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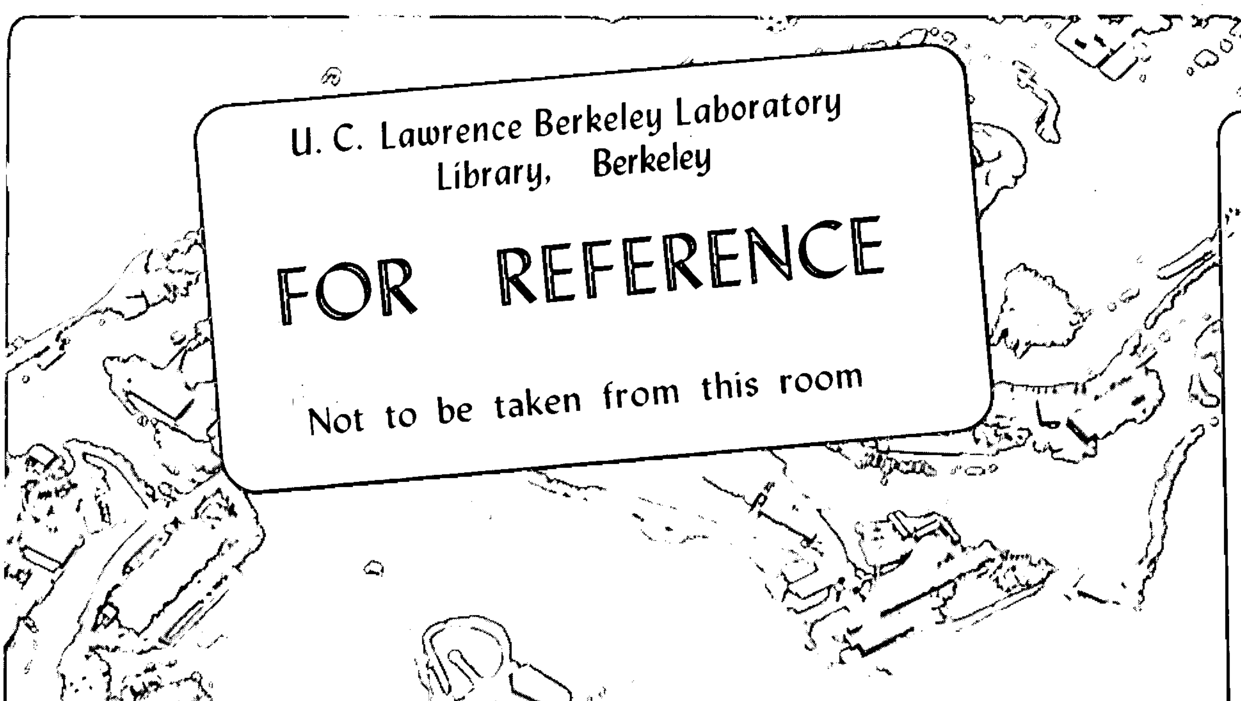

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Introduction to the Differential Geometry of Quantum Groups[*] [†]

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Abstract

An introduction to the noncommutative differential calculus on quantum groups. The invariant group average is also discussed.

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1. Introduction

In this lecture I shall try to describe those aspects of the mathematics of quantum groups which I believe may be relevant for the formulation of new physical theories based on noncommutative calculus and noncommutative geometry. As a consequence, there is no attempt here to cover the standard applications of the theory to integrable systems, knot theory and conformal field theory in two dimensions.

A very helpful review of the more conventional mathematical approach can be found in [1], which contains numerous references. This standard approach is based on the concept of quasitriangular Hopf algebras. It provides a satisfactory all-embracing theory of quantum groups but it ignores the existence of the quantum exterior differential calculus of Woronowicz on quantum groups and therefore the connection with noncommutative differential geometry [2]. Here I shall underplay the importance of Hopf algebras and concentrate instead on the calculus aspect. Comultiplication is barely mentioned, in (2.17) and (3.4). Instead I emphasize equations such as (3.5),(4.2) and (4.15) from which the comultiplication rule can be derived. The word antipode will never appear here, although plenty of inverse quantum matrices will play a role. I shall emphasize the computational aspect which I believe will be important in the "new physics". The quantum groups $SL_q(2)$ and $SU_q(2)$ provide useful simple examples in the following. This restricted choice of material can also be justified by space limitations. Many details are contained in the papers quoted here, where numerous additional references can also be found.

