

Theorem 1.6.2 *Let $f_k \in {}_1\mathcal{M}_g^A(\Gamma^-, \Gamma^+)$, $k = 1, \dots$, be a sequence of stable holomorphic curves in a (complete) directed symplectic cobordism W . Then there exists a chain of directed symplectic cobordisms*

$$A_1, \dots, A_a, W, B_1, \dots, B_b,$$

where all cobordisms A_i and B_i are cylindrical, and a stable curve f_∞ of height $a+b+1$ in this chain such that a subsequence of $\{f_i\}$ converges to f_∞ . See Fig. 2.

Theorem 1.6.3 *Let W be a completed directed symplectic cobordism, $V \subset W$ a contact hypersurface, and J_k a sequence of compatible almost complex structures on W which realizes the splitting of W along V into two directed symplectic cobordisms*

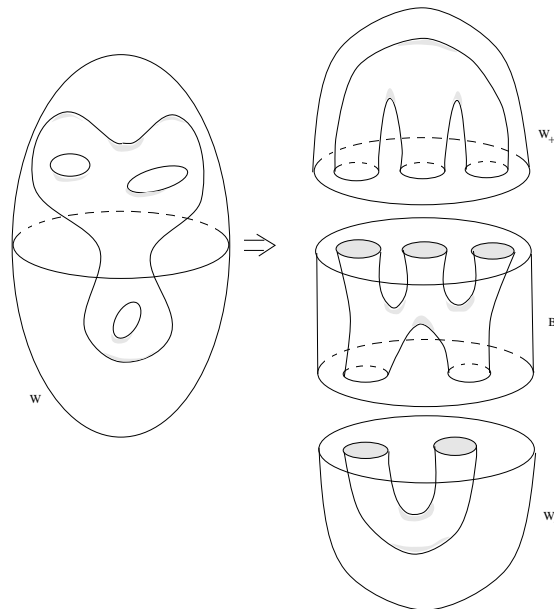


Figure 3: A possible splitting of a sequence of holomorphic curve when $J_k \rightarrow J_\infty$

W_-^∞ and W_+^∞ (see Section 1.4 above). Let f_k be a sequence of stable J_k -holomorphic curves from ${}_1\mathcal{M}_g^A(\Gamma^-, \Gamma^+; W, J_k)$. Then there exists a chain of directed symplectic cobordisms

$$A_1, \dots, A_a, W_-, B_1, \dots, B_b, W_+, C_1, \dots, C_c$$

where all cobordisms A_i, B_j, C_l are cylindrical, such that a subsequence of $\{f_i\}$ converges to a stable curve of height $a + b + c + 2$ in the chain

$$A_1, \dots, A_a, W_-, B_1, \dots, B_b, W_+, C_1, \dots, C_c.$$

See Fig. 3. The reader may consult [40] for the analysis of splitting $\mathbb{C}P^2$ along the boundary of a tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$.

The definition of convergence can be extended in an obvious way to a sequences of stable curves of height $l > 1$. Namely, we say that a sequence of l -story curves $f^k = (f_1^k, \dots, f_l^k)$, $k = 1, \dots, \infty$, in a chain (W_1, \dots, W_l) converge to a stable L -story, $L = m_1 + \dots + m_l$, curve $f = (f_{11}, \dots, f_{1m_1}, \dots, f_{l1} \dots f_{lm_l})$ in a chain

$$(W_{11}, \dots, W_{1m_1}, \dots, W_{l1} \dots W_{lm_l})$$

if for each $i = 1, \dots, l$ the cobordism W_i splits into the composition

$$W_i = W_{i1} \odot \dots \odot W_{im_i}$$

and the sequence f_i^k , $k = 1, \dots, \infty$, of stable curves of height 1 converges to the m_i -story curve $f_i = (f_{i1}, \dots, f_{im_i})$ in the chain $(W_{i1}, \dots, W_{im_i})$ in the sense of Definition 1.6.1.

It is important to combine Theorems 1.6.2 and 1.6.3 with the following observation which is a corollary of Stokes' theorem combined with Proposition 1.5.1.

Proposition 1.6.4 *A holomorphic curve in an exact directed symplectic cobordism (and in particular in a cylindrical one) must have at least one positive puncture.*

In particular, we have

Corollary 1.6.5 *Let $f^n \in \mathcal{M}_0(W, J)$ be a sequence of rational holomorphic curves with one positive, and possibly several negative punctures. Suppose that the sequence converges to a stable curve*

$$F = \{g_1, \dots, g_a, f, h_1, \dots, h_b\}$$

of height $a + b + 1$ in a chain

$$A_1, \dots, A_a, W, B_1, \dots, B_b.$$

Then the W -component $f \in \mathcal{M}_0(W, J)$ of the stable curve F has precisely one positive puncture as well.

1.7 Dimension of the moduli spaces $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$

One has the following index formula for the corresponding $\bar{\partial}$ -problem which compute the dimension of the moduli space $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ for a generic choice of J .

Proposition 1.7.1

$$(1) \quad \dim \mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J) = \sum_1^{s^+} \text{CZ}(\gamma_i^+) - \sum_1^{s^-} \text{CZ}(\gamma_k^-) \\ + (n-3)(2-2g-s^+-s^-) + 2c_1(A) + 2r,$$

where s^\pm are the cardinalities of the sets Γ^\pm , and $c_1 \in H^2(W)$ is the first Chern class of the almost complex manifold (W, J)

Making the moduli spaces non-singular by picking generic J is needed for the purpose of curve counting but does not always work properly. It is therefore crucial that the moduli spaces of stable J -holomorphic curves are non-singular *virtually*. This means that for *any* J the moduli spaces, being generally speaking singular, can be equipped with some canonical additional structure that make them *function in the theory the same way as if they were orbifolds with boundary and had the dimension prescribed by the Fredholm index*. In particular, the moduli spaces come equipped with rational fundamental cycles relative to the boundary (called *virtual fundamental cycles*) which admit pairing with suitable de Rham cochains and allow us to use the Stokes integration formula.

Technically the virtual smoothness is achieved by a finite-dimensional reduction of the following picture: a singular moduli space is the zero locus of a section defined by the Cauchy-Riemann operator in a suitable orbi-bundle over a moduli orbifold

of stable C^∞ -maps. More general *virtual transversality* properties for families of J 's also hold true (cf. [20, 21, 49, 48, 59, 61, 52] at al.). We are reluctant to provide in this quite informal exposition precise formulations because of numerous not entirely innocent subtleties this would entail. Fortunately, what we intend to say in the rest of this paper does not depend much on the details we are omitting.

As it was explained in Section 1.6 above the moduli space $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$ can be compactified by adding strata which consist of stable holomorphic curves of different height. This compactification looks quite similar to the Gromov-Kontsevich compactification of moduli spaces of holomorphic curves in a closed symplectic manifold with a compatible almost complex structure. There is, however, a major difference. In the case of a closed manifold all the strata which one needs to add to compactify the moduli space of smooth holomorphic curves have (modulo virtual cycle complications) codimension ≥ 2 . On the other hand in our case the codimension one strata are present *generically*. Thus in this case the boundary of the compactified moduli space, rather than the moduli space itself, carries the fundamental cycle.

In particular, this boundary is tiled by codimension one strata represented by stable curves (f_-, f_+) of *height two*. Each such a stratum can be described by the constraint matching the positive ends of f_- with the negative ends of f_+ in the Cartesian product of the moduli spaces \mathcal{M}_\pm corresponding to the curves f_\pm separately. Proposition 1.7.2 below describes these top-dimensional boundary strata more precisely in two important for our purposes situations. Let us point out that 1.7.2 literally holds only under certain transversality conditions. Otherwise it should be understood only virtually.

Proposition 1.7.2 *1. Let $(W = V \times \mathbb{R}, J)$ be a cylindrical cobordism. Then any top-dimensional stratum \mathcal{S} on the boundary of the compactified moduli space $\overline{\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)}/\mathbb{R}$ consists of stable curves (f_-, f_+) of height two, $f_\pm \in \mathcal{M}_\pm/\mathbb{R}$, where*

$$\mathcal{M}_- = \widetilde{\mathcal{M}}_{g_-, r_-}^{A_-}(\Gamma^-, \Gamma; W, J)/\mathbb{R}, \quad \mathcal{M}_+ = \widetilde{\mathcal{M}}_{g_+, r_+}^{A_+}(\Gamma, \Gamma^+; W, J)/\mathbb{R},$$

$$g = g_- + g_+, \quad r = r_- + r_+, \quad A = A_- + A_+, \quad \Gamma = \{\gamma_1, \dots, \gamma_l\} \subset \mathcal{P}.$$

All but one connected components of each of the curves f_- and f_+ are trivial cylinders (i.e. have the form $\gamma \times \mathbb{R}, \gamma \in \mathcal{P}$) without marked points.

2. Let $(W = \overline{V^- V^+}, J)$ be any cobordism, and $(W_\pm, J_\pm) = (V^\pm \times \mathbb{R}, J_\pm)$ be the cylindrical cobordisms associated to its boundary. Then any top-dimensional strata \mathcal{S} on the boundary of the compactified moduli space $\overline{\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)}$ consists of stable curves (f_-, f_+) of height two, $f_\pm \in \mathcal{M}_\pm$, where either

$$\mathcal{M}_- = \widetilde{\mathcal{M}}_{g_-, r_-}^{A_-}(\Gamma^-, \Gamma; W_-, J_-)/\mathbb{R}, \quad \mathcal{M}_+ = \widetilde{\mathcal{M}}_{g_+, r_+}^{A_+}(\Gamma, \Gamma^+; W, J) \quad \text{and } \Gamma \subset \mathcal{P}^-,$$

or

$$\mathcal{M}_+ = \widetilde{\mathcal{M}}_{g_+, r_+}^{A_+}(\Gamma, \Gamma^+; W_+, J_+)/\mathbb{R}, \quad \mathcal{M}_- = \widetilde{\mathcal{M}}_{g_-, r_-}^{A_-}(\Gamma^-, \Gamma; W, J) \quad \text{and } \Gamma \subset \mathcal{P}^+.$$

In both cases we have

$$g = g_- + g_+, \quad r = r_- + r_+, \quad \text{and} \quad A = A_- + A_+.$$

The part of the stable curve (f_-, f_+) which is contained in W_{\pm} must have precisely one non-cylindrical connected component, while there are no restrictions on the number and the character of connected components or the other part of the stable curve.

In both cases the stratum $\mathcal{S} = \mathcal{S}(\Gamma, g_-|g_+, r_-|r_+, A_-|A_+)$ is diffeomorphic to a κ -multiple cover of the product $\mathcal{M}_- \times \mathcal{M}_+$, where the multiplicity κ is determined by the multiplicities of periodic orbits from Γ .

Proposition 1.7.2 is not quite sufficient for our purposes, as we also need to know the structure of the boundary of moduli spaces of 1-parametric families of holomorphic curves. However, we are not formulating the corresponding statement in this paper, because it is intertwined in a much more serious way with the virtual cycle techniques and terminology. An algebraic description of this boundary is given in Theorem 2.4.2 below.

Let us consider some special cases of the formula (1). Suppose, for instance, that W is the cotangent bundle of a manifold L . Then W is a symplectic manifold which has only a positive cylindrical end. If L is orientable then there is a canonical way to define Conley-Zehnder indices. Namely, one takes any trivialization along orbits, which is tangent to vertical Lagrangian fibers. The resulting index is independent of a particular trivialization. For this trivialization, and a choice of a contact form corresponding to a metric on L we have

Proposition 1.7.3 *Periodic orbits of the Reeb flow are lifts of closed geodesics, and if L is orientable their Conley-Zehnder indices are equal to Morse indices of the corresponding geodesics and we have*

$$\dim \mathcal{M}_g^A(\Gamma^+) = \sum_i \text{Morse}(\gamma_i^+) + (n-3)(2-2g-s^+).$$

Notice that for a metric on L of non-positive curvature we have $s^+ > 1$, because in this case there are no contractible geodesics. Moreover, if the metric has negative curvature then all geodesics have Morse indices equal to 0. Hence, we get

Corollary 1.7.4 *In the cotangent bundle of a negatively curved manifold of dimension > 2 there could be only isolated holomorphic curves. If, in addition, $n \neq 3$ then these curves are spheres with two positive punctures. Each of these curves is asymptotically cylindrical at punctures over lifts of the same geodesic with opposite orientations.*

Let us point out that the orientability is not required in Corollary 1.7.4. The corresponding result for a non-orientable manifold follows from 1.7 applied to its orientable double cover.

Absence of hyperbolic Lagrangian submanifolds in uniruled manifolds

As the first application of the above compactness theorems let us prove here the following theorem of C. Viterbo. Let us recall that a complex projective manifold W is called uniruled, if there is a rational holomorphic curve through each point of W . For instance, according to Y. Myaoka–S. Mori [53] and J. Kollar [44] Fano manifolds are uniruled.

Theorem 1.7.5 (C. Viterbo, [65]) *Let W be a uniruled manifold of complex dimension > 2 , ω its Kähler symplectic form, and $L \subset W$ an embedded Lagrangian submanifold. Then L does not admit a Riemannian metric of negative sectional curvature.*

Proof. J. Kollar [44] and in a more general case Y. Ruan [59] proved that there exists a homology class $A \in H_2(W)$, such that for any almost complex structure compatible with ω and any point $z \in W$ there exists $f \in \mathcal{M}_{0,1}^A(W, J)$ with $f(y) = z$, where y is the marked point. Let us identify a neighborhood U of L in W with a neighborhood of the zero-section in $T^*(L)$. Suppose L admits a Riemannian metric of negative curvature. We can assume that U is the round neighborhood of radius 1 in $T^*(L)$. Let us consider a sequence J^m of almost complex structures on W , which realizes the splitting along the contact type hypersurface ($V = \partial U, \alpha = pdq|_V$). (see Section 1.4). Then according to Example 1.3.1 W splits into $W_- = T^*(L)$ and $W_+ = W \setminus L$. The almost complex structure J_- on $T^*(L)$ is compatible at infinity with the contact 1-form $\alpha = pdq|_V$. According to Corollary 1.7.4 for any choice $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ and any $g \geq 0$ the moduli spaces $\mathcal{M}_g(\Gamma; W_-, J_-)$ are empty, or 0-dimensional. On the other hand, Theorem 1.6.3 together with Ruan’s theorem guarantee the existence of a rational holomorphic curve with punctures through every point of L . This contradiction proves that L cannot admit a metric of negative curvature. \blacktriangleright

1.8 Coherent orientation of the moduli spaces of holomorphic curves

To get started with the algebraic formalism, one first needs to orient moduli spaces $\mathcal{M}(\Gamma^-, \Gamma^+)$ of holomorphic curves with punctures. This problem is much easier in the case of moduli spaces of closed holomorphic curves, because in that case moduli spaces are even-dimensional and carry a canonical almost complex structure (see Section 1.8.2 below). In our case we have to adapt the philosophy of *coherent orientations* of the moduli spaces borrowed from Floer homology theory (see [19]). We sketch this approach in this section.

1.8.1 Determinants

In order to separate the problems of orientation and transversality we are going to orient the determinant line bundles of the linearized $\bar{\partial}$ -operators, rather than the moduli spaces themselves.

For a linear Fredholm operator $F : A \rightarrow B$ between Banach spaces we can define its determinant line $\det(F)$ by

$$\det(F) = (\Lambda^{\max} \text{Ker}(F)) \otimes (\Lambda^{\max} \text{Coker}(F))^*.$$

We note that for the trivial vector space $\{0\}$ we have $\Lambda^{max}\{0\} = \mathbb{R}$. An orientation for F is by definition an orientation for the line $\det(F)$. In particular, given an isomorphism F we can define a canonical orientation given through the vector $1 \otimes 1^* \in \mathbb{R} \otimes \mathbb{R}^*$.

Given a continuous family $F = \{F_y\}_{y \in Y}$ of Fredholm operators, parameterized by a topological space Y , the determinants of operators F_y form a line bundle $\det(F) \rightarrow Y$. The fact that this is a line bundle in a natural way might be surprising since the dimensions of kernel and cokernel vary in general. This is however a standard fact, see for example [19].

1.8.2 Cauchy-Riemann Type Operators on Closed Riemann Surfaces

Let (S, j) be a closed, not necessarily connected Riemann surface and $E \rightarrow S$ a complex vector bundle. Denote by $X_E \rightarrow S$ the complex n -dimensional vector bundle whose fiber over $z \in S$ consists of all complex ant-linear maps

$$\phi : T_z S \rightarrow E_z, \quad z \in S, \quad \text{i. e. } J \circ \phi + \phi \circ j = 0,$$

where J is the complex structure on E . Fixing a connection ∇ and a smooth $a \in \text{Hom}_{\mathbb{R}}(E, X_E)$ we can define a Cauchy-Riemann type operator

$$L : C^\infty(E) \rightarrow C^\infty(X_E)$$

by the formula

$$(Lh)(X) = \nabla_X h + J \nabla_{jX} h + (ah)(X),$$

where X is an arbitrary vector field on S . Since the space of connections is an affine space we immediately see that the set \mathcal{O}_E of all Cauchy-Riemann type operators on E is convex. For a proper functional analytic set-up, where we may chose Hölder or Sobolev spaces, the operator L is Fredholm. By elliptic regularity theory the kernel and cokernel would be spanned always by the same smooth functions, regardless which choice we have made. The index of L is given by the Riemann-Roch formula

$$\text{ind}(L) = (1 - g) \dim_{\mathbb{R}}(E) + 2c(E),$$

where $c(E)$ the first Chern number $c_1(E)(S)$ of E . Here we assume S to be a connected closed surface of genus $g = g(S)$.

Let $\phi : (S, j) \rightarrow (T, i)$ be a biholomorphic map and $\Phi : E \rightarrow F$ a \mathbb{C} -vector bundle isomorphism covering ϕ . Then Φ induces an isomorphism

$$\Phi_* : \mathcal{O}_E \rightarrow \mathcal{O}_F$$

in the obvious way. The operators (E, L) and (F, K) are called isomorphic if there exists $\Phi : E \rightarrow F$, so that $\Phi_*(L) = K$. We will denote by $[E, L]$ the equivalence class of an operator (E, L) which consists of operators (F, K) , equivalent to (E, L) under isomorphisms, *isotopic to the identity*, and by $[[E, L]]$ the equivalence class under the action of the full group of isomorphisms. The moduli space of equivalence classes $[[E, L]]$ will be denoted by \mathcal{CR} , and the ‘‘Teichmüller space’’ which consists of classes $[E, L]$ will be denoted by $\widetilde{\mathcal{CR}}$. An isomorphism Φ induces an isomorphism

between the kernel (cokernel) of L and Φ_*L for every $L \in \mathcal{O}_E$, and hence one can canonically associate the determinant line to an isomorphism class, and thus define the *determinant line bundle* \mathcal{V} over the moduli space \mathcal{CR} . Given an orientation o for L we obtain an induced orientation $\Phi_*(o)$. Let us note the following

Lemma 1.8.1 *The bundle \mathcal{V} is orientable.*

Proof. The lift $\tilde{\mathcal{V}}$ of the bundle \mathcal{V} to the Teichmuller space $\tilde{\mathcal{CR}}$ is obviously orientable, because each connected component of the space $\tilde{\mathcal{CR}}$ is contractible. However, one should check that an arbitrary isomorphism $\Phi : (E, L) \rightarrow (F, K)$ preserves the orientation. This follows from the following observation. Any connected component of $\tilde{\mathcal{CR}}$ contains an isomorphism class of a complex linear operator (E, L_0) , and any two complex linear operators representing points in a given component of $\tilde{\mathcal{CR}}$ are homotopic in the class of complex linear operators. The determinant of (E, L_0) can be oriented canonically by observing that its kernel and cokernel are complex spaces. Any isomorphism maps a complex linear operator to a complex linear operator and preserves its complex orientation. Hence, it preserves an orientation of the determinant line of any operator (E, L) . \blacktriangleright

We will call an orientation of \mathcal{V} *complex* if it coincides with the complex orientation of determinants of complex linear operators.

The components of the space \mathcal{CR} are parameterized by the topological type of the underlying surface S and the isomorphism class of the bundle E . It turns out that the complex orientation of \mathcal{V} satisfies three *coherency* Axioms A1–A3 which we formulate below. They relate orientations of \mathcal{V} over different components of \mathcal{CR} . Conversely, we will see that these axioms determine the orientation uniquely up to a certain normalization.

Given (E, L) and (F, K) over surfaces Σ_0 and Σ_1 we define a *disjoint union*

$$[E, L] \dot{\cup} [F, K] := [G, M]$$

of (E, L) and (F, K) as a pair (G, M) , where G is a bundle over the disjoint union $\Sigma = \Sigma_0 \amalg \Sigma_1$, so that $(G, M)|_{\Sigma_0}$ is isomorphic to (E, L) and $(G, M)|_{\Sigma_1}$ is isomorphic to (F, K) . Clearly, the isomorphism class of a disjoint union is uniquely determined by the classes of (E, L) and (F, K) . Thus, we have a well-defined construction called *disjoint union*: The determinant $\det \Sigma$ is canonically isomorphic to $\det L \otimes \det K$, and hence the orientations o_K and o_L define an orientation $o_K \otimes o_L$ of Σ . Our first axiom reads

Axiom C1. For any disjoint union $[G, M] = [E, L] \dot{\cup} [F, K]$ the orientation o_M equals $o_K \otimes o_L$.

Given (E, L) and (F, K) , where E and F are bundles over S of possibly different rank, we can define an operator $(E \oplus F, L \oplus K)$. There is a canonical map

$$\det(L) \otimes \det(K) \rightarrow \det(L \oplus K),$$

and thus given orientations o_L and o_K we obtain $o_L \oplus o_K$.

Axiom C2.

$$o_{L \oplus K} = o_L \oplus o_K.$$

To formulate the third axiom, we need a construction, called *cutting and pasting*.

Let (E, L) be given and assume that $\gamma_1, \gamma_2 : S^1 \rightarrow S$ be real analytic embeddings with mutually disjoint images. Assume that $\Phi : E|_{\gamma_1} \rightarrow E|_{\gamma_2}$ is a complex vector bundle isomorphism covering $\sigma = \gamma_2 \circ \gamma_1^{-1}$. The maps γ_1 and γ_2 extends as holomorphic embeddings $\bar{\gamma}_j : [-\varepsilon, \varepsilon] \times S^1 \rightarrow S$ for a suitable small $\varepsilon > 0$, so that the images are still disjoint. Locally, near γ_j we can distinguish the left and the right side of γ_j . These sides correspond to the left or the right part of the annulus $[-\varepsilon, \varepsilon] \times S^1$. Cutting S along the curves γ_j we obtain a compact Riemann surface \bar{S} with boundary. Its boundary components are γ_j^\pm , $j = 1, 2$, where γ_j^\pm is canonically isomorphic to γ_j . The vector bundle E induces a vector bundle $\bar{E} \rightarrow \bar{S}$. We define a space of smooth sections $\Gamma_\Delta(\bar{E})$ as follows. It consists of all smooth sections \bar{h} with the property that

$$\bar{h}|_{\gamma_j^-} = \bar{h}|_{\gamma_j^+} \text{ for } j = 1, 2.$$

Then L induces an operator $\bar{L} : \Gamma_\Delta(\bar{E}) \rightarrow \Gamma(X_{\bar{E}})$. The operators L and \bar{L} have naturally isomorphic kernel and cokernel. So an orientation o of $\det(L)$ induces one of $\det(\bar{L})$. The boundary condition Δ can be written in the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Phi(\gamma_1(t))\bar{h} \circ \gamma_1^-(t) \\ \bar{h} \circ \gamma_2^-(t) \end{bmatrix} = \begin{bmatrix} \Phi(\gamma_1(t))\bar{h} \circ \gamma_1^+(t) \\ \bar{h} \circ \gamma_2^+(t) \end{bmatrix}$$

We introduce a parameter depending boundary condition by

$$\begin{bmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{bmatrix} \cdot \begin{bmatrix} \Phi(\gamma_1(t))\bar{h} \circ \gamma_1^-(t) \\ \bar{h} \circ \gamma_2^-(t) \end{bmatrix} = \begin{bmatrix} \Phi(\gamma_1(t))\bar{h} \circ \gamma_1^+(t) \\ \bar{h} \circ \gamma_2^+(t) \end{bmatrix}$$

for $\tau \in [0, \frac{\pi}{2}]$. For all these boundary conditions L induces an operator, which is again Fredholm of the same index. For every τ we obtain a Cauchy-Riemann type operator from $\Gamma_{\Delta_\tau}(\bar{E})$ to $\Gamma(X_{\bar{E}})$. Note that for a section h satisfying the boundary condition Δ_τ the section ih satisfies the same boundary condition. On the other hand for $\tau = \frac{\pi}{2}$ we obtain a Fredholm operator whose kernel and cokernel naturally isomorphic to the kernel on cokernel of a Fredholm operator on a new closed surface. Namely identify γ_1^+ with γ_2^- and γ_2^+ with γ_1^- . For the bundle \bar{E} we identify the part above γ_1^+ via Φ with the part over γ_2^- and we identify the part above γ_1^- with $-\Phi$ to the part above γ_2^+ . The latter surface and bundle we denote by $E_\Phi \rightarrow S_\Phi$ and the corresponding operator by L_Φ . Letting the parameter run we obtain starting with an orientation o for L an orientation o_Φ for L_Φ . If o is the complex orientation it is easily verified that o_Φ is the complex orientation as well. We say that the operator L_Φ is an operator obtained from L by cutting and pasting. This operator L_Φ has the same index as L , and the component of $[L, \Phi]$ in $\widehat{\mathcal{CR}}$ depends only the isotopy classes of the embeddings γ_1 and γ_2 .

Axiom C3.

$$o_{L_\Phi} = o_\Phi.$$

Note that we have to require here that the parts of L over the curves γ_1 and γ_2 are isomorphic via the gluing data. It is straightforward to check that

Theorem 1.8.2 *The complex orientation of \mathcal{V} is coherent, i.e. it satisfies Axioms C1–C3.*

Let us point out a simple

Lemma 1.8.3 *Let (E, L) be an isomorphism then the orientation by $1 \otimes 1^*$ of the $\det L = \mathbb{R} \otimes \mathbb{R}^*$ defines the complex orientation of \mathcal{V} over the component of $[E, L]$.*

The following theorem gives the converse of Theorem 1.8.2.

Theorem 1.8.4 *Suppose that a coherent orientation of \mathcal{V} coincides with the complex orientation for the trivial line bundle over S^2 and for the line bundle over S^2 with Chern number 1. Then the orientation is complex.*

Proof. Let us first observe that according to Theorem 1.8.2 the disjoint union, direct sum and cutting and pasting procedures preserve the class of complex orientations. Consider the pair (E_0, L_0) , where E_0 is the trivial bundle $S^2 \times \mathbb{C} \rightarrow S^2$ and L_0 is the standard Cauchy-Riemann operator. Then the $\text{ind } L_0 = 2$. Take small loops around north pole and south pole on S^2 and identify the trivial bundles over these loops. Now apply the cutting and pasting procedure and Axiom C3 to obtain the disjoint union of the trivial bundle over the torus and the trivial bundle over S^2 . Hence we can use Axiom C1 to obtain an induced orientation for the Cauchy-Riemann operator on the trivial bundle over T^2 . Taking appropriate loops we obtain orientations for all trivial line bundles over Riemann surfaces of arbitrary genus. Using direct sums and disjoint unions constructions, and applying Axioms C1 and C2 we see that the orientation of all trivial bundles of arbitrary dimensions over Riemann surfaces of arbitrary genus are complex. Let E_1 be the bundle over S^2 with Chern number 1. Then we can use C3 to glue two copies of (E_1, L_1) to obtain the complex orientation of the disjoint union of a Cauchy Riemann operator on the trivial bundle and one on the bundle with Chern number 2. Now it is clear that the given coherent orientation has to be complex over all components of the moduli space \mathcal{CR} . \blacktriangleright

In the next section we extend the coherent orientation from Cauchy-Riemann type operators over closed surfaces to a special class of Cauchy-Riemann type operators on Riemann surfaces with punctures.

1.8.3 A special class of Cauchy-Riemann type operators on punctured Riemann surfaces

Let us view \mathbb{C}^n as a real vector space equipped with the Euclidean inner product which is the real part of the standard Hermitian inner product. We define a class of self-adjoint operators as follows. Their domain in $L^2(S^1, \mathbb{C}^n)$ is $H^{1,2}(S^1, \mathbb{C}^n)$ of Sobolev maps $h : S^1 \rightarrow \mathbb{C}^n$. The operators have the form

$$(2) \quad (Ax)(t) = -i \frac{dx}{dt} - a(t)x,$$

where $a(t)$ is a smooth loop of real linear self-adjoint maps. We assume that A is non-degenerate in the sense that $Ah = 0$ only has the trivial solution, which just means that the time-one map $\psi(1)$ of the Hamiltonian flow

$$(3) \quad \begin{aligned} \dot{\psi}(t) &= ia(t)\psi(t), \\ \psi(0) &= \text{Id} \end{aligned}$$

has no eigenvalues equal to 1. In particular, $A : H^{1,2} \rightarrow L^2$ is an isomorphism.

Given a smooth vector bundle $E \rightarrow S^1$ we can define $H^{1,2}(E)$ and $L^2(E)$ and a class of operators B by requiring that $A = \Phi B \Phi^{-1}$ for an Hermitian trivialization Φ of the bundle E . We shall call such operators *asymptotic*, for reasons which will become clear later.

As it was defined in Section 1.5 above, an asymptotically marked punctured Riemann surface is a triplet $(S, \mathbf{x}, \boldsymbol{\mu})$, where $S = (S, j)$ is a closed Riemann surface, $\mathbf{x} = \{x_1, \dots, x_s\}$ is the set of punctures, some of them called positive, some negative, and $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_s\}$ is the set of asymptotic markers, i.e. tangent rays, or equivalently oriented tangent lines at the punctures.

One can introduce near each puncture $x_k \in \mathbf{x}$ a holomorphic parameterization, i.e. a holomorphic map $h_k : D \rightarrow S$ of the unit disk D such that $h_k(0) = x_k$ and the asymptotic marker μ_k is tangent to the ray $h_k(r)$, $r \geq 0$. We assume that the coordinate neighborhoods $\mathcal{D}_k = h_k(D)$ of all the punctures are disjoint. Then we define $\sigma_k : \mathbb{R}^+ \times S^1 \rightarrow \mathcal{D} \setminus \{0\}$ by

$$\sigma_k(s, t) = h_k(e^{\pm 2\pi(s+it)}),$$

where the sign $-$ is chosen if the puncture x_k is positive, and the sign $+$ for the negative puncture. We will refer to σ_k as holomorphic polar coordinates adapted to (x_k, μ_k) . Given two adapted polar coordinate systems σ and σ' near the same puncture $x \in \mathbf{x}$ we observe that the transition map (defined for R large enough)

$$\sigma^{-1} \circ \sigma' : [R, \infty) \times S^1 \rightarrow [0, \infty) \times S^1$$

satisfies for every multi-index α

$$D^\alpha[\sigma^{-1} \circ \sigma'(s, t) - (c + s, t)] \rightarrow 0$$

uniformly for $s \rightarrow \infty$, where c is a suitable constant. The main point is the fact that there is no phase shift in the t -coordinate.

Given $(S, \mathbf{x}, \boldsymbol{\mu})$ we associate to it a smooth surface \bar{S} with boundary compactifying the punctured Riemann surface $S \setminus \mathbf{x}$ by adjoining a circle for every puncture. Each circle has a distinguished point $0 \in S^1 = \mathbb{R}/\mathbb{Z}$. Namely for each positive puncture we compactify $\mathbb{R}^+ \times S^1$ to $[0, \infty) \times S^1$, where $[0, \infty)$ has the smooth structure making the map

$$[0, \infty) \rightarrow [0, 1] : s \rightarrow s(1 + s^2)^{-\frac{1}{2}}, \quad \infty \rightarrow 1$$

a diffeomorphism. We call $S_k^+ = \{\infty\} \times S^1$ the circle at infinity associated to (x_k, μ_k) . For negative punctures we compactify at $-\infty$ in a similar way.

Definition 1.8.5 *A smooth complex vector bundle $E \rightarrow (S, \mathbf{x}, \mu)$ is a smooth vector bundle over \hat{S} together with Hermitian trivializations*

$$\Phi_k : E|_{S_k} \rightarrow S^1 \times \mathbb{C}^n.$$

An isomorphism between two bundles E and F over surfaces S and T is a complex vector bundle isomorphism $\Psi : E \rightarrow F$ which covers a biholomorphic map $\phi : (S, j) \rightarrow (T, i)$, preserves punctures and the asymptotic markers (their numbering and signs) and respects the asymptotic trivializations.

Define as in Section 1.8.2 above the bundle $X_E \rightarrow \bar{S}$. Set $\dot{S} = S \setminus \Gamma$. We introduce the Sobolev space $H^1(E)$ which consists of all sections h of $E \rightarrow \dot{S}$ of class $H_{loc}^{1,2}$ with the following behavior near punctures. Suppose, that x is a positive puncture and σ is an adapted system of holomorphic polar coordinates. Pick a smooth trivialization ψ of $E \rightarrow \bar{S}$ over $[0, \infty] \times S^1$ (in local coordinates) compatible with the given asymptotic trivialization. Then the map $(s, t) \rightarrow \psi(s, t)h \circ \sigma(s, t)$ is assumed to belong to $H^{1,2}(\mathbb{R}^+ \times S^1, \mathbb{C}^n)$. A similar condition is required for negative punctures. In a similar way we define the space $L^2(X_E)$. Observe that defacto we use measures which are infinite on \dot{S} and that the neighborhoods of punctures look like half-cylinders.

A Cauchy-Riemann type operator L on E has the form

$$(Lh)X = \nabla_X h + J\nabla_{jX} h + (ah)X,$$

where X is a vector field on S . We require, however a particular behaviour of L near the punctures. Namely, regarding E as a trivial bundle $[0, \infty] \times \mathbb{C}^n$ with respect to the chosen polar coordinates and trivialization near say a positive puncture we require that

$$(Lh)(s, t) \left(\frac{\partial}{\partial s} \right) = \frac{\partial h}{\partial s} - A(s)h,$$

where $A(s) \rightarrow A_\infty$ for an asymptotic operator A_∞ , as it was previously introduced.

Theorem 1.8.6 *The operator L is Fredholm.*

The index of L can be computed in terms of Maslov indices of the asymptotic operators (and, of course, the first Chern class of E and the topology of S).

Similar to the case of closed surfaces we define the notion of isomorphic pairs (E, L) and (F, K) , where we emphasize the importance of the compatibility of the asymptotic trivializations, define the moduli space $\mathcal{CR}_{\text{punct}} \supset \mathcal{CR}_{\text{closed}}$ and the Teichmüller spaces $\widetilde{\mathcal{CR}}_{\text{punct}} \supset \widetilde{\mathcal{CR}}_{\text{closed}}$, and extend the determinant line bundle \mathcal{V} to $\mathcal{CR}_{\text{punct}}$ and $\widetilde{\mathcal{V}}$ to $\widetilde{\mathcal{CR}}_{\text{punct}}$. The bundle $\widetilde{\mathcal{V}}$ is orientable by the same reason as in the case of closed surfaces: each component of the space $\widetilde{\mathcal{CR}}_{\text{punct}}$ is contractible. However, unlike the closed case, there is no canonical (complex) orientation of $\widetilde{\mathcal{V}}$. Still due to the requirement that isomorphisms preserves the end structure of the operators, one can deduce the fact that even isotopically non-trivial isomorphisms preserve the orientation of $\widetilde{\mathcal{V}}$, which shows that the bundle \mathcal{V} over $\mathcal{CR}_{\text{punct}}$ is orientable.

Let us review now Axioms C1–C3 for the line bundle \mathcal{V} over $\mathcal{CR}_{\text{punct}}$. The formulation of Disjoint Union Axiom C1 should be appended by the following requirement.

Let (E, L) and (F, K) be operators over the punctured Riemann surfaces $(S, \mathbf{x} = \{x_1, \dots, x_s\})$ and $(T, \mathbf{y} = \{y_1, \dots, y_t\})$, respectively. Then $(E, L) \dot{\cup} (F, K)$ is an operator over the surface $S \amalg T$ with the set of punctures $\mathbf{z} = \{x_1, \dots, x_s, y_1, \dots, y_t\}$. The disjoint union operation is associative, but not necessarily commutative (unlike the case of closed surfaces). Axioms C2 and C3 we formulate without any changes compared to the closed case. By a coherent orientation of the bundle \mathcal{V} over $\mathcal{CR}_{\text{punct}}$ we will mean any orientation of \mathcal{V} which satisfies Axioms C1–C3.

Take the trivial (and globally trivialized) line bundle $E_0 = \mathbb{C} \times \mathbb{C}$ over the 1-punctured Riemann sphere $\mathbb{C} = \mathbb{C}P^1 \setminus \infty$. For any admissible asymptotic operator A we choose a Cauchy-Riemann operator L_A^\pm on E_0 which has A as its asymptotics at ∞ . The superscript \pm refers to the choice of ∞ as the positive or negative puncture. Note, that the component of $([E_0, L_A^\pm])$ in the moduli spaces \mathcal{CR} is uniquely determined by the homotopy class $[A]$ of the asymptotic operator A in the space of *non-degenerate* asymptotic operators.

The following theorem describes all possible coherent orientations of the line bundle \mathcal{V} over \mathcal{CR} .

Theorem 1.8.7 *Let us choose an orientation o_A^\pm of the operator (E_0, L_A^\pm) for a representative A of each homotopy class $[A]$ of non-degenerate asymptotic operators. Then this choice extends to the unique coherent orientation of the bundle \mathcal{V} over $\mathcal{CR}_{\text{punct}}$, which coincide with the complex orientation over $\mathcal{CR}_{\text{closed}}$.*

Thus there are infinitely many coherent orientations of \mathcal{V} over $\mathcal{CR}_{\text{punct}}$ unlike the case of closed surfaces, when there are precisely four.

We sketch below the proof of Theorem 1.8.7. First, similar to the case of closed surfaces, it is sufficient to consider only operators on the trivial, and even globally trivialized bundles. Next take the disjoint union of (E_0, L_A^-) and (E_0, L_A^+) , consider two circles γ^\pm around the punctures in the two copies of \mathbb{C} and apply the cutting/pasting construction along these circles. As the result we get a disjoint union of an operator \tilde{L}_A on the trivial line bundle over the closed Riemann sphere, and an operator \bar{L}_A over the cylinder $C = S^1 \times \mathbb{R}$, which we view as the Riemann sphere with two punctures $x_1 = \infty$ and $x_2 = 0$ and consider x_1 as a positive puncture and x_2 as a negative one. The operator \bar{L}_A has the same asymptotic operator A at both punctures. Then Axioms C1 and C3 determine the orientation of \bar{L}_A , because for the operator \tilde{L}_A we have chosen the complex orientation. Notice that if one glue L_A^\pm in the opposite order, then we get an operator \bar{L}'_A which has the reverse numbering of the punctures. The orientation of \bar{L}'_A determined by the gluing may be the same, or opposite as for the operator \bar{L}_A , depending on the parity of the Conley-Zehnder index of the asymptotic operator A .⁴

Consider now an arbitrary operator (E, L) acting on sections of a complex line bundle E over a punctured Riemann surface $(S, \mathbf{x}, \boldsymbol{\mu})$ with $\mathbf{x} = \{x_1, \dots, x_s\}$, $E = S \times \mathbb{C}^n$, and the asymptotic operators A_1, \dots, A_s at the corresponding punctures. For each $i = 1, \dots, s$ consider an operator $(E_0, L_i = L_{A_i}^\pm)$, where $E_0 = \mathbb{C} \times \mathbb{C}^n$, the

⁴ The operator \bar{L}_A is homotopic to an isomorphism, and thus has a canonical orientation $1 \otimes 1^*$. If we insist on that normalization, than our construction would determine the orientation of L_A^- in terms of L_A^+ .

sign $+$ is chosen if the puncture x_i is negative, and the sign $-$ is chosen otherwise. Using Axiom C1 we orient the operator $(E, L) \dot{\cup} (E_0, L_s)$, and then choosing circles around the puncture x_s and ∞ apply the cutting/pasting procedure. As the result we get the disjoint union of an operator L' over the Riemann surface with punctures (x_1, \dots, x_{s-1}) and the operator \bar{L}_A , or \bar{L}'_A depending on whether the puncture x_s was negative, or positive. Hence Axioms C1 and C3 determine the orientation of L' in terms of the orientation of L . Repeating the procedure for the punctures x_{s-1}, \dots, x_1 we express the orientation of L in terms of the complex orientation of an operator over the closed surface.

It remains to observe that if E is a trivial complex bundle of rank $r > 1$, then any asymptotic operator A can be deformed through non-degenerate asymptotic operators to an operator \tilde{A} which is split into the direct sum of asymptotic operators on the trivial complex line. Hence we can use the direct sum axiom C2 to orient determinants of operators acting on bundles of arbitrary rank.

1.8.4 Remark about the coherent orientation for asymptotic operators with symmetries

Let A be an asymptotic operator given by the formula (2), where the loop $a(t)$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$, of symmetric matrices has a symmetry $a(t + 1/2) = a(t)$, $t \in \mathbb{R}/\mathbb{Z}$. Let L be a Cauchy-Riemann type operator on a bundle $E \rightarrow S$, which has A as its asymptotic operator at a puncture $x \in S$ with an asymptotic marker μ . Let L' be an operator which differs from L by rotating by the angle π the marker μ to a marker μ' , with the corresponding change of the trivialization near the puncture. Let $h : S \rightarrow S$ be a diffeomorphism which rotates the polar coordinate neighborhood \mathcal{D} of the punctures x by π , and is fixed outside a slightly larger neighborhood. Then the operator h_*L' has the same asymptotic data as L and the isomorphism classes $[E, L]$ and $[E, h_*L']$ belongs to the same component of the space $\widehat{\mathcal{CR}}$. Given a coherent orientation of \mathcal{V} , do the orientations o_L and o_{h_*L} coincide? It turns out that

Lemma 1.8.8 *Let Ψ be the time-one map of the linear Hamiltonian flow $\psi(t)$, defined by the equation (3). The orientations o_L and o_{h_*L} coincide if and only if the number of real eigenvalues of Ψ (counted with multiplicities) from the interval $(-1, 0)$ is even.*

This lemma is the reason why we excluded certain periodic orbits from \mathcal{P} in Section 1.2 above. See also Remarks 1.9.2 and 1.9.6.

1.8.5 Coherent orientations of moduli spaces

The moduli spaces of holomorphic curves which we need to orient are zero sets of nonlinear Cauchy-Riemann type operators, whose linearizations are related to operators of the kind we described (see below for more details). In general, the moduli spaces are neither manifolds nor orbifolds, due to the fact that Fredholm sections cannot be made transversal to the zero section by changing natural parameters like the almost complex structure or the contact form. Such a transversality will only be achievable by making abstract perturbations, leading to virtual moduli spaces. Those

virtual spaces will be the moduli spaces which will provide us with the data for our constructions. Nevertheless the Fredholm operators occurring in the description of the virtual moduli spaces will only be compact perturbations of the Cauchy-Riemann type operators, and hence the orientation scheme for these virtual moduli spaces does not differ from the case of moduli spaces of holomorphic curves.

A moduli space $\mathcal{M}(\Gamma^+, \Gamma^-; W, J)$ of holomorphic curves in a directed symplectic cobordism $(W = \overline{V^-V^+}, J)$ is a fiber bundle over the corresponding moduli space of Riemann surfaces. Its base is a complex orbifold, and hence it is canonically oriented, while the fiber over a point S , where S is a Riemann surface with a fixed conformal structure and positions of punctures, can be viewed as the space solutions of the $\bar{\partial}_J$ -equation. If the transversality is achieved then the tangent bundle of a moduli space $\mathcal{M}(\Gamma^+, \Gamma^-; W, J)$ arise as the kernel of the linearized surjective operator $\bar{\partial}_J$. The linearization of $\bar{\partial}_J$ at a point $f \in \mathcal{M}(\Gamma^+, \Gamma^-; W, J)$ is a Fredholm operator in a suitable functional analytic setting. This set-up involves Sobolev spaces with suitable asymptotic weights derived from the non-degeneracy properties of the periodic orbits. It is a crucial observation, again a corollary of the behaviour near the punctures, that up to a compact perturbation, the operator L splits into two operators L' and L'' , where L' is a complex linear operator acting on the complex line bundle $T(S)$ of the Riemann surface S , and L'' is a Cauchy-Riemann type on the bundle E , such that $T(S) \oplus E = f^*(TM)$. This operator is usually only real linear, but most importantly it is of the kind we just described in our linear theory. The trivialization of E near the punctures is determined by the chosen in 1.5 trivialization of the contact structure near periodic orbit of the Reeb vector fields on V^\pm , and the asymptotic operators are determined by the linearized Reeb flow near the periodic orbits. We have $\det L = \det L' \otimes \det L''$. But $\det L'$ has a canonical complex orientation, and hence the orientation for $\det L$ is determined by the orientation of $\det L''$. Therefore, a choice of a coherent orientation of \mathcal{V} over \mathcal{CR} determines in the transversal case the orientation of all the moduli spaces $\mathcal{M}(\Gamma^+, \Gamma^-; W, J)$.

1.9 First attempt of algebraization: Contact Floer homology

1.9.1 Recollection of finite-dimensional Floer theory

Let us first recall the basic steps in defining a Floer homology theory in the simplest case of a Morse function f on a finite-dimensional orientable closed manifold M . We refer the reader to Floer's original papers (see, for instance, [18]), as well as an excellent exposition by D. Salamon [62] for the general theory.

First, one forms a graded complex $C(f, g)$ generated by critical points c_1, \dots, c_N of f , where the grading is given by the Morse index of critical points. Next, we choose a generic Riemannian metric g on M which satisfy the Morse-Smale condition of transversality of stable and unstable varieties of critical points. This enables us to define a differential $d = d_{f,g} : C(f, g) \rightarrow C(f, g)$ by counting gradient trajectories connecting critical points of neighboring indices:

$$d(c_i) = \sum L_i^j c_j,$$

where the sum is taken over all critical points c_j with $\text{ind } c_j = \text{ind } c_i - 1$. The coefficient

L_i^j is the *algebraic number* of trajectories connecting c_i and c_j . This means that the trajectories are counted with signs. In the finite-dimensional case the signs could be determined as follows. For each critical point we orient arbitrarily its stable manifold. Together with the orientation of M this allows us to orient all unstable manifolds, as well as the intersections of stable and unstable ones. If $\text{ind } c_j = \text{ind } c_i - 1$ then the stable manifold of c_i and the unstable manifold of c_j intersect along finitely many trajectories which we want to count, and hence each of these trajectories gets an orientation. Comparing this orientation with the one given by the direction of the gradient ∇f we can associate with every trajectory a sign.⁵

To show that $d^2 = 0$, which then would allow us to define the homology group $H_*(C(f, g), d)$, we proceed as follows. Let us observe that the coefficients K_i^j in the expansion $d^2(c_i) = \sum K_i^j c_j$ count the algebraic number of broken gradient trajectories $(\delta_{il}, \delta_{lj})$ passing through an intermediate critical point c_l , $l = 1, \dots, N$. But each broken trajectory $(\delta_{il}, \delta_{lj})$, which connects critical points whose indices differ by 2, is a boundary point of the 1-dimensional manifold of smooth trajectories connecting c_i and c_j . The algebraic number of boundary points of a compact 1-dimensional manifold is, of course, equal to 0. Hence $K_i^j = 0$, and thus $d^2 = 0$.

Next we want to show that the homology group $H_*(C(f, g), d)$ is an invariant of the manifold M (of course, in the case we consider it is just $H_*(M)$), i.e. it is independent of the choice of the function f and the Riemannian metric g . The proof of the invariance consists of three steps.

Step 1. Let us show that given a homotopy of functions $F = \{f_t\}_{t \in [0,1]}$, and a homotopy of Riemannian metrics $G = \{g_t\}_{t \in [0,1]}$, one can define a homomorphism $\Phi = \Phi_{F,G} : C(f_1, g_1) \rightarrow C(f_0, g_0)$ which commutes with the boundary homomorphisms $d_0 = d_{f_0, g_0}$ and $d_1 = d_{f_1, g_1}$, i.e.

$$(4) \quad \Phi \circ d_1 - d_0 \circ \Phi = 0.$$

To construct Φ we consider the product $W = M \times \mathbb{R}$ and, assuming that the homotopies $\{f_t\}$ and $\{g_t\}$ are extended to all $t \in \mathbb{R}$ as independent of t on $(-\infty, -1] \cup [1, \infty)$, we define on W a function, still denoted by F , by the formula

$$F(x, t) = \begin{cases} f_0(x) + ct, & t \in (\infty, 0); \\ f_t(x) + ct, & t \in [0, 1]; \\ f_1(x) + ct, & t \in (0, \infty), \end{cases}$$

where the constant c is chosen to ensure that $\frac{\partial F}{\partial t} > 0$. Similarly, we use the family of Riemannian metrics g_t to define a metric G on W which is equal to g_t on $M \times t$ for all $t \in \mathbb{R}$, and such that $\frac{\partial}{\partial t}$ is the unit vector field orthogonal to the slices $M \times t$, $t \in \mathbb{R}$.

⁵The generalization of this procedure to an infinite-dimensional case is not straightforward, because stable and unstable manifolds not only can become infinite-dimensional, but in most interesting cases cannot be defined at all. On the other hand, the moduli spaces of gradient trajectories connecting pairs of critical points (which in the finite-dimensional case coincide with the intersection of stable and unstable manifolds of the critical points) are often defined, and one can use the coherent orientation scheme, similar to the one described in Section 1.8 above for the moduli spaces of holomorphic curves, to define their orientation.

The gradient trajectories of ∇F converge to critical points of f_1 at $+\infty$, and to the critical points of f_0 at $-\infty$. For a generic choice of G the moduli space of the (unparameterized) trajectories connecting two critical points, c^1 of f_1 and c^0 of f_0 , is a compact k -manifold with boundary with corners, where $k = \text{ind } c^1 - \text{ind } c^0$. Hence, similarly to the above definition of the differential d , we can define a homomorphism $\Phi : C(f_1, g_1) \rightarrow C(f_0, g_0)$ by taking an algebraic count of gradient trajectories between the critical point of f_1 and f_0 of the same Morse index, i.e. $\Phi(c_j^1) = \sum \tilde{L}_j^i c_j^0$. The identity (4) comes from the description of the boundary of the 1-dimensional moduli spaces of trajectories of ∇F . Notice that the function F has no critical points, and hence a family of gradient trajectories cannot converge to a broken trajectory in a usual sense. However, this can happen *at infinity*. Let us recall that the function F and the metric G are cylindrical outside of $M \times [-1, 1]$. Hence away from a compact set a gradient trajectory of F projects to a gradient trajectory of f_0 or f_1 . When the projection, say at $+\infty$, of a sequence $\delta_n : \mathbb{R} \rightarrow W$ of trajectories of ∇F converges to a broken trajectory of ∇f_1 this can be interpreted as a splitting at $+\infty$. This phenomenon is very similar to the one described for the moduli spaces of holomorphic curves in Section 1.6. Namely, there exist gradient trajectories $\delta : \mathbb{R} \rightarrow W$ of ∇F , and $\delta' : \mathbb{R} \rightarrow M_1$ of ∇f_1 , such that

- $\delta_n \rightarrow \delta$ uniformly on $(-\infty, C]$ for all C ;
- there exists a sequence $C_n \rightarrow +\infty$ such that $\delta'_n(t) = \delta_n(t + C_n)$ converges to $(\delta'(t), t)$ uniformly on all subsets $[-C, \infty)$.

In this sense broken trajectories of the form (δ, δ') and (δ'', δ) , where δ'' is a trajectory of ∇f_0 form the boundary of the 1-dimensional moduli spaces of trajectories of ∇F connecting critical points c^1 of f_1 and c^0 of f_0 with $\text{ind } c^1 - \text{ind } c^0 = 1$. Therefore the algebraic number of these trajectories equals 0. On the other hand, this number is equal to $\Phi \circ d_1 - d_0 \circ \Phi$ which yields the identity (4).

Step 2. Our next goal is to check that if (F_u, G_u) , $u \in [0, 1]$, is a homotopy of homotopies which is constant outside of a compact subset of W , then the homomorphisms $\Phi_0 = \Phi_{F_0, G_0}$ and $\Phi_1 = \Phi_{F_1, G_1}$ are related via the chain homotopy formula

$$(5) \quad \Phi_1 - \Phi_0 = K \circ d_1 + d_0 \circ K,$$

for a homomorphism $K : C(f_1, g_1) \rightarrow C(f_0, g_0)$. The space of all homotopies (F, G) connecting given pairs (f_0, g_0) and (f_1, g_1) is contractible, and hence (5) implies that the homomorphism $\Phi_* : H_*(C(f_1, g_1), d_1) \rightarrow H_*(C(f_0, g_0), d_0)$ is independent of the choice of a homotopy (F, G) .

To prove (5) one studies moduli spaces of gradient trajectories of the whole 1-parametric family of functions F_u . For a generic choice of the homotopy one has isolated critical values of the parameter u when appear *handle-slides*, i.e. gradient connections between critical points with the index difference -1 . By counting these trajectories one can then define a homomorphism $K : C(f_1, g_1) \rightarrow C(f_0, g_0)$ in exactly the same way as the homomorphism Φ was defined in Step 1 by counting trajectories with the index difference 0.

The identity (5) expresses the fact that the broken trajectories of the form (δ, δ') and (δ'', δ) , where δ is a handle-slide trajectory and δ' is a trajectory of ∇f_1 , form

the boundary of the moduli space of index 0 trajectories in the family (F_u, G_u) . The difference in signs in formulas (4) and (5) is a reflection of the fact that the homomorphism K raises the grading by 1, while Φ leaves it unchanged.

Step 3. Finally we need to show that

$$(6) \quad (\Phi_{F,G})_* = (\Phi_{F',G'})_* \circ (\Phi_{F'',G''})_*,$$

if $(F, G) = \{f_t, g_t\}_{t \in [0,2]}$ is the composition of homotopies $(F'', G'') = \{f''_t, g''_t\}_{t \in [0,1]}$ and $(F', G') = \{f'_t, g'_t\}_{t \in [1,2]}$. To prove this we view, as in Step 1, the homotopy (F, G) as a function and a metric on the cylinder $W = M \times \mathbb{R}$. Consider a deformation (F_T, G_T) of F, G , by cutting W open along $M \times 1$ and inserting a cylinder $M \times [0, T]$ of growing height T with the function and the metric independent of the coordinate t . When $T \rightarrow +\infty$ the gradient trajectories of F_T with respect to G_T split in a sense, similar to the one explained in Step 2,⁶ into a "broken trajectory" (δ'', δ') , where δ' (resp. δ'') is a trajectory of $\nabla_{G'} F'$ (resp. $\nabla_{G''} F''$). Consider the 1-dimensional moduli space \mathcal{M} of trajectories of $\nabla_{G_T} F_T$, $T \in [0, \infty)$, connecting a fixed critical point $c = c^2$ of f_2 with an arbitrary critical point c^0 of f_0 with $\text{ind } c - \text{ind } c^0 = 1$. Then the boundary of \mathcal{M} consists of

- a) all the trajectories of $\nabla_{G_0} F_0 = \nabla_G F$ connecting c^0 and c ; they are given by the expression $\Phi(c)$;
- b) all the broken trajectories (δ'', δ') described above, such that δ'' begins at c^0 and ends at a critical point c^1 of f^1 which is, necessarily, of the same Morse index as c^0 and c^2 , δ' begins at c^1 and ends at c ; these broken trajectories are described by the expression $\Phi_{F',G'}(\Phi_{F'',G''}(c))$;
- c) broken trajectories defined according to Step 2 for the 1-dimensional family $F_T, T \in [0, \infty)$; they are described by the expression $K(d_0(c)) + d_2(K(c))$ for some homomorphism $K : C(f_2, g_2) \rightarrow C(f_0, g_0)$.

Thus the sum (taken with appropriate signs) of the three expressions defined in a)–c) equals 0, and thus we get

$$\Phi_{F',G'}(\Phi_{F'',G''}(c)) - \Phi(c) = K \circ d_0(c) + d_2 \circ K(c),$$

i.e. the homomorphisms Φ and $\Phi_{F',G'} \circ \Phi_{F'',G''}$ are chain homotopic, which yields formula (6).

We can finish now the proof that the homology group $H_*(C(f, g), d)$ is independent of the choice of f and g as follows. Given two pairs (f_0, g_0) and (f_1, g_1) we first take any homotopy (F, G) connecting (f_0, g_0) with (f_1, g_1) , and also take the inverse homotopy (\bar{F}, \bar{G}) connecting (f_1, g_1) with (f_0, g_0) . The composition (\tilde{F}, \tilde{G}) of the homotopies (F, G) and (\bar{F}, \bar{G}) connects the pair (f_0, g_0) with itself. According to Step 3 we have $(\Phi_{\tilde{F}, \tilde{G}})_* = (\Phi_{F,G})_* \circ (\Phi_{\bar{F}, \bar{G}})_*$. On the other hand, we have shown in Step 3 that the homomorphism $(\Phi_{\tilde{F}, \tilde{G}})_*$ is independent of the choice of a homotopy, connecting

⁶See also the discussion of a similar phenomenon for the moduli spaces of holomorphic curves in Section 1.6 above.

(f_0, g_0) with itself, and hence it equals the identity. Therefore, we conclude that $(\Phi_{F,G})_*$ is surjective, while $(\Phi_{\overline{F},\overline{G}})_*$ is injective. Taking the composition of homotopies $(\overline{F}, \overline{G})$ and (F, G) in the opposite order we prove that both homomorphisms are bijective.

A. Floer discovered that the finite-dimensional scheme which we explained in this section works, modulo some analytic complications, for several geometrically interesting functional on infinite-dimensional spaces. For instance, in the symplectic Floer homology theory one deals with critical points of the action functional. Its critical points are periodic orbits of a Hamiltonian system, while for an appropriate choice of a metric and an almost complex structure the gradient trajectories can be interpreted as holomorphic cylinders which connect these trajectories. The role of broken trajectories is played here by split holomorphic cylinders, and finite-dimensional compactness theorems are replaced by the highly non-trivial Gromov compactness theorem for holomorphic curves.

In the rest of this section we explore the Floer-theoretic approach for the problem of defining invariants of contact manifolds. We will see that this approach works only in a very special and restrictive situation. However, the general algebraic formalism of SFT, though quite different, has a distinctive flavor of a Floer homology theory.

1.9.2 Floer homology for the Action functional

Let us make an attempt to define invariants of contact manifolds in the spirit of Floer homology theory. Let (V, ξ) be a contact manifold with a fixed contact form α and an almost complex structure $J : \xi \rightarrow \xi$, compatible with the symplectic form $d\alpha|_\xi$. Then J and $d\alpha$ define a Riemannian metric on the vector bundle ξ by the formula $g(X, Y) = d\alpha(X, JY)$ for any vectors $X, Y \in \xi$. We extend g to the whole tangent bundle $T(V)$ by declaring the vector field R_α to be the unit normal field to ξ . Consider the free loop space

$$\Lambda(V) = \{u : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow V\},$$

and define the *action functional*

$$(7) \quad S : \Lambda(V) \rightarrow \mathbb{R} \quad \text{by the formula} \quad S(\gamma) = \int_\gamma \alpha.$$

The least action principle tells us that the critical points of the functional S are, up to parameterization, the periodic orbits of the Reeb field R_α .

The metric g on $T(V)$ defines a metric on $\Lambda(V)$ and thus allows us to consider gradient trajectories of the action functional connecting critical points of V . The gradient direction $\nabla S(u)$, $u \in \Lambda(V)$, is given by the vector field $J\pi(\frac{du}{dt})$, where $\pi : T(V) \rightarrow \xi$ is the projection along the Reeb direction, so that a gradient trajectory $u(t, s)$, $t \in \mathbb{R}/\mathbb{Z}$, $s \in \mathbb{R}$, is given by the equation

$$(8) \quad \frac{\partial u}{\partial s}(t, s) = J\pi\left(\frac{\partial u}{\partial t}(t, s)\right).$$

Equation (8) has a flavor of a Cauchy-Riemann equation. We want to modify it into a genuine one. Namely, consider the Cauchy-Riemann equation

$$\frac{\partial U}{\partial t}(t, s) = J \frac{\partial U}{\partial s}(t, s)$$

for $U(s, t) = (u(s, t), \varphi(s, t)) \in V \times \mathbb{R}$. It can be rewritten as a system

$$(9) \quad \begin{aligned} \frac{\partial u}{\partial s}(t, s) &= J\pi\left(\frac{\partial u}{\partial t}(t, s)\right) + \frac{\partial \varphi}{\partial t}(t, s)R_\alpha(u(t, s)) \\ \frac{\partial \varphi}{\partial s}(s, t) &= -\left\langle \frac{\partial u}{\partial t}(t, s), R_\alpha(u(t, s)) \right\rangle. \end{aligned}$$

Notice that $dS(\nabla S + \psi R_\alpha) \geq 0$ for any function $\psi(t, s)$. Hence, the first equation of the system (9) can be viewed as the flow equation of the gradient-like vector-field $\nabla S + \frac{\partial \varphi}{\partial t} R_\alpha$. Trajectories of this gradient like field connecting critical points γ^-, γ^+ of the action functional correspond to elements of the moduli space $\mathcal{M}_0(\gamma^-, \gamma^+; W, J)$, and therefore the Floer homology philosophy ([18]), which we described above in the finite-dimensional case, suggests the following construction.

Let us associate a variable q_γ with every periodic orbit $\gamma \in \mathcal{P}_\alpha$ and assign to it the grading

$$\deg q_\gamma = \text{CZ}(\gamma) + (n - 3).$$

The choice of the constant $n - 3$ is not important for purposes of this definition, but it will become important for generalizations considered in the second part of this paper.

Let A be the group algebra $\mathbb{C}[H_2(V)]$. We will fix a basis A_1, \dots, A_N of $H_2(V; \mathbb{C})$ and identify each homology class $\sum d_i A_i$ with its *degree* $d = (d_1, \dots, d_N)$. Thus we can view the algebra A as the algebra of Laurent polynomials of N variables z_1, \dots, z_N with complex coefficients, and write its elements in the form $\sum a_d z^d$, where $z^d = z_1^{d_1} \dots z_N^{d_N}$. The variables z_i are also considered graded, $\deg z_i = -2c_1(A_i)$, $i = 1, \dots, N$. Consider a complex \mathfrak{F} generated by the (infinitely many) graded variables q_γ with coefficients in the graded algebra A , and define a differential $\partial : \mathfrak{F} \rightarrow \mathfrak{F}$ by the formula:

$$(10) \quad \partial q_\gamma = \sum_{\gamma', d} \frac{n_{\gamma, \gamma', d}}{\kappa_{\gamma'}} z^d q_{\gamma'},$$

where $\kappa_{\gamma'}$ denotes the multiplicity of the orbit γ' , the sum is taken over all trajectories $\gamma' \in \mathcal{P}_\alpha$ and $d = (d_1, \dots, d_N)$ with

$$\text{CZ}(\gamma') = \text{CZ}(\gamma) + 2\langle c_1, d \rangle - 1,$$

and the coefficient $n_{\gamma, \gamma', d}$ counts the algebraic number of components of the 0-dimensional moduli space $\mathcal{M}_0^d(\gamma', \gamma; W, J)/\mathbb{R}$.⁷ Notice that the Liouville flow of the

⁷ Let us recall that according to our definition of the moduli space $\mathcal{M}_0^d(\gamma', \gamma; W, J)/\mathbb{R}$ the coefficient $n_{\gamma, \gamma', d}$ counts equivalence classes of holomorphic curves *with asymptotic markers*, and hence each holomorphic cylinder connecting γ and γ' is counted $\kappa_\gamma \kappa_{\gamma'}$ times, unless the cylinder itself is multiply covered. The role of the denominators $\kappa_{\gamma'}$ in formula (10), as well in a similar formula (11) below, is to correct this “over-counting”.

vector field $\frac{\partial}{\partial t}$ defines a \mathbb{R} -action on the moduli spaces $\mathcal{M}_0^d(\gamma', \gamma; W, J)$, which makes the 1-dimensional components of the moduli spaces canonically oriented. Comparing this orientation with the coherent orientation we produce signs which we use in the formula (10).

To simplify the assumptions in the propositions which we formulate below we will assume for the rest of this section that $c_1|_{\pi_2(V)} = 0$. This assumption allows us to define for any contractible periodic orbit γ the Conley-Zehnder index $\text{CZ}_{\text{disk}}(\gamma)$ computed with respect to *any disk* Δ spanned by γ in V . We denote $\text{deg}_{\text{disk}}(\gamma) = \text{CZ}_{\text{disk}}(\gamma) + n - 3$.

Proposition 1.9.1 *If for a contact form α the Reeb field R_α has no contractible periodic orbits $\gamma \in \mathcal{P}_\alpha$ with $\text{deg}_{\text{disk}}(\gamma) = 1$, then $\partial^2 = 0$.*

Sketch of the proof. Similarly to the finite-dimensional case considered in Section 1.9.1 above the identity $\partial^2 = 0$ in Floer homology is equivalent to the fact that the codimension 1 stratum of the compactified moduli spaces $\mathcal{M}_0^d(\gamma', \gamma)$ consists of broken trajectories, which in our case are represented by the height 2 stable curves (f_1, f_2) , $f_1 \in \mathcal{M}_0^{d'}(\gamma', \gamma'')/\mathbb{R}$, $f_2 \in \mathcal{M}_0^{d''}(\gamma'', \gamma)/\mathbb{R}$, where $d = d' + d''$. However, in the general case a sequence of holomorphic cylinders in $\mathcal{M}_0^d(\gamma', \gamma)$ can split into curves different from cylinders, as it is stated in Proposition 1.7.2 and Corollary 1.6.5. But if this happens then the first-floor curve f_1 must have a component which is conformally equivalent to \mathbb{C} and asymptotically cylindrical over a contractible orbit at $+\infty$. Moreover, if (f_1, f_2) belongs to a top-dimensional stratum of the boundary of the moduli space $\mathcal{M}_0^d(\gamma', \gamma)$, then $\text{deg}_{\text{disk}}(\gamma) = 1$, which contradicts our assumption.

Remark 1.9.2 Let us recall that we excluded from \mathcal{P} certain “bad” periodic orbits (see the footnote in Section 1.2). However on the boundary of the moduli space $\mathcal{M}_0^d(\gamma', \gamma)$ there could be a stratum which consists of height 2 stable curves (f_1, f_2) , $f_1 \in \mathcal{M}_0^{d'}(\gamma', \gamma'')$, $f_2 \in \mathcal{M}_0^{d''}(\gamma'', \gamma')$, where the orbit γ'' is one of the bad orbits which we excluded from \mathcal{P} . The orbit γ'' has even multiplicity $2k$, and hence on the boundary of $\mathcal{M}_0^d(\gamma', \gamma)$ there are $2k$ strata which correspond to $2k$ different possible positions of the asymptotic marker at the punctures mapped to γ'' . The Poincaré return map of the Reeb flow along the orbit γ'' has an odd number of eigenvalues in the interval $(-1, 0)$, and hence according to Lemma 1.8.8 the coherent orientation will automatically assign to these orbits opposite signs, which means that these strata will not contribute to the sum (10). This explains why the exclusion of bad orbits is *possible*. Remark 1.9.6 below explains why this exclusion is *necessary*.

Now we follow Steps 1–3 in Section 1.9.1 above to show the independence of the homology group

$$\oplus H_k(\mathfrak{F}, \partial) = \text{Ker} \partial / \text{Im} \partial,$$

graded by the degree k , of the choice of a nice contact form α and a compatible almost complex structure J .

Suppose now that we have a directed symplectic cobordism $W = \overline{V^- V^+}$, and J is a compatible almost complex structure on W . Suppose that the inclusions $V^\pm \hookrightarrow W$

induce isomorphisms on 2-dimensional homology. Then we can define a homomorphism $\Phi = \Phi_W : \mathfrak{F}^+ \rightarrow \mathfrak{F}^-$ by the formula

$$(11) \quad \Phi(q_\gamma) = \sum_{\gamma', d} \frac{1}{\kappa_{\gamma'}} n_{\gamma, \gamma', d} z^d q_{\gamma'},$$

where the sum is taken over all trajectories $\gamma' \in \mathcal{P}^-$ and d with $\text{CZ}(\gamma') = \text{CZ}(\gamma) + 2\langle c_1, d \rangle$, and the coefficient $n_{\gamma, \gamma', d}$ counts the algebraic number of points of the compact 0-dimensional moduli space $\mathcal{M}_0^d(\gamma', \gamma; W, J)$. If the condition on the second homology is not satisfied then the above construction gives us only a correspondence, rather than a homomorphism. See Section 2.5 for the discussion of a more general case.

Proposition 1.9.3 *Suppose that the contact forms α^\pm associated to the ends satisfy the condition $\deg_{\text{disk}}(\gamma) \neq 0, 1$ for any contractible in W periodic orbit $\gamma \in \mathcal{P}^\pm$. Then the homomorphism Φ_W commutes with ∂ .*

Proposition 1.9.4 *Let $J_t, t \in [0, 1]$, be a family of almost complex structures compatible with the directed symplectic cobordism $W = \overline{V^- V^+}$. Suppose that the forms α^\pm associated to the ends satisfies the condition $\deg_{\text{disk}}(\gamma) \neq -1, 0, 1$ for any contractible in W periodic orbit $\gamma \in \mathcal{P}^\pm$. Then the homomorphisms $\Phi_0 = \Phi_{W, J_0}$ and $\Phi_1 = \Phi_{W, J_1}$ are chain homotopic, i.e there exists a homomorphism $\Delta : \mathfrak{F}^+ \rightarrow \mathfrak{F}^-$ such that $\Phi_1 - \Phi_0 = \partial\Delta + \Delta\partial$.*

Proposition 1.9.5 *Given two cobordisms W_1 and W_2 , and a compatible almost complex structure J on the composition $W_1 \odot W_2$, the homomorphism $\Phi_{W_1 \odot W_2}$ is chain-homotopic to $\Phi_{W_1} \circ \Phi_{W_2}$.*

Together with an obvious remark that for the cylindrical cobordism W_0 the homomorphism Φ_{W_0} is the identity, Propositions 1.9.1–1.9.4 imply that if a contact structure ξ on V admits a *nice* contact form, i.e. a form without contractible periodic orbits of index $-1, 0$ and 1 , then the *contact homology group*

$$\oplus HC_k(V, \xi) = \oplus H_k(\mathfrak{F}, \partial)$$

is well defined and independent of the choice of a nice contact form and a compatible almost complex structure (however, if $H_2(V) \neq 0$ and/or $H_1(V) \neq 0$ it depends on a choice of spanning surfaces F_γ and the framing of the bundle ξ over basic loops). Similarly to what was explained in the sketch of the proof Proposition 1.9.1 the “niceness” assumptions guarantees that the top codimension strata on the boundary of the involved moduli spaces consist of height 2 cylindrical curves, and thus the proofs of Propositions 1.9.3–1.9.5 may precisely follow the standard scheme of the Floer theory (see [18, 62]).

Remark 1.9.6 Similarly to what we explained in Remark 1.9.2 the coefficient $n_{\gamma, \gamma'}$ in the definition (11) of Φ equals 0 if at least one of the orbits γ, γ' is “bad”. Hence, in the presence of “bad” orbits the homomorphism Φ could never be equal to the identity, even for the cylindrical cobordism. This explains why the exclusion of “bad” periodic orbits is *necessary*.

Besides the degree (or Conley-Zehnder) grading, the contact homology group is graded by elements of $H_1(V)$, because the boundary operator preserves the homology class of a periodic orbit. We will denote the part of $HC_*(V, \xi)$ which correspond to a class $a \in H_1(V)$ by $HC_*(V, \xi|_a)$. One can similarly construct a contact homology group $HC_*^{\text{contr}}(V, \xi)$, generated only by contractible periodic orbits, which is another invariant of the contact manifold (V, ξ) .

Contact structures which admit nice contact forms do exist, as it is illustrated by examples in Section 1.9.3 below. However, the condition of existence of a nice form is too restrictive. The general case leads to an algebraic formalism developed in Sections 2.2–2.5 below.