2. Quantum groups and Lie algebras

This section is a review of some very well known facts about quantum groups which are needed in the following. We follow mostly the approach of Faddeev, Reshetikin and Takhtajan [3]. A quantum group can be defined in terms of the defining representation of the corresponding Lie group. For instance, $GL_q(2)$ can be defined in terms of the two by two matrix

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{2.1}$$

The space of the group parameters α, β, γ and δ can be quantized by turning the parameters into non commuting quantities satisfying the commutation relations

$$\begin{array}{ll} \alpha\beta = q\beta\alpha & \alpha\gamma = q\gamma\alpha \\ \beta\delta = q\delta\beta & \gamma\delta = q\delta\gamma \\ \alpha\delta - \delta\alpha = \lambda\beta\gamma & \beta\gamma = \gamma\beta, \end{array} \tag{2.2}$$

where q is a generic complex number and we shall consistently use the abbreviation

$$\lambda = q - q^{-1}. \tag{2.3}$$

Using these commutation relations it is easy to check that the "quantum determinant"

$$\det_q T = \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma \tag{2.4}$$

is central, i.e. it commutes with α, β, γ and δ. We shall call a matrix like (2.1) whose elements satisfy (2.2) a q-matrix.

The commutation relations (2.2) have the following remarkable property. Let $\alpha', \beta', \gamma', \delta'$ commute with $\alpha, \beta, \gamma, \delta$ and let them satisfy the same commutation relations (2.2). So

$$\alpha'\beta' = q\beta'\alpha' \quad etc...,\tag{2.5}$$

which means that

$$T' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \tag{2.6}$$

is a q-matrix with elements commuting with those of T. Then the matrix

$$T'' = TT' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta' \end{pmatrix} \tag{2.7}$$

obtained from T and T' by matrix multiplication, rows by columns, is also a q-matrix. We refer to this fact as to the quantum group property of (2.2).

The commutation relations (2.2) can be written compactly in terms of the R-matrix of the quantum group [3]. Consider the tensor product of two vector spaces and let the matrix T operate on it as

$$T_1 = T \otimes 1, \tag{2.8}$$

i.e. T on the first space and the identity on the second, or explicitly

$$(T_1)^{ik}_{j\ell} = T^i_{\ j}\delta^k_{\ \ell}. \tag{2.9}$$

Similarly define

$$T_2 = 1 \otimes T, \tag{2.10}$$

i.e.

$$(T_2)^{ik}_{j\ell} = \delta^i_{\ j}T^k_{\ \ell}. \tag{2.11}$$

If the matrix elements of T were commuting quantities, T_1 and T_2 would commute. Because of (2.2) their products in opposite order are not equal but they are related by conjugation in the tensor product space

$$R_{12}T_1T_2 = T_2T_1R_{12}. \tag{2.12}$$

The matrix R_{12} operates in the tensor product space and is given for $GL_q(2)$ by

$$(R_{12})^{ik}_{j\ell} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{2.13}$$

where the rows (and columns) are numbered as 11,12,21,22. We shall call (2.12) the "RTT equation". Using it, it is very easy to prove the quantum group property, since T' also satisfies an equation like (2.12) and T_1' and T_2 commute.

The R-matrix satisfies the "Yang-Baxter equation" in the triple tensor product space

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.14}$$

which ensures consistency of the RTT equation (2.12). In this approach classifying all possible quantum groups is equivalent to finding all solutions of the Yang-Baxter equation[3].

Since the determinant (2.4) is central, one can set it equal to unity:

$$\alpha\delta - q\beta\gamma = 1. \tag{2.15}$$

The quantum group $GL_q(2)$ is then restricted to the "subgroup" $SL_q(2)$. However beware: quantum groups are not groups.

For quantum groups other than $GL_q(n)$ or $SL_q(n)$ in addition to giving the relevant R matrix one must impose further restrictions compatible with (2.12) and (2.14), such as orthogonality or other conditions for the quantum matrices.

See reference [3] where the cases of quantum orthogonal and symplectic groups are described in detail.