1.9.3 Examples

1. Contact homology of the standard contact sphere S^{2n-1} .

Take the 1-form $\alpha = \frac{1}{2} \sum (x_i dy_i - y_i dx_i)$, which is a primitive of the standard symplectic structure in \mathbb{R}^{2n} . Its restriction to a generic ellipsoid

$$S = \left\{ \sum \frac{x_i^2 + y_i^2}{a_i^2} = 1 \right\}$$

is a nice contact form for the standard contact structure ξ on the sphere $S = S^{2n-1}$. The form $\alpha|_S$ has precisely one periodic orbit for each Conley-Zehnder index $n + 2i - 1$ for $i = 1, \dots, n$. Hence the contact homology group $HC_*(S, \xi)$ has one generator in each dimension $2i, i \geq n - 1$. See also the discussion in Section 2.9.2 below.

2. Contact homology of Brieskorn spheres.

Ilya Ustilovsky computed ([63]) the contact homology of certain Brieskorn spheres. Let us consider the Brieskorn manifold

$$\Sigma(p, \underbrace{2, \dots, 2}_n) = \{z_0^p + \sum_1^n z_j^2 = 0\} \cap \left\{ \sum_0^n |z_j|^2 = 1 \right\} \subset \mathbb{C}^{n+1}.$$

$\Sigma(p, \underbrace{2, \dots, 2}_n)$ carries a canonical contact structure as a strictly pseudo-convex hypersurface in a complex manifold.

Suppose that $n = 2m + 1$ is odd, and $p \equiv 1 \pmod{8}$. Under this assumption $\Sigma(p, \underbrace{2, \dots, 2}_n)$ is diffeomorphic to S^{2n-1} (see [7]). However, the following theorem of

Ustilovsky implies that the contact structures on Brieskorn spheres $\Sigma(p, \underbrace{2, \dots, 2}_n)$ and

$\Sigma(p', \underbrace{2, \dots, 2}_n)$ are not isomorphic, unless $p = p'$. This result should be confronted with

a computation of Morita ([54]), which implies that the formal homotopy class (see Section 1.1 above) of the contact structure on $\Sigma(p, \underbrace{2, \dots, 2}_n)$ is standard, provided $p \equiv$

$1 \pmod{2(2m!)}$. Hence, Ustilovsky's theorem provides infinitely many non-isomorphic contact structures on S^{4m+1} in the standard formal homotopy class.

Theorem 1.9.7 (I. Ustilovsky, [63]) *The contact homology*

$$HC_* \left(\Sigma(p, \underbrace{2, \dots, 2}_n) \right)$$

is defined, and the dimension

$$c_k = \dim HC_k \left(\Sigma(p, \underbrace{2, \dots, 2}_n) \right)$$

is given by the formula

$$c_k = \begin{cases} 0, & k \text{ is odd or } k < 2n - 4, \\ 2, & k = 2 \left[\frac{2N}{p} \right] + 2(N + 1)(n - 2), \text{ for } N \geq 1, 2N + 1 \notin p\mathbb{Z}, \\ 1, & \text{in all other cases.} \end{cases}$$

3. Contact homology of boundaries of subcritical Stein manifolds.

A co-oriented contact manifold (V, ξ) is called Stein-fillable if it can be realized as a strictly pseudoconvex boundary of a complex manifold W , whose interior is Stein, and if the co-oriented contact structure ξ coincides with the canonical contact structure of a strictly pseudo-convex hypersurface. We say that (V, ξ) admits a subcritical Stein filling if the corresponding Stein manifold $\text{Int}W$ admits an exhausting plurisubharmonic function without critical points of dimension $\dim_{\mathbb{C}}(W)$. If $\dim V > 3$ then one can equivalently require that W deformation retracts to a CW-complex of dimension $< \dim_{\mathbb{C}} W$ (see [13]).

Mei-Lin Yau studied in her PhD thesis [69] contact homology of contact manifolds admitting a subcritical Stein filling. Here is her result.

Theorem 1.9.8 (Mei-Lin Yau, [69]) *Let (V, ξ) be a contact manifold of dimension $2n-1$ which admits a subcritical Stein filling W . Suppose that $c_1(V) = 0$ and $H_1(V) = 0$. Let c_1, \dots, c_k be generators of $H_*(W)$. Then the contact homology $HC_*(V)$ is defined and generated by elements $q_{i,j}$ of degree $\deg q_{i,j} = 2(n+i-2) - \dim c_j$, where $j = 1, \dots, k$, and $i \geq 1$.*

4. Contact homology of T^3 and its coverings.

Set $\alpha_n = \cos 2\pi n z dx + \sin 2\pi n z dy$. This contact form descends to the 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ and defines there a contact structure ξ_n . The structure ξ_1 is just the canonical contact structure on T^3 as the space of co-oriented contact elements of T^2 . The form α_n for $n > 1$ is equal to the pull-back $\pi_n^*(\alpha_1)$, where $\pi_n : T^3 \rightarrow T^3$ is the covering $(x, y, z) \mapsto (x, y, nz)$. Notice that all structures ξ_n are homotopic as plane field to the foliation $dz = 0$.

Theorem 1.9.9 *The contact homology group $HC_*(T^3, \xi_n|w)$, where w is the homology class $(p, q, 0) \in H_1(T^3)$, is isomorphic to \mathbb{Z}^{2n} .*

In particular we get as a corollary a theorem of E. Giroux:

Corollary 1.9.10 (E. Giroux, [26]) *The contact structures ξ_n , $n = 1, \dots$, are pairwise non-isomorphic.*

The contact manifold (T^3, ξ_1) is foliated by pre-Lagrangian tori $L_{p,q}$, indexed by simple homology classes $(p, q) \in H_1(T^2)$. Each torus $L_{p,q}$ is foliated by the S^1 -family of lifts of closed geodesics which represent the class (p, q) . Thus for any given $(p, q) \in H_1(T^2)$ (even when (p, q) have common divisors) the set of closed orbits in \mathcal{P}_{α_1} which represent the class $(p, q, 0) \in H_1(T^3)$ is a circle $S_{p,q}$, and for any $n \geq 1$ the set of closed orbits in \mathcal{P}_{α_n} which represent the class $(p, q, 0) \in H_1(T^3)$ consists of n copies $S_{p,q}^1, \dots, S_{p,q}^n$ of such circles. The forms α_n have no contractible periodic orbits, but of course, they are degenerate. To compute the contact homology groups, one can either work directly with these degenerate forms, as it is explained in Section 2.9.2 below, and show that $HC_*(T^3, \xi_n|_w) = H_*\left(\bigcup_1^n S_{p,q}^i\right) = \mathbb{Z}^{2n}$, or first perturb the form α_1 , and respectively all its covering forms $\alpha_n = \pi_n^*(\alpha_1)$, in order to substitute each circle $S_{p,q}^i$ by two non-degenerate periodic orbits, and then show that the orbits from each of these pairs are connected by precisely two holomorphic cylinders, which cancel each other in the formula for the boundary operator ∂ .

1.9.4 Relative contact homology and contact non-squeezing theorems

Let us observe that the complex (\mathfrak{F}, ∂) is filtrated by the values of the action functional S , $\mathfrak{F} = \bigcup_{a \in \mathbb{R}} \mathfrak{F}^a$, where the complex \mathfrak{F}^a is generated by variables q_γ with $S(\gamma) \leq a$. The differential ∂ respects this filtration, and hence descends to $\mathfrak{F}^b/\mathfrak{F}^a$, $a < b$. Hence, one can define the homology $H_*^{(a,b]}(\mathfrak{F}, \partial) = H_*(\mathfrak{F}^b/\mathfrak{F}^a, \partial)$ in the window $(a, b] \subset \mathbb{R}$. Of course, $H_*^{(a,b]}$ depends on a choice of a particular nice form α . If $\alpha > \beta$ then we have a map $\Phi_* : H_*^{(a,b]}(\mathfrak{F}, \partial; \alpha) \rightarrow H_*^{(a,b]}(\mathfrak{F}, \partial; \beta)$. We write H^a instead of $H^{(-\infty, a]}$.

Consider now a contact manifold (V, ξ) which is either closed, or satisfies the following *pseudo-convexity* condition at infinity. A contact manifold $(V, \xi = \{\alpha = 0\})$ with a fixed contact form α is called pseudo-convex at infinity if there exists a compatible almost complex structure J on the symplectization $V \times \mathbb{R}$ for which V can be exhausted by compact domains V_i with smooth pseudo-convex boundary. A sufficient condition for pseudo-convexity is existence of an exhaustion $V = \bigcup V_i$, such that for each $i = 1, \dots$, trajectories of the Reeb field $R_\alpha|_{V_i}$ do not have interior tangency points with ∂V_i . For instance, for the standard contact form $\alpha = dz - \sum y_i dx_i$ on \mathbb{R}^{2n+1} the latter condition is satisfied for an exhaustion of \mathbb{R}^{2n+1} by round balls, and hence the standard contact form on \mathbb{R}^{2n+1} is pseudo-convex at infinity.

Our goal is to define a relative contact homology group $HC_*(V, U, \xi)$ for a relatively compact open subset $U \subset V$, so that this group would be invariant under a contact isotopy of U in V .

Let us fix a contact form α on V which satisfies the above pseudo-convexity condition. Let us denote by $\mathcal{F}_{U,\alpha}$ the set of C^∞ -functions $f : V \rightarrow [0, \infty)$ which are ≤ 1

on U , and for which the contact form $f\alpha$ is nice and pseudo-convex at infinity.⁸ Take a strictly increasing sequence of functions $f_i \in \mathcal{F}_{U,\alpha}$, such that

- a) $\max_K f_i \xrightarrow{i \rightarrow \infty} \infty$ for each compact set $K \subset (V \setminus \overline{U})$;
- b) $f_i|_U \xrightarrow{i \rightarrow \infty} 1$ uniformly on compact sets.

Proposition 1.9.11 *The limit*

$$(12) \quad HC_*(V, U, \xi) = \lim_{a \rightarrow +\infty} \lim_{\leftarrow} HC_*^a(V, f_i \alpha)$$

is independent of α , and thus it is an invariant of the contact pair (V, U) . A contact isotopy $f_t : V \rightarrow V$ induces a family of isomorphisms $(f_t)_* : H_*(V, U) \rightarrow H_*(V, f_t(U))$. An inclusion $i : U_1 \hookrightarrow U_2$ induces a homomorphism

$$i_* : HC_*(V, U_1, \xi) \rightarrow HC_*(V, U_2, \xi).$$

One of the most celebrated results in Symplectic topology is Gromov's non-squeezing theorem which states that one cannot symplectically embed a $2n$ -ball of radius 1 into $D_r^2 \times \mathbb{R}^{2n-2}$ for $r < 1$. Here D_r^2 denotes a 2-disk of radius r and $D_r^2 \times \mathbb{R}^{2n-2}$ is endowed with the product of standard symplectic structures. Because of the conformal character of contact geometry one cannot expect as strong non-squeezing results for contact manifolds: one can embed any domain in the standard \mathbb{R}^{2n-1} in an arbitrary small ball. However, it turns out that it is not always possible to realize a contact squeezing via a contact isotopy inside a manifold with a non-trivial first Betti number.

As an example, consider the 1-jet bundle

$$V = J^1(\mathbb{R}^n, S^1) = T^*(\mathbb{R}^n) \times S^1$$

of S^1 -valued functions with its standard contact structure ξ , given by the contact form $\alpha = dz - \sum y_i dx_i$, $(x, y) \in \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$, $z \in S^1 = \mathbb{R}/\mathbb{Z}$. The contact form α satisfies the condition of pseudo-convexity at infinity and it is nice: the Reeb field equals $\frac{\partial}{\partial z}$, and thus it has no contractible periodic orbits. Let us consider the class \mathcal{N} of domains $\Omega \subset V$ which are images of the split domains $U \times S^1 \subset V$ under a contact isotopy of V , where U is connected. For any $\Omega \in \mathcal{N}$ the relative contact homology group $HC_*(V, \Omega)$ is well defined because for any function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ the form $f(x, y)\alpha$ is nice.

Let us denote by $\mathcal{E}_r(\Omega)$, $\Omega \in \mathcal{N}$, the space of contact embeddings $D_r \times S^1 \rightarrow \Omega \times S^1$, contact isotopic in V to the inclusion

$$D_r \times S^1 \hookrightarrow \mathbb{R}^{2n} \times S^1 = V.$$

Notice, that for any two embeddings $f, g \in \mathcal{E}_r(\Omega)$ there exists a positive $\rho \leq r$, such that the restrictions $f|_{D_\rho \times S^1}$ and $g|_{D_\rho \times S^1}$ are isotopic via a *contact* isotopy.

⁸ Of course, the set $\mathcal{F}_{U,\alpha}$ may be empty, because the niceness condition is very restrictive. In this case one needs to employ a more general construction from Section 2.2.3.

Given a contact embedding $f \in \mathcal{E}_r(\Omega)$ there is defined a homomorphism

$$f_* : HC_*(V, D_r \times S^1, \xi) \rightarrow HC_*(V, \Omega, \xi).$$

Let us choose a symplectic trivialization of the contact bundle ξ induced by the projection $V \rightarrow \mathbb{R}^{2n}$. We will assume that indices of periodic orbits, and hence the grading of contact homology groups, are associated with this trivialization.

For each homology class $k \in \mathbb{Z} = H_1(D_r \times S^1)$ let us consider the maximal $l = l(f, k)$ such that

$$\text{Ker}(f_*|_{HC_l(V, D_r \times S^1, \xi|_k)}) \neq 0,$$

and define an invariant $w_{\text{cont}}(V, \Omega)$, called the relative *contact width* by the formula

$$(13) \quad w_{\text{cont}}(V, \Omega) = \sup_{k, r > 0, f \in \mathcal{E}_r(U)} \frac{2k}{l(f, k)}.$$

S.-S. Kim has computed $w_{\text{cont}}(V, \Omega)$ for certain domains Ω . In particular, she proved

Proposition 1.9.12

$$\begin{aligned} w_{\text{cont}}(V, D_r^{2n} \times S^1) &= \pi r^2; \\ w_{\text{cont}}(V, D_r^2 \times D_R^{2n-2} \times S^1) &= \pi r^2, \end{aligned}$$

if $R \geq r$.

The contact width is clearly a monotone invariant, i.e.

$$w_{\text{cont}}(V, U_1 \times S^1) \leq w_{\text{cont}}(V, U_2 \times S^1)$$

if $U_1 \subset U_2$. Hence Proposition 1.9.12 implies

Corollary 1.9.13 *Suppose that $r < \min(r', R)$. Then there is no contact isotopy $f_t : D_{r'}^{2n} \times S^1 \rightarrow V$ such that f_0 is the inclusion, and*

$$f_1(D_{r'}^{2n} \times S^1) \subset D_r^2 \times D_R^{2n-2} \times S^1.$$

Problem 1.9.14 *Suppose there exists a contact isotopy $f_t : V = \mathbb{R}^{2n} \times S^1 \rightarrow V$ with $f_0 = \text{Id}$ and $f_1(U_1 \times S^1) \subset U_2 \times S^1$. Does there exist a Hamiltonian isotopy $g_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $g_0 = \text{Id}$ and $g_1(U_1) \subset U_2$?*

Notice that the converse is obviously true.

2 Algebraic formalism

2.1 Informal introduction

The Floer-theoretic formalism described in Section 1.9 is applicable only to a very limited class of contact manifolds. As it follows from Theorem 1.6.2 the boundary

of the moduli space of holomorphic cylinders in the symplectization may consist of stable curves, different from broken cylinders; for instance, it may contain height 2 stable curves which consist of a pair of pants on the upper level, and a copy of \mathbb{C} plus a trivial cylinder at the bottom one. Hence the minimal class of holomorphic curves in symplectizations which has the property that the stable curves of height > 1 on the boundary of the corresponding moduli space are made of curves from the same class, must contain all rational curves with one positive and an arbitrary number of negative punctures. The counting of curves with one positive and arbitrary number of negative punctures can still be interpreted as a differential, but this time defined not on the *vector space* generated by periodic trajectories but on the graded *algebra* which they generate. Thus this leads to a straightforward generalization of the Floer type formalism considered in Section 1.9 when instead of the additive Floer complex \mathfrak{F} generated by the variables q_γ , one considers a graded commutative algebra \mathfrak{A} generated by these variables, and when instead of the formula (10) the differential ∂q_γ is defined as a polynomial of a higher degree, rather than a linear expression as in the Floer homology case. Namely, we define

$$(14) \quad \partial q_\gamma = \sum \frac{n_{\Gamma, I, d}}{k! \prod_1^k i_j! \kappa_{\gamma_j}^{i_j}} q_{\gamma_1}^{i_1} \dots q_{\gamma_k}^{i_k} z^d,$$

where the sum is taken over all ordered⁹ sets of different periodic orbits $\Gamma = \{\gamma_1, \dots, \gamma_k\}$, multi-indices $d = (d_1, \dots, d_N)$ and $I = (i_1, \dots, i_k)$, $i_j \geq 0$, and where the coefficient $n_{\Gamma, I, d}$ counts the algebraic number of elements of the moduli space

$$\mathcal{M}_0^d(\gamma; \underbrace{\gamma_1, \dots, \gamma_1}_{i_1}, \dots, \underbrace{\gamma_k, \dots, \gamma_k}_{i_k}) / \mathbb{R},$$

if this space is 0-dimensional, and equals 0 otherwise. The differential ∂ extends to the algebra \mathfrak{F} according to the graded Leibnitz rule. Roughly speaking, ∂q_γ is a polynomial, whose monomials $q_{\gamma_1} \dots q_{\gamma_l}$ are in 1-1 correspondence with rigid, up to translation, rational holomorphic curves with one positive cylindrical end over γ and l negative cylindrical ends over trajectories $\gamma_1, \dots, \gamma_l$.

It turns out that the quasi-isomorphism class of the differential algebra (\mathfrak{A}, ∂) is independent of all extra choices (see Section 2.2.3 below). In particular, the *contact homology algebra* $H_*(\mathfrak{A}, \partial)$ is an invariant of the contact manifold (M, ξ) .

Having included into the picture the moduli spaces of rational curves with one positive and several negative punctures, one may wonder, what is the role of rational curves with an arbitrary number of positive and negative punctures. One can try to interpret the counting of rational holomorphic curves with fixed number of positive and an arbitrary number of negative punctures as a sequence of bracket type operations on the algebra \mathfrak{A} . These operations satisfy an infinite system of identities, which remind the formalism of homotopy Lie algebras.

However, there is a more adequate algebraic formalism for this picture. Let us associate with each periodic orbit γ two graded variables p_γ and q_γ of the same parity (but of different integer grading, as we will see in Section 2.2.2 below), and consider an algebra \mathfrak{P} of formal power series $\sum f_\Gamma(q) p^\Gamma$, where $f_\Gamma(q)$ are polynomials

⁹The coefficient $\frac{1}{k!}$ is the price we pay for taking *ordered* sets of periodic orbits.

of $q = \{q_\gamma\}$ with coefficients in (a completion of) the group algebra of $H_2(V)$. It is useful to think about the algebra \mathfrak{P} as the graded Poisson algebra of functions on the infinite-dimensional symplectic super-space \mathbf{V} with the even symplectic form $\sum_{\gamma \in \mathcal{P}} \kappa_\gamma^{-1} dp_\gamma \wedge dq_\gamma$, or rather on its formal analog along the 0-section $\{p = \{p_\gamma\} = 0\}$.

With each 0-dimensional moduli space $\mathcal{M}_0^d(\Gamma^-, \Gamma^+)/\mathbb{R}$,

$$\Gamma^\pm = \left\{ \underbrace{\gamma_1^\pm, \dots, \gamma_1^\pm}_{i_1^\pm}, \dots, \underbrace{\gamma_{s^\pm}^\pm, \dots, \gamma_{s^\pm}^\pm}_{i_{s^\pm}^\pm} \right\},$$

we associate a monomial

$$\frac{n_{\Gamma^-, \Gamma^+, d}}{s^-! s^+! (i_1^-)! \dots (i_{s^-}^-)! (i_1^+)! \dots (i_{s^+}^+)!} q_{\gamma^-}^{I^-} p_{\gamma^+}^{I^+} z^d,$$

where $q_{\gamma^-}^{I^-} = (q_{\gamma_1^-})^{i_1^-} \dots (q_{\gamma_{s^-}^-})^{i_{s^-}^-}$, $p_{\gamma^+}^{I^+} = (p_{\gamma_1^+})^{i_1^+} \dots (p_{\gamma_{s^+}^+})^{i_{s^+}^+}$, and $n_{\Gamma^-, \Gamma^+, d}$ is the algebraic number of elements of the moduli space $\mathcal{M}_0^d(\Gamma^-, \Gamma^+)/\mathbb{R}$.

The sum of all these monomials over all 1-dimensional moduli spaces $\mathcal{M}_0^d(\Gamma^-, \Gamma^+)$ for all ordered sets Γ^-, Γ^+ of periodic orbits is an *odd* element $\mathbf{h} \in \mathfrak{P}$. All the operations on the algebra \mathfrak{A} which we mentioned above appear as the expansion terms of \mathbf{h} with respect to p -variables. It turns out that the infinite system of identities for operations on \mathfrak{A} , which we mentioned above, and which is defined by counting holomorphic curves with a certain fixed number of positive punctures, can be encoded into a single equation $\{\mathbf{h}, \mathbf{h}\} = 0$. Then the differentiation with respect to the Hamiltonian vector field, defined by the Hamiltonian \mathbf{h} :

$$d^{\mathbf{h}}(g) = \{\mathbf{h}, g\}, \quad g \in \mathfrak{P},$$

defines a differential $d = d^{\mathbf{h}} : \mathfrak{P} \rightarrow \mathfrak{P}$, which satisfies the equation $d^2 = 0$. Thus one can define the homology $H_*(\mathfrak{P}, d^{\mathbf{h}})$ which inherits the structure of a graded Poisson algebra.

The identities, like $d^2 = 0$ and $\partial^2 = 0$, encode in algebraic terms information about the structure of the top-dimensional strata on the boundary of compactified moduli spaces of holomorphic curves, as it is described in Proposition 1.7.2 above. For instance, the codimension 1 strata on the boundary of the moduli space $\mathcal{M}_0^d(\Gamma^-, \Gamma^+)/\mathbb{R}$ consists of height two stable rational curves (f_1, f_2) . Each floor of this curve may be disconnected, but precisely one of its components differs from the straight cylinder. Each connected component of f_1 can be glued with a component of f_1 only along one of their ends. One can easily see that the combinatorics of such gluing precisely corresponds to the Poisson bracket formalism and that the algebraic sum of the monomials associated to all stable curves of height two equals $\{\mathbf{h}, \mathbf{h}\}$. On the other hand, the algebraic number of such height 2 curves equals 0 because they form the boundary of the a compactified 1-dimensional moduli space of holomorphic curves. Hence we get the identity $\{\mathbf{h}, \mathbf{h}\} = 0$. The identity is not tautological due to the fact that \mathbf{h} is odd. In view of the super-Jacobi identity it is equivalent to the identity $(d^{\mathbf{h}})^2 = 0$.

One can go further and include into the picture moduli spaces of punctured holomorphic curves of higher genus. Introducing a new variable, denoted \hbar , to keep track

of the genus, one can associate with each 0-dimensional moduli space $\mathcal{M}_g^d(\Gamma^-, \Gamma^+)/\mathbb{R}$ a monomial

$$\frac{n_{\Gamma^-, \Gamma^+, d, g}}{s^-! s^+! (i_1^-)! \dots (i_{s^-}^-)! (i_1^+)! \dots (i_{s^+}^+)!} q_{\gamma^-}^{I^-} p_{\gamma^+}^{I^+} \hbar^{g-1} z^d,$$

and form a generating function $\mathbf{H} = \hbar^{-1} \sum_{g=0}^{\infty} \mathbf{H}_g \hbar^g$ counting all rigid holomorphic curves of arbitrary genus, whose term \mathbf{H}_0 coincides with \mathbf{h} . Again, the codimension 1 strata of the boundary of the moduli spaces $\mathcal{M}_g(\Gamma^-, \Gamma^+)$ consists of height 2 stable curves, but unlike the case of rational curves, two connected components on different levels can be glued together along an arbitrary number of ends. The combinatorics of such gluing can be described by the formalism of algebra of higher order differential operators. Fig. 4 illustrates how the composition of differential operators can be interpreted via gluing of Riemann surfaces with punctures. A letter p_i in the picture represents a differential operator $\hbar \frac{\partial}{\partial q_i}$, and a surface of genus g with upper punctures p_{i_1}, \dots, p_{i_k} and lower punctures q_{j_1}, \dots, q_{j_l} represents a differential operator

$$\hbar^{g-1} q_{j_1} \dots q_{j_l} p_{i_1} \dots p_{i_k} = \hbar^{g-1} q_{j_1} \dots q_{j_l} \left(\hbar \frac{\partial}{\partial q_{i_1}} \right) \dots \left(\hbar \frac{\partial}{\partial q_{i_k}} \right).$$

Thus we are led to consider \mathbf{H} as an element of the Weyl super-algebra \mathfrak{W} . This algebra should be viewed as a quantization of the Poisson algebra \mathfrak{P} , so that the description of the boundary of the moduli spaces is given by the equation $[\mathbf{H}, \mathbf{H}] = 0$, where $[\ , \]$ denotes the commutator in \mathfrak{W} . As in the rational case, this identity is equivalent to the identity $D_{\mathbf{H}}^2 = 0$ for the differential $D^{\mathbf{H}}(f) = [H, f]$. Hence we can define the homology algebra $H_*(\mathfrak{W}, D^{\mathbf{H}})$, which also turns out to be an invariant of the contact manifold (V, ξ) . Similarly to the standard Gromov-Witten theory for closed symplectic manifolds one can develop an even more general formalism by encoding in \mathbf{H} information about higher-dimensional moduli spaces of holomorphic curves. This leads to a deformation of the differential algebra $(\mathfrak{W}, D^{\mathbf{H}})$ along the space of closed forms on V . The corresponding family of homology algebras is then parameterized by $H^*(V)$.

After going that far it is natural to make the above algebraic structure for contact manifolds a part of a formalism in the spirit of topological field theory, which we call *Symplectic Field Theory*, and which also includes the theory of Gromov-Witten invariants of closed manifolds. To do that one considers moduli spaces of holomorphic curves with cylindrical ends in directed symplectic cobordisms $W = \overline{V^- V^+}$. The generating function counting rational holomorphic curves in W can be naturally written as a function $\mathbf{f}(q^-, p^+)$ of p^+ -variables associated with the positive end V^+ , and q^- -variables associated with the negative end V^- of the cobordism W . It turns out that the Lagrangian submanifold in $(-\mathbf{V}^-) \times \mathbf{V}^+$ generated by the function \mathbf{f} defines a *Lagrangian correspondence* $\mathbf{L}_W \subset (-\mathbf{V}^-) \times (\mathbf{V}^+)$ which transforms the Hamiltonian functions \mathbf{h}^+ and \mathbf{h}^- to each other, i.e.

$$(15) \quad (\mathbf{h}^-(p^-, q^-) - \mathbf{h}^+(p^+, q^+))|_{\mathbf{L}_W} = 0,$$

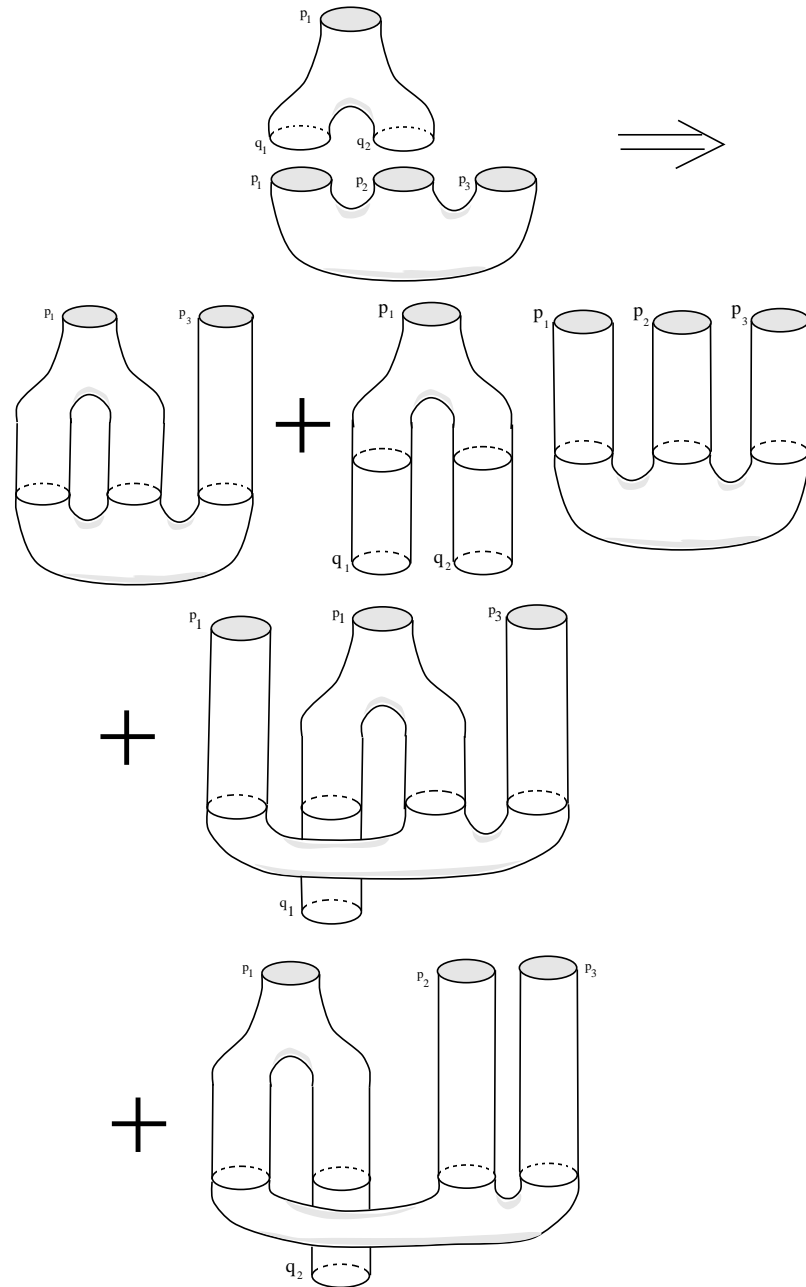


Figure 4: There are four different way to glue the lower and upper surfaces on the picture along their matching ends, i.e. the ends denoted by p 's and q 's with the same index. These 4 ways correspond to 4 terms in the composition formula for differential operators: $(\hbar^{-1}p_1p_2p_3) \circ (\hbar^{-1}q_1q_2p_1) = p_1p_3 + \hbar^{-2}q_1q_2p_1^2p_2p_3 + \hbar^{-1}q_1p_1^2p_3 + \hbar^{-1}q_2p_1p_2p_3$. We are ignoring here the sign issues and assuming all the boundary components to be simple orbits.

where

$$L_W = \begin{cases} q_{\gamma^+}^+ = \kappa_{\gamma^+} \frac{\partial \mathbf{f}}{\partial p_{\gamma^+}^+}(q^-, p^+); \\ p_{\gamma^-}^- = \kappa_{\gamma^-} \frac{\partial \mathbf{f}}{\partial q_{\gamma^-}^-}(q^-, p^+). \end{cases}$$

We recall that κ_{γ^\pm} denotes the multiplicity of the orbit γ^\pm .

The composition of symplectic cobordisms produces the composition of Lagrangian correspondences, so that if one consider a ‘‘Heegard splitting’’ of a closed symplectic manifold W along a contact hypersurface V , then the computation of Gromov-Witten invariants of W can be viewed as a Lagrangian intersection problem in the symplectic super-space \mathbf{V} associated to the contact manifold V .

After what was said it should not come as a surprise that in the quantized picture Lagrangian correspondences are being replaced by Fourier integral operators, and the composition of correspondences by the convolution of the corresponding operators.

We describe below the SFT-formalism with more details. We treat contact manifolds in Section 2.2 and symplectic cobordisms in Section 2.3. Section 2.4 is devoted to the SFT-version of the chain homotopy statement in Floer homology theory. In Section 2.5 we introduce the composition formula for the SFT-invariants of symplectic cobordisms. In Section 2.6 we discuss how the introduced algebraic structures of contact manifolds depend on extra choices. Section 2.7 is devoted to a differential equation for the potential \mathbf{F} of a directed symplectic cobordism with a *non-empty* boundary. Together with the gluing formula from Section 2.5 this equation provides an effective tool for computing Gromov-Witten invariants. The remainder of the paper has even more sketchy character than the rest of the paper. Section 2.8 is devoted to invariants of Legendrian submanifolds via SFT. Section 2.9 is devoted to various examples and possible generalizations of SFT. In particular, in Section 2.9.2 we discuss how one can adapt the theory to include an important for applications, though non-generic, case of contact forms with continuous families of periodic orbits. In Section 2.9.3 we describe a new recursive procedure for computing rational Gromov-Witten invariants of $\mathbb{C}P^n$. Finally, in Section 2.9.4 we just touch the wealth of other invariants which exist in Symplectic Field Theory.

2.2 Contact manifolds

2.2.1 Evaluation maps

Let (V, ξ) be a contact manifold with a fixed contact form α , $(W = V \times \mathbb{R}, d(e^t \alpha))$ the symplectization of (V, ξ) , and J a compatible almost complex structure. As in Section 1.5 we denote by f_V and $f_{\mathbb{R}}$ the V - and \mathbb{R} -components of a J -holomorphic curve f in W , and by $\mathcal{M}_{g,r,s^-,s^+}(W, J)$ the disjoint union

$$\bigcup \mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+),$$

which is taken over all $A \in H_2(V)$, and all sets $\Gamma^-, \Gamma^+ \subset \mathcal{P}_\alpha$ of cardinalities s^\pm .

Let us view the set $\mathcal{P} = \mathcal{P}_\alpha$ of periodic orbits of the Reeb fields R_α as a discrete topological space. It naturally splits into the disjoint union

$$\prod_{k=1}^{\infty} \mathcal{P}_k,$$

of identical subspaces, where \mathcal{P}_k is the space of periodic orbits of multiplicity k .

Consider now three sets of evaluation maps:

$$ev_i^0 : \mathcal{M}_{g,r,s^-,s^+}/\mathbb{R} \rightarrow V, \quad i = 1, \dots, r,$$

$$ev_j^+ : \mathcal{M}_{g,r,s^-,s^+}/\mathbb{R} \rightarrow \mathcal{P}, \quad j = 1, \dots, s^+,$$

and

$$ev_k^- : \mathcal{M}_{g,r,s^-,s^+}/\mathbb{R} \rightarrow \mathcal{P}, \quad k = 1, \dots, s^-,$$

where ev_i^0 is the evaluation map $f_V(y_i)$ at the i -th marked point y_i , while ev_j^\pm are the evaluation maps at asymptotic markers μ_j^{\pm} . More precise, let

$$\bar{f} = (f, j, \mathbf{x}^-, \mathbf{x}^+, \mathbf{y}, \mu^{\mathbf{x}^-}, \mu^{\mathbf{x}^+}) \in \mathcal{M}_{g,r,s^-,s^+},$$

and f be asymptotically cylindrical over a periodic orbit $\gamma_j^\pm \in \mathcal{P}$ at $\pm\infty$ at the puncture x_j^\pm . Then $ev_j^\pm(\bar{f})$ is a point of \mathcal{P} representing the orbit γ_j^\pm (comp. Section 2.9.2 below).

All the above evaluation maps can be combined into a map

$$ev : \mathcal{M}_{g,r,s^-,s^+}/\mathbb{R} \rightarrow V^r \times (\mathcal{P}^-)^{s^-} \times (\mathcal{P}^+)^{s^+},$$

which extends to the compactified moduli space $\overline{\mathcal{M}_{g,r,s^-,s^+}/\mathbb{R}}$.

2.2.2 Correlators

Now we are ready to define correlators. Given r differential forms $\theta_1, \dots, \theta_r$ on V and s^\pm (0-dimensional) cohomology classes $\alpha_1^\pm, \dots, \alpha_{s^\pm}^\pm \in H^*(\mathcal{P})$ we define the degree -1 , or contact *correlator*

$$(16) \quad \begin{aligned} & -1 \langle \theta_1, \dots, \theta_r; \alpha_1^-, \dots, \alpha_{s^-}^-; \alpha_1^+, \dots, \alpha_{s^+}^+ \rangle_g^A = \\ & \int_{\overline{\mathcal{M}_{g,r,s^-,s^+}/\mathbb{R}}} ev^*(\theta_1 \otimes \dots \otimes \theta_r \otimes \alpha_1^- \otimes \dots \otimes \alpha_{s^-}^- \otimes \alpha_1^+ \otimes \dots \otimes \alpha_{s^+}^+). \end{aligned}$$

Usually we will assume that the forms $\theta_1, \dots, \theta_r$ are closed, but even in this case the above correlator depends on the actual forms, and not just their cohomology classes, because the domain of integration may have a boundary. As we will see below the superscript -1 in $^{-1}\langle \dots \rangle$ corresponds to the grading of the generating function for these correlators. It also refers to the enumerative meaning of the correlators: they count components of 1-dimensional moduli spaces of holomorphic curves. We will

consider below also correlators ${}^0\langle \dots \rangle$ and ${}^1\langle \dots \rangle$, counting 0-dimensional and -1 -dimensional (i.e. appearing in 1-dimensional families) moduli spaces of holomorphic curves.