Jimbo [4] and Drinfeld [5] have provided us with a consistent quantum deformation of any simple Lie algebra. For the case of $SL(2)$ one has the generators H, X_+ and X_- and their commutation relations are deformed to

$$
\begin{aligned}
[H, X_\pm] &= \pm 2 X_\pm \\
[X_+, X_-] &= \frac{q^H - q^{-H}}{q - q^{-1}}.
\end{aligned}
\tag{2.16}
$$

The comultiplication is given by

$$
\begin{aligned}
\Delta(H) &= H \otimes 1 + 1 \otimes H \\
\Delta(X_\pm) &= X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm.
\end{aligned}
\tag{2.17}
$$

It is easy to check that (2.16) and (2.17) are consistent with each other. For $q \to 1$ they become the standard relations for $SL(2)$ (or $SU(2)$ if appropriate reality conditions are imposed).

3. Relation between the quantum group and the quantum Lie algebra.

In this section we follow again the approach of reference [3] with some minor modification. We continue to use the example of $SL(2)$.

Define the upper diagonal matrix

$$
L^+ = \begin{pmatrix} q^{-H/2} & \lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}
\tag{3.1}
$$

and the lower diagonal matrix

$$
L^- = \begin{pmatrix} q^{H/2} & 0 \\ -\lambda X_- & -q^{-H/2} \end{pmatrix}.
\tag{3.2}
$$

It is easy to see that the algebra (2.16) can be written compactly as

$$
\begin{aligned}
R_{12} L_2^+ L_1^+ &= L_1^+ L_2^+ R_{12} \\
R_{12} L_2^- L_1^- &= L_1^- L_2^- R_{12} \\
R_{12} L_2^+ L_1^- &= L_1^- L_2^+ R_{12}.
\end{aligned}
\tag{3.3}
$$

The first two equations (3.3) state that L^+ and L^- are q^{-1}-matrices. From their definition (3.1) they have quantum determinant equal to one. The comultiplication rule (2.17) is equivalent to

$$
\begin{aligned}
\Delta((L^+)^i{}_j) &= (L^+)^i{}_k \otimes (L^+)^k{}_j \\
\Delta((L^-)^i{}_j) &= (L^-)^i{}_k \otimes (L^-)^k{}_j
\end{aligned}
\tag{3.4}
$$

(sum over repeated indices).

The matrix elements of L^+ and L^- can be identified with quantum differential operators on group space. Their action on the group variables is given by the equations

$$
\begin{aligned}
L_1^+ T_2 &= T_2 \mathcal{R}_{21} L_1^+ \\
L_1^- T_2 &= T_2 \mathcal{R}_{12}^{-1} L_1^-.
\end{aligned}
\tag{3.5}
$$

Here $\mathcal{R}$ is a suitably normalized R-matrix. For $SL(n)$

$$
\mathcal{R} = q^{-1/n} R.
\tag{3.6}
$$

The normalization factor is determined by the property that L^+ and L^- have q-determinant equal to one. It is easy to check, using (2.14) that (3.5) are consistent with (2.12), i.e. the action on the group preserves the quantum structure of the group. The algebra (3.3) and the comultiplication law (3.4) can actually be derived from (3.5), which can be taken as the basic relations.

Equations (3.3-5) are general. Consistent relations among the matrix elements of L^+ and among those of L^- (or an appropriate ansatz generalizing (3.1)) must be given for different groups so that the number of independent generators agrees with that of the classical Lie algebra (see [3], [6]).

Let us define a right "vacuum"> and a left "vacuum" < such that

$$
L^+ >= L^- >= I >
\tag{3.7}
$$

and

$$
< T = < I
\tag{3.8}
$$

where I is the unit matrix. Using (3.5) one can compute vacuum values of products. For instance

$$
< L_0^+ T_1 >=< T_1 \mathcal{R}_{10} L_0^+ >= \mathcal{R}_{10}.
\tag{3.9}
$$

More generally

$$\begin{aligned} < L_0^+ T_1 T_2 ... T_n > &= \mathcal{R}_{10}\mathcal{R}_{20}...\mathcal{R}_{n0} \\ < L_0^- T_1 T_2 ... T_n > &= \mathcal{R}_{01}^{-1}\mathcal{R}_{02}^{-1}...\mathcal{R}_{0n}^{-1}. \end{aligned} \tag{3.10}$$

The knowledge of all vacuum values is equivalent to the basic relations (3.5). We see that the enveloping algebra of the quantum Lie algebra is dual to the algebra of functions on the quantum group and consists of "regular" linear functionals of these functions.

4. Bicovariant calculus

The bicovariant calculus on quantum groups is due to Woronowicz [7]. Here we follow the approach of Jurčo [9], which provides a direct connection between the calculus of Woronowicz and the work of Faddeev, Reshetikin and Takhtajan [3].