If we are given K linearly independent differential forms $\Theta_1, \dots, \Theta_K$, then it is convenient to introduce a “general form” $t = \sum_1^K t_i \Theta_i$ from the space $L = L(\Theta_1, \dots, \Theta_K)$ generated by the chosen forms, and view t_i as graded variables with $\deg(t_i) = \deg(\Theta_i) - 2$. In particular, all terms in the sum $\sum_1^K t_i \Theta_i$ have even degrees.

Let us consider two copies \mathcal{P}^+ and \mathcal{P}^- of the 0-dimensional space \mathcal{P} , one associated with the positive end of W , the other with the negative one. Cohomology classes in \mathcal{P}_+ we will denote by p , and in \mathcal{P}_- by q , and will write

$$p = \sum_{\gamma \in \mathcal{P}} \frac{1}{\kappa_\gamma} p_\gamma [\gamma], \quad q = \sum_{\gamma \in \mathcal{P}} \frac{1}{\kappa_\gamma} q_\gamma [\gamma],$$

where κ_γ is the multiplicity of γ , and the cohomology classes $[\gamma]$ form the canonical basis of $H^*(\mathcal{P})$, dual to points in \mathcal{P} . Of course, speaking about cohomology classes of a discrete space may sound somewhat ridiculous. However, this point of view is useful, especially in preparation for a more general case when some periodic orbits may be degenerate and thus the spaces \mathcal{P}^\pm need not to be anymore discrete, see Section 2.9.2 below. We will also fix a basis A_1, \dots, A_N of $H_2(V)$. The coordinate vector $d = (d_1, \dots, d_N)$ of a class A is called the degree. Here d_j are integers, while we consider t, p, q as graded variables, where the degrees of the variables p, q are defined by the formulas

$$\begin{aligned} \deg(p_\gamma) &= -\text{CZ}(\gamma) + (n - 3), \\ \deg(q_\gamma) &= +\text{CZ}(\gamma) + (n - 3). \end{aligned}$$

The choice of grading, somewhat strange at the first glance, is explained by Proposition 2.2.1 below.

The correlators

$$\langle \underbrace{t, \dots, t}_r; \underbrace{q, \dots, q}_{s^-}; \underbrace{p, \dots, p}_{s^+} \rangle_g^d$$

with different r, d, g determine all the correlators involving forms from the space L .

2.2.3 Three differential algebras

Similar to the theory of Gromov-Witten invariants of closed symplectic manifolds we will organize all correlators into a generating function, called *Hamiltonian*,

$$\mathbf{H} = \frac{1}{\hbar} \sum_{g=0}^{\infty} \mathbf{H}_g \hbar^g,$$

where,

$$(17) \quad \mathbf{H}_g = \sum_d \sum_{r, s^\pm=0}^{\infty} \frac{1}{r!s^-!s^+!} {}^{-1} \langle \underbrace{t, \dots, t}_r; \underbrace{q, \dots, q}_{s^-}; \underbrace{p, \dots, p}_{s^+} \rangle_g^d z^d,$$

and $t = \sum_1^K t_i \Theta_i$. We will assume throughout the paper, that all forms $\Theta_1, \dots, \Theta_K$ are closed (see, however, Remarks 2.2.3 and 2.3.4, and Section 2.7 below). The variables \hbar and $z = (z_1, \dots, z_N)$ are also considered as graded with $\deg \hbar = 2(n-3)$ and $\deg(z_i) = -2c_1(A_i)$, where c_1 is the first Chern class of the almost complex structure J .

Proposition 2.2.1 *a) For each $g = 0, \dots$ the series \mathbf{H}_g can be viewed as formal power series in variables p_γ with coefficients which are polynomials of variables q_γ and formal power series of t_i ¹⁰ with coefficients in the group algebra $\mathbb{C}[H_2(V)]$ (which we identify with the algebra of Laurent polynomials of z with complex coefficients);*

b) All terms of \mathbf{H} have degree -1 ;

c) $\mathbf{H}|_{p=0} = \mathbf{H}_{\text{const}}$, where

$$\mathbf{H}_{\text{const}} = \hbar^{-1} \sum_{g,r=0}^{\infty} \frac{1}{r!} \underbrace{^{-1}(t, \dots, t)}_r^0 \hbar^g$$

accounts for the contribution of constant holomorphic curves. In particular, $\mathbf{H}|_{p=0}$ is independent of q and z .

The polynomial dependence of \mathbf{H}_g on variables q_γ and z in a geometric language just means that the union $\overline{\mathcal{M}}_g^d(\Gamma^+)$, $\Gamma^+ = \gamma_1 \dots \gamma_{s^+}$, of the compactified moduli spaces of holomorphic curves of a fixed genus of any degree with prescribed positive ends $\gamma_1^+, \dots, \gamma_{s^+}$ is compact, and in particular that there are only finitely many possibilities for the degrees and the negative ends of these curves. This follows from the fact that for each curve $C \in \mathcal{M}_g(\Gamma^-, \Gamma^+)$ we have

$$(18) \quad 0 \leq \int_C d\alpha = \sum_{\gamma_i \in \Gamma^+} \int_{\gamma_i} \alpha - \sum_{\gamma_j \in \Gamma^-} \int_{\gamma_j} \alpha \leq \sum_{\gamma_i \in \Gamma^+} \int_{\gamma_i} \alpha,$$

the fact that there exists a constant $m > 0$ such that $\int_\gamma \alpha > m$ for any periodic orbit $\gamma \in \mathcal{P}_\alpha$ and Theorem 1.6.2 above. Proposition 2.2.1b) follows from the formula (1) for the dimension of the moduli spaces of holomorphic curves, our degree convention and the fact that a correlator $^{-1}\langle \theta_1, \dots, \theta_r; \gamma_1^-, \dots, \gamma_{s^-}^-; \gamma_1^+, \dots, \gamma_{s^+}^+ \rangle_g^A$ may be different from 0 only if the total dimension of the forms $\theta_1, \dots, \theta_r$ equals the dimension of the moduli space $\mathcal{M}_{g,r}^A(\gamma_1^-, \dots, \gamma_{s^-}^-; \gamma_1^+, \dots, \gamma_{s^+}^+)/\mathbb{R}$. Proposition 2.2.1c) just means that every non-constant holomorphic curves should have at least one positive end, which follows from inequality (18), or alternatively the maximum principle for holomorphic curves.

¹⁰ In fact, \mathbf{H}_g depends polynomially on all variables t_i of degree $\neq 0$. The degree 0 variables, i.e. the variables associated with 2-forms, enter into the constant part of \mathbf{H}_g (i.e. the part describing constant holomorphic curves) polynomially, while the non-constant part of \mathbf{H}_g depends polynomially on e^{t_i} . This fact is similar to the standard Gromov-Witten theory and will not be discussed in the present paper.

Let us consider the *Weyl super-algebra* $\mathfrak{W} = \{\sum_{\Gamma,g} f_{\Gamma,g}(q,t)p^\Gamma \hbar^g\}$, where

$$\Gamma = (\gamma_1, \dots, \gamma_a), \gamma_i \in \mathcal{P}, a = 1, \dots, p^\Gamma = p_{\gamma_1} \dots p_{\gamma_a},$$

and $f_{\Gamma,g}(q,t)$ are polynomials of variables q_γ and formal power series of t_i ¹¹. Proposition 2.2.1a) states that $\mathbf{H} \in \hbar^{-1}\mathfrak{W}$.

The product operation $F \circ G$ in \mathfrak{W} is associative and satisfies the following commutation relations: all variables are super-commute (i.e. commute or anti-commute according to their grading), except p_γ and q_γ which correspond to the same periodic orbit γ . For these pairs of variables we have the following commutation relation:

$$(19) \quad [p_\gamma, q_\gamma] = p_\gamma \circ q_\gamma - (-1)^{\deg p_\gamma \deg q_\gamma} q_\gamma \circ p_\gamma = \kappa_\gamma \hbar$$

where κ_γ is the multiplicity of the orbit γ . The algebra \mathfrak{W} can be represented as an algebra of formal differential operators with respect to q -variables acting *on the left* on the space of polynomials $f(q, z, \hbar)$, by setting

$$p_\gamma = \kappa_\gamma \hbar \overrightarrow{\frac{\partial}{\partial q_\gamma}}.$$

Alternatively by setting

$$q_\gamma = \kappa_\gamma \hbar \overleftarrow{\frac{\partial}{\partial p_\gamma}}$$

we can represent \mathfrak{W} as an algebra of polynomial differential operators acting *on the right* on the algebra $\{\sum_{\Gamma,g} f_{\Gamma,g}(q,z)\hbar^g p^\Gamma\}$ of formal power series of \hbar and the p -variables.

Notice that the commutator $[F, G]$ of two homogeneous elements $F, G \in \mathfrak{W}$ equals $F \circ G - (-1)^{\deg F \deg G} G \circ F$, and hence if $F \in \mathfrak{W}$ is an odd element (i.e. all its summands are odd) then $[F, F] = 2F \circ F$, and $[F, F] = 0$ if F is even. For any two elements $F, G \in \mathfrak{W}$ the commutator $[F, G]$ belongs to the ideal $\hbar\mathfrak{W}$. According to Proposition 2.2.1 the Hamiltonian \mathbf{H} can be viewed as an element of $\frac{1}{\hbar}\mathfrak{W}$, and hence the above remark shows that for $F \in \mathfrak{W}$ we have $[\mathbf{H}, F] \in \mathfrak{W}$.

Theorem 2.2.2 *The Hamiltonian \mathbf{H} satisfies the identity*

$$(20) \quad \mathbf{H} \circ \mathbf{H} = 0.$$

This theorem (as well as Theorems 2.3.3, 2.4.2 and 2.5.3 below) follows from the description of the boundary of the corresponding moduli spaces of holomorphic curves. As it was stated in Proposition 1.7.2 this boundary is tiled by codimension one strata represented by stable curves of height 2, so that the (virtual) fundamental cycles of the boundary of the compactified moduli spaces can be symbolically written as $\partial[\mathcal{M}] = \kappa \sum [\mathcal{M}_-] \times [\mathcal{M}_+]$, where $[\mathcal{M}_\pm]$ are chains represented by the corresponding moduli spaces and where the coefficient κ depends on multiplicities of orbits along which the two levels of the corresponding stable curve are glued. Together with the Stokes formula $\int_{[\mathcal{M}]} d\omega = \int_{\partial[\mathcal{M}]} \omega$, and the fact that the integrand is a closed form, we obtain identity (20).

¹¹See the previous footnote.

Remark 2.2.3 The same argument shows that when the forms Θ_i generating the space L are not necessarily closed we get the following equation

$$(21) \quad d\mathbf{H} + \frac{1}{2}[\mathbf{H}, \mathbf{H}] = 0,$$

which generalizes (20) and can be interpreted as the zero-curvature equation for the connection $d + [\mathbf{H}, \cdot]$. We denote here by d the de Rham differential, i.e.

$$(22) \quad \begin{aligned} d\mathbf{H} &= d\left(\sum_{d,g} \sum_{r,s^\pm=0}^{\infty} \frac{1}{r!s^-!s^+!} \right. \\ &\quad \left. \left\langle \underbrace{\sum_1^K t_i \Theta_i, \dots, \sum_1^K t_i \Theta_i}_r; \underbrace{q, \dots, q}_{s^-}; \underbrace{p, \dots, p}_{s^+} \right\rangle_g^d z^d \hbar^{g-1} \right) \\ &= \sum_{d,g} \sum_{s^\pm=0}^{\infty} \frac{1}{(r-1)!s^-!s^+!} \\ &\quad \left\langle \sum_1^K t_i d\Theta_i, \underbrace{\sum_1^K t_i \Theta_i, \dots, \sum_1^K t_i \Theta_i}_{r-1}; \underbrace{q, \dots, q}_{s^-}; \underbrace{p, \dots, p}_{s^+} \right\rangle_g^d z^d \hbar^{g-1}. \end{aligned}$$

The identity $\mathbf{H} \circ \mathbf{H} = 0$ is equivalent to $[\mathbf{H}, \mathbf{H}] = 0$, because \mathbf{H} is an odd element. Let us define the differential $D = D^{\mathbf{H}} : \mathfrak{W} \rightarrow \mathfrak{W}$ by the formula

$$(23) \quad D^{\mathbf{H}}(f) = [\mathbf{H}, f] \quad \text{for } f \in \mathfrak{W}.$$

Then Theorem 2.2.2 translates into the identity $D^2 = 0$. The differential $D^{\mathbf{H}}$ satisfies the Leibnitz rule, and thus (\mathfrak{W}, D) is a differential Weyl (super-)algebra. In particular, one can define the homology algebra $H_*(\mathfrak{W}, D)$, which inherits its multiplication operation from the Weyl algebra \mathfrak{W} . The differential $D^{\mathbf{H}}$ extends in an obvious way to the modules $\hbar^{-k}\mathfrak{W}$, $k = 1, \dots$.

Example 2.2.4 Let $V = S^1$. We have in this case

$$(24) \quad \mathbf{H} = \hbar^{-1} \left(\frac{t_1 t_0^2}{2} + t_1 \sum p_k q_k - \frac{t_1 \hbar}{24} \right),$$

where $t = t_0 1 + t_1 d\phi$ is a general harmonic differential form on S^1 , so that $\deg t_1 = -1$, $\deg t_0 = -2$, $\deg \hbar = -4$ and $\deg p_k, \deg q_k = -2$, which corresponds to the convention that the Maslov index of any path in the 1-point group $Sp(0)$ equals 0. The term $t_1 t_0^2 / 2 = \int_{S^1} t^{\wedge 3} / 6$ is the contribution of the moduli space S^1 of constant maps $\mathbb{C}P^1 \rightarrow \mathbb{R} \times S^1$ with 3 marked points. The term $-\frac{t_1 \hbar}{24}$ is accounted for the

contribution of constant curves of genus 1 (see [68]), and the term $t_1 p_k q_k$ represents the contribution $t_1 = \int_{S^1} t$ of trivial curves of multiplicity k with one marking. All other curves do not contribute to \mathbf{H} for dimensional reasons and because $t_1^2 = 0$.

Notice that if we organize all variables p_k, q_k into formal Fourier series (comp. [25])

$$(25) \quad u(x) = \sum_{k=1}^{\infty} (p_k e^{ixk} + q_k e^{-ixk}),$$

then the term accounting for the contribution of rational curves in the formula (24) takes the form

$$(26) \quad \frac{t_1}{4\pi} \int_0^{2\pi} (t_0 + u(x))^2 dx,$$

see further discussion of this u -formalism in Section 2.9.2 below.

We will associate now with (\mathfrak{W}, D) two other differential algebras, (\mathfrak{P}, d) and (\mathfrak{A}, ∂) , which can be viewed as *semi-classical* and *classical* approximations of the Weyl differential algebra.

Let us denote by

- \mathfrak{P} – a graded Poisson algebra of formal power series in variables p_γ with coefficients which are polynomials of q_γ, z_j, z_j^{-1} , and formal power series of t_i ,¹² and by
- \mathfrak{A} – a graded commutative algebra generated by variables $q = \{q_\gamma\}_{\gamma \in \mathcal{P}}$ with coefficients in the algebra $\mathbb{C}[H_2(V)][[t]]$.

The Poisson bracket on \mathfrak{P} is defined by the formula

$$(27) \quad \{h, g\} = \sum_{\gamma} \kappa_{\gamma} \left(\frac{\partial h}{\partial p_{\gamma}} \frac{\partial g}{\partial q_{\gamma}} - (-1)^{\deg h \deg g} \frac{\partial g}{\partial p_{\gamma}} \frac{\partial h}{\partial q_{\gamma}} \right),$$

assuming that h and g are *monomials*. When computing partial derivatives, like $\frac{\partial h}{\partial q_{\gamma}}$, one should remember that we are working in the super-commutative environment, and in particular the operator $\frac{\partial}{\partial q_{\gamma}}$ has the same parity as the variable q_{γ} .

Remark 2.2.5 Notice, that if similarly to Example 2.2.4 above we organize the variables $p_k = p_{\gamma_k}, q_k = q_{\gamma_k}$ corresponding to multiples of each simple periodic orbit $\gamma = \gamma_1$ into a Fourier series

$$u_{\gamma} = \sum_{k=1}^{\infty} (p_k e^{ixk} + q_k e^{-ixk}),$$

then the value of the Poisson tensor (27) on covectors $\delta u, \delta v$ takes the form

$$(28) \quad \frac{1}{2\pi i} \int_0^{2\pi} (\delta u)' \delta v dx.$$

¹²See the first footnote in Section 2.2.3.

In order to define differentials on the algebras \mathfrak{A} and \mathfrak{P} let us first make the following observation.

Lemma 2.2.6 *We have*

$$[\mathbf{H}, \mathbf{H}] = \frac{1}{\hbar} \{\mathbf{H}_0, \mathbf{H}_0\} + \dots,$$

and for any $f = \frac{1}{\hbar} \sum f_g \hbar^g \in \frac{1}{\hbar} \mathfrak{W}$ we have

$$D^{\mathbf{H}}(f) = \frac{1}{\hbar} \{\mathbf{H}_0, f_0\} + \dots$$

In particular, \mathbf{H}_0 satisfies the equation $\{\mathbf{H}_0, \mathbf{H}_0\} = 0$.

To cope with a growing number of indices we will rename \mathbf{H}_0 into \mathbf{h} . Lemma 2.2.6 allows us to define the differential $d = d^{\mathbf{h}} : \mathfrak{P} \rightarrow \mathfrak{P}$ by the formula

$$(29) \quad dg = \{\mathbf{h}, g\} \quad \text{for } g \in \mathfrak{P}.$$

Theorem 2.2.2 then implies

Proposition 2.2.7 *We have $d^2 = 0$ and $d\{f, g\} = \{df, g\} + (-1)^{\deg f} \{f, dg\}$ for any homogeneous element $f \in \mathfrak{P}$. In other words, (\mathfrak{P}, d) is a graded differential Poisson algebra with unit.*

Proposition 2.2.7 enables us to define the homology $H_*(\mathfrak{P}, d)$ which inherits from \mathfrak{P} the structure of a graded Poisson algebra with unit.

Let us recall that according to 2.2.1 $\mathbf{h}|_{p=0} = \mathbf{h}_{const}$, where \mathbf{h}_{const} accounts for constant rational holomorphic curves, and thus it is independent of q -variables. In fact,

$$\mathbf{h}_{const}(t) = \frac{1}{6} \sum_{i,j,k=1}^K c^{ijk} t_i t_j t_k,$$

where $c^{ijk} = \int_V \Theta_i \wedge \Theta_j \wedge \Theta_k$ are the structural coefficients of the cup-product. Hence,

$$\mathbf{h} = \mathbf{h}_{const} + \sum h_\gamma(q, t, z) p_\gamma + \dots,$$

where \dots denote terms at least quadratic in p_γ . Thus we have

$$(30) \quad \{\mathbf{h}, \mathbf{h}\} = 2 \sum_{\gamma, \gamma' \in \mathcal{P}} \kappa_{\gamma'} h_{\gamma'}(q, t) \frac{\partial h_\gamma}{\partial q_{\gamma'}}(q, t) p_\gamma + o(p) = 0.$$

Therefore,

$$(31) \quad \sum_{\gamma' \in \mathcal{P}} \kappa_{\gamma'} h_{\gamma'}(q, t) \frac{\partial h_\gamma}{\partial q_{\gamma'}}(q, t) = 0$$

for all t and all $\gamma \in \mathcal{P}$.

Let us define a differential $\partial : \mathfrak{A} \rightarrow \mathfrak{A}$ by the formula

$$(32) \quad \partial f = \{\mathbf{h}, f\}|_{\{p=0\}} = \sum_{\gamma \in \mathcal{P}} \kappa_\gamma h_\gamma \frac{\partial f}{\partial q_\gamma}.$$

Then the equation (31) is equivalent to

Proposition 2.2.8 $\partial^2 = 0$, and hence (\mathfrak{A}, ∂) is a graded commutative differential algebra with unit.

The homology group $H_*(\mathfrak{A}, \partial)$ inherits the structure of a graded commutative algebra with unit.

As it was already mentioned in Section 2.1, it is convenient to view the Poisson algebra \mathfrak{P} as an algebra of functions on an infinite-dimensional symplectic super-space \mathbf{V} with the even symplectic form $\omega = \sum k_\gamma^{-1} dp_\gamma \wedge dq_\gamma$. Then the differential $d^{\mathbf{h}}$ is the Hamiltonian vector field on \mathbf{V} generated by the Hamiltonian function \mathbf{h} . One should remember, however, that the p -variables are formal, so all that we have is the infinite jet of the symplectic space \mathbf{V} along the 0-section. The equation $\mathbf{h}|_{p=0} = \mathbf{h}_{const}$ translates into the fact that the vector field $d^{\mathbf{h}}$ is tangent to the 0-section, and the differential ∂ is just the restriction of this vector field to the 0-section. The higher order terms in the expansion of \mathbf{h} with respect to p -variables define a sequence of (co-)homological operations on the algebra \mathfrak{A} .

Notice also that the differentials D, d and ∂ do not involve any differentiation with respect to t . Hence the differential algebras $(\mathfrak{W}, D^{\mathbf{H}})$, $(\mathfrak{P}, d^{\mathbf{h}})$ and (\mathfrak{A}, ∂) can be viewed as *families* of differential algebras, parameterized by $t \in H^*(V)$, and in particular, one can compute the homology at any fixed $t \in H^*(V)$. We will sometimes denote the corresponding algebras and their homology groups with the subscript t , i.e. $(\mathfrak{W}, D)_t$, $H_*(\mathfrak{P}, d)_t$, etc., and call them *specialization* at the point $t \in H^*(V)$. We will also use the notation

$$H_*^{\text{SFT}}(V, \xi | J, \alpha), \quad H_*^{\text{RSFT}}(V, \xi | J, \alpha), \quad \text{and} \quad H_*^{\text{cont}}(V, \xi | J, \alpha)$$

instead of $H_*(\mathfrak{W}, \partial)$, $H_*(\mathfrak{P}, \partial)$ and $H_*(\mathfrak{A}, \partial)$, and will usually omit the extra data J, α from the notation: as we will see in Section 2.5 below all these homology algebras are independent of J, α and other extra choices, like closed forms representing cohomology classes of V , a coherent orientation of the moduli spaces, etc. The abbreviation RSFT stands here for *Rational Symplectic Field Theory*.

Remark 2.2.9 It is important to observe that the algebras $\mathfrak{W}, \mathfrak{P}$ and \mathfrak{A} have an additional grading by elements of $H_1(V)$ (comp. Section 1.9 above). This grading is also inherited by the corresponding homology algebras. However, this grading carries a non-trivial information only when we consider homology of algebras, specialized at points $t = \sum_1^K t_i \Theta_i$ with $t_i = 0$ for at least some of the coordinates t_i corresponding 1-dimensional forms. Otherwise all cycles in these algebras are graded by the 0-class from $H_1(V)$.

2.3 Symplectic cobordisms

2.3.1 Evaluation maps and correlators

Let us now repeat the constructions of the previous section for a general directed symplectic cobordism $W = \overline{V^- V^+}$ between two contact manifolds V^- and V^+ with fixed contact forms α^- and α^+ . As in Section 2.2.1 we consider the sets \mathcal{P}^\pm of periodic orbits of the Reeb fields $R^\pm = R_{\alpha^\pm}$ as discrete topological spaces.

We denote by $\mathcal{M}_{g,r,s^-,s^+}^A(W, J)$ the disjoint union

$$\bigcup \mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J),$$

where the union is taken over all sets (Γ^-, Γ^+) of cardinality (s^-, s^+) , and consider three sets of evaluation maps:

$$ev_i^0 : \mathcal{M}_{g,r,s^-,s^+}^A(W, J) \rightarrow W, \quad i = 1, \dots, r,$$

$$ev_j^\pm : \mathcal{M}_{g,r,s^-,s^+}^A(W, J) \rightarrow \mathcal{P}^\pm, \quad j = 1, \dots, s^\pm,$$

where ev_i^0 is the evaluation map $f(y_i)$ of the map f at the i -th marked point y_i , while ev_j^\pm are the evaluation maps at asymptotic markers $\mu_j^{\mathbf{x}^\pm}$, i.e. $ev_j^\pm(f)$ is a point of \mathcal{P}^\pm representing the orbit γ_j^\pm , which contains the image of the corresponding marker. The evaluation maps ev_i^0 and ev_j^\pm can be combined into a map

$$ev : \mathcal{M}_{g,r,s^-,s^+}^A(W, J) \rightarrow W^r \times (\mathcal{P}^+)^{s^+} \times (\mathcal{P}^-)^{s^-}.$$

Now we are ready to define degree 0, or symplectic correlators. We will have to consider on W differential forms, which do not necessarily have compact support, but have, however, *cylindrical ends*. We say that a differential form θ on W is said to have cylindrical ends if it satisfies the following condition:

there exists $C > 0$ such that

$$\theta|_{V^- \times (-\infty, -C)} = (\pi^-)^*(\theta^-) \quad \text{and} \quad \theta|_{V^+ \times (C, \infty)} = (\pi^+)^*(\theta^+),$$

where θ^\pm are forms on V^\pm , and π^\pm are the projections of the corresponding ends to V^\pm . We will denote the forms θ^\pm also by $\text{restr}^\pm(\theta)$, or $\theta|_{V^\pm}$. In what follows we assume that all considered differential forms on W have cylindrical ends.

Given r differential forms $\theta_1, \dots, \theta_r$ on W and s^\pm cohomology classes

$$\alpha_1^\pm, \dots, \alpha_{s^\pm}^\pm \in H^*(\mathcal{P}^\pm) = H_0^*(\mathcal{P}^\pm)$$

we define the degree 0 correlator

$$(33) \quad \int_{\mathcal{M}_{g,r,s^-,s^+}^A} \langle \theta_1, \dots, \theta_r; \alpha_1^-, \dots, \alpha_{s^-}^-; \alpha_1^+, \dots, \alpha_{s^+}^+ \rangle_g^A =$$

$$\int_{\mathcal{M}_{g,r,s^-,s^+}^A} ev^*(\theta_1 \otimes \dots \otimes \theta_r \otimes \alpha_1^- \otimes \dots \otimes \alpha_{s^-}^- \otimes \alpha_1^+ \otimes \dots \otimes \alpha_{s^+}^+).$$

Similar to Section 2.2.1 above, we denote the cohomology classes in $H^*(\mathcal{P}^+) = H^0(\mathcal{P}^+)$ (resp. in $H^*(\mathcal{P}^-) = H^0(\mathcal{P}^-)$) by p^+ (resp. q^-), and write

$$p^+ = \sum_{\gamma \in \mathcal{P}^+} k_g^{-1} p_\gamma^+[\gamma] \quad (\text{resp.} \quad q^- = \sum_{\gamma \in \mathcal{P}^-} k_g^{-1} q_\gamma^-[\gamma]).$$

We will also fix a basis A_1, \dots, A_N of $H_2(W)$ and denote by $d = (d_1, \dots, d_N)$ the degree of A in this basis.

Let us call a system of linearly independent closed forms $\theta_1, \dots, \theta_m$ on W with cylindrical ends *basic*, if

$$(34) \quad \begin{aligned} &\text{a) the image } \text{restr}^\pm(L(\theta_1, \dots, \theta_m)) \\ &\quad \text{generates } \text{Im}(H^*(W) \rightarrow H^*(V^\pm)); \\ &\text{b) the homomorphism } \text{Ker}((\text{restr}^+ \oplus \text{restr}^-)|_L) \rightarrow H_{\text{comp}}^*(W) \\ &\quad \text{is bijective.} \end{aligned}$$

Here we denote by $L(\theta_1, \dots, \theta_m)$ the subspace generated by the forms $\theta_1, \dots, \theta_m$, and by $H_{\text{comp}}^*(W)$ the cohomology with compact support. Equivalently, one can say that a basic system of forms consists of a basis of $H^*(W)$ together with a basis of $\text{Ker}(H_{\text{comp}}^*(W) \rightarrow H^*(W))$.

A general point $t \in L(\theta_1, \dots, \theta_m)$ we will write in the form $t = \sum_1^m t_i \theta_i$. The grading of the variables t, p^+, q^- is defined as in Section 2.2.2:

$$(35) \quad \begin{aligned} \deg(t_i) &= \deg(\theta_i) - 2; \\ \deg(p_\gamma^+) &= -\text{CZ}(\gamma^+) + (n - 3), \\ \deg(q_\gamma^-) &= \text{CZ}(\gamma^-) + (n - 3) \end{aligned}$$

2.3.2 Potentials of symplectic cobordisms

Let us now organize the correlators into a generating function, called the *potential* of the symplectic cobordism $(W = \overline{V^- V^+}, J, \alpha^\pm)$

$$(36) \quad \begin{aligned} \mathbf{F} &= \mathbf{F}_{W, J, \alpha^\pm} = \frac{1}{\hbar} \sum_{g=0}^{\infty} \mathbf{F}_g \hbar^g, \\ &\text{where} \\ \mathbf{F}_g &= \sum_d \sum_{r, s^\pm=0} \frac{1}{r! s^+! s^-!} \underbrace{0 \langle t, \dots, t \rangle_r}_{r} \underbrace{\langle q^-, \dots, q^- \rangle_{s^-}}_{s^-} \underbrace{\langle p^+, \dots, p^+ \rangle_{s^+}}_{s^+} \underbrace{\rangle_g^d}_{g} z^d. \end{aligned}$$

When W is a closed symplectic manifold, then the potential F is just the Gromov-Witten invariant of the symplectic manifold W . However, if W is not closed, then F itself is not an invariant. It depends on particular forms θ_i , rather than their cohomology classes, on J , on a coherent orientation, and several other choices. We will see, however, that the *homotopy class* of F , which we define in Section 2.4 below, is independent of most of these choices.

In order to make sense out of the expression for \mathbf{F} let us consider a graded commutative algebra $\mathfrak{D} = \mathfrak{D}(W, \alpha^\pm)$ which consists of power series of the form

$$(37) \quad \sum_{\Gamma, d, g} \varphi_{\Gamma, d, g}(q^-, t) z^d (p^+)^{\Gamma} \hbar^g,$$

where $\varphi_{\Gamma, d, g}$ are polynomials of q_γ , formal power series of variables t_i ,¹³ and where Γ and d satisfies the following Novikov type inequality:

$$(38) \quad [\omega](d) = \sum_{A_i} d_i \int_{A_i} \omega > -|\Gamma| = -\sum_{i=1}^k |\gamma_i|,$$

where $\Gamma = \{\gamma_1, \dots, \gamma_k\}$, and $|\gamma_i| = \int_{\gamma_i} \alpha^+$ is the period of the periodic orbit $\gamma_i \in \mathcal{P}^+$, or equivalently its action. Recall that $(p^+)^{\Gamma} = p_{\gamma_1}^+ \dots p_{\gamma_k}^+$.

Proposition 2.3.1 *We have*

$$\mathbf{F}_{W, J, \alpha^\pm} \in \frac{1}{\hbar} \mathfrak{D}(W, \alpha^\pm).$$

Let us also consider a bigger algebra $\mathfrak{D}\mathfrak{D}$ which consists of series

$$(39) \quad \sum_{\Gamma, d} \varphi_{\Gamma, d}(q^-, t, \hbar) z^d (p^+)^{\Gamma},$$

where $\varphi_{\Gamma, d}$ are polynomials of q_γ^- , formal power series of t_i and formal *Laurent series* of \hbar , while Γ and d still satisfy condition (38). For instance, for any element $f \in \hbar^{-1} \mathfrak{D}$ we have $e^f \in \mathfrak{D}\mathfrak{D}$.

The algebra $\mathfrak{D}\mathfrak{D} = \mathfrak{D}\mathfrak{D}(W, J, \alpha^\pm)$ has a structure of a *left D-module* over the Weyl algebra $\mathfrak{W}^- = \mathfrak{W}(V^-, J, \alpha^-)$, and of a *right D-module* over the Weyl algebra $\mathfrak{W}^+ = \mathfrak{W}(V^+, J, \alpha^+)$. Indeed, we first associate with an element

$$\Delta^- = \sum_{\Gamma = \{\gamma_1, \dots, \gamma_m\}, \Gamma', d, g} \delta_{\Gamma', \Gamma, d, g}^-(t) (q^-)^{\Gamma'} (p^-)^{\Gamma} z^d \hbar^g \in \mathfrak{W}^-$$

a differential operator

$$(40) \quad \sum_{\Gamma = \{\gamma_1, \dots, \gamma_m\}, \Gamma', d, g} \delta_{\Gamma', \Gamma, d, g}^-(t) (q^-)^{\Gamma'} \hbar^{m+g} \prod_{i=1}^m \kappa_{\gamma_i} \overrightarrow{\frac{\partial}{\partial q_{\gamma_i}}} z^d,$$

then change the coefficient ring via the inclusion homomorphism $H_2(V^-) \rightarrow H_2(W)$, and finally lift functions $\delta_{\Gamma', \Gamma, d, g}^-(t)$ to the space of forms with cylindrical ends on W via the restriction map $t \mapsto t|_{V^-}$. We will denote the resulting operator by $\overrightarrow{\Delta^-}$. Similarly we associate with $\Delta^+ \in \mathfrak{W}^+$ an operator $\overleftarrow{\Delta^+}$ by first quantizing

¹³See the first footnote in Section 2.2.3.

$q_\gamma^+ \Rightarrow \hbar \kappa_\gamma \overleftarrow{\frac{\partial}{\partial p_\gamma^+}}$ and then making an appropriate change of the coefficient ring. It is straightforward to verify that for $f \in \mathfrak{D}$ we have $\overrightarrow{\Delta} f, f \overleftarrow{\Delta} \in \mathfrak{D}$, and for $F \in \mathfrak{D}\mathfrak{D}$ we have $\overrightarrow{\Delta} F, F \overleftarrow{\Delta} \in \mathfrak{D}\mathfrak{D}$.

Let us denote the Hamiltonians (see Section 2.2.3 above) in \mathfrak{W}^\pm by \mathbf{H}^\pm and define a map $D_W = D_W : \mathfrak{D}\mathfrak{D} \rightarrow \mathfrak{D}\mathfrak{D}$ by the formula

$$(41) \quad D_W(G) = \overrightarrow{\mathbf{H}} G - (-1)^{\deg G} G \overleftarrow{\mathbf{H}}, \quad G \in \mathfrak{D}\mathfrak{D},$$

where we assume G dimensionally homogeneous. Clearly, Theorem 2.2.2 implies that $D_W^2 = 0$. However, the differential algebra $(\mathfrak{D}\mathfrak{D}, D_W)$ is too big and instead of considering its homology we will define a differential on the algebra \mathfrak{D} , or which is equivalent but more convenient, on the module $\hbar^{-1}\mathfrak{D}$.