Define the matrix

$$Y = L^+(L^-)^{-1} \tag{4.1}$$

which is neither upper nor lower triangular. It is not hard to see that (3.5) imply (see the explicit example (6.8) below)

$$Y_1 T_2 = T_2 \mathcal{R}_{21} Y_1 \mathcal{R}_{12} \tag{4.2}$$

and that the algebra relations (3.3) imply

$$\mathcal{R}_{21} Y_1 \mathcal{R}_{12} Y_2 = Y_2 \mathcal{R}_{21} Y_1 \mathcal{R}_{12}. \tag{4.3}$$

These equations have the remarkable property that they are covariant under the transformation

$$T \to T'T, \quad Y \to Y, \tag{4.4}$$

as well as under

$$T \to TT', \quad Y \to (T')^{-1} Y T'. \tag{4.5}$$

The matrix elements of T' are taken to commute with those of T as well as with those of Y. Furthermore, the matrix T' satisfies

$$R_{12} T_1' T_2' = T_2' T_1' R_{12}.$$

We shall say that (4.2) and (4.3) are "left-invariant" (because of (4.4)) and "right-covariant" (because of (4.5)). The matrix elements of Y are the differential operators of a "bicovariant" calculus on the quantum group.

Take the case of $SL(2)$ and write

$$Y = \begin{pmatrix} y_1 & y_+ \\ y_- & y_2 \end{pmatrix}. \tag{4.6}$$

Using (4.3) one can verify that

$$D = y_1 y_2 - q^2 y_+ y_- \tag{4.7}$$

commutes with y_1, y_2, y_+ and y_- and that

$$Y^{-1} = \frac{1}{D}\begin{pmatrix} y_2 & -q^2 y_+ \\ -q^2 y_- & q^2 y_1 + (1-q^2) y_2 \end{pmatrix}. \tag{4.8}$$

Furthermore, when Y is given by (4.1), it follows from (3.3) that

$$D = (\det_{q^{-1}} L^+)(\det_{q^{-1}} L^-)^{-1}. \tag{4.9}$$

The special matrices L^+ and L^- given in (3.1) and (3.2) have q^{-1}-determinant equal one, therefore

$$D = 1. \tag{4.10}$$

We could have used Y^{-1} instead of Y, but we see from (4.8) that the corresponding operators are linear combinations of the elements of Y. The calculus of Y^{-1} is completely equivalent to that of Y. An alternative choice is given by $(L^+)^{-1}L^-$ or its reciprocal. This gives operators which belong to the same enveloping algebra but are not linearly related to those of Y and Y^{-1}. The resulting calculus is right invariant and left covariant.

The "determinant" D, given by (4.7), commutes not only with the elements of Y but also with the matrix elements of T, as one can check using (4.2). Therefore (4.10) is a consistent condition. In general there are other quantities which commute with the elements of Y but not with those of T. These correspond to the classical Casimir operators, which are central in the Lie algebra, but have a nontrivial action on the group. For $SL(2)$ the Casimir operator is given, by

$$C = y_1 + q^{-2} y_2. \tag{4.11}$$

A general formula for Casimir operators [3] can be given in terms of the invariant quantum trace discussed at the beginning of the next section.It is

$$C_k = Tr(\mathcal{D}^{-1}Y^k) \tag{4.12}$$

where $k = 1, 2, ...r$ and r is the rank of the group.

As $q \to 1$, the matrices L^+, L^- and Y tend to the unit matrix I. To establish a connection with the classical Lie algebra, define the matrix X by

$$Y = I - \lambda X. \tag{4.13}$$

The matrix elements of

$$X = \begin{pmatrix} \chi_1 & \chi_+ \\ \chi_- & \chi_2 \end{pmatrix} \tag{4.14}$$

correspond to the generators of the classical Lie algebra (differential operators on the group). From (4.2) and (4.3), one can easily obtain the analogous relations for X. They are

$$X_1T_2 = T_2\mathcal{R}_{21}X_1\mathcal{R}_{12} - \frac{1}{\lambda}T_2(\mathcal{R}_{21}\mathcal{R}_{12} - I_{12}) \tag{4.15}$$

and

$$\begin{aligned} &\mathcal{R}_{21}X_1\mathcal{R}_{12}X_2 \ - X_2\mathcal{R}_{21}X_1\mathcal{R}_{12} \\ &= \frac{1}{\lambda}(\mathcal{R}_{21}\mathcal{R}_{12}X_2 \ - X_2\mathcal{R}_{21}\mathcal{R}_{12}). \end{aligned} \tag{4.16}$$

The condition (4.10) becomes, in terms of X,

$$\chi_1 + \chi_2 - \lambda\chi_1\chi_2 + q^2\lambda\chi_+\chi_- = 0. \tag{4.17}$$

Using this equation one can eliminate, for instance, χ_1. The Casimir operator (4.11) becomes then

$$C = 1 + q^{-2} + q^{-2}\lambda^2(1 - \lambda\chi_2)^{-1}[q\chi_2 + \chi_2^2 + q^4\chi_+\chi_-]. \tag{4.18}$$

As $q \to 1$ the expression in square brackets tends to the well known classical expression for the $SL(2)$ Casimir operator.

Although we started from he matrices L^+ and L^- and defined Y by (4.1) in terms of them, we can now formulate the bicovariant calculus directly in terms

of Y and the eqs. (4.2), (4.3) (4.7) and (4.10) which it satisfies. Then X will still be defined from (4.13). The matrices L^+ and L^- are left-invariant but (because of the triangularity conditions they satisfy) they transform in a very complicated way under the right multiplication (4.5). Nevertheless, as shown in [6], one can always decompose a matrix like Y into a product like (4.1) where L^+ is upper and L^- lower triangular. It seems that the triangularity properties of L^+ and L^- should be considered as special choices of gauge.

5. Quantum differential forms

Just as for ordinary Lie groups, one can introduce exterior differential forms [7]. We shall derive their properties from those of the differential operators on the group. We first notice that, if a matrix M transforms as

$$M \to (T')^{-1}MT' = M' \tag{5.1}$$

where T' is a q-matrix whose elements commutes with those of M, then the quantum trace

$$Tr(\mathcal{D}^{-1}M) = Tr(\mathcal{D}^{-1}M') \tag{5.2}$$

is invariant, where $\mathcal{D}$ is a suitable matrix and Tr denotes the usual trace. In general, $\mathcal{D}$ must satisfy, for any q-matrix T,

$$\mathcal{D}^t(T^{-1})^t(\mathcal{D}^t)^{-1} = (T^t)^{-1} \tag{5.3}$$

where t denotes the ordinary transposed of a matrix. It turns out that $\mathcal{D}$ can be chosen diagonal. For $SL(n)$,

$$\mathcal{D} = diag\left(1, q^2, q^4, ...q^{2(n-1)}\right). \tag{5.4}$$

Let us now introduce a matrix of differential one-forms

$$\Omega = \begin{pmatrix} \omega^1 & \omega^- \\ \omega^+ & \omega^2 \end{pmatrix}. \tag{5.5}$$

The exterior differential

$$\begin{aligned} d &= Tr(\mathcal{D}^{-1}\Omega X) \\ &= \omega^1\chi_1 + \omega^-\chi_- + q^{-2}\omega^+\chi_+ + q^{-2}\omega^2\chi_2 \end{aligned} \tag{5.6}$$

is invariant if one transforms

$$X \to (T')^{-1} X T' \tag{5.7}$$

(see (4.5) and (4.13)) and

$$\Omega \to (T')^{-1} \Omega T'. \tag{5.8}$$

At the same time the quantum trace of Ω

$$\xi = Tr(\mathcal{D}^{-1}\Omega) = \omega^1 + q^{-2}\omega^2 \tag{5.9}$$

is invariant.

The exterior differential is required to satisfy the standard underformed relations

$$d^2 = 0, \quad d(\text{constant}) = 0 \tag{5.10}$$

and

$$d(fg) = dfg + (-1)^{p(f)} f dg, \tag{5.11}$$

where $p(f)$ is the parity of f. The properties of the differential one-forms can be derived from those of the differential operators by using (5.10) and (5.11). For instance, it must be

$$\begin{aligned} 0 &= d^2 = d\omega^1 \chi_1 + d\omega^- \chi_- + q^{-2} d\omega^+ \chi_+ + q^{-2} d\omega^2 \chi_2 \\ &- \omega^1 d\chi_1 - \omega^- d\chi_- - q^{-2}\omega^+ d\chi_+ - q^{-2}\omega^2 d\chi_2. \end{aligned} \tag{5.12}$$