For an *even* element $F \in \hbar^{-1}\mathfrak{D}$ let us define a map $D^F = T_F D_W : \hbar^{-1}\mathfrak{D} \rightarrow \hbar^{-1}\mathfrak{D}$ by the formula

$$(42) \quad D^F(g) = e^{-F} [D_W, g](e^F) = e^{-F} (D_W(g e^F) - (-1)^{\deg g} g D_W(e^F)), \quad g \in \hbar^{-1}\mathfrak{D}.$$

The map $T_F D_W$ is the linearization of the map $\tilde{D}_W : \hbar^{-1}\mathfrak{D} \rightarrow \hbar^{-1}\mathfrak{D}$, defined by the formula

$$\tilde{D}_W(F) = e^{-F} D_W(e^F), \quad F \in \hbar^{-1}\mathfrak{D}.$$

at the point $F \in \hbar^{-1}\mathfrak{D}$. Notice that if $D_W(e^F) = 0$ then $D^F(g) = e^{-F} D_W(g e^F)$. Let us first state a purely algebraic

Proposition 2.3.2 *Suppose that for $F \in \hbar^{-1}\mathfrak{D}$ we have $D_W(e^F) = 0$. Then*

1. $(D^F)^2 = 0$;
2. *The homology algebra $H_*(\mathfrak{D}, D^F)$ inherits the structure of a left module over the homology algebra $H_*(\mathfrak{W}^-, D^-)$, and the structure of a right module over the homology algebra $H_*(\mathfrak{W}^+, D^+)$;*
3. *The homomorphisms $F^\pm : \mathfrak{W}^\pm \rightarrow \mathfrak{D}$, defined by the formulas*

$$(43) \quad \begin{aligned} f &\mapsto e^{-F} \overrightarrow{f} e^F, \quad f \in \mathfrak{W}^-, \quad \text{and} \\ f &\mapsto e^F \overleftarrow{f} e^{-F}, \quad f \in \mathfrak{W}^+, \end{aligned}$$

commute with the boundary maps of chain complexes, i.e.

$$F^\pm \circ D^\pm = D^F \circ F^\pm,$$

and thus induce homomorphisms of homology

$$F_*^\pm : H_*(\mathfrak{W}^\pm, D^\pm) \rightarrow H_*(\mathfrak{D}, D^F),$$

as modules over $H_(\mathfrak{W}^\pm, D^\pm)$.*

Theorem 2.3.3 *The potential $\mathbf{F} \in \hbar^{-1}\mathfrak{D}$ defined above by the formula (36) satisfies the equation*

$$(44) \quad D_W e^{\mathbf{F}} = 0,$$

and hence all conclusions of Proposition 2.3.2 hold for \mathbf{F} .

The appearance of $e^{\mathbf{F}}$ in equation (44) has the following reason. Similar to Theorem 2.2.2 above, equation (44) follows from the description of codimension 1 strata on the boundary of the moduli space of holomorphic curves in the cobordism W , see Proposition 1.7.2 above. Notice that $e^{\mathbf{F}}$ is the generating function counting possibly disconnected holomorphic curves in W . Thus the identity

$$\overrightarrow{\mathbf{H}} e^{\mathbf{F}} - e^{\mathbf{F}} \overleftarrow{\mathbf{H}} = 0$$

asserts, in agreement with Proposition 1.7.2, that the codimension 1 strata on the boundary of the moduli space $\mathcal{M}(W)$ of not necessarily connected curves in W correspond to stable curves (f_1, f_2) of height 2, where one of the curves f_1, f_2 belongs to $\widetilde{\mathcal{M}}(W)$, while the second one is contained in the symplectization of V^\pm and has precisely one component different from the straight cylinder over a periodic orbit from \mathcal{P}^\pm .

Remark 2.3.4 (Comp. Remark 2.2.3 above) The potential \mathbf{F} , extended to the space of all, not necessarily closed differential forms satisfies the equation

$$(45) \quad d(e^{\mathbf{F}}) = D_W e^{\mathbf{F}},$$

where d is the de Rham differential. This equation generalizes equation (44).

Following the scheme of Section 2.2.3 above we will associate now with the cobordism W two other left-right modules, one over the Poisson algebras \mathfrak{P}^\pm , and another over the graded differential algebras \mathfrak{A}^\pm .

Consider the graded commutative algebra $\mathfrak{L} = \mathfrak{L}(W, \alpha^\pm)$ of power series of the form

$$(46) \quad \sum_{\Gamma, d} \varphi_{\Gamma, d}(q^-, t) z^d (p^+)^\Gamma,$$

where $\varphi_{\Gamma, d}$ are polynomials of q_γ^- and formal power series of t_i , while Γ and d satisfies the above inequality (38). Let us also consider the larger graded commutative algebra

$$(47) \quad \widehat{\mathfrak{L}} = \left\{ \sum_{\Gamma^+, \Gamma^-, d} \varphi_{\Gamma^+, \Gamma^-, d}(q^-, q^+, t) z^d (p^+)^\Gamma (p^-)^{\Gamma'} \right\},$$

where the Novikov condition (38) is satisfied for both pairs (d, Γ) and (d, Γ') . The algebra $\widehat{\mathfrak{L}}$ has a natural Poisson bracket so that the homomorphisms $f \mapsto \widehat{f}$, where we denote by \widehat{f} the image in $\widehat{\mathfrak{L}}$ of an element $f \in \mathfrak{P}^\pm$ under the coefficient homomorphism,

are Poisson homomorphisms. We set $\widehat{\mathbf{h}} = \widehat{\mathbf{h}}^- - \widehat{\mathbf{h}}^+$, and for any $f \in \mathcal{L}$ denote by L_f the ‘‘Lagrangian variety’’

$$\{p_\gamma^- = \kappa_\gamma \frac{\partial \widehat{f}}{\partial q_\gamma^-}, q_\gamma^+ = \kappa_\gamma \frac{\partial \widehat{f}}{\partial p_\gamma^+}\}.$$

Strictly speaking L_f is an ideal in the Poisson algebra \widehat{L} . However, it is useful to think about L_f as a Lagrangian variety in the symplectic super-space $(\mathbf{V}^-) \oplus \mathbf{V}^+$ with the symplectic form

$$\sum \kappa_{\gamma^-}^{-1} dp_{\gamma^-}^- \wedge dq_{\gamma^-}^- + \kappa_{\gamma^+}^{-1} dq_{\gamma^+}^+ \wedge dp_{\gamma^+}^+,$$

and with an appropriate change of the coefficient ring.

For any function $f \in \mathcal{L}$, which satisfies the Hamilton-Jacobi equation

$$(48) \quad \widehat{\mathbf{h}}|_{L_f} = 0$$

the Hamiltonian vector field defined by the Hamiltonian $\widehat{\mathbf{h}}$ is tangent to L_f , and hence the differential $d^f : \mathcal{L} \rightarrow \mathcal{L}$, defined by the formula

$$d^f(g) = \{\widehat{\mathbf{h}}, g\}|_{L_f}$$

has the following meaning: we identify \mathcal{L} with the space of functions on L_f and differentiate them along the Hamiltonian vector field determined by $\widehat{\mathbf{h}}$.

Here is an analog of Proposition 2.3.5 for the algebra \mathcal{L} .

Proposition 2.3.5 *Suppose that $\widehat{\mathbf{h}}|_{L_f} = 0$. Then*

1. $(d^f)^2 = 0$;
2. The maps $f^\pm : \mathfrak{P}^\pm \rightarrow \mathcal{L}$, defined by the formula $g \mapsto \widehat{g}|_{L_f}$, are homomorphisms of chain complexes, i.e.

$$d^f \circ f^\pm = f^\pm \circ d^\pm;$$

3. If $g_1, g_2 \in \mathfrak{P}^\pm$ Poisson commute with \mathbf{h}^\pm or, in other words, if $g_1, g_2 \in \text{Ker } d^\pm$ then

$$\{f^\pm(g_1), f^\pm(g_2)\} = f^\pm(\{g_1, g_2\}).$$

where the left-side Poisson bracket is taken in the algebra $\widehat{\mathcal{L}}$.

Let us recall that $\mathbf{F} = \mathbf{F}_W \in \mathfrak{D}$ has the form $\mathbf{F} = \hbar^{-1} \sum_{g=0}^{\infty} \mathbf{F}_g \hbar^g$. Again, to simplify the notation we will write \mathbf{f} instead of \mathbf{F}_0 . The following theorem is the reduction of Theorem 2.3.3 to the level of rational Gromov-Witten theory.

Theorem 2.3.6 *The series $\mathbf{f}(q^-, p^+, t)$ belongs to the algebra \mathcal{L} and satisfies the equation*

$$(49) \quad \widehat{\mathbf{h}}|_{L_f} = 0.$$

In particular, all statements of the above Proposition 2.3.5 hold for \mathbf{f} , and this allows us to define the homology

$${}_{\mathbf{f}}H_*^{\text{RSFT}}(W|J, \alpha^\pm) = H_*(\mathfrak{L}, d^{\mathbf{f}}).$$

The chain maps \mathbf{f}^\pm induce homomorphism of Poisson homology algebras

$$(50) \quad (\mathbf{f}^\pm)_* : H_*^{\text{RSFT}}(V^\pm|J, \alpha^\pm) = H_*(\mathfrak{P}^\pm, d^\pm) \rightarrow {}_{\mathbf{f}}H_*^{\text{RSFT}}(W|J, \alpha^\pm).$$

For the rest of this section we assume that W is a rational homology cobordism, i.e. the restriction maps

$$H^*(V^-; \mathbb{R}) \leftarrow H^*(W; \mathbb{R}) \rightarrow H^*(V^+, \mathbb{R})$$

are isomorphisms. Equivalently, this means that the inclusions $V^\pm \rightarrow W$ induce isomorphisms of rational homology groups.

The potential $\mathbf{f} \in \mathfrak{L}$ which we defined above can be written in the form

$$(51) \quad \mathbf{f} = \sum_i \sum_{|\Gamma^+|=i} f_\Gamma^i(q^-, t)(p^+)^{\Gamma^+}.$$

Notice that the assumption that W is a homology cobordism implies that $f^0(q^-, t)$ is independent of q^- . Let us now define a homomorphism $\Psi : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$ by the formula

$$(52) \quad \Psi(q_\gamma^+) = f_\gamma^1(q^-, t) \in \mathfrak{A}^-$$

on the generators $q_\gamma^+, \gamma \in \mathcal{P}^+$, of the algebra \mathfrak{A}^+ and then extend by linearity.

Theorem 2.3.7 *The homomorphism $\Psi : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$ commutes with the boundary operators ∂^\pm , i.e. $\partial^- \circ \Psi = \Psi \circ \partial^+$, and in particular defines a homomorphism of homology algebras*

$$(\Psi)_* : H_*(\mathfrak{A}^+, \partial^+) \rightarrow H_*(\mathfrak{A}^-, \partial^-).$$

Without the assumption that W is a homology cobordism one gets only a correspondence between the algebras \mathfrak{A}^+ and \mathfrak{A}^- , similar to the ‘‘semi-classical’’ case considered above.

2.4 Chain homotopy

Let $W = \overline{V^-V^+}$ be a directed symplectic cobordism with fixed contact forms α^\pm on V^\pm . We will discuss in this section how the function $\mathbf{F} = \mathbf{F}_{W, J, \alpha^\pm}(p^+, q^-, t)$ and other associated structures change when one replaces J with another compatible almost complex structure J' and replaces t with a form $t' = t + d\theta$ where θ has compact support in W .

Let us begin with some algebraic preliminaries. Two series $F_0, F_1 \in \hbar^{-1}\mathfrak{D}$ are called *homotopic*, if they can be included into a family $F_s \in \hbar^{-1}\mathfrak{D}$, $s \in [0, 1]$, which satisfies the following differential equation

$$(53) \quad \frac{dF_s}{ds} = e^{-F_s} \left(\overrightarrow{[\mathbf{H}^-, K_s]} e^{F_s} + e^{F_s} \overleftarrow{[K_s, \mathbf{H}^+]} \right), \quad s \in [0, 1],$$

for a family $K_s \in \hbar^{-1}\mathfrak{D}$. Here $[\mathbf{H}^-, K_s]$ and $[K_s, \mathbf{H}^+]$ are commutators in the algebra $\hbar^{-1}\widehat{\mathfrak{D}}$, defined similar to $\widehat{\mathfrak{L}}$ in (47) above, i.e.

$$(54) \quad \widehat{\mathfrak{D}} = \left\{ \sum_{\Gamma^+, \Gamma^-, d, g} f_{\Gamma^+, \Gamma^-, d}(q^-, q^+, t) z^d \hbar^g (p^+)^{\Gamma^+} (p^-)^{\Gamma^-} \right\},$$

where the Novikov condition (38) is satisfied for both pairs (d, Γ^+) and (d, Γ^-) . In other words, we view K_s as an operator on $\hbar^{-1}\mathfrak{D}$, acting by the multiplication by the series K_s , and view \mathbf{H}^- and \mathbf{H}^+ as left and right differential operators.

Notice that the family $K_s \in \hbar^{-1}\mathfrak{D}$, $s \in [0, 1]$, defines a flow $\Phi^s = \Phi_K^s : \mathfrak{D}\mathfrak{D} \rightarrow \mathfrak{D}\mathfrak{D}$, by a differential equation

$$(55) \quad \frac{d\Phi^s(G)}{ds} = \mathcal{K}_s(\Phi^s(G)),$$

where we set

$$\mathcal{K}_s(G) = \left(\overrightarrow{[\mathbf{H}^-, K_s]}G + G\overleftarrow{[K_s, \mathbf{H}^+]} \right), \quad s \in [0, 1].$$

The linear operators Φ^s preserve the ‘‘submanifold’’ $\mathfrak{E} = e^{\hbar^{-1}\mathfrak{D}_{\text{even}}}$, where $\mathfrak{D}_{\text{even}}$ is the even part of \mathfrak{D} , and we have

$$\Phi^s(e^{F_0}) = e^{F_s},$$

where the family F_s satisfies the equation (53).

The tangent space to \mathfrak{E} at a point e^F , $F \in \hbar^{-1}\mathfrak{D}_{\text{even}}$, consists of series fe^F , $f \in \hbar^{-1}\mathfrak{D}_{\text{even}}$, and thus it is naturally parameterized by $\hbar^{-1}\mathfrak{D}_{\text{even}}$. With respect to this parameterization the differential of the flow $\Phi^s|_{\mathfrak{E}}$ defines a family of maps $T_F^s : \hbar^{-1}\mathfrak{D}_{\text{even}} \rightarrow \hbar^{-1}\mathfrak{D}_{\text{even}}$, $F \in \hbar^{-1}\mathfrak{D}_{\text{even}}$, by the formula

$$(56) \quad T_F^s(f) = e^{-F_s} \Phi^s(fe^F), \quad \text{where } F_s = \Phi^s(F);$$

We extend T_F^s to the whole $\hbar^{-1}\mathfrak{D}$ by the same formula (56). Let us list some properties of the flows Φ^s and T_F^s

Proposition 2.4.1 *Suppose that for an element $F \in \hbar^{-1}\mathfrak{D}_{\text{even}}$ we have*

$$D_W(e^F) = \overrightarrow{\mathbf{H}^-} e^F - e^F \overleftarrow{\mathbf{H}^+} = 0.$$

Then

1. *The flow $T_F^s : \hbar^{-1}\mathfrak{D} \rightarrow \hbar^{-1}\mathfrak{D}$ satisfies the equation*

$$(57) \quad T_F^s \circ D^F = D^{F_s} \circ T_F^s.$$

for all $s \in [0, 1]$. In particular, $D_W(F_s) = 0$ for all $s \in [0, 1]$, and T_F^s defines a family of isomorphisms $H_(\mathfrak{D}, D^F) \rightarrow H_*(\mathfrak{D}, D^{F_s})$.*

2. *The homology class $[e^{F_s}] \in H_*(\mathfrak{D}\mathfrak{D}, D_W)$ is independent of s .*

3. The diagram

$$\begin{array}{ccc} \mathfrak{D} & \xrightarrow{T_F^s} & \mathfrak{D} \\ \searrow F^\pm & & \nearrow F_s^\pm \\ & \mathfrak{W}^\pm & \end{array}$$

homotopically commutes, i.e. there exist operators $A_s^\pm : \mathfrak{W}^\pm \rightarrow \mathfrak{D}$, such that

$$(58) \quad (T_F^s)^{-1} \circ F_s^\pm - F^\pm = D^F \circ A_s^\pm + A_s^\pm \circ D^\pm.$$

In particular, this diagram commutes on the level of homology algebras.

The proof of this proposition is a straightforward computation by differentiating the corresponding equations. To illustrate the argument, let us verify (58) in 2.4.1.

Take, for instance, $f \in \mathfrak{W}^-$ and set

$$\begin{aligned} G_s(f) &= (T_F^s)^{-1}(e^{-F_s} \vec{f} e^{F_s}) - e^{-F} \vec{f} e^F \\ &= e^{-F} (\Phi^s)^{-1} (\vec{f} e^{F_s}) - e^{-F} \vec{f} e^F. \end{aligned}$$

Then we have $G_0(f) = 0$ and

$$\begin{aligned} \frac{dG_s(f)}{ds} &= -e^{-F} \left((\Phi^s)^{-1} (\mathcal{K}_s \vec{f} e^{F_s}) \right) + e^{-F} (\Phi^s)^{-1} (\vec{f} \mathcal{K}_s (e^{F_s})) \\ &= e^{-F} (\Phi^s)^{-1} \left([\mathcal{K}_s, \vec{f}] e^F \right). \end{aligned}$$

Now remember that $\mathcal{K}_s = [\widehat{\mathbf{H}}, K_s]$, where $\widehat{\mathbf{H}} = \overrightarrow{\mathbf{H}} - \overleftarrow{\mathbf{H}}$, and using the Jacobi identity we get

$$\begin{aligned} \frac{dG_s(f)}{ds} &= e^{-F} (\Phi^s)^{-1} \left([[\widehat{\mathbf{H}}, K_s], \vec{f}] e^{F_s} \right) \\ &= e^{-F} (\Phi^s)^{-1} \left([[K, \vec{f}], \widehat{\mathbf{H}}] e^{F_s} \right) \\ &\quad + e^{-F} (\Phi^s)^{-1} \left([[\vec{f}, \widehat{\mathbf{H}}], K_s] e^{F_s} \right). \end{aligned}$$

Let us define now a linear operator $B_s : \hbar^{-1}\mathfrak{W}_- \rightarrow \hbar^{-1}\mathfrak{D}$ by the formula

$$(59) \quad B_s(g) = e^{-F} (\Phi^s)^{-1} \left(\overrightarrow{[g, K_s]} e^{F_s} \right).$$

for $g \in \hbar^{-1}\mathfrak{W}_-$. Recall that $D_W(e^F) = \widehat{\mathbf{H}}e^F = 0$, and observe that $\widehat{\mathbf{H}}$ commutes with $\mathcal{K}_s = [\widehat{\mathbf{H}}, K_s]$ because $\widehat{\mathbf{H}} \circ \widehat{\mathbf{H}} = 0$. We also have $[\widehat{\mathbf{H}}, f] = D^- f$ for $f \in \hbar^{-1}\mathfrak{W}_-$. Hence $\frac{dG_s(f)}{ds}$ can be rewritten as

$$(60) \quad \frac{dG_s(f)}{ds} = D^F(B_s(f)) + B_s(D^-(f))$$

Finally we integrate B_s into the required linear operator $A^- : \hbar^{-1}\mathfrak{W}^- \rightarrow \hbar^{-1}\mathfrak{D}$:

$$(61) \quad A_s^-(g) = \int_0^s B_s(g) ds, \quad \text{for } g \in \hbar^{-1}\mathfrak{W}^-.$$

In view of equation (60) we get

$$(62) \quad (\Phi^s)^{-1} \circ (F^s)^- - (F)^- = D^F \circ A_s^- + A_s^- \circ D^-.$$



Let us consider now a generic family J^τ , $\tau \in [0, 1]$, of compatible almost complex structures on W connecting $J^0 = J$ with $J^1 = J'$. We assume that the deformation J^τ is fixed outside of a compact subset of W .

Set

$$(63) \quad \mathcal{M}_{g,r,s^+,s^-}(W, \{J^\tau\}) = \bigcup_{\tau \in [0,1]} \mathcal{M}_{g,r,s^+,s^-}(W, J^\tau).$$

The evaluation maps defined for each τ can be combined into the evaluation map

$$(64) \quad ev : \mathcal{M}_{g,r,s^+,s^-}(W, \{J^\tau\}) \rightarrow (W \times I)^r \times (\mathcal{P}^-)^{s^-} \times (\mathcal{P}^+)^{s^+}.$$

Consider closed forms $\widehat{\theta}_1, \dots, \widehat{\theta}_r$ on $W \times I$, such that $\widehat{\theta}_i = \widetilde{\theta}_i + d\beta_i$, $i = 1, \dots, r$, where $\widetilde{\theta}_i$ is the pull-back of a form θ_i on W with cylindrical ends, and β_i has compact support in $W \times \text{Int}I$.

Similarly to correlators of degree -1 and 0 (see 2.2.2 and 2.3.1) we can define correlators of degree 1 , or 1 -parametric correlators by the formula

$$(65) \quad {}^1\langle \widehat{\theta}_1, \dots, \widehat{\theta}_r; \theta_1^-, \dots, \theta_{s^-}^-; \theta_1^+, \dots, \theta_{s^+}^+ \rangle_g^A = \frac{\int}{\mathcal{M}_{g,r,s^+,s^-}(W, \{J^\tau\})} ev^*(\widehat{\theta}_1 \otimes \dots \otimes \widehat{\theta}_r \otimes \theta_1^- \otimes \dots \otimes \theta_{s^-}^- \otimes \theta_1^+ \otimes \dots \otimes \theta_{s^+}^+),$$

for a homology class $A \in H_2(W)$, and cohomology classes

$$\theta_i^\pm \in H^*(\mathcal{P}^\pm), \quad i = 1, \dots, s^\pm.$$

Consider a closed form $T = \widetilde{t} + d\beta$, where the notation \widetilde{t} and β have the same meaning as above, i.e. \widetilde{t} is the pull-back of a form t on W with cylindrical ends, and β has compact support in $W \times \text{Int}I$. We can organize the correlators

$${}^1\langle T, \dots, T; q^-, \dots, q^-, p^+, \dots, p^+ \rangle_g^A$$

into a generating function $\mathbf{K} = \frac{1}{\hbar} \sum_{g=0}^{\infty} \mathbf{K}_g \hbar^g \in \frac{1}{\hbar}\mathfrak{D}$, defined by the formula

$$(66) \quad \mathbf{K} = \sum_d \sum_{g,r,s^\pm=0}^{\infty} \frac{1}{r!s^+!s^-!} {}^1\langle \underbrace{T, \dots, T}_r; \underbrace{q^-, \dots, q^-}_{s^-}; \underbrace{p^+, \dots, p^+}_{s^+} \rangle_g^A \hbar^g z^d$$

Let us define an operator $\mathcal{K} : \mathfrak{D}\mathfrak{D} \rightarrow \mathfrak{D}\mathfrak{D}$ by the formula

$$(67) \quad \mathcal{K}(G) = \overrightarrow{[\mathbf{H}^+, \mathbf{K}]}G + G\overleftarrow{[\mathbf{K}, \mathbf{H}^-]}, \quad G \in \mathfrak{D}\mathfrak{D}.$$

Similar to Theorems 2.2.2 and 2.3.3 the next theorem can be viewed as an algebraic description of the boundary of the compactified moduli space $\overline{\mathcal{M}_{g,r,s^+,s^-}(W, \{J^\tau\})}$.

Theorem 2.4.2 *For a generic family $J_\tau, \tau \in [0, 1]$, of compatible almost complex structures on W we have*

$$(68) \quad e^{\mathbf{F}^1} = e^{\mathcal{K}}(e^{\mathbf{F}^0}),$$

where $\mathbf{F}^0 = \mathbf{F}_{W, J_0}(T|_{W \times 0})$, $\mathbf{F}^1 = \mathbf{F}_{W, J_1}(T|_{W \times 1})$, and $\mathcal{K} = \mathcal{K}(T)$.

Notice that if we define \mathbf{F}^s , $s \in [0, 1]$, by the formula

$$(69) \quad e^{\mathbf{F}^s} = e^{s\mathcal{K}}(e^{\mathbf{F}^0}),$$

then the flow $\Phi^s(F) = F^s$ satisfies the differential equation (55) with $K(s) \equiv \mathbf{K}$. Hence Φ^0 and Φ^1 are homotopic, and therefore Theorem 2.4.2 and Proposition 2.4.1 imply

Corollary 2.4.3 *1. The homology class $[e^{\mathbf{F}}] \in H_*(\mathfrak{D}\mathfrak{D}, D_W)$ is independent of the choice of a compatible almost complex structure J on W and of the choice of $t \bmod (d(\Omega_{\text{comp}}(W)))$, where $\Omega_{\text{comp}}(W)$ is the space of forms with compact support.*

2. For a generic compatible deformation J^τ , $\tau \in [0, 1]$, the isomorphism $T : \mathfrak{D} \rightarrow \mathfrak{D}$ defined by the formula

$$(70) \quad T(f) = e^{-\mathbf{F}^1} e^{\mathcal{K}}(f e^{\mathbf{F}^0})$$

satisfies the equation

$$(71) \quad T \circ D^{\mathbf{F}^0} = D^{\mathbf{F}^1} \circ T,$$

and thus defines an isomorphism $H_(\mathfrak{D}, D^{\mathbf{F}^0}) \rightarrow H_*(\mathfrak{D}, D^{\mathbf{F}^1})$.*

3. The diagram

$$\begin{array}{ccc} \mathfrak{D} & \xrightarrow{T} & \mathfrak{D} \\ \swarrow_{(\mathbf{F}^0)^\pm} & & \nearrow_{(\mathbf{F}^1)^\pm} \\ & \mathfrak{W}^\pm & \end{array}$$

homotopically commutes, i.e. there exist operators $A^\pm : \mathfrak{W}^\pm \rightarrow \mathfrak{D}$, such that

$$(72) \quad T^{-1} \circ (\mathbf{F}^1)^\pm - (\mathbf{F}^0)^\pm = D^{\mathbf{F}^0} \circ A^\pm + A^\pm \circ D^\pm.$$

Consider now the equivalence relation for rational potentials.

Two series $f_0, f_1 \in \mathcal{L}$ are called *homotopic* if there exist families $f_s, k_s \in \mathcal{L}$, $s \in [0, 1]$, so that the family f_s connects f_0 and f_1 and satisfies the following Hamilton-Jacobi differential equation

$$(73) \quad \frac{\partial f_s(p^+, q^-, t)}{\partial s} = \mathbf{G}(p^+, \frac{\partial f_s(p^+, q^-, t)}{\partial p^+}, \frac{\partial f_s(p^+, q^-, t)}{\partial q^-}, q^-),$$

where

$$(74) \quad \mathbf{G}(p^+, q^+, p^-, q^-, t) = \{\mathbf{h}^+ - \mathbf{h}^-, k_s\} = \sum_{\gamma^+ \in \mathcal{P}^+, \gamma^- \in \mathcal{P}^-} \kappa_{\gamma^-} \frac{\partial \mathbf{h}^-(p^-, q^-, t)}{\partial p_{\gamma^-}^-} \frac{\partial k_s(p^+, q^-, t)}{\partial q_{\gamma^-}^-} + \kappa_{\gamma^+} \frac{\partial k_s(p^+, q^-, t)}{\partial p_{\gamma^+}^+} \frac{\partial \mathbf{h}^+(p^+, q^+, t)}{\partial q_{\gamma^+}^+}.$$

Here κ_γ denotes, as usual, the multiplicity of γ .

We can view the correspondence

$$f(p^+, q^-, t) \mapsto f_s(p^+, q^-, t),$$

where f_s is the solution of the above equation (73) with the initial data $f_0 = f$, as a non-linear operator $S^s : \mathcal{L} \rightarrow \mathcal{L}$. Let us denote by T_f^s the linearization of S^s at a point f . The next proposition is a rational version of Proposition 2.4.1. It can be either deduced from 2.4.1, or similarly proven by differentiation with respect to the parameter s . Denote by \mathcal{S} the subspace of \mathcal{L} which consists of solutions of the Hamilton-Jacobi equation (48), i.e.

$$\widehat{\mathbf{h}}|_{L_f} = 0, \quad \text{where } L_f = \{p_{\gamma^-}^- = \kappa_{\gamma^-} \frac{\partial f}{\partial q_{\gamma^-}^-}, q_{\gamma^+}^+ = \kappa_{\gamma^+} \frac{\partial f}{\partial p_{\gamma^+}^+}\}.$$

Proposition 2.4.4 1. *The subspace $\mathcal{S} \subset \mathcal{L}$ is invariant under the flow S^s .*

2. *For $f \in \mathcal{S}$ the isomorphism $T_f^s : \mathcal{L} \rightarrow \mathcal{L}$ satisfies the equation*

$$(75) \quad T_f^s \circ d^f = d^{S^s(f)} \circ T^s,$$

and thus defines an isomorphism $H_(\mathcal{L}, d^f) \rightarrow H^*(\mathcal{L}, d^{S^s(f)})$.*

3. *For $f \in \mathcal{S}$ the diagram*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{T_f^s} & \mathcal{L} \\ \nwarrow (f)^\pm & & \nearrow (S^s(f))^\pm \\ & \mathfrak{P}^\pm & \end{array}$$

homotopically commutes.

Theorem 2.4.2 reduces on the level of rational curves to the following

Theorem 2.4.5 *Let \mathbf{F}^0 , \mathbf{F}^1 and \mathbf{K} be as in Theorem 2.4.2. Set $\mathbf{f}^0 = \mathbf{F}_0^0$, $\mathbf{f}^1 = \mathbf{F}_0^1$, $\mathbf{k} = \mathbf{K}_0$. Then \mathbf{f}_0 and \mathbf{f}_1 are homotopic, i.e. they can be included into a family \mathbf{f}_s , $s \in [0, 1]$, such that the Hamilton-Jacobi equation (73) holds with $f_s = \mathbf{f}_s$ and $k_s \equiv \mathbf{k}$.*

Hence, Proposition 2.4.4 implies

Corollary 2.4.6 *For a generic compatible deformation J_s , $s \in [0, 1]$, we have*

1. *The operator $S^1 : \mathfrak{L} \rightarrow \mathfrak{L}$ defines an automorphism of the space of solutions of (48);*
2. *The linearization $T^1 = T_{\mathbf{f}_0}^1$ of S^1 at the point \mathbf{f}_0 satisfies the equation*

$$(76) \quad T^1 \circ d^{\mathbf{f}^0} = d^{\mathbf{f}^1} \circ T^1,$$

and thus defines an isomorphism $H_(\mathfrak{L}, d^{\mathbf{f}^0}) \rightarrow H^*(\mathfrak{L}, d^{\mathbf{f}^1})$.*

2. *The diagram*

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{T^1} & \mathfrak{L} \\ \swarrow_{(\mathbf{f}^0)^\pm} & & \nearrow_{(\mathbf{f}^1)^\pm} \\ & \mathfrak{P}^\pm & \end{array}$$

homotopically commutes.

To formulate the ‘‘classical level’’ corollary of Theorem 2.4.5 we assume, as usual, that W is a homology cobordism.

Theorem 2.4.7 *The homomorphisms $\Psi_{J_0}^1, \Psi_{J_1}^1 : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$ associated to two compatible almost complex structures J_0 and J_1 are homotopic, i.e. there exists a map $\Delta : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$ such that*

$$\Psi_{J_1}^1 - \Psi_{J_0}^1 = \partial^- \circ \Delta + \Delta \circ \partial^+.$$

The map Δ can be expressed through the function $\mathbf{k} \in \mathfrak{L}$. However, unlike the case of usual Floer homology theory, Δ and \mathbf{k} are related via a first order non-linear PDE (which can be deduced from the equation (73)), and thus one cannot write a general explicit formula relating Δ and \mathbf{k} .

2.5 Composition of cobordisms

In this section we study the behavior of potentials and associated algebraic structures under the operation of composition of directed symplectic cobordisms.

Let us recall (see Section 1.3) that given a dividing contact type hypersurface V in a directed symplectic cobordism $W = \overline{V^- V^+}$ one can split W into a composition

$W = W_- \circledast W_+$ of cobordisms $W_- = \overrightarrow{V^-V}$ and $W_+ = \overrightarrow{VV^+}$. From the point of view of an almost complex structure the process of splitting consists of deforming an original almost complex structure $J = J^0$ to an almost complex structure J^∞ , such that the restrictions $J_\pm = J^\infty|_{W_\pm}$ are compatible with the structure of (completed) directed symplectic cobordisms W_\pm .

Conversely, directed symplectic cobordisms $W_- = \overrightarrow{V^-V}$ and $W_+ = \overrightarrow{VV^+}$ with matching data on the common boundary can be glued into a cobordism $W = \overrightarrow{V^-V^+}$ in the following sense: there exists a family J^τ of almost complex structures on W which in the limit splits W into the composition of cobordisms $W_- = \overrightarrow{V^-V}$ and $W_+ = \overrightarrow{VV^+}$.

In order to write the formula relating the potentials of W and W_\pm we first need to make more explicit the relation between 2-dimensional homology classes realized by holomorphic curves in W_\pm and W . We will keep assuming that there are no torsion elements in H_1 . Let us recall (see Sections 1.2 and 1.5 above) that we have chosen curves $C_-^i \subset W_-$, $i = 1, \dots, m_-$, $C_+^j \subset W_+$, $j = 1, \dots, m_+$, and $C^k \subset W$, $k = 1, \dots, m$, which represent bases of first homology of the respective cobordisms. We also have chosen for every periodic orbit $\gamma \in \mathcal{P}_\alpha$ of the Reeb field R_α on V a surface F_\pm^γ realizing homology in W_\pm between γ and a linear combination of basic curves C_\pm^i . For our current purposes we have to make one extra choice: for each curve C_\pm^i we choose a surface S_\pm^i which realizes homology in W between C_\pm^i and the corresponding linear combination of the curves C^1, \dots, C^m . All the choices enable us to associate with every orbit $\gamma \in \mathcal{P}_\alpha$ a homology class C^γ which is realized by the chain

$$F_+^\gamma - F_-^\gamma + \sum_1^{m_+} n_j^+ S_+^j - \sum_1^{m_-} n_i^- S_-^i,$$

where $\partial F_\pm^\gamma = [\gamma] - \sum_1^{m_\pm} n_j^\pm C_\pm^j$. We will denote by d^γ the degree of C^γ , i.e. the string of its coordinates in the chosen basis $A_1, \dots, A_N \in H_2(W)$.

Let us define an operation

$$(77) \quad \star : \mathfrak{D}\mathfrak{D}_- \otimes \mathfrak{D}\mathfrak{D}_+ \rightarrow \mathfrak{D}\mathfrak{D},$$

where $\mathfrak{D}\mathfrak{D}_\pm = \mathfrak{D}\mathfrak{D}_{W_\pm}$ and $\mathfrak{D}\mathfrak{D} = \mathfrak{D}\mathfrak{D}_W$. For $F = \sum_\Gamma f_\Gamma(t^-, q^-, \hbar, z^-) p^\Gamma \in \mathfrak{D}\mathfrak{D}_-$ and $G = \sum_{\Gamma^+} g_{\Gamma^+}(t^+, q, \hbar, z^+) (p^+)^{\Gamma^+} \in \mathfrak{D}\mathfrak{D}_+$ we set

$$(78) \quad F \star G(t, q^-, p^+, \hbar, z) = \left(\sum_\Gamma \tilde{f}_\Gamma(t, q^-, \hbar, z) \hbar^s \prod_{i=1}^s \kappa_{\gamma_i} z^{d^{\gamma_i}} \frac{\overrightarrow{\partial}}{\partial q_{\gamma_i}} \sum_{\Gamma^+} \tilde{g}_{\Gamma^+}(t, q, \hbar, z) (p^+)^{\Gamma^+} \right) \Big|_{q=0}.$$

Here we denote by \tilde{f}_Γ and \tilde{g}_{Γ^+} the images of f_Γ and g_{Γ^+} under the coefficient homomorphisms $H_2(W_\pm) \rightarrow H_2(W)$. Let us explain what happens with the variables

t and z in more details. Let $A_1^\pm, \dots, A_{N^\pm}^\pm$, and A_1, \dots, A_N , be the chosen bases in $H_2(W_\pm)$ and $H_2(W)$. Then we have

$$i_*^\pm(A_k^\pm) = \sum_{j=1}^{N^\pm} n_{kj}^\pm A_j,$$

where $k = 1, \dots, N$, (n_{kj}^\pm) are integer matrices, and $i^\pm : W_\pm \rightarrow W$ the inclusion maps.

We have

$$\begin{aligned} f_\Gamma(t^-, q^-, \hbar, z^-) &= \sum_{d=(d_1, \dots, d_{N^-})} f_{\Gamma, d}(t^-, q^-, \hbar)(z_1^-)^{d_1} \dots (z_{N^-}^-)^{d_{N^-}}, \\ g_{\Gamma^+}(t^+, q, \hbar, z^+) &= \sum_{d=(d_1, \dots, d_{N^+})} g_{\Gamma^+, d}(t^+, q, \hbar)(z_1^+)^{d_1} \dots (z_{N^+}^+)^{d_{N^+}}, \end{aligned}$$

where we denote by z_\pm the “ z -variables” in W_\pm . Then

$$\begin{aligned} \tilde{f}_\Gamma(t, q^-, \hbar, z) &= \sum_{d=(d_1, \dots, d_{N^-})} \tilde{f}_{\Gamma, d}(t|_{W_-}, q^-, \hbar) z_1^{M_1^-} \dots z_N^{M_N^-}, \\ \tilde{g}_{\Gamma^+}(t, q, \hbar, z) &= \sum_{d=(d_1, \dots, d_{N^+})} \tilde{g}_{\Gamma^+, d}(t|_{W_+}, q, \hbar) z_1^{M_1^+} \dots z_N^{M_N^+}, \end{aligned}$$

where $M_j^\pm = \sum_{k=1}^{N^\pm} n_{kj}^\pm d_k$, $j = 1, \dots, N$.

Let us observe

Lemma 2.5.1 1. *The operation \star is associative.*

2. *For $F \in \hbar^{-1}\mathfrak{D}_-$, $G \in \hbar^{-1}\mathfrak{D}_+$ there exists a unique function $H \in \hbar^{-1}\mathfrak{D}$, such that*

$$e^H = e^F \star e^G.$$

We will denote this H by $F \diamond G$, so that we have $e^{F \diamond G} = e^F \star e^G$. We will also consider the maps

$$\diamond G : \hbar^{-1}\mathfrak{D}_- \rightarrow \hbar^{-1}\mathfrak{D}, \quad \diamond G(F) = F \diamond G, \quad F \in \mathfrak{D}_-,$$

and

$$F \diamond : \hbar^{-1}\mathfrak{D}_+ \rightarrow \hbar^{-1}\mathfrak{D}, \quad F \diamond(G) = F \diamond G, \quad G \in \mathfrak{D}_+,$$

and for even F, G their linearizations:

$$\begin{aligned} T_F(\diamond G) &: \hbar^{-1}\mathfrak{D}_- \rightarrow \hbar^{-1}\mathfrak{D}, \\ T_F(\diamond G)(f) &= \left. \frac{d(F + \varepsilon f) \diamond G}{d\varepsilon} \right|_{\varepsilon=0} = e^{-F \diamond G}((fe^F) \star e^G), \quad f \in \hbar^{-1}\mathfrak{D}_-, \end{aligned}$$

and

$$\begin{aligned} T_G(F \diamond) &: \hbar^{-1}\mathfrak{D}_+ \rightarrow \hbar^{-1}\mathfrak{D}, \\ T_G(F \diamond)(g) &= \left. \frac{d(F \diamond(G + \varepsilon g))}{d\varepsilon} \right|_{\varepsilon=0} = e^{-F \diamond G}(e^F \star (ge^G)), \quad g \in \hbar^{-1}\mathfrak{D}_+, \end{aligned}$$

Let us first formulate an algebraic

Proposition 2.5.2 *Suppose that $F \in \hbar^{-1}\mathfrak{D}_-$ and $G \in \hbar^{-1}\mathfrak{D}_+$ are even elements, which satisfy the equations*

$$D_{W_-}(e^F) = \overrightarrow{\mathbf{H}} e^F - e^F \overleftarrow{\mathbf{H}} = 0$$

and

$$D_{W_+}(e^G) = \overrightarrow{\mathbf{H}} e^G - e^G \overleftarrow{\mathbf{H}} = 0,$$

where $\mathbf{H}^\pm = \mathbf{H}_{V^\pm}$, $\mathbf{H} = \mathbf{H}_V$. Then we have

1. $D_W(e^{F \diamond G}) = \overrightarrow{\mathbf{H}} e^{F \diamond G} - e^{F \diamond G} \overleftarrow{\mathbf{H}} = 0.$
2. The homomorphisms $T_G(F \diamond) : \hbar^{-1}\mathfrak{D}_+ \rightarrow \hbar^{-1}\mathfrak{D}$ and $T_F(\diamond G) : \hbar^{-1}\mathfrak{D}_- \rightarrow \hbar^{-1}\mathfrak{D}$ satisfy the equations

$$T_G(F \diamond) \circ D^G = D^{F \diamond G} \circ T_G(F \diamond)$$

and

$$T_F(\diamond G) \circ D^F = D^{F \diamond G} \circ T_F(\diamond G),$$

and in particular they define homomorphisms of the corresponding homology algebras:

$$(T_G(F \diamond))_* : H_*(\mathfrak{D}_+, D^G) \rightarrow H_*(\mathfrak{D}, D^{F \diamond G})$$

and

$$(T_F(\diamond G))_* : H_*(\mathfrak{D}_-, D^G) \rightarrow H_*(\mathfrak{D}, D^{F \diamond G}).$$

3.

$$T_F(\diamond G) \circ F^- = (F \diamond G)^-$$

and

$$T_G(F \diamond) \circ G^+ = (F \diamond G)^+.$$

4. Suppose we are given three cobordisms W_1, W_2, W_3 with matching ends so that we can form the composition $W_{123} = W_1 \odot W_2 \odot W_3$, and series $F_i \in \hbar^{-1}\mathfrak{D}_i = \hbar^{-1}\mathfrak{D}_{W_i}$, $i = 1, 2, 3$, such that $D_{W_i} e^{F_i} = 0$, $i = 1, 2, 3$. Then

$$(79) \quad T_{F_1 \diamond F_2}(\diamond F_3) \circ T_{F_1}(\diamond F_2) = T_{F_1}(\diamond(F_2 \diamond F_3)).$$

The proof of this proposition is immediate from the definition of the corresponding operations. For instance, to prove 2.5.2.1 we write

$$\begin{aligned} \overrightarrow{\mathbf{H}} (e^{F \diamond G}) - (e^{F \diamond G}) \overleftarrow{\mathbf{H}} &= \overrightarrow{\mathbf{H}} e^F \star e^G - e^F \star e^G \overleftarrow{\mathbf{H}} \\ &= \left(\overrightarrow{\mathbf{H}} e^F \right) \star e^G - e^F \star \left(e^G \overleftarrow{\mathbf{H}} \right) \\ &= \left(e^F \overleftarrow{\mathbf{H}} \right) \star e^G - e^F \star \left(\overrightarrow{\mathbf{H}} e^G \right). \end{aligned}$$

To finish the argument let us consider a cylindrical cobordism $W_0 = V \times \mathbb{R}$, take the function $I = \sum \kappa_\gamma^{-1} p_\gamma q_\gamma$. Taking into account associativity of \star (see 2.5.1) we have

$$(80) \quad \overrightarrow{f} e^G = (f e^I) \star e^G, \quad \text{and} \quad e^F \overleftarrow{f} = e^F \star (f e^I).$$

Hence, we have

$$\begin{aligned} & \left(e^F \overleftarrow{\mathbf{H}} \right) \star e^G - e^F \star \left(\overrightarrow{\mathbf{H}} e^G \right) \\ &= e^F \star (\mathbf{H} e^I) \star e^G - e^F \star (\mathbf{H} e^I) \star e^G = 0. \end{aligned}$$

Any cohomology class from $H^*(W)$ can be represented by a form t which splits into the sum of forms t_{\pm} with cylindrical ends on W_{\pm} , so that we have $t_{\pm}|_V = t_V$. Let us define now a series $\mathbf{F}^{\infty} \in \hbar^{-1}\mathfrak{D}$ by the formula

$$(81) \quad \mathbf{F}^{\infty}(q^-, p^+, t) = \mathbf{F}_-(q^-, p, t_-) \diamond \mathbf{F}_+(q, p^+, t_+),$$

where p, q are variable associated to the space $H^*(\mathcal{P})$ of periodic orbits of the Reeb vector field R_{α} of the contact form α on V .

The following theorem is the main claim of this section, and similar to Theorems 2.2.2 and 2.3.3 and 2.4.2 it is a statement about the boundary of an appropriate moduli space of holomorphic curves. This time we deal with limits of J^s -holomorphic curves in W when $s \rightarrow \infty$, i.e. when the family J^s realizes the splitting of W into the composition $W_- \odot W_+$, see Theorem 1.6.3 above.

Theorem 2.5.3 *The element \mathbf{F}^{∞} is homotopic to the potential $\mathbf{F} = \mathbf{F}_{W, J, \alpha^{\pm}}$ for any generic compatible almost complex structure J on W .*

Let us now describe the above results on the level of rational potentials. Let $W_- = \overrightarrow{V^-} \overleftarrow{V}$, $W_+ = \overleftarrow{V} \overrightarrow{V^+}$ and $W = W_- \odot W_+ = \overrightarrow{V^-} \overleftarrow{V} \overrightarrow{V^+}$ be as above. Set $\mathbf{h}^{\pm} = \mathbf{h}_{V^{\pm}}$, $\mathbf{h} = \mathbf{h}_V$, $\widehat{\mathbf{h}}_- = \mathbf{h}^- - \mathbf{h}$, $\widehat{\mathbf{h}}_+ = \mathbf{h} - \mathbf{h}^+$, $\widehat{\mathbf{h}} = \mathbf{h}^+ - \mathbf{h}^-$, $\mathfrak{L}_{\pm} = \mathfrak{L}_{W_{\pm}}$ and $\mathfrak{L} = \mathfrak{L}_W$.

The operation $\diamond : \hbar^{-1}\mathfrak{D}_- \times \hbar^{-1}\mathfrak{D}_+ \rightarrow \hbar^{-1}\mathfrak{D}$ defined above reduces on the rational level to the operation

$$\sharp : \mathfrak{L}_- \times \mathfrak{L}_+ \rightarrow \mathfrak{L},$$

defined as follows. For $f_{\pm} \in \mathfrak{L}_{\pm}$ we set

$$(82) \quad f_- \sharp f_+(q^-, p^+) = \left(f_-(q^-, p) + f_+(q, p^+) - \sum_{\gamma \in \mathcal{P}} \kappa_{\gamma}^{-1} z^{-d_{\gamma}} q_{\gamma} p_{\gamma} \right) \Big|_L,$$

where

$$L = \begin{cases} q_{\gamma} = \kappa_{\gamma} z^{d_{\gamma}} \frac{\partial f_-}{\partial p_{\gamma}}; \\ p_{\gamma} = \kappa_{\gamma} z^{d_{\gamma}} \frac{\partial f_+}{\partial q_{\gamma}}. \end{cases}$$

Notice that given series

$$F_- = \hbar^{-1} \sum_0^{\infty} (F_-)_g \hbar^g \in \hbar^{-1}\mathfrak{D}^- \quad \text{and} \quad F_+ = \hbar^{-1} \sum_0^{\infty} (F_+)_g \hbar^g \in \hbar^{-1}\mathfrak{D}^+$$

with $F_- \diamond F_+ = \hbar^{-1} \sum_0^\infty (F_- \diamond F_+)_g \hbar^g \in \hbar^{-1} \mathfrak{D}$ then

$$(F_- \diamond F_+)_0 = (F_-)_0 \sharp (F_+)_0.$$

We will also consider the operations

$$\sharp f_+ : \mathfrak{L}_- \rightarrow \mathfrak{L}, \quad \sharp f_+(f_-) = f_- \sharp f_+ \quad \text{and} \quad \sharp f_- : \mathfrak{L}_+ \rightarrow \mathfrak{L}, \quad \sharp f_-(f_+) = f_- \sharp f_+,$$

and their linearizations

$$\begin{aligned} T_{f_-}(\sharp f_+) : \mathfrak{L}_- &\rightarrow \mathfrak{L}, & T_{f_-}(\sharp f_+)(g) &= (g|_{L_+}) \sharp f_+ \quad \text{and} \\ T_{f_+}(\sharp f_-) : \mathfrak{L}_+ &\rightarrow \mathfrak{L}, & T_{f_+}(\sharp f_-)(g) &= f_- \sharp (g|_{L_-}). \end{aligned}$$

Here we view $g|_{L_+}$ (resp. $g|_{L_-}$) as an element of \mathfrak{L}_+ which depends on variables q^- as parameters (resp. an element of \mathfrak{L}_- , which depends on p^+ as parameters). We have the following rational version of Theorem 2.5.2.

Proposition 2.5.4 *Suppose that even elements $f_\pm \in \mathfrak{L}_\pm = \mathfrak{L}_{W_\pm}$ satisfy equation (48), i.e.*

$$\widehat{\mathbf{h}}_\pm|_{L_{f_\pm}} = 0,$$

where

$$L_{f_-} = \begin{cases} q_\gamma &= k_\gamma \frac{\partial f_-}{\partial p_\gamma^-}; \quad \gamma \in \mathcal{P} \\ p_{\gamma^-}^- &= \kappa_{\gamma^-} \frac{\partial f_-}{\partial q_{\gamma^-}^-}, \quad \gamma^- \in \mathcal{P}^-, \end{cases}$$

and

$$L_{f_+} = \begin{cases} p_\gamma &= k_\gamma \frac{\partial f_+}{\partial q_\gamma^+}; \quad \gamma \in \mathcal{P} \\ q_{\gamma^+}^+ &= \kappa_{\gamma^+} \frac{\partial f_+}{\partial p_{\gamma^+}^+}, \quad \gamma^+ \in \mathcal{P}^+. \end{cases}$$

Then

1. The function $f_- \sharp f_+$ satisfies the Hamilton-Jacobi equation

$$\widehat{\mathbf{h}}_{L|_{f_- \sharp f_+}} = 0;$$

2. The homomorphisms $T_{f_-}(\sharp f_+) : \mathfrak{L}_- \rightarrow \mathfrak{L}$ and $T_{f_+}(f_- \sharp) : \mathfrak{L}_+ \rightarrow \mathfrak{L}$ satisfy the equations

$$\begin{aligned} T_{f_-}(\sharp f_+) \circ d^{f_-} &= d^{f_- \sharp f_+} \circ T_{f_-}(\sharp f_+), \\ T_{f_+}(f_- \sharp) \circ d^{f_+} &= d^{f_- \sharp f_+} \circ T_{f_+}(f_- \sharp), \end{aligned}$$

and hence define homomorphisms of the corresponding homology algebras:

$$\begin{aligned} (T_{f_-}(\sharp f_+))_* &: H_*(\mathfrak{L}_-, d^{f_-}) \rightarrow H_*(\mathfrak{L}, d^{f_- \sharp f_+}), \\ (T_{f_+}(f_- \sharp))_* &: H_*(\mathfrak{L}_+, d^{f_+}) \rightarrow H_*(\mathfrak{L}, d^{f_- \sharp f_+}); \end{aligned}$$

3.

$$\begin{aligned} T_{f_-}(\#f_+) \circ (f_-)^- &= (f_- \# f_+)^-, \\ T_{f_+}(\#f_-) \circ (f_+)^- &= (f_- \# f_+)^+; \end{aligned}$$

4. Suppose we are given three cobordisms W_1, W_2, W_3 with matching ends so that we can form the composition $W_{123} = W_1 \odot W_2 \odot W_3$, and series $f_i \in \hbar^{-1}\mathfrak{L}_i = \mathfrak{L}_{W_i}$ which satisfy Hamilton-Jacobi equations $\widehat{\mathbf{h}}_{W_i}|_{L_{f_i}} = 0$, $i = 1, 2, 3$. Then

$$T_{f_2}(\#f_3) \circ T_{f_1}(\#f_2) = T_{f_1}(\#(f_2\#f_3)).$$

Let t be a closed form on W which is split into two forms t_{\pm} on W_{\pm} with cylindrical ends. Set $\mathbf{f}_{\pm} = \mathbf{f}_{W_{\pm}}$ and

$$(83) \quad \mathbf{f}^{\infty}(q^-, p^+, t) = \mathbf{f}_-(q^-, p, t_-) \# \mathbf{f}_+(q, p^+, t_+).$$

Alternatively \mathbf{f}^{∞} can be defined as the first term in the expansion $\mathbf{F}^{\infty} = \hbar^{-1} \sum_0^{\infty} \mathbf{F}_g^{\infty} \hbar^g$.

The following theorem is a rational analog, and a direct corollary of Theorem 2.5.3.

Theorem 2.5.5 *The series $\mathbf{f}^{\infty}(q^-, p^+, t)$ and $\mathbf{f}_{W,J}(q, p, t)$ are homotopic for any generic compatible almost complex structure J on W .*

Coming down to the ‘‘classical’’ level, let us assume that W, W_- and W_+ are homology cobordisms (see 2.2.3 above). Thus there are defined the homomorphisms $\Psi : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$, $\Psi_+ : \mathfrak{A}^+ \rightarrow \mathfrak{A}$ and $\Psi_- : \mathfrak{A} \rightarrow \mathfrak{A}^-$, see Section 2.3.2 above. Set $\Psi_{\infty} = \Psi_+ \circ \Psi_-$. Then we have

Theorem 2.5.6 *For any generic compatible almost complex structure J on W homomorphisms $\Psi_1 = \Psi_J, \Psi_{\infty} : \mathfrak{A}^+ \rightarrow \mathfrak{A}^-$ are chain homotopic.*

2.6 Invariants of contact manifolds

Theorem 2.5.3 allows us to define SFT-invariants of contact manifolds. Let (V, ξ) be a contact manifold, and α^+ and α^- two contact forms for ξ , such that $\alpha^+ > \alpha^-$, i.e. $\alpha^+ = f\alpha^-$, for a function $f > 1$. Then for an appropriately chosen function $\zeta : V \times \mathbb{R} \rightarrow (0, \infty)$ the form $\omega = d(\zeta\alpha^-)$ on $W = V \times \mathbb{R}$ is symplectic, and (W, ω) is a directed symplectic cobordism between (V, α^-) and (V, α^+) . Let t^{\pm} be two cohomologous forms on V , and t be a closed form on W with cylindrical ends which restricts to t^{\pm} on V^{\pm} . Suppose we are also given almost complex structures J^{\pm} on V , compatible with α^{\pm} , which are extended to a compatible almost complex structure J on W . We will call a directed symplectic cobordism (W, J, t) , chosen in the above way, a *concordance* between the data on its boundary. Notice, that concordance becomes an equivalence relation if we identify contact forms proportional with a constant factor. A concordance (W, J, t) is called *trivial* if $W = V \times \mathbb{R}$, the almost complex structure J is translationally invariant, and t is the pull-back of a form t_+ under the projection $W \rightarrow V$.

Let us denote, as usual, by $(\mathfrak{W}^\pm, D^\pm)$ the differential Weyl algebras associated to the data at the ends of the cobordism W , by (\mathfrak{D}, D_W) the D -module $\mathfrak{D}(W, J, \alpha^\pm)$, by $\mathbf{F} \in \mathfrak{W}$ the potential of the cobordism W , and by $\mathbf{F}^\pm : (\mathfrak{W}^\pm, D^\pm) \rightarrow (\mathfrak{D}, D_W)$ the corresponding homomorphisms of differential algebras defined in (43).

Theorem 2.6.1 *For any concordance (W, J, t) the homomorphisms*

$$\mathbf{F}^\pm : (\mathfrak{W}^\pm, d^\pm) \rightarrow (\mathfrak{D}, D_W)$$

are quasi-isomorphisms of differential algebras. In particular, the homology algebras $H_(\mathfrak{W}^-, D^-)$ and $H_*(\mathfrak{W}^+, D^+)$ are isomorphic.*

Proof. We will prove 2.6.1 in three steps.

Step 1. Let us begin with the trivial concordance (W, J, t) . In this case \mathfrak{D} can be identified with \mathfrak{W}^\pm and we have $\mathbf{F}(q, p, t) = \hbar^{-1} \sum \kappa_\gamma^{-1} q_\gamma p_\gamma$. Hence

$$\mathbf{F}^-(f) = e^{-\mathbf{F}} \overrightarrow{f} e^{\mathbf{F}} = f = e^{-\mathbf{F}} \left(e^{\mathbf{F}} \overleftarrow{f} \right) = \mathbf{F}^+(f).$$

Step 2. If we add now to t a form $d\theta$, where θ has a compact support, and change J in a compact part of W then according to Theorem 2.4.2 the potential $\mathbf{F}_{W, J}(t+d\theta)$ remains the same up to homotopy, and hence Corollary 2.4.3 implies that the homomorphisms \mathbf{F}_*^\pm induced on homology remain unchanged.

Step 3. Now assume that $(W, J, t) = (W^1, J^1, t^1)$ is a general concordance. Consider the reversed concordance (W^2, J^2, t^2) , so that the compositions

$$(W^{12} = W^1 \circledast W^2, J^{12} = J^1 \circledast J^2, t^{12} = t_1 \circledast t_2)$$

and

$$(W^{21} = W^2 \circledast W^1, J^{21} = J^2 \circledast J^1, t^{21} = t_2 \circledast t_1)$$

of concordances (W^1, J^1, t^1) and (W^2, J^2, t^2) are as in Step 2. Then according to Theorem 2.5.3 $\mathbf{F}_{W^{12}, J^{12}}(t^{12})$ is homotopic to

$$\mathbf{F}_{W^1, J^1}(t^1) \diamond \mathbf{F}_{W^2, J^2}(t^2) = \mathbf{F}^1 \diamond \mathbf{F}^2$$

and $\mathbf{F}^{21} = \mathbf{F}_{W^{21}, J^{21}}(t^{21})$ is homotopic to

$$\mathbf{F}_{W^2, J^2}(t^2) \diamond \mathbf{F}_{W^1, J^1}(t^1) = \mathbf{F}^2 \diamond \mathbf{F}^1.$$

Hence Proposition 2.5.2 implies

$$\text{Id} = (\mathbf{F}_{W^1 \circledast W^2})_*^- = (T_{\mathbf{F}^1}(\diamond \mathbf{F}^2))_* \circ (\mathbf{F}^1)_*^-$$

and

$$\begin{aligned} \text{Id} &= (T_{\mathbf{F}^1}(\diamond(\mathbf{F}^2 \diamond \mathbf{F}^1)))_* \\ &= (T_{\mathbf{F}^1 \diamond \mathbf{F}^2}(\diamond \mathbf{F}^1))_* \circ (T_{\mathbf{F}^1}(\diamond \mathbf{F}^2))_* \end{aligned}$$

Hence $(T_{\mathbf{F}^1}(\diamond \mathbf{F}^2))_*$, $(\mathbf{F}^1)_*^-$, and similarly $(\mathbf{F}^1)_*^+$ are isomorphisms. \blacktriangleright

The following rational and classical analogs of Theorem 2.6.1 can be either deduced directly from Theorem 2.6.1, or alternatively can be proven similarly to 2.6.1 using 2.4.5 (resp. 2.4.7) and 2.5.5 (resp. 2.5.6).

Theorem 2.6.2 For any concordance (W, J, t) the homomorphisms

$$\mathbf{f}^\pm : (\mathfrak{P}^\pm, D^\pm) \rightarrow (\mathfrak{L}, D_W)$$

are quasi-isomorphisms of differential algebras. In particular, Poisson homology algebras $H_*(\mathfrak{P}^\pm, d^\pm)$ are isomorphic.

Theorem 2.6.3 For any concordance (W, J, t) the homomorphism

$$\Psi : (\mathfrak{A}^+, \partial^+) \rightarrow (\mathfrak{A}^-, \partial^-)$$

is a quasi-isomorphism of differential algebras.

The definition of the differential algebras (\mathfrak{W}, D) , (\mathfrak{P}, d) and (\mathfrak{A}, ∂) depends on the choice of a coherent orientation (see Section 1.8), and of spanning surfaces and framings of periodic orbits (see Section 1.2). As it is stated in Theorem 1.8.7 a coherent orientation is determined by a choice of asymptotic operators associated with each periodic orbit γ . Let \mathbf{H}' be the new Hamiltonian which one gets by changing the orientation of the asymptotic operator associated with a fixed periodic orbit γ . One can then check that the change of variables $(p_\gamma, q_\gamma) \mapsto (-p_\gamma, -q_\gamma)$ is an isomorphism between the differential algebras $(\mathfrak{W}, D^{\mathbf{H}})$ and $(\mathfrak{W}, D^{\mathbf{H}'})$. Different choices for spanning surfaces and framings of periodic orbits do not affect mod 2 grading but change the integer grading of the differential algebras.

Remark 2.6.4 An accurate introduction of virtual cycle techniques would reveal that even more choices should be made. However, an independence of all these extra choices can be also established following the scheme of this section.

2.7 A differential equation for potentials of symplectic cobordisms

In this section we describe differential equations for the potentials \mathbf{F}_W and \mathbf{f}_W of a directed symplectic cobordism with a non-empty boundary. These equations completely determine the potentials, and in combination with gluing Theorems 2.5.3 and 2.5.5 they provide in many cases an effective recursive procedure for computing potentials \mathbf{F}_W and \mathbf{f}_W , and even Gromov-Witten invariants of closed symplectic manifolds W (see some examples in Section 2.9.3 below).

Let us assume for simplicity that W has only a positive end $E^+ = V \times (0, \infty)$, and choose a basic system $\Delta_1, \dots, \Delta_k, \Theta_1, \dots, \Theta_m$ of closed forms so that the following conditions are satisfied:

- a) $\Delta_1, \dots, \Delta_k$ form a basis of $H^*(W)$, and the restrictions $\delta_i = \Delta_i|_V$, $i = 1, \dots, l$ for $l \leq k$ form a basis of $\text{Im}(H^*(W) \rightarrow H^*(V))$;
- b) $\Theta_1, \dots, \Theta_m$ are compactly supported and represent a basis of $\text{Ker}(H_{\text{comp}}^*(W) \rightarrow H^*(W))$,

- c) there exist forms $\theta_1, \dots, \theta_m$ on V and a compactly supported 1-form ρ on $(0, +\infty)$, such that $\Theta_j = e_*(\rho \wedge \pi^*(\theta_j))$, $j = 1, \dots, m$, where π is the projection $E = V \times (0, \infty) \rightarrow V$ and $e : E \hookrightarrow W$ is the inclusion. In other words, Θ_j equals $\rho \wedge \pi^*(\theta_j)$ viewed as a form on W .

Theorem 2.7.1 *Let $\mathbf{H} = \mathbf{H}_{V,\alpha,J}$ be the Hamiltonian associated with the contact manifold V . Set*

$$\mathbf{H}^j(t_1, \dots, t_l, q, p) = \left(\frac{\partial \mathbf{H}}{\partial s_j} \left(\sum_{i=1}^l t_i \delta_i + s_j \theta_j, q, p \right) \right) \Big|_{s_j=0}, \quad j = 1, \dots, m,$$

$$\mathbf{F}^0(t_1, \dots, t_k, p) = \mathbf{F}_{W,J} \left(\sum t_i \Delta_i, p \right),$$

and define $\mathbf{F}(t_1, \dots, t_k, \tau_1, \dots, \tau_m, p)$ by the formula

$$(84) \quad e^{\mathbf{F}(t_1, \dots, t_k, \tau_1, \dots, \tau_m, p)} = e^{\mathbf{F}^0(t_1, \dots, t_k, p)} \overleftarrow{\mathbf{G}}(t_1, \dots, t_l, \tau_1, \dots, \tau_m, p),$$

where we denote by $\overleftarrow{\mathbf{G}}$ the operator obtained from

$$(85) \quad \mathbf{G}(t_1, \dots, t_l, \tau_1, \dots, \tau_m, q, p) = e^{\tau_m \mathbf{H}^m(t_1, \dots, t_l, q, p)} \dots e^{\tau_1 \mathbf{H}^1(t_1, \dots, t_l, q, p)}$$

by quantizing $q_\gamma = \kappa_\gamma \hbar \frac{\overleftarrow{\partial}}{\partial p_\gamma}$. Then $\mathbf{F}(t_1, \dots, t_k, \tau_1, \dots, \tau_m, p)$ is homotopic to

$$\mathbf{F}_{W,J} \left(\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^m \tau_r \Theta_r, p \right).$$

Proof. Set

$$T^j(s) = \mathbf{F}_{W,J} \left(\sum t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j, p \right).$$

We have

$$T^j(1) = T^{j+1}(0) \quad \text{for } j = 1, \dots, m-1,$$

$$T^m(1) = \mathbf{F}_{W,J} \left(\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^m \tau_r \Theta_r, p \right),$$

and

$$T^1(0) = \mathbf{F}_{W,J} \left(\sum t_i \Delta_i, p \right) = \mathbf{F}^0(t_1, \dots, t_k, p).$$

Let $S^j \in \hbar^{-1} \mathfrak{D}$ be defined from the equation

$$e^{S^j} = e^{T^j(0)} e^{\overleftarrow{\tau_j \mathbf{H}^j(t_1, \dots, t_l, q, p)}}.$$

It is enough to prove that $T^j(1)$ is homotopic to S^j for $j = 1, \dots, m$. We have

$$\begin{aligned} \frac{\partial e^{T^j(s)}}{\partial s} &= \frac{\partial T^j(s)}{\partial s} e^{T^j(s)} \\ &= e^{T^j(s)} \sum_d \sum_{g,u,v=0}^{\infty} \frac{1}{u!v!} \langle \tau_j \Theta_j, \underbrace{\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j}_u; \underbrace{p, \dots, p}_v \rangle_g z^d \hbar^{g-1}. \end{aligned}$$

The compactly supported form Θ_j is exact in W ,

$$\Theta_j = d\tilde{\Theta}_j,$$

where $\tilde{\Theta}_j$ is closed at infinity, has a cylindrical end, and $\tilde{\Theta}_j|_V = \theta_j$. Hence,

$$\begin{aligned} \frac{\partial T^j(s)}{\partial s} &= \sum_d \sum_{g,u,v=0}^{\infty} \frac{1}{u!v!} \langle \tau_j d\tilde{\Theta}_j, \underbrace{\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \Theta_j}_u; \underbrace{p, \dots, p}_v \rangle_g z^d \hbar^{g-1} \\ &= d \left(\sum_d \sum_{g,u,v=0}^{\infty} \frac{1}{u!v!} \langle \tau_j \tilde{\Theta}_j, \underbrace{\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j}_u; \underbrace{p, \dots, p}_v \rangle_g z^d \hbar^{g-1} \right) \\ &= \frac{\partial}{\partial u} \left(d \left(\sum_d \sum_{g,u,v=0}^{\infty} \frac{1}{u!v!} \langle \underbrace{\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j + u \tau_j \tilde{\Theta}_j}_u; \underbrace{p, \dots, p}_v \rangle_g z^d \hbar^{g-1} \right) \right) \Big|_{u=0} \\ &= \frac{\partial}{\partial u} \left(d(\mathbf{F}_{W,J}(\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j + u \tau_j \tilde{\Theta}_j, p)) \right) \Big|_{u=0}, \end{aligned}$$

where d denotes the de Rham differential. Using equation (45) we get

$$\begin{aligned} d(\mathbf{F}_{W,J}(\sum_{i=1}^k t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j + u \tau_j \tilde{\Theta}_j, p)) &= \\ e^{-\mathbf{F}_{W,J}} (e^{\mathbf{F}_{W,J}} \tilde{\mathbf{H}}(\sum_{i=1}^l t_i \Delta_i + u \tau_j \theta_j, p)), & \end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial T^j(s)}{\partial s} &= \frac{\partial}{\partial u} \left(e^{-\mathbf{F}w, j} \left(e^{T^j(s)} \overleftarrow{\mathbf{H}} \left(\sum_{i=1}^l t_i \Delta_i + u \tau_j \theta_j, q, p \right) \right) \right) \Big|_{u=0} \\
&= \tau_j e^{-T^j(s)} \left(-\mathbf{F}^j(t_1, \dots, t_k, \tau_1, \dots, \tau_j, s, p) \left(e^{T^j(s)} \overleftarrow{\mathbf{H}} \left(\sum_{i=1}^l t_i \Delta_i, q, p \right) \right) \right. \\
&\quad + \left(e^{T^j(s)} \mathbf{F}^j(t_1, \dots, t_k, \tau_1, \dots, \tau_j, s, p) \right) \overleftarrow{\mathbf{H}} \left(\sum_{i=1}^l t_i \Delta_i, q, p \right) \\
&\quad \left. + e^{T^j(s)} \overleftarrow{\mathbf{H}}^j(t_1, \dots, t_l, q, p) \right) = \tau_j e^{-T^j(s)} \left(\left(e^{T^j(s)} [\mathbf{F}^j, \overleftarrow{\mathbf{H}}] + e^{T^j(s)} \overleftarrow{\mathbf{H}}^j \right) \right),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{F}^j(t_1, \dots, t_k, \tau_1, \dots, \tau_j, s, p) \\
&= \frac{\partial}{\partial u} \left(\mathbf{F} \left(\sum t_i \Delta_i + \sum_{r=1}^{j-1} \tau_r \Theta_r + s \tau_j \Theta_j + u \tilde{\Theta}_j, p \right) \right) \Big|_{u=0}
\end{aligned}$$

and

$$\mathbf{H}^j(t_1, \dots, t_l, q, p) = \frac{\partial}{\partial u} \left(\mathbf{H} \left(\sum t_i \Delta_i + u \theta_j, q, p \right) \right) \Big|_{u=0}.$$

Therefore,

$$\frac{\partial e^{T^j(s)}}{\partial s} = e^{T^j(s)} \frac{\partial T^j(s)}{\partial s} = \tau_j e^{T^j(s)} \left([\mathbf{F}^j, \overleftarrow{\mathbf{H}}] + \overleftarrow{\mathbf{H}}^j \right).$$

Let us define now a family $U^j(s) \in \hbar^{-1}\mathfrak{D}$, $s \in [0, 1]$ by the formula

$$(86) \quad e^{U^j(s)} = e^{T^j(s)} e^{(1-s)\tau_j \overleftarrow{\mathbf{H}}^j(t_1, \dots, t_l, q, p)}.$$

Then $U^j(s)$ is a homotopy between S^j and $T^j(1)$. Indeed, we have $U^j(0) = S^j$ and $U^j(1) = T^j(1)$. On the other hand we get an equation

$$\frac{\partial e^{U^j(s)}}{\partial s} = \tau_j e^{U^j(s)} \left(-\overleftarrow{\mathbf{H}}^j + [\mathbf{F}^j, \overleftarrow{\mathbf{H}}] + \overleftarrow{\mathbf{H}}^j \right) = e^{U^j(s)} [\tau_j \mathbf{F}^j, \overleftarrow{\mathbf{H}}],$$

which is the definition of homotopy (see Section 2.4 above). ▶

We formulate now a version of Theorem 2.7.1 for rational potentials.

Set

$$\mathbf{h}^j(t_1, \dots, t_l, q, p) = \frac{\partial \mathbf{h}}{\partial s_j} \left(\sum t_i \delta_i + s_j \theta_j, q, p \right), \quad j = 1, \dots, m,$$

and for any $\mathbf{g} \in \mathfrak{L}$ we denote by $L_{\mathbf{g}}$ the Lagrangian variety of \mathbf{V} , defined by the equation

$$(87) \quad L_{\mathbf{g}} = \left\{ q_\gamma = \kappa_\gamma \frac{\partial \mathbf{g}}{\partial p_\gamma} \right\}.$$

Theorem 2.7.2 *Let $\mathbf{f}(t_1, \dots, t_k, \tau_1, \dots, \tau_m, p)$ be the solution of the Hamilton-Jacobi equation*

$$(88) \quad \frac{\partial \mathbf{f}}{\partial \tau_j}(t_1, \dots, t_k, \tau_1, \dots, \tau_m) = \mathbf{h}^j(t_0, \dots, t_i, q, p)|_{L_\tau}$$

with the initial condition

$$\mathbf{f}|_{\tau_j=0} = \mathbf{f}_{W,J}(\sum t_i \Delta_i + \sum_{r \neq j} \tau_r \Theta_r, p).$$

Then

$$\mathbf{f}(t_1, \dots, t_k, \tau_1, \dots, \tau_m, p)$$

is homotopic to

$$\mathbf{f}_{W,J}(\sum t_i \Delta_i + \sum_{r=1}^m \tau_r \Theta_r, p).$$

2.8 Invariants of Legendrian knots

Symplectic Field Theory can be extended to include Gromov-Witten invariants of pairs (W, L) , where L is a Lagrangian submanifold of a symplectic manifold W . The corresponding relative object is a pair (W, L) , where $W = \overrightarrow{V^- V^+}$ is a directed symplectic cobordism between contact manifolds (V^\pm, α^\pm) , and L is a Lagrangian cobordism between Legendrian submanifolds $\Lambda^\pm \subset V^\pm$. More precisely, we assume that Lagrangian submanifold L is cylindrical at infinity over Λ^\pm , i.e. there exists $C > 0$, such that $L \cap V^- \times (-\infty, -C] = \Lambda^- \times (-\infty, -C]$ and $L \cap V^+ \times (C, \infty] = \Lambda^+ \times (C, \infty]$. In other words, we require L to coincide at infinity with symplectizations of Legendrian submanifolds Λ^\pm .