Substitute (5.6) in the last four terms of this equation. We know the commutation relations among the χ's from (4.16). With a little work one obtains the commutation relations for the one-forms

$$\begin{aligned} \omega^+\omega^- + \omega^-\omega^+ &= 0 \\ \omega^1\omega^+ + \omega^+\omega^1 &= 0 \\ \omega^1\omega^- + \omega^-\omega^1 &= 0 \\ \omega^2\omega^+ + q^2\omega^+\omega^2 &= q\lambda\omega^+\omega^1 \\ \omega^2\omega^- + q^{-2}\omega^-\omega^2 &= -q^{-1}\lambda\omega^-\omega^1 \\ \omega^1\omega^2 + \omega^2\omega^1 &= -q^{-1}\lambda\omega^+\omega^- \\ (\omega^1)^2 = (\omega^+)^2 &= (\omega^-)^2 = 0 \\ (\omega^2)^2 &= q\lambda\omega^+\omega^- \end{aligned} \tag{5.13}$$

and the quantum Maurer-Cartan equations

$$\begin{aligned} d\omega^1 &= -q^{-3}\omega^+\omega^- \\ d\omega^2 &= q^{-1}\omega^+\omega^- \\ d\omega^+ &= q^{-1}\omega^+(\omega^1 - \omega^2) \\ d\omega^- &= q^{-1}(\omega^1 - \omega^2)\omega^-. \end{aligned} \tag{5.14}$$

These are the bicovariant relations of Woronowicz [7] for the case of $SL(2)$. They have a very interesting property as pointed out by Woronowicz in general. Although the group $SL(2)$ has only three parameters, there are four one-forms, one too many, it would seem at first. However, the invariant form ξ of (5.9) can be easily seen to satisfy

$$\xi^2 = 0, \quad d\xi = 0 \tag{5.15}$$

and, for all $\omega^\alpha, \alpha = 1, -, +, 2$

$$\xi\omega^\alpha + \omega^\alpha\xi = \lambda d\omega^\alpha. \tag{5.16}$$

As $q \to 1, \lambda \to 0$ and ξ decouples. What happens to $\lambda^{-1}\xi$ in the limit? We leave this as an interesting exercise for the reader.

Finally, it is clear that (4.15) and (5.6) allow us to derive the commutation relations between the one-forms and the matrix elements of the matrix T. One finds, in general

$$\Omega_1 T_2 = T_2 \mathcal{R}_{12}^{-1} \Omega_1 \mathcal{R}_{21}^{-1}. \tag{5.17}$$

The derivation of this equation requires the identity, satisfied by the $\mathcal{R}$ matrix,

$$\left((\mathcal{R}_{12}^{-1})^{t_1}\right)^{-1} = \mathcal{D}_1^{-1} \mathcal{R}_{12}^{t_1} \mathcal{D}_1, \tag{5.18}$$

where t_1 denotes transposition in the space 1 of the tensor product. For $SL(2)$ one obtains, from (5.17),

$$\begin{aligned} \alpha\omega^1 &= q\omega^1\alpha \\ \alpha\omega^- &= \omega^-\alpha + \lambda\omega^1\beta \\ \alpha\omega^+ &= \omega^+\alpha \\ \alpha\omega^2 &= q^{-1}\omega^2\alpha + q^{-1}\lambda\omega^+\beta \end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
\beta\omega^1 &= q^{-1}\omega^1\beta \\
\beta\omega^- &= \omega^-\beta \\
\beta\omega^+ &= \omega^+\beta + q^2\lambda\omega^1\alpha \\
\beta\omega^2 &= q\omega^2\beta + q\lambda^2\omega^1\beta + q\lambda\omega^-\alpha.
\end{aligned} \tag{5.20}$$

The same equations are valid with α replaced by γ and β replaced by δ. For functions on the group the invariant form ξ plays a similar role as in (5.16) for forms: if f is a function of α, β, γ and δ, one has

$$\xi f - f\xi = \lambda df. \tag{5.21}$$

Notice that, from (5.19)and (5.20), all one-forms ω^α, for $\alpha = 1, -, +, 2$, commute with the quantum determinant (2.4) of T. Therefore the invariant form ξ also commute with it and, by (5.21), it is identically

$$d(\det_q T) \equiv 0. \tag{5.22}$$

As a consequence one can impose (2.15).