The moduli space of holomorphic curves to be considered in this case consists of equivalence classes of holomorphic curves with boundary which can have punctures of two types, interior and at the boundary. The boundaries of holomorphic curves are required to be mapped to the Lagrangian submanifold L , the holomorphic curves should be cylindrical over periodic orbits from \mathcal{P}^\pm at interior punctures, while at boundary punctures we require them to be asymptotically cylindrical over Reeb chords connecting points of the Legendrian submanifolds $\Lambda^\pm \subset V^\pm$. A more precise definition is given below. The algebraic structure arising from the stratification of boundaries of these moduli spaces is more complicated than in the closed case. First of all, unlike the interior punctures the punctures at the boundary are cyclically ordered, which leads to associative, rather than graded commutative algebras. Second, the ‘‘usual’’ cusp degenerations of curves with boundary at boundary points (see [35]) has in this case codimension 1, rather than 2 as in the closed case, and hence the combinatorics of such degenerations should also be a part of the algebraic formalism.

We will sketch in this paper only the simplest of three cases of SFT, namely the ‘‘classical case’’, which corresponds to the theory of moduli spaces of holomorphic disks with only 1 positive puncture at the boundary.

Let $(V, \xi = \{\alpha = 0\})$ be a contact manifold with a fixed contact form α , $W = V \times \mathbb{R}$ its symplectization with a compatible almost complex structure J , $\Lambda \subset V$ a compact

Legendrian submanifold, and $L = \Lambda \times \mathbb{R} \subset W$ the symplectization of Λ , i.e. the corresponding Lagrangian cylinder in W . We assume that all periodic orbits of the Reeb vector field R_α are non-degenerate and fix a marker on every periodic orbit. We also consider the set \mathcal{C} of Reeb chords connecting points on Λ , and impose an extra non-degeneracy condition along the chords from \mathcal{C} by requiring that the linearized flow of R_α along a chord $c \in \mathcal{C}$ connecting points $a, b \in \Lambda$ sends the tangent space $T_a(\Lambda)$ to a space transversal to $T_b(\Lambda)$. We also require that none of the chords from \mathcal{C} be a part of an orbit from \mathcal{P} . Under these assumptions, the set \mathcal{C} is finite: $\mathcal{C} = \{c_1, \dots, c_m\}$.

We will restrict the consideration to the case when

$$(89) \quad \pi_1(V) = 0, \quad \pi_2(V, \Lambda) = 0, \quad \text{and} \quad w_2(\Lambda) = 0.$$

First two assumptions are technical and can be removed (comp. Section 1.2 above). However, the third one is essential for orientability of the involved moduli spaces of holomorphic curves. Moreover, the invariants we define depends on a particular choice of a spin-structure on Λ .¹⁴

As in Section 1.2 we choose capping surfaces F_γ for $\gamma \in \mathcal{P}$, and for each chord $c \in \mathcal{C}$ we also choose a surface G_c which is bounded by a curve $c \cup \delta_c$, where $\delta_c \subset \Lambda$. The choice of surfaces F_γ , $\gamma \in \mathcal{P}$, allows us to define Conley-Zehnder indices of periodic orbits (see Section 1.2 above). Similarly, surfaces G_c enable us to define *Maslov indices* $\mu(c)$, $c \in \mathcal{C}$. Namely, let us consider a Lagrangian subbundle of $\xi|_{\partial G_c}$, which consists of the Lagrangian sub-bundle $T\Lambda|_{\delta_c} \subset \xi|_{\delta_c}$ over δ_c , together with the family of Lagrangian planes $T_u \subset \xi_u$, $u \in c$, which are images of $T_a(\Lambda)$ under the linearized flow of the Reeb field R_α . Choose a symplectic trivialization of $\xi|_{\partial G_c}$ which extends to G_c . With respect to this trivialization the above sub-bundle can be viewed as a path of Lagrangian planes in a symplectic vector space. The Maslov index of such path is defined as in [56].

Consider a unit disk $D \subset \mathbb{C}$ with punctures

$$\left(\{z^+, z_1^-, \dots, z_\sigma^-\} \cup \{x_1^-, \dots, x_s^-\} \right),$$

where $\mathbf{z} = \{z^+, z_1^-, \dots, z_\sigma^-\}$, $0 \leq \sigma \leq m$, is a counter-clockwise ordered set of punctures on ∂D , and $\mathbf{x} = \{x_1^-, \dots, x_s^-\}$ is an ordered set of interior punctures. As usual we provide interior punctures with asymptotic markers.

Let us denote by $\mathcal{M}^A(\{c_{i_1}, \dots, c_{i_\sigma}\}, \{\gamma_1, \dots, \gamma_s\}, c_i; W, \Lambda, J)$ the moduli space of J -holomorphic maps

$$(D \setminus (\mathbf{z} \cup \mathbf{x}), \partial(D \setminus (\mathbf{z} \cup \mathbf{x}))) \rightarrow (W, L),$$

which are asymptotically cylindrical at the negative end over the periodic orbit γ_k^- at the puncture $x_{i_k}^-$, and over the chord c_{i_k} at the puncture z_k , asymptotically cylindrical at the positive end over the chord c_i at the puncture z^+ , and which send asymptotic markers of interior punctures to the markers on the corresponding periodic orbits.

Two maps are called equivalent if they differ by a conformal map $D \rightarrow D$ which preserves all punctures, marked points and asymptotic markers. Each moduli space

¹⁴We thank K. Fukaya for pointing this out.

$\mathcal{M}^A(c_i, \{c_{i_1}, \dots, c_{i_\sigma}\}, \{\gamma_1, \dots, \gamma_s\}; W, \Lambda, J)$ is invariant under translations $V \times \mathbb{R} \rightarrow V \times \mathbb{R}$ along the factor \mathbb{R} , and we denote the corresponding quotient moduli space by

$$\mathcal{M}^A(c_i, \{c_{i_1}, \dots, c_{i_\sigma}\}, \{\gamma_1, \dots, \gamma_s\}; W, \Lambda, J)/\mathbb{R}.$$

Let $(\mathfrak{A}, \partial) = (\mathfrak{A}(V, \alpha), \partial_J)$ be the graded commutative differential algebra defined in Section 2.2.3 above, or rather its specialization at the point 0. Consider a graded associative algebra $\mathfrak{K} = \mathfrak{K}(V, \Lambda, \alpha)$ generated by elements $c_i \in \mathcal{C}$ with coefficients in the algebra \mathfrak{A} . We define a differential $\partial_\Lambda = \partial_{\Lambda, J}$ on \mathfrak{K} first on the generators c_i by the formula

$$(90) \quad \partial_\Lambda(c_i) = \sum \frac{n_{\Gamma, I, d}}{k! \prod_1^k \kappa_{\gamma_j}^{i_j} i_j!} c_{j_1} \cdots c_{j_\sigma} q_{\gamma_1}^{i_1} \cdots q_{\gamma_k}^{i_k} z^d,$$

where the sum is taken over all $d \in H_2(V)$, all ordered sets of different periodic orbits $\Gamma = \{\gamma_1, \dots, \gamma_k\}$, all multi-indices $J = (j_1, \dots, j_\sigma)$, $1 \leq j_i \leq m$, and $I = (i_1, \dots, i_k)$, $i_j \geq 0$, and where the coefficient $n_{\Gamma, I, d}$ counts the algebraic number of elements of the moduli space

$$\mathcal{M}^d(c_i, \{c_{j_1}, \dots, c_{j_\sigma}\}, \underbrace{\{\gamma_1, \dots, \gamma_1\}}_{i_1}, \dots, \underbrace{\{\gamma_k, \dots, \gamma_k\}}_{i_k})/\mathbb{R},$$

if this space is 0-dimensional, and equals 0 otherwise. The differential extends to the whole algebra \mathfrak{K} by the graded Leibnitz rule. However, it does not treat coefficients as constants: we have $\partial_\Lambda(q_\gamma) = \partial(q_\gamma)$, where ∂ is the differential defined on the algebra \mathfrak{A} .

Then we have

Proposition 2.8.1

$$\partial_\Lambda^2 = 0.$$

Given a family of contact forms $\Lambda_\tau, \alpha_\tau, J_\tau$ $\tau \in [0, 1]$ of Legendrian submanifolds, contact forms, and compatible almost complex structures one can define, similar to the case of closed contact manifolds (see Sections 2.3.2 and Section 2.4 above) a homomorphism of differential algebras

$$\Psi_S : \mathfrak{K}(V, \Lambda_0, \alpha_0) \rightarrow \mathfrak{K}(V, \Lambda_1, \alpha_1),$$

which is independent up to homotopy of the choice of a connecting homotopy. Composition of homotopies generates composition of homomorphisms, and hence one conclude

Proposition 2.8.2 *The quasi-isomorphism type of the differential algebra*

$$(\mathfrak{K}(V, \Lambda, \alpha), \partial_{\Lambda, J})$$

depends only on the contact structure ξ and the Legendrian isotopy class of Λ . The Legendrian contact homology algebra $H_(\mathfrak{K}, \partial_\Lambda)$ has a structure of a module over the contact homology algebra $H_*^{\text{cont}}(V, \xi) = H_*(\mathfrak{A}, d)$, and it is an invariant of the Legendrian knot (or link) Λ .*

The theory looks especially simple when the contact structure ξ on V admits a contact form α such that the Reeb vector field R_α has no closed periodic orbits. If, in addition the space of trajectories is a manifold M (e.g. when $V = J^1(N) = T^*(N) \times \mathbb{R}$ with a contact form $dz + pdq$), then W is automatically symplectic, and the projection $\pi : W \rightarrow V$ sends the Legendrian submanifold $\Lambda \subset V$ to an immersed *Lagrangian* submanifold $L \looparrowright M$ with transverse self-intersection points. These points correspond to Reeb chords c_i connecting points on Λ . Hence, the algebra \mathfrak{K} in this case is just a free associative algebra, generated over \mathbb{C} (or \mathbb{C}) by the self-intersection points of L . It is possible to choose a compatible almost complex structures J on the symplectization $W = V \times \mathbb{R}$ and J_M on M to make the projection $W \rightarrow M$ holomorphic (comp. Section 2.9.2 below). Then punctured holomorphic disks in W from moduli spaces $\mathcal{M}^A(c_i, \{c_{i_1}, \dots, c_{i_\sigma}\}; W, \Lambda, J)$ project to J_M -holomorphic disks in M with boundary in the immersed Lagrangian manifold L . Conversely, one can check that each such disk lifts to a disk from the corresponding moduli space $\mathcal{M}^A(c_i, \{c_{i_1}, \dots, c_{i_\sigma}\}; W, \Lambda, J)$, uniquely, up to translation along the \mathbb{R} -factor in $W = V \times \mathbb{R}$. This is especially useful when $\dim M = 2$. In this case L is an immersed curve, and the holomorphic disks are precisely immersed, or branched disks with their boundaries in L . Moreover, branched disks are never rigid, because the branching point may vary. Hence, the differential $\partial : \mathfrak{K} \rightarrow \mathfrak{K}$ can be defined in this case in a pure combinatorial way, just summing the terms corresponding to all appropriate immersed disks whose boundary consists of arcs of L , and which are locally convex near their corner.

Yu. Chekanov independently realized (see [8]) this program for Legendrian links in the standard contact \mathbb{R}^3 . He was also motivated by the hypothetical description of the compactification of the moduli spaces of holomorphic discs, but has chosen to prove the invariance of the quasi-isomorphism type of the differential algebra (\mathfrak{K}, ∂) in a pure combinatorial way. In fact, he proved a potentially stronger form of equivalence of differential algebras of isotopic Legendrian links, which he called stable tamed isomorphism. Stable tame isomorphism implies quasi-isomorphism, but we do not know whether it is indeed stronger. Let also note that Chekanov considered a \mathbb{Z}_2 -version of the theory. In some examples it works better the \mathbb{Q} -version, which is provided by our formalism. J. Etnyre–J. Sabloff ([16]) and L. Ng ([55]) worked out the combinatorial meaning of signs dictated by the coherent orientation theory (see Section 1.8 above), and proved invariance of the stable tame type of the differential algebra (\mathfrak{K}, ∂) defined over \mathbb{Z} . Note that Chekanov’s paper [8] also contains examples which show that the stable tame \mathbb{Z}_2 -isomorphism type do distinguish some Legendrian knots, which could not be previously distinguished.

Similar to the absolute case of SFT, one can define further invariants of Legendrian submanifolds by including in the formalism higher-dimensional moduli spaces. For instance, by introducing marked points on the boundary of the disk one gets a non-commutative deformation of Legendrian contact homology along the homology of Legendrian manifolds. This is useful, in particular, to define invariants of Legendrian links with ordered components. However, the full-scale generalization of Symplectic Field Theory to directed symplectic-Lagrangian cobordisms between pairs of contact manifolds and their Legendrian submanifolds, which would formalize information about moduli spaces of holomorphic curves of arbitrary genus and arbitrary number of positive and negative punctures, is not straightforward due to existence

of different type of codimension 1 components on the boundary of the corresponding moduli spaces. We will discuss this theory in one of our future papers.

2.9 Remarks, examples, and further algebraic constructions in SFT

2.9.1 Dealing with torsion elements in H_1

Let us discuss grading issues for a contact manifold $(V, \xi = \{\alpha = 0\})$ in the case when the torsion part of $H_1(V)$ is non-trivial. As we will see it is impossible to assign in a coherent way an integer grading to torsion elements and still keep the property that the Hamiltonian \mathbf{H} has total grading -1 . We will deal with this problem by assigning to some elements fractional degrees, and thus obtain a rational grading, incompatible with the canonical \mathbb{Z}_2 -grading. In fact the term “grading” is misleading in this case, and more appropriately one should talk about an Euler vector field with rational coefficients.

Let us split $H_1(V)$ as $T \oplus F$, where T and F are the torsion and free parts, respectively. As in Section 1.2 above let us choose curves C_1, \dots, C_k representing a basis of F , fix a trivialization of the bundles ξ_{C_i} , for any periodic orbit $\gamma \in \mathcal{P}_\alpha$ with $[\gamma] \in F$ choose a surface F_γ which realizes the homology between $[\gamma]$ and a linear combinations $\sum n_i[C_i]$, and trivialize the bundle $\xi|_{F_\gamma}$ accordingly. For any other periodic orbit γ let γ_l be its smallest multiple which belong to F . In particular, the bundle ξ_{γ_l} is already trivialized by a framing f . The problem is that in general there is no framing over γ which would produce f over γ_l . Choose then an arbitrary framing g over γ and denote by g_l the resulting framing over γ_l . Let $2m(g_l, f) \in \pi_1(Sp(2n-2, \mathbb{R})) = \mathbb{Z}$ be the Maslov class of the framing g_l with respect to f . The Conley-Zehnder indices of γ_l with respect to these two gradings are then related by the formula

$$\text{CZ}(\gamma|f) = \text{CZ}(\gamma|g_l) + 2m(g_l, f).$$

We then assign to γ the fractional degree

$$(91) \quad \text{deg } \gamma = \text{CZ}(\gamma|g) - \frac{2m(g_l, f)}{l}.$$

With this modification SFT can be extended to the case of contact manifolds with no restrictions on H_1 . However, the price we pay is that this grading, even if integer, may not be compatible with the universal \mathbb{Z}_2 -grading which determines the sign rules.

2.9.2 Morse-Bott formalism

Our assumption that all periodic orbits from \mathcal{P}_α for the considered contact forms α are non-degenerate, though generic, but is very inconvenient for computations: in many interesting examples periodic orbits come in continuous families. Sometimes the Reeb flow is periodic, and it sounds quite stupid to destroy this beautiful symmetry.

In fact the above formalism can be adapted to this “Morse-Bott” case. We sketch below how this could be done for the periodic Reeb flow of an S^1 -invariant form of

a pre-quantization space. We consider below only the “semi-classical” case which concerns moduli spaces of rational holomorphic curves.

Let (M, ω) be a symplectic manifold of dimension $2n - 2$ with an integral cohomology class $[\omega] \in H^2(M)$. We will assume for simplicity that $H_1(M) = 0$. The pre-quantization space V is a circle bundle over M with first Chern class equal to $[\omega]$. The fibration $\pi : V \rightarrow M$ admits a S^1 -connection form α whose curvature is ω . It defines a S^1 -invariant contact structure ξ on V , transversal to the fibers of the fibration. The Reeb flow of R_α is periodic, so all its trajectories are closed and coincide with the fibers of the fibration π , or their multiples.

The fiber of the fibration V is a torsion element in $H_1(V)$, and if l is the greatest divisor of the class $[\omega]$ then the l -multiple of the fiber is homological to 0.

Consider the cylindrical cobordism (the symplectization) $W = V \times \mathbb{R}$ with an almost complex structure J compatible with α and denote by $\mathcal{M}_{0,r}(s|W, J, \alpha)$ the moduli space of rational holomorphic curves in W with s punctures and r marked points. Near punctures the curves are required to be asymptotically cylindrical over some fibers of V , or their multiples. However, we do not specify to which particular fiber they are being asymptotic, or whether this fiber is considered on the positive, or negative end of W . We do not equip curves from $\mathcal{M}_{0,r}(s|W, J, \alpha)$ with asymptotic markers of punctures, because we cannot fix in a continuous way points on each simple periodic orbit, as we did in the non-degenerate case.

As it was already mentioned in Section 1.4 above, W can be viewed as the total space of the complex line bundle L associated with the S^1 -fibration $V \rightarrow M$, with the zero-section removed, and the almost complex structure J can be chosen compatible with the structure of this bundle, so that the projection $W \rightarrow M$ becomes holomorphic with respect to a certain almost complex structure J_M on M compatible with ω . Then automatically the bundle induced over any holomorphic curve in the base has a structure of a holomorphic line bundle. With this choice of J each holomorphic curve $f \in \mathcal{M}_{0,r}(s|W, J, \alpha)$ projects to a J_M -holomorphic sphere $\bar{f} : \mathbb{C}P^1 \rightarrow M$, and can be viewed as a meromorphic section of the induced bundle $(\bar{f})^*L$ over $\mathbb{C}P^1$. This bundle is ample, and hence poles of its sections correspond to the *negative* ends of f , while zeroes correspond to the positive ones. Notice that although the moduli spaces $\mathcal{M}_{0,r}(s|W, J, \alpha)$ can be identified with the moduli spaces of closed holomorphic curves in a $\mathbb{C}P^1$ -bundle over M with prescribed tangencies to two divisors, their compactifications are different, and in particular the compactification of the first moduli space may have codimension one strata on its boundary.

The correspondence $f \mapsto \bar{f}$ define a fibration

$$\text{pr} : \mathcal{M}_{0,r}(s|W, J, \alpha)/\mathbb{R} \rightarrow \mathcal{M}_{0,r+s}(M, J_M).$$

The fiber $\text{pr}^{-1}(\bar{f})$ is the union of (an infinite number of) disjoint circles, which are indexed by sequences of integers (k_1, \dots, k_{s+r}) with $\sum k_i = d_0 = \int_A \omega$, where $A \in H_2(M)$ is the homology class realized by \bar{f} , and where there are precisely s non-zero coefficients k_i .

Let us consider two copies \mathcal{P}^\pm of the space $\mathcal{P} = \mathcal{P}_\alpha$ of periodic orbits, as we need to differentiate between positive and negative ends of holomorphic curves. We will

write $\ddot{\mathcal{P}} = \mathcal{P}^+ \cup \mathcal{P}^-$ and define the evaluation maps:

$$(92) \quad ev_0 : \mathcal{M}_{0,r}(s|W, J, \alpha)/\mathbb{R} \rightarrow V^{\times r} \quad \text{and} \quad ev_{\pm}^+ : \mathcal{M}_{0,r}(s|W, J, \alpha)/\mathbb{R} \rightarrow \ddot{\mathcal{P}}^{\times s}.$$

Here ev_{\pm}^+ associates with each puncture the corresponding point of $\ddot{\mathcal{P}}$. The space \mathcal{P}^{\pm} can be presented as $\coprod_{k=1}^{\infty} \mathcal{P}_k^{\pm}$, where each \mathcal{P}_k^{\pm} is a copy of M , associated with k -multiple orbits.

We will denote forms on \mathcal{P}^+ by p , on \mathcal{P}^- by q , denote by p_k, q_k their restrictions to \mathcal{P}_k^{\pm} , and organize them into Fourier series $u = \sum_{k=1}^{\infty} (p_k e^{ikx} + q_k e^{-ikx})$. If we are given a basis of $H^*(M)$ represented by forms $\Delta_1, \dots, \Delta_a$ we will consider only forms from the space generated by this basis, and write $p_k = \sum_{i=1}^a p_{k,i} \Delta_i$, $q_k = \sum_{i=1}^a q_{k,i} \Delta_i$ and denote by u_i the Δ_i -component of u , i.e.

$$u_i = \sum_{k=1}^{\infty} (p_{k,i} e^{ikx} + q_{k,i} e^{-ikx}) \quad \text{and} \quad u = \sum_1^a u_i \Delta_i.$$

Given a closed form t on V and a class $A \in H_2(V)$ we define the correlator

$$(93) \quad \frac{\int_{\mathcal{M}_{0,r}^A(s|W, J, \alpha)/\mathbb{R}} ev_0^*(t \otimes \dots \otimes t) \wedge (ev_{\pm}^+)^*(u \otimes \dots \otimes u) \Big|_{x=0}}{\int_{\mathcal{M}_{0,r}^A(s|W, J, \alpha)/\mathbb{R}} ev_0^*(t \otimes \dots \otimes t) \wedge (ev_{\pm}^+)^*(u \otimes \dots \otimes u) \Big|_{x=0}} =$$

Let us choose a basis A_0, \dots, A_N in $H_2(M)$ in such a way that $\int \omega = l > 0$ and $\int \omega = 0$ for $i = 1, \dots, N$. Then the classes $A_i, i \geq 1$, lift to classes $\tilde{A}_i \in H_2(V)$ which under the assumption $H_1(M) = 0$ form a basis of $H_2(V)$. The degree $d = (d_1, \dots, d_N)$ of a class $A \in H_2(V)$ is a vector of its coordinates in this basis.

To associate an absolute homology class with a holomorphic curve we pick the l -multiple (recall that l denotes the greatest divisor of ω) of the fiber γ over a point $x \in M$ and choose a lift of the surface representing the class A_0 with $\int \omega = l$ as a spanning surface F_{γ} . Any other m -multiple of γ we will cap with the chain $\frac{m}{l}[F_{\gamma}]$. However, to fix a spanning surface for a fiber over any other point $y \in M$ or its multiples, one needs to make some extra choices, for instance fix a path connecting x and y . The condition $H_1(M) = 0$ guarantees independence of the homology class of the resulting surface of the choice of this connecting path. Notice that with this choice, the degree of $f \in \mathcal{M}_{0,r}(s|W, J, \alpha)/\mathbb{R}$ equals (d_1, \dots, d_N) , if the degree of its projection $\text{pr}(f) \in \mathcal{M}_{0,r+s}(M, J_m)$ is equal to (d_0, d_1, \dots, d_N) .

In this notation the rational Hamiltonian $\mathbf{h} = \mathbf{h}_{V, J, \alpha}$ is defined by the formula

$$(94) \quad \mathbf{h}(t, u) = \sum_d \sum_{r, s=0}^{\infty} \frac{1}{r!s!} {}^{-1} \langle \underbrace{t, \dots, t}_r; \underbrace{u, \dots, u}_s \rangle_0^d z^d.$$

Suppose that a basis of $H^*(M)$, represented by closed forms $\Delta_1, \dots, \Delta_a$, is chosen in such a way that for $b \leq a$ the system forms $\tilde{\Delta}_j = \pi^*(\Delta_j)$, $j = 1, \dots, b$, generate the image $\pi^*(H^*(M)) \subset H^*(V)$, and the forms $\tilde{\Theta}_1, \dots, \tilde{\Theta}_c$, complete it to a basis of $H^*(V)$. We will denote (graded) coordinates in the space generated by the forms $\tilde{\Delta}_j$, $j = 1, \dots, b$ and $\tilde{\Theta}_1, \dots, \tilde{\Theta}_c$ by $(t, \tau) = (t_1, \dots, t_b, \tau_1, \dots, \tau_c)$.

As usual, the Hamiltonian \mathbf{h} is viewed as an element of a graded commutative Poisson algebra \mathfrak{P} , which consists of formal power series of coordinates of vectors p_k and $T = (t, \tau) = (t_1, \dots, t_b, \tau_1, \dots, \tau_c)$ with coefficients which are polynomials of coordinates of vectors $q_k = (q_{k,1}, \dots, q_{k,a})$. The coefficients of these polynomials belong to a certain completion (see condition (38) above) of the group algebra of $H_2(V)$. All the variables $p_{k,i}, q_{k,i}$ have in this case the same parity as forms Δ_i , the parity of variables t_i and τ_j is the same as the degree of the corresponding forms $\tilde{\Delta}_i$ and $\tilde{\Theta}_j$. If $l = 1$, i.e. when $H_1(V) = 0$, then the integer grading of variables which corresponds to the choice of capping surfaces described above is defined as follows:

$$(95) \quad \begin{aligned} \deg t_i &= \deg \tilde{\Delta}_i - 2; \\ \deg \tau_i &= \deg \tilde{\Theta}_i - 2; \\ \deg q_{k,i} &= \deg \Delta_i - 2 + 2ck; \\ \deg p_{k,i} &= \deg \Delta_i - 2 - 2ck; \\ \deg z_i &= 2c_1(A_i) \end{aligned}$$

where $c = c_1(A_0)$. As it was explained in Section 2.9.1 if $l > 1$ one can only define fractional degrees, given by the above formulas (95) with $c = \frac{c_1(A_0)}{l}$.

The following proposition is useful for applications (see below the discussion of Biran-Cieliebak conjecture about subcritical symplectic manifolds). It follows from the fact that all the moduli spaces $\mathcal{M}_{g,r}(s|W, J, \alpha)$ which we defined above are even-dimensional.

Proposition 2.9.1 *Let (V, ξ) be the contact pre-quantization space for a symplectic manifold (M, ω) . Then all contact homology algebras*

$$H_*^{\text{SFT}}(V, \xi)|_{t=0}, H_*^{\text{RSFT}}(V, \xi)|_{t=0}, H_*^{\text{cont}}(V, \xi)|_{t=0}$$

specialized at $0 \in H^(V)$ are free graded, respectively Weyl, Poisson, or commutative algebras, generated by elements*

$$p_{k,i}, q_{k,i}, \quad i = 1, \dots, a, \quad k = 1, \dots.$$

In particular, the parts of all these homology algebras graded by the homology class $w \in H_1(V)$ (see 2.2.9 above) are non-trivial.

The Poisson tensor on \mathfrak{P} is determined in the “ u -notation” by the following generalization of the formula (28):

$$(96) \quad \frac{1}{2\pi i} \int_0^{2\pi} \langle (\delta u)', \delta v \rangle dx,$$

where $\langle \cdot, \cdot \rangle$ denotes Poincaré pairing on cohomology $H^*(M)$, which is given in the basis $\Delta_1, \dots, \Delta_a$ by the matrix

$$\eta_{ij} = \langle \Delta_i, \Delta_j \rangle = \int_M \Delta_i \wedge \Delta_j.$$

The Poisson tensor can be written in (p, q) -coordinates as

$$\sum_{k=1}^{\infty} k \sum_{i,j=1}^a \eta_{ij} \frac{\partial}{\partial p_{k,i}} \wedge \frac{\partial}{\partial q_{k,j}}.$$

It can be shown that the above Hamiltonian \mathbf{h} satisfies the identity $\{\mathbf{h}, \mathbf{h}\} = 0$, and that the differential Poisson algebra $(\mathfrak{P}, d^{\mathbf{h}})$ is quasi-isomorphic to the corresponding differential Poisson algebra defined in Section 2.2.3 for any non-degenerate contact form for the same contact structure ξ on V .

The following formula (97), which sometimes allows to compute the Hamiltonian \mathbf{h} of V in terms of the Gromov-Witten invariant $\mathbf{f} = \mathbf{f}_{M, J_M}$ of M , emerged in a discussion of the authors with T. Coates and F. Bourgeois.

Proposition 2.9.2 *Set*

$$\mathbf{h}_{W,J}^j(t, q, p, z) = \frac{\partial \mathbf{h}}{\partial \tau_j} \left(\sum_1^b t_i \tilde{\Delta}_i + \tau_j \tilde{\Theta}_j, q, p, z \right) \Big|_{\tau_j=0},$$

and

$$\widehat{\mathbf{f}}^j(t, z) = \frac{\partial \mathbf{f}}{\partial s} \left(\sum_1^a t_i \Delta_i + s \pi_* \tilde{\Theta}_j, z \right) \Big|_{s=0},$$

for $j = 1, \dots, c$ (comp. Theorem 2.7.2). Then we have

$$(97) \quad \begin{aligned} & \mathbf{h}_{W,J}^j(t_1, \dots, t_b, q, p, z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \widehat{\mathbf{f}}^j(t_1 + u_1(x), \dots, t_b + u_b(x), u_{b+1}(x), \dots, u_a(x), \tilde{z}) dx \end{aligned}$$

where $z = (z_1, \dots, z_N)$, $\tilde{z} = (e^{-ilx}, z_1, \dots, z_N)$ and l is the greatest divisor of ω .

To prove (97) one just observes that the correlator

$$^{-1} \langle \tilde{\Delta}_{j_1}, \dots, \tilde{\Delta}_{j_r}, \tilde{\Theta}_j; u_{i_1} \Delta_{i_1}, \dots, u_{i_v} \Delta_{i_v} \rangle_0^d$$

equals the Fourier coefficient with e^{ilx} of the correlator

$$^0 \langle \Delta_{j_1}, \dots, \Delta_{j_r}, \pi_* \tilde{\Theta}_j, u_{i_1} \Delta_{i_1}, \dots, u_{i_v} \Delta_{i_v} \rangle_0^{\tilde{d}}.$$

Notice that if $\tilde{\Theta}_j$ is an odd form, then

$$\mathbf{h}\left(\sum_1^b t_i \tilde{\Delta}_i + \tau_j \tilde{\Theta}_j, q, p, \tilde{z}\right) = \tau_j \mathbf{h}^j(t, q, p, z),$$

because all terms of \mathbf{h} must contain τ_j . In particular, for $M = \mathbb{C}P^{n-1}$ the manifold V is a rational homology sphere, and thus a volume form Θ on S^{2n-1} generates the odd part of $H^*(V; \mathbb{R})$. Hence, the formula (97) completely determines \mathbf{h} . Namely, let $\mathbf{f}(t, z)$ be the Gromov-Witten invariant of $\mathbb{C}P^{n-1}$, and let $\Delta_{2i}, i = 0, \dots, n-1$, be closed forms generating $H^*(\mathbb{C}P^{n-1})$, so that $\Delta_{2n-2} = \pi_*(\tilde{\Theta})$. Set $\tilde{\Delta}_0 = \pi^*(\Delta_0)$ and

$$\hat{\mathbf{f}}(t, z) = \widehat{\mathbf{f}}^{2n-2}(t, z) = \frac{\partial \mathbf{f}(t, z)}{\partial t_{2n-2}}.$$

Then we have

$$(98) \quad \mathbf{h}(t_0 \tilde{\Delta}_0 + \tau \Theta, q, p) = \frac{\tau}{2\pi} \int_0^{2\pi} \hat{\mathbf{f}}(t_0 \Delta_0 + u, e^{-ix}) dx.$$

Let us consider some applications of the formula (98).

Contact homology of the standard contact 3-sphere

Let $\pi : V = S^3 \rightarrow M = \mathbb{C}P^1$ be the Hopf fibration. V is the pre-quantization space for (S^2, ω) with $\int_{S^2} \omega = 1$. The 0-form $\Delta_0 = 1$ and the symplectic 2-form $\Delta_2 = \omega$ generate $H^*(M)$, the 0-form $\tilde{\Delta}_0 = \pi^*(\Delta_0)$ and the volume form $\tilde{\Theta}_3$ with $\pi_*(\tilde{\Theta}_3) = \Delta_2$ on S^3 generate $H^*(S^3)$. Thus we have functional variables

$$u_j(w) = \sum_{k=1}^{\infty} (p_{k,j} e^{ikx} + q_{k,j} e^{-ikx}),$$

associated to the classes $\Delta_j, j = 0, 2$, and variables t_0 and τ associated to $\tilde{\Delta}_0$ and $\tilde{\Theta}_3$. According to (95) we have

$$\begin{aligned} \deg q_{k,0} &= -2 + 4k, & \deg q_{k,2} &= 4k, & \deg p_{k,0} &= -2 - 4k, \\ \deg p_{k,2} &= -4k, & \deg t_0 &= -2, & \deg \tau &= 1. \end{aligned}$$

The potential \mathbf{f} for $M = \mathbb{C}P^1$ can be written, as it well known (see also Section 2.9.3 below), as

$$(99) \quad \mathbf{f} = \frac{t_0^2 t_2}{2} + e^{t_2} z,$$

and hence

$$\hat{\mathbf{f}} = \frac{t_0^2}{2} + e^{t_2} z,$$

Thus applying (97) we get the following expression for the rational Hamiltonian \mathbf{h} for S^3 :

$$(100) \quad \mathbf{h} = \frac{\tau}{2\pi i} \int_0^{2\pi} \left(\frac{(t_0 + u_0)^2}{2} + e^{u_2 - ix} \right) dx = \tau \left(\frac{t_0^2}{2} + \sum_{k \geq 1} q_{k,0} p_{k,0} \right. \\ \left. + \sum_{t,s \geq 0} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ j_1, \dots, j_t \geq 0 \\ \sum_1^s l_i - \sum_1^t m_j = 1}} \frac{q_{1,2}^{i_1} \cdots q_{s,2}^{i_s} p_{1,2}^{j_1} \cdots p_{t,2}^{j_t}}{i_1! \cdots i_s! j_1! \cdots j_t!} \right).$$

Let us use (100) to compute the contact homology algebra

$$H_*^{\text{cont}}(S^3, \xi_0) = H_*(\mathfrak{A}(S^3, J, \alpha), \partial).$$

The part of \mathbf{h} linear in the p -variables has the form

$$\tau \sum_1^\infty (p_{k,0} q_{k,0} + p_{k,2} h_k(q_{1,2}, \dots, q_{k-1,2})),$$

so that the differential $\partial : \mathfrak{A} \rightarrow \mathfrak{A}$ is given by the formulas

$$\partial q_{k,2} = k\tau q_{k,0}, \quad \partial q_{k,0} = k\tau h_k(q_{1,2}, \dots, q_{k-1,2}).$$

Here are few first polynomials h_k :

$$\begin{aligned} h_1 &= 1, \\ h_2 &= q_{1,2}, \\ h_3 &= q_{2,2} + \frac{1}{2} q_{1,2}^2, \\ h_4 &= q_{3,2} + q_{2,2} q_{1,2} + \frac{1}{6} q_{1,2}^3. \end{aligned}$$

Notice that $\text{Im} \partial$ coincides with the ideal $I(\tau)$ generated by τ . Hence, the homology algebra $H_*(\mathfrak{A}, \partial)$ specialized over a point $t = (t_0, 0)$ is a free graded commutative algebra \mathfrak{A}_0 generated by variables $q_{k,0}, q_{k,2}$, $k = 1, \dots$, and in particular it has one generator in each even dimension. On the other hand, over any point $t = (t_0, \tau)$ with $\tau \neq 0$ the algebra $H_*(\mathfrak{A}, \partial)$ is isomorphic to a proper subalgebra \mathfrak{A}_1 of \mathfrak{A}_0 . It has its first non-trivial generator $g_1 = q_{1,2} - \frac{1}{2} q_{1,0}^2$ in dimension 4.

Remark 2.9.3 The contact homology of the Lens space $V = L(l, 1)$ which is the pre-quantization space for (S^2, ω) with $\int_{S^2} \omega = l$ can be computed similarly. The variables $p_{k,0}, q_{k,0}, p_{k,2}, q_{k,2}, t_0$ and τ have the same \mathbb{Z}_2 -grading, as in the case of S_3 ,

i.e. all of them, except τ are even. However, the grading assigned by the Euler field to $p_{k,0}, q_{k,0}, p_{k,2}, q_{k,2}$ is fractional in this case and given by formulas

$$\begin{aligned} \deg q_{k,0} &= -2 + \frac{4k}{l}, & \deg q_{k,2} &= \frac{4k}{l}, & \deg p_{k,0} &= -2 - \frac{4k}{l}, \\ \deg q_{k,0} &= -\frac{4k}{l}. \end{aligned}$$

The formula for \mathbf{h} takes the form

$$(101) \quad \mathbf{h} = \frac{\tau}{2\pi i} \int_0^{2\pi} \left(\frac{(t_0 + u_0)^2}{2} + e^{u_2 - ilx} \right) dx$$

We will not carry on here the computation of the contact homology of the Lens space $L(l, 1)$, and only note, that as in the case of S^3 the homology algebra $H_*(\mathfrak{A}, \partial)$ specialized over a point $t = 0$ is a free graded commutative algebra \mathfrak{A}_0 generated by variables $q_{k,0}, q_{k,2}$, $k = 1, \dots$, and over any point $t \neq 0$ the algebra $H_*(\mathfrak{A}, \partial)$ is isomorphic to a proper subalgebra \mathfrak{A}_1 of \mathfrak{A}_0 . In particular, *over any point the contact homology algebra $H_*(\mathfrak{A}, \partial)$ has no odd elements.*

Distinguishing contact structures on pre-quantizations spaces

Formula (97) can be used for distinguishing contact structures on pre-quantization spaces of certain symplectic manifolds, which have different Gromov-Witten invariants. Here is an example.

Proposition 2.9.4 *Let (M_1, ω_1) and (M_2, ω_2, J_2) be two symplectic 4-manifolds with integral cohomology classes of their symplectic forms. Suppose that for compatible almost complex structures J_1 on M_1 and J_2 on M_2 there are no non-constant J_1 -holomorphic spheres in M_1 , while M_2 contains an embedded J_2 -holomorphic (-1) -sphere S . Then the pre-quantization spaces (V_1, ξ_1) and (V_2, ξ_2) are not contactomorphic.¹⁵*

Remark 2.9.5 Even when the manifolds M_1 and M_2 are homeomorphic, the pre-quantization spaces V_1 and V_2 are not diffeomorphic (even not homotopy equivalent!) for most choices of symplectic forms ω_1 and ω_2 , and hence the statement of the theorem is trivial in these cases. However, one can show that for homeomorphic M_1 and M_2 the symplectic forms can always be deformed in the class of symplectic forms compatible with the chosen almost complex structures J_1 and J_2 , in order to make V_1 and V_2 diffeomorphic.

To prove Proposition 2.9.4 we will show that the ‘‘classical’’ contact homology algebras $H_*^{\text{cont}}(V_1, \xi_1)$ and $H_*^{\text{cont}}(V_2, \xi_2)$ are not isomorphic.

¹⁵ Yongbin Ruan proved in [59] that under the assumptions of Proposition 2.9.4 the symplectic manifolds $(M_1, \omega_1) \times \mathbb{C}P^1$ and $(M_2, \omega_2) \times \mathbb{C}P^1$ are not symplectomorphic (and not even deformationally equivalent), despite the fact that for a certain choice of M_1 and M_2 (e.g. M_1 is the Barlow surface and $M_2 = \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$), and for appropriate symplectic forms ω_1 and ω_2 the underlying manifolds V_1 and V_2 are diffeomorphic.

Let $S_0 = S, S_1, \dots, S_m$ be the exceptional J_2 -holomorphic spheres in M_2 . Then the cohomology classes $S_0^*, \dots, S_m^*, [\omega]$ are linearly independent, where we denote by S_i^* the cohomology class Poincaré-dual to $[S_i] \in H_2(M_2)$. Hence there exists a class $X \in H^2(M_2)$, such that

$$(102) \quad XS_0^* = 1, \quad X[\omega] = 0, \quad \text{and} \quad XS_i^* = 0 \quad \text{for} \quad 1 \leq i \leq m.$$

Then the potential $\mathbf{f}_{M_2, J_2}(tX)$ coincides with

$$\mathbf{f}_{S, J_2|_S}(tX|_S) = e^t z.$$

Let us choose a basis of closed forms $\Delta_i, i = 0, \dots, a$, generating $H^*(M_2)$, so that one of the forms, say Δ_1 represents the class X . The form Δ_1 then lifts to a form $\tilde{\Delta}_1$ such that $\pi_* \tilde{\Delta}_1 = \Delta_1$. According to the formula (98) we have

$$(103) \quad \mathbf{h}_V(\tau_1 \tilde{\Delta}_1, u) = \frac{\tau_1}{2\pi} \int_0^{2\pi} e^{u_1 - ilx} dx,$$

where $u = \sum_0^a u_j \Delta_j$, $u_j = \sum_1^\infty (p_{k,j} e^{ikx} + q_{k,j} e^{-ikx})$, $l = \int_S \omega$.

Hence, the contact homology algebra $H_*^{\text{cont}}(M_2, \xi_2)$, specialized at the point $\tau \tilde{\Delta}$ for $\tau \neq 0$ is isomorphic to the contact homology algebra of the standard contact Lens space $L(l, 1) = \pi^{-1}(S) \subset V$, specialized at the volume form $\tau \tilde{\Delta}|_{L(l,1)}$. It follows then from Remark 2.9.3 that $H_*^{\text{cont}}(V_2, \xi_2)$, specialized at any point has no odd elements. On the other hand, for any 2-dimensional cohomology class $t \in H_2(M_1)$ we have $\mathbf{f}_{M_1, J_1}(t) = 0$, and hence for any 3-form $\tilde{\Delta}$ on V_1 , the formula (98) implies that the Hamiltonian $\mathbf{h}_{V_1}(\tilde{\Delta}, u)$ vanishes as well, and therefore the contact homology algebra $H_*^{\text{cont}}(M_1, \xi_1)$, specialized at the point $\tau \tilde{\Delta}, \tau \neq 0$ is a free graded commutative algebra generated by the odd variable τ , and even variables $p'_{k,j}, q'_{k,j}$, which correspond to even dimensional generators of $H^*(M_1)$. Hence the contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are not isomorphic. \blacktriangleright

Subcritical symplectic manifolds

The content of this example is a result of our discussion with P. Biran and K. Cieliebak.

In [11] S. Donaldson generalized the Kodaira embedding theorem by proving that for any closed symplectic manifold (W, ω) with an integral cohomology class of the symplectic form there exists an integer $l > 0$ such that the homology class dual to $l[\omega]$ can be represented by an embedded symplectic hypersurface W_0 . In fact, S. Donaldson proved a stronger result, which together with an improvement by Biran-Cieliebak asserts that for a sufficiently large l the hypersurface W_0 can be chosen in such a way that in the complement $W \setminus W_0$ there exists a vector field X with the following properties:

$L_X \omega = -\omega$, where L_X denotes the Lie derivative along X ; in other words, X is conformally symplectic and contracting;

X is forward integrable;

X is gradient-like for a Morse function $\phi : W \setminus W_0 \rightarrow \mathbb{R}_+$, which coincides with $-\log \text{dist}^2$ near W , where $\text{dist}(x)$ is the distance function from a point x to W_0 with respect to some Riemannian metric.

The vector field X retracts $W \setminus W_0$ to the Morse complex K of the function ϕ , which is automatically isotropic for the symplectic form ω (see [14]), and, in particular, $\dim K \leq n$. Biran-Cieliebak call the pair (W, W_0) *subcritical* if $\dim K < n$. They constructed in [6] several interesting examples of subcritical pairs, and conjectured that *if (W, W_0) is subcritical, then $l = 1$* . We sketch below the proof of this conjecture.

First, let us observe that the contact structure ξ , defined by the contact form $\alpha = X \lrcorner \omega$ on the boundary V of a small tubular neighborhood of W_0 , is equivalent to the contact structure which is defined on V as the pre-quantization space of the symplectic manifold (W_0, ω) . On the other hand, the condition, that the pair (W, W_0) is sub-critical implies that the contact manifold (V, ξ) is itself subcritical in the sense of Example 1.9.3.4 above, i.e. it is isomorphic to the strictly pseudo-convex boundary of a sub-critical Stein (or Weinstein) manifold with its canonical complex structure. Let us recall (see 2.2.9 above) that all SFT-objects, in particular Floer contact homology $HC_*(V, \xi)$ and the contact homology algebra $H_*^{\text{cont}}(V, \xi)$ are graded by elements of $H_1(V)$. Using arguments as in the theorem of Mei-Lin Yau (see 1.9.8 above) one can show that all non-trivial elements in the contact homology algebra $H_*^{\text{cont}}(V, \xi)$ of a subcritical contact manifold (V, ξ) may correspond only to $0 \in H_1(V)$. On the other hand, it follows from Proposition 2.9.1 above that $H_*^{\text{cont}}(V, \xi)$ specialized at $0 \in H^*(V)$ has non-trivial elements which correspond to the homology class of the fiber in $H_1(V)$. Therefore, $l = 1$.

2.9.3 Computing rational Gromov-Witten invariants of $\mathbb{C}P^n$

We will show in this section how SFT can be used for computing rational Gromov-Witten invariants of $\mathbb{C}P^n$. Our method differs from traditional ways (see [46, 23, 33, 64, 60]) for this computation. We will be simultaneously computing the rational potential of \mathbb{C}^n and the rational Gromov-Witten invariant of $\mathbb{C}P^n$ by a recursion using Theorem 2.7.2

Let us choose basic forms in \mathbb{C}^n as in the previous section, i.e $\Delta = 1$, and Θ is a volume form with compact support in $\mathbb{C}^n \setminus 0 = S^{2n-1} \times (0, \infty)$ with $\int_{\mathbb{C}^n} \Theta = 1$.

We denote by δ the restriction of Δ to S^{2n-1} . We also assume that Θ splits into a product $\hat{\theta} \wedge \rho$, where $\hat{\theta}$ is pull-back of a unit volume form θ on S^{2n-1} , and ρ is a compactly supported form in $(0, \infty)$. Set

$$\begin{aligned}
 \mathbf{h}^1(t_0, q, p) &= \frac{\partial \mathbf{h}}{\partial \tau}(t_0 \delta + \tau \theta, q, p)|_{\tau=0} \\
 (104) \qquad &= \frac{1}{2\pi} \int_0^{2\pi} \widehat{\mathbf{f}}_{\mathbb{C}P^{n-1}}(t_0 + u_0(x), u_2, \dots, u_{2n-2}, e^{-ix}) dx,
 \end{aligned}$$

where

$$\widehat{\mathbf{f}}_{\mathbb{C}P^{n-1}}(t_0, \dots, t_{2n-2}, z) = \frac{\partial \mathbf{f}_{\mathbb{C}P^{n-1}}}{\partial t_{2n-2}}(t_0, \dots, t_{2n-2}, z),$$

$\mathbf{f}_{\mathbb{C}P^{n-1}}(t_0, \dots, t_{2n-2}, z)$ is the rational Gromov-Witten invariant of $\mathbb{C}P^{n-1}$, and

$$u_{2j}(x) = \sum_1^{\infty} p_{k,2j} e^{ikx} + q_{k,2j} e^{-ikx}.$$

Then the equation (88), which determines $\mathbf{f}(t_0, t_{2n}, p) = \mathbf{f}_{\mathbb{C}^n}(t_0\Delta + t_{2n}\Theta, p)$ takes the form

$$(105) \quad \frac{\partial \mathbf{f}}{\partial t_{2n}}(t_0, t_{2n}, p) = \frac{1}{2\pi} \int_0^{2\pi} \widehat{\mathbf{f}}_{\mathbb{C}P^{n-1}}(t_0 + u_0(x), u_2, \dots, u_{2n-2}, e^{-ix}) dx \Big|_{L_{\mathbf{f}}},$$

where

$$L_{\mathbf{f}} = \left\{ q_{k,2j} = k \frac{\partial \mathbf{f}}{\partial p_{k,2n-2j-2}}(t_0, t_{2n}, p) \right\}.$$

Together with the initial data

$$(106) \quad \mathbf{f}(t_0, 0, p) = \begin{cases} p_{1,0}, & \text{if } n = 1; \\ 0, & \text{otherwise} \end{cases}$$

the equation (105) provides a recursive procedure for computing coefficients $f_j(t_0, p)$ of the expansion

$$\mathbf{f}(t_0, t_{2n}, p) = \sum_0^{\infty} f_j(t_0, p) t_{2n}^j.$$

For instance for $n = 1$ we have (see Example 2.2.4) $\mathbf{h}^1 = \frac{t_0^2}{2} + \sum_1^{\infty} p_k q_k$, where we write p_k, q_k instead of $p_{k,0}, q_{k,0}$, and hence the equation (105) takes the form

$$(107) \quad \frac{\partial \mathbf{f}}{\partial t_2}(t_0, t_2, p) = \frac{t_0^2}{2} + \sum_0^{\infty} k p_k \frac{\partial \mathbf{f}}{\partial p_k}(t_0, t_2, p)$$

with the initial data $\mathbf{f}(t_0, 0, p) = p_1$. This linear first order PDE is straightforward to solve, and we get

$$\mathbf{f}(t_0, t_2, p) = \frac{t_2 t_0^2}{2} + e^{t_2} p_1.$$

For $n = 2$ the Hamiltonian \mathbf{h} is given by the formula (100), and we have

$$\mathbf{h}(t_0, \tau, p) = \tau \mathbf{h}^1(t_0, p).$$

Thus the equation for the potential of \mathbb{C}^2 has the form

$$(108) \quad \begin{aligned} \frac{\partial \mathbf{f}}{\partial t_4}(t_0, t_4, p) &= \frac{t_0^2}{2} + \sum_{k \geq 1} k \frac{\partial \mathbf{f}}{\partial p_{k,2}} p_{k,0} \\ &+ \sum_{t,s \geq 0} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ j_1, \dots, j_t \geq 0 \\ \sum_1^s l_i - \sum_1^t m_j = 1}} \frac{\left(\frac{\partial \mathbf{f}}{\partial p_{1,0}}\right)^{i_1} \dots \left(s \frac{\partial \mathbf{f}}{\partial p_{s,0}}\right)^{i_s} p_{1,2}^{j_1} \dots p_{t,2}^{j_t}}{i_1! \dots i_s! j_1! \dots j_t!}; \\ \mathbf{f}(t_0, 0, p) &= 0. \end{aligned}$$

Hence, we get

$$(109) \quad \begin{aligned} \mathbf{f}(t_0, t_4, p) &= t_4 \left(\frac{t_0^2}{2} + p_{1,2} \right) + \frac{t_4^2}{2!} p_{1,0} + \frac{t_4^3}{3!} (p_{2,2} + \frac{1}{2} p_{1,2}^2) \\ &+ \frac{t_4^4}{4!} (2p_{2,0} + p_{1,2} p_{1,0}) + \dots \end{aligned}$$

To compute $\mathbf{f}_{\mathbb{C}^n}$ for $n > 2$ we need to know $\mathbf{f}_{\mathbb{C}P^{n-1}}$. So to complete the recursion we will explain now how to express the rational Gromov-Witten invariant $\mathbf{f}_{\mathbb{C}P^{n-1}}$ through $\mathbf{f}_{\mathbb{C}^n}$.

First of all we split, as it is described in Example 1.3.2 above, $\mathbb{C}P^n$ along the boundary of a tubular neighborhood of $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ into two completed symplectic cobordism $W_1 = \mathbb{C}^n$ and $W_2 = \mathbb{C}P^n \setminus x$, where we introduce on W_2 a complex structure of the holomorphic line bundle over $\mathbb{C}P^{n-1}$ determined by the hyperplane section $\mathbb{C}P^{n-2} \subset \mathbb{C}P^{n-1}$. We will denote by \mathbf{f}_1 and \mathbf{f}_2 the potentials for W_1 and W_2 , respectively.

Let $\Delta_0, \dots, \Delta_{2n-2}$ be closed forms representing the standard basis of $H^*(\mathbb{C}P^{n-1})$. We will keep the same notation for the pull-backs of these forms to W_2 . Let Δ_{2n} be a closed $2n$ -form with a compact support, which generates

$$\text{Ker}(H_{\text{comp}}^*(W_2) \rightarrow H^*(W_2)).$$

We are interested in the potential $\mathbf{f}_2(t_0, \dots, t_{2n}, q) = \mathbf{f}_2(\sum_{i=1}^n t_{2i} \Delta_{2i}, q)$. First of all notice that by dimensional reasons the moduli spaces of holomorphic curves which project to non-constant curves in $\mathbb{C}P^{n-1}$ do not contribute to the potential

$$\mathbf{f}_2(t_0, \dots, t_{2n-2}, 0, q),$$

and hence we have

$$(110) \quad \mathbf{f}_2\left(\sum_{i=1}^{n-1} t_{2i} \Delta_{2i}, q\right) = z \sum_{i=0}^{n-1} q_{1,2i} \sum_{\substack{(s_1, \dots, s_{n-1}) \\ \sum s_j(j-1) = n-i-1}} \prod_{j=1}^{n-1} \frac{t_{2j}^{s_j}}{s_j!}.$$

In particular, for $n = 2$ we get

$$\mathbf{f}_2(t_0\Delta_0 + t_2\Delta_2, q) = ze^{t_2}q_{1,2}$$

One can recover $\mathbf{f}_2(t_0, \dots, t_{2n}, q)$ for $t_{2n} \neq 0$ using the equation (88), as we did it above for $W_1 = \mathbb{C}^n$. However, for the purpose of our computation of Gromov-Witten invariant $\mathbf{f}_{\mathbb{C}P^n}$ this is not necessary, as we can proceed as follows.

Notice that the above chosen forms Δ_2, Δ_{2n-2} extend to CP^n . On the other hand, we will choose a volume form Δ_{2n} on $\mathbb{C}P^n$ to be supported in the affine part, so that the restriction $\Delta_{2n}|_{\mathbb{C}^n}$ coincides with the form Θ introduced above. Then Theorem (2.5.5) implies that

$$(111) \quad \mathbf{f}_{\mathbb{C}P^n} \left(\sum_{i=1}^n t_{2i} \Delta_{2i} \right) = \left(\mathbf{f}_1(t_0, t_{2n}, p) + \mathbf{f}_2(t_0, \dots, t_{2n-2}, 0, q) - \sum_{i+j=n-1} \sum_{k=1}^{\infty} \frac{1}{k} p_{k,2i} q_{k,2j} \right) \Big|_L,$$

where

$$(112) \quad L = \begin{cases} p_{1,2i} = z \sum_{\substack{(s_1, \dots, s_{n-1}) \\ \sum s_j(j-1)=i}} \prod_{j=1}^{n-1} \frac{t_{2j}^{s_j}}{s_j!}; \\ p_{k,2i} = 0, \text{ if } k > 1; \\ q_{k,2i} = k \frac{\partial \mathbf{f}_1}{\partial p_{k,2(n-i-1)}}(t_0, t_{2n}, p). \end{cases}$$

Plugging expressions from (112) into equation (111) we get

$$(113) \quad \mathbf{f}_{\mathbb{C}P^n}(t_0, \dots, t_{2n}) = \mathbf{f}_{\mathbb{C}^n}(t_0, t_{2n}, p) \Big|_{L_1},$$

where

$$L_1 = \begin{cases} p_{1,2i} = z \sum_{\substack{(s_1, \dots, s_{n-1}) \\ \sum s_j(j-1)=i}} \prod_{j=1}^{n-1} \frac{t_{2j}^{s_j}}{s_j!}; \\ p_{k,2i} = 0 \text{ if } k > 1. \end{cases}$$

Indeed, two last terms in the formula (111) cancel each other (as it always happens when \mathbf{f}_2 is linear with respect to q -variables). For instance, for $n = 1$ we get

$$(114) \quad \mathbf{f}_1(t_0, t_2, p) = \left(\frac{t_0^2 t_2}{2} + e^{t_2} p_1 \right) \Big|_{p_1=z} = \frac{t_0^2 t_2}{2} + e^{t_2} z.$$

For $n = 2$ we have

$$(115) \quad L_1 = \begin{cases} p_{1,0} = ze^{t_2}; \\ p_{k,i} = 0, \text{ for all other } k, i, \end{cases}$$

and hence

$$(116) \quad \mathbf{f}_{\mathbb{C}P^2}(t_0, t_2, t_4) = \mathbf{f}_{\mathbb{C}^n}(t_0, t_4, ze^{t_2}, 0, \dots).$$

Remark 2.9.6 The method which we used above for computing of the rational potential of $\mathbb{C}P^n$, when applied to an arbitrary symplectic manifold W , allows us to express \mathbf{f}_W through the potential of the affine part $W \setminus M$. The latter computation seems tractable when the Weinstein manifold $W \setminus M$ is subcritical (see Section 1.3 above), i.e. when its isotropic skeleton does not have Lagrangian cells. On the other hand, when Lagrangian cells are present this problem is related to central questions of Symplectic topology.

2.9.4 Satellites

Let $(V, \xi = \{\alpha = 0\})$ be a contact manifold, $(W = V \times \mathbb{R}, d(e^t \alpha))$ its symplectization, and J a compatible translation-invariant almost complex structure on W . In this section we will show that the homological Poisson super-algebra $H_*(\mathfrak{P}, d^{\text{h}})$ comes equipped with some additional structures, rather unfamiliar in abstract Poisson geometry. Namely, the counting of genus g curves with a fixed complex structure and with a fixed configuration of n points gives rise to an odd n -linear totally symmetric poly-form $\mathbf{h}^{g,n}$ on the Poisson super-space \mathbf{V} underlying \mathfrak{P} . The poly-form descends well to the homology and thus yields another invariant of the contact structure which we call the *genus- g n -point satellite* of the Poisson structure.

Let us denote by $\overline{\mathcal{M}}_{g,m}(V)/\mathbb{R}$ the compactified moduli space of stable connected J -holomorphic curves in W which are characterized by the arithmetical genus g and by the total number m of punctures and marked points numbered somehow by the indices $1, \dots, m$ (see Section 1.6 above). We emphasize that the moduli space in question contains equivalence classes of all such curves, and in particular, may have infinitely many connected components corresponding to different homotopy types of curves in W and different numbering of the m markings. Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of the moduli space of genus g Riemann surfaces with n marked points. For any g, n with $2g - 2 + n > 0$ and $l \geq 0$ there is a natural *contraction map* $\text{ct} : \overline{\mathcal{M}}_{g,n+l}(V)/\mathbb{R} \rightarrow \overline{\mathcal{M}}_{g,n}$ defined by forgetting the map to W and the last l markings followed by the contraction of those components of the curve which have become unstable. Given a differential form τ on $\overline{\mathcal{M}}_{g,n}$ we will denote by $\text{ct}^* \tau$ its pull-back to $\overline{\mathcal{M}}_{g,m}(V)/\mathbb{R}$.

Let $u = (p, q, t)$ denote a point in \mathbf{V} , that is p, q and t are (closed) differential forms on $\mathcal{P}^-, \mathcal{P}^+$ and V respectively. We will denote $\text{ev}_i^* u$, $i = 1, \dots, m$, the pull-back by the evaluation map

$$\text{ev}_i : \overline{\mathcal{M}}_{g,m}(V)/\mathbb{R} \rightarrow (\mathcal{P}^- \cup \mathcal{P}^+ \cup V)$$

at the i -th marking. Let us emphasize the point that we are treating here the marked points and punctures on equal footing.

Let $\delta u \in \mathbf{V}$ be a tangent vector to \mathbf{V} at a point $u \in \mathbf{V}$. We introduce the formal function

$$(117) \quad \mathbf{h}_\tau^{g,n} := \frac{1}{n!} \sum_{l=0}^{\infty} \frac{1}{l!} \int_{\overline{\mathcal{M}}_{g,n+l}(V)/\mathbb{R}} \text{ct}^* \tau \wedge \text{ev}_1^* \delta u \wedge \dots \wedge \text{ev}_n^* \delta u \wedge \text{ev}_{n+1}^* u \wedge \dots \wedge \text{ev}_{n+l}^* u.$$

It is a super-symmetric n -linear form in δu with coefficients depending on the application point u .

Let $d^{\mathbf{h}}(f)$ denote the Lie derivative of a tensor field f along the odd Hamiltonian vector field $d^{\mathbf{h}}$ on \mathbf{V} with the Hamilton function \mathbf{h} .

Proposition 2.9.7 *Let τ be a top degree form on $\overline{\mathcal{M}}_{g,n}$. Then $d^{\mathbf{h}}(\mathbf{h}_{\tau}^{g,n}) = 0$. If the top degree form $\tau = d\alpha$ is exact then $\mathbf{h}_{g,n}^{\tau} = d^{\mathbf{h}}(\mathbf{h}_{g,n}^{\alpha})$. In particular, the tensor field $\mathbf{h}_{\tau}^{g,n}$ descends to the homology algebra $H_*(\mathfrak{P}, \partial)$ into a satellite which depends only on the total volume of τ .*

This follows from the Stokes formula applied to $\mathbf{h}_{g,n}^{d\tau} = 0$ and respectively to $\mathbf{h}_{g,n}^{d\alpha}$. Codimension 1 boundary strata of the moduli space $\mathcal{M}_{g,m}(V)/\mathbb{R}$ are formed by stable curves of height 2. Most of the strata do not contribute to the Stokes formula since they are mapped by the contraction map to complex codimension 1 strata of the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$, where τ and α restrict to 0 for dimensional reasons. Exceptions to this rule occur only if one of the two curves which form the stable curve is to be contracted. It is therefore a sphere with glued to the other level of the stable curve along at precisely one end, and which have at most one marked points or ends with the index $\leq n$, and with any number of ends or marked points with indices $> n$. Contributions of such curves to the Stokes formula is expressed bi-linearly via the 1-st or 2-nd derivatives of the Hamilton function \mathbf{h} and the satellite. It is easy to see that the whole expression is interpreted correctly as the Lie derivative of the tensor field $\mathbf{h}_{g,n}^{\tau}$ along the Hamiltonian vector field $d^{\mathbf{h}}$. \blacktriangleright

We will assume further on that τ is normalized to the total volume 1 and will often drop it from the notation for the satellite $\mathbf{h}^{g,n}$.

Let us consider now a directed symplectic cobordism $W = \overline{V_- V_+}$ between two contact boundaries V_{\pm} . Then we have the Hamilton function $\widehat{\mathbf{h}} = \mathbf{h}^+ - \mathbf{h}^-$ and the satellites $\widehat{\mathbf{h}}^{g,n} = (\mathbf{h}^{g,n})^+ - (\mathbf{h}^{g,n})^-$ defined as elements of the algebra $\widehat{\mathfrak{L}}$, which in the case when the cobordism is a concordance just equal to the tensor product of the Poisson algebras \mathfrak{P}_{\pm} . Also, we have the potential $\mathbf{f}(p_-, q_+, t)$ counting rational J -holomorphic curves in W which defines a Lagrangian correspondence between \mathfrak{P}^{\pm} invariant under the vector field $d^{\widehat{\mathbf{h}}}$ with the Hamilton function $\widehat{\mathbf{h}}$. Finally, using the moduli spaces $\mathcal{M}_{g,m}(W)$ of J -holomorphic curves in the cobordism, we can introduce the satellites $\mathbf{f}_{\tau}^{g,n}$ as symmetric n -forms on the space (p_-, q_+, t) -space parameterizing the Lagrangian correspondence. Then the arguments similar to the above proof of the proposition, but applied this time to $\mathbf{f}_{d\tau}^{g,n} = 0$, show that *the restriction of $\widehat{\mathbf{h}}_{\tau}^{g,n}$ to the Lagrangian correspondence defined by \mathbf{f} coincides with the Lie derivative of $\mathbf{f}_{\tau}^{g,n}$ along the vector field $d^{\widehat{\mathbf{h}}}$ restricted to the Lagrangian correspondence* (comp. Theorem 2.3.6 above). In this sense the Lagrangian correspondences defined by symplectic cobordisms preserve the satellite structures defined by $(\mathbf{h}^{g,n})^{\pm}$ on the homology $H_*(\mathfrak{P}_{\pm}, d^{\mathbf{h}^{\pm}})$. In particular, *the satellite structures of a contact manifold V on the homology $H_*(\mathfrak{P}, d^{\mathbf{h}})$ depend only on the contact structure.*

The following discussion is the first steps in the study of the geometric structure defined by the satellites.

Let $\mathbf{h}_{\alpha_1, \dots, \alpha_n}^{g,n}$ denote components of the satellite tensors on \mathfrak{P} . Using the Poisson tensor $\pi^{\mu\nu}$ we can couple two satellites with respect to some indices:

$$\mathbf{h}_{\dots\mu}^{g',n'+1} \pi^{\mu\nu} \mathbf{h}_{\nu\dots}^{g'',n''+1}.$$

Similarly, we can couple two indices in $\mathbf{h}^{g-1,n+2}$ with two indices in the 2-nd differential $\delta^2 \mathbf{h}$ of the Hamilton function \mathbf{h} .

Proposition 2.9.8 *If $g = g' + g'' > 0$ then the coupling of $\mathbf{h}^{g',n'}$ and $\mathbf{h}^{g'',n''}$ is a Lie derivative along ∂ and thus vanishes in the homology $H_*(\mathfrak{P}, \partial)$. Similarly, the coupling of $\mathbf{h}^{g-1,n+2}$ with $\delta^2 \mathbf{h}$ vanishes in the homology $H_*(\mathfrak{P}, \partial)$.*

The proof is based on some famous but non-trivial property of the spaces $\overline{\mathcal{M}}_{g,n}$ with $g > 0$ to have complex codimension one strata homologically independent. Such strata correspond to different ways of cutting a (g, n) -surface along one circle and can be identified either with $\overline{\mathcal{M}}_{g',n'+1} \times \overline{\mathcal{M}}_{g'',n''+1}$ where $g' + g'' = g, n' + n'' = n$ or with $\overline{\mathcal{M}}_{g-1,n+2}$. The independence property implies that a volume form τ on the stratum, say $\tau' \otimes \tau''$ in the first case, can be obtained as the restriction of a closed codegree two form ω on $\overline{\mathcal{M}}_{g,n}$ which have exact (or even zero, for suitable choices of τ) restrictions to all other codimension-1 strata in $\overline{\mathcal{M}}_{g,n}$. Applying the Stokes formula to $0 = \mathbf{h}_{d\omega}^{g,n}$ we find that the coupling of $\mathbf{h}_{\tau'}^{g',n'+1}$ and $\mathbf{h}_{\tau''}^{g'',n''+1}$ (or — in the second case — of $\mathbf{h}_{\tau}^{g-1,n+2}$ and $\delta^2 \mathbf{h}$) equals $d^{\mathbf{h}}(\mathbf{h}_{\omega}^{g,n})$.

Remark 2.9.9 To the contrary, coupling $\mathbf{h}^{0,3}$ with itself via one index is not, in general, a $d^{\mathbf{h}}$ -derivative, but instead the following triple sum is:

$$\begin{aligned} & \mathbf{h}_{\alpha\beta\mu}^{0,3} \pi^{\mu\nu} \mathbf{h}_{\nu\gamma\delta}^{0,3} + (-1)^{(\deg \alpha + \deg \beta) \deg \gamma} \mathbf{h}_{\gamma\alpha\mu}^{0,3} \pi^{\mu\nu} \mathbf{h}_{\nu\beta\delta}^{0,3} + \\ & (-1)^{\deg \alpha (\deg \beta + \deg \gamma)} \mathbf{h}_{\beta\gamma\mu}^{0,3} \pi^{\mu\nu} \mathbf{h}_{\nu\alpha\delta}^{0,3} \equiv 0. \end{aligned}$$

This follows from the property of the 3 boundary strata in $\overline{\mathcal{M}}_{0,4}$ to represent the same homology class (use the Stokes formula for $\omega = 1$). In fact $\mathbf{h}^{0,3}$ coincides with the 3-rd differential $\delta^3 \mathbf{h}/6$ of the Hamilton function, and the above Jacobi-like identity can be derived by 4 differentiations of $\{\mathbf{h}, \mathbf{h}\} = 0$ in $\alpha, \beta, \gamma, \delta$. One can interpret the integrability property $(d^{\mathbf{h}})^2 = 0$ of the odd vector field $d^{\mathbf{h}}$ on \mathbf{V} as a homotopy Lie super-algebra structure on $\Pi \mathbf{V}^*$, the dual space with changed parity. The identity in question corresponds to the Jacobi identity for the remnant Lie super-algebra structure in homology.

It is sometimes convenient to extend the definition of genus 0 satellites to unstable values of n by $\mathbf{h}^{0,n} = \delta^n \mathbf{h}/n!$ for $n = 0, 1, 2$. Also, one can define the function $\mathbf{h}^{1,0}$ at least locally as a potential for $\mathbf{h}^{1,1}$, using the following

Proposition 2.9.10 *The differential 1-form $\mathbf{h}^{1,1}$ is closed.*

Indeed, the partial derivatives $\delta_{\mu} \mathbf{h}_{\nu}^{1,1}$ and $\delta_{\nu} \mathbf{h}_{\mu}^{1,1}$ are identified with the satellites $(\mathbf{h}_{\omega}^{1,2})_{\mu\nu}$ corresponding to the 2-form ω on $\overline{\mathcal{M}}_{1,2}$ pulled-back from $\overline{\mathcal{M}}_{1,1}$ by forgetting

the 1-st and respectively the 2-nd marked point. But the two maps $\overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{1,1}$ coincide.

It would be interesting to study other general properties of satellites which may depend on more sophisticated geometry of Deligne-Mumford compactifications. For instance, what can be said about Poisson brackets among the functions $\mathbf{h}^{g,0}$?

We complete the section with the computation of the satellites in the example $V = S^1$. Let $t = t_0 1 + t_1 d\phi$ denote the general harmonic form on S^1 , $\delta t = \tau_0 1 + \delta t_1 \delta\phi$, $u(x) = t_0 + \sum p_k e^{ikx} + q_k e^{-ikx}$, $\delta u = \delta t_0 + \sum \delta p_k e^{ikx} + \delta q_k e^{-ikx}$.

Proposition 2.9.11 *For $2g - 2 + n \geq 0$ we have*

$$\mathbf{h}^{g,n+1} = \frac{\delta t_1}{2\pi n!} \int_0^{2\pi} (u_{xx})^g (\delta u)^n dx.$$

Let us begin with the remark that the formula does not (and cannot) contain $t = t_0 + t_1 \phi$ because $\deg t < 2$, and therefore pushing forward from $\mathcal{M}_{g,m+1}(V)/\mathbb{R} \rightarrow \mathcal{M}_{g,m}(V)/\mathbb{R}$ by forgetting the corresponding marked point would send t to 0. Exceptions to this rule could occur only if $\mathcal{M}_{g,m}(V)$ were ill-defined, that is only in the case of constant maps with "unstable" indices, $2g - 2 + m \leq 0$, which has no effect on the satellites with "stable" indices. On the other hand the factor δt_1 is (and must be) present in the formula since the dimension of the moduli spaces is odd. With this information in mind, the enumerative question equivalent to computation of the satellites can be described as follows. On a Riemann surface Σ of genus g , we are given a divisor D of n distinct points with (possibly zero) multiplicities m_1, \dots, m_n . We have to count the divisors $l_1 P_1 + \dots + l_g P_g$ which in the sum with D form the divisor of a rational function. (In particular, the degree $\sum m_i + \sum l_j$ of the total divisor must vanish.) The answer to this question is equal to the degree of the Abel-Jacobi map $\Sigma^g \rightarrow J_\Sigma$ defined by integration of holomorphic differentials $\vec{\omega} = (\omega_1, \dots, \omega_g)$ on Σ as

$$(P_1, \dots, P_g) \mapsto l_1 \int^{P_1} \vec{\omega} + \dots + l_g \int^{P_g} \vec{\omega}.$$

When the multiplicities $(l_1, \dots, l_g) = (1, \dots, 1)$, the degree equals $g!$ (it is well-known that $S^g \Sigma^g \rightarrow J_\Sigma$ is a bi-rational isomorphism). For arbitrary (l_1, \dots, l_g) the Abel-Jacobi map has the Jacobi matrix $[l_j \omega_i(P_j)]$. Thus the degree equals $l_1^2 \dots l_g^2 g!$. Taking these answers as the coefficients in the generating function on the variables t_0, p_l, q_{-l} corresponding to $l = 0, l > 0$ and $l < 0$ we arrive at the factor u_{xx}^g . The other factor $(\delta u)^n/n!$ is similarly accountable for all possible choices of multiplicities m_1, \dots, m_n in the divisor D . The contour integration of the product couples the choices with $m_1 + \dots + m_n + l_1 + \dots + l_g = 0$.

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