There is an alternative way to introduce differential forms on a quantum group, which is perhaps closer to that which one often follows in the classical case. Define the new matrix of forms

$$\tilde{\Omega} = T^{-1}dT, \tag{5.23}$$

which satisfies

$$d\tilde{\Omega} = -\tilde{\Omega}^2. \tag{5.24}$$

Clearly $\tilde{\Omega}$ is left-invariant (under $T \to T'T$) and right-covariant, i.e. under $T \to TT'$ it transforms as

$$\tilde{\Omega} \to (T')^{-1}\tilde{\Omega}T'. \tag{5.25}$$

How are Ω and $\tilde{\Omega}$ related? Since they transform in the same way, we are led to the identification

$$\tilde{\Omega} = c_1\Omega + c_2\xi I \tag{5.26}$$

where c_1 and c_2 are constants. For $SL(n)$ one has

$$c_1 = -q^{2/n-2n+1}, \quad c_2 = q\left[\frac{1}{n}\right], \tag{5.27}$$

where in general

$$[x] \equiv \frac{1-q^{2x}}{1-q^2}. \tag{5.28}$$

6. Invariant measure

The general properties of a left and right invariant Haar measure for compact quantum groups were discussed by Woronowicz [8]. Here we consider briefly the case of $SU_q(2)$ and, using a different technique, we compute explicitly the invariant measure.

For $SU_q(2)$ the unitarity condition

$$T^\dagger = T^{-1} \tag{6.1}$$

for the matrix (2.1) gives

$$\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}, \tag{6.2}$$

where the bar indicates an involution of the algebra of functions on the group which changes the order of factors in a product. For consistency it must be

$$\overline{q} = q. \tag{6.3}$$

In the following we shall keep in mind that

$$\delta = \overline{\alpha}, \quad \beta = -q\overline{\gamma} \tag{6.4}$$

but we shall continue to use the letters β and δ for those matrix elements. Ordered monomials in α, β, γ and δ can be taken as a basis for functions on the group or at least for polynomials. Using the determinant condition

$$\alpha\delta - q\beta\gamma = 1 = \delta\alpha - q^{-1}\beta\gamma \tag{6.5}$$

and the commutation relations (2.2) one can transform any monomial to the form $\alpha^k\beta^\ell\gamma^m$ or to the form $\delta^k\beta^\ell\gamma^m$, where the exponents take all integer values $0, 1, 2, \ldots$. We shall take these monomials as a complete basis.

We wish to associate to a function $\langle f\rangle$ on the group a real number, its invariant group average, which we shall denote as $\langle f\rangle$. There are different ways to state the invariance of the group average. For algebraic manipulations a convenient way is to require that

$$\langle \chi_\alpha f\rangle = 0, \tag{6.6}$$

where χ_α are the differential operators of a bicovariant calculus, the matrix elements of the matrix X in (4.13). For $q \neq 1$, we can use (4.13) and rewrite (6.6) in the very convenient form

$$\langle Y^i{}_j f\rangle = \delta^i{}_j. \tag{6.7}$$

This condition actually allows us to compute the average for all basic monomials, by means of the commutation relations (4.2).

We write explicitly the commutation relations (4.2) for the example of $SU_q(2)$. They are

$$\begin{aligned}
y_+\alpha &= \alpha y_+ + q^{-1}\lambda\beta y_2\\
y_-\alpha &= \alpha y_-\\
y_2\alpha &= q^{-1}\alpha y_2\\
y_+\beta &= \beta y_+\\
y_-\beta &= \beta y_- + q^{-1}\lambda\alpha y_2\\
y_2\beta &= q\beta y_2,
\end{aligned} \tag{6.8}$$

together with the equations obtained by replacing α with γ and β with δ..We have not written the relations involving y_1, which we consider as defined by (4.7), (4.10). Using (6.8) one can easily compute

$$\langle y_2\alpha^k\beta^\ell\gamma^m\rangle = q^{-k+\ell-m}\alpha^k\beta^\ell\gamma^m. \tag{6.9}$$

Together with (6.7), which is now

$$\langle y_2\alpha^k\beta^\ell\gamma^m\rangle = 1 \tag{6.10}$$

this implies that $\langle\alpha^k\beta^\ell\gamma^m\rangle$ must vanish unless $k+m-\ell=0$. Computing $\langle y_-\alpha^k\beta^\ell\gamma^m\rangle$ one finds, since (6.7) gives

$$\langle y_-\alpha^k\beta^\ell\gamma^m\rangle = 0, \tag{6.11}$$

that

$$\langle\alpha^k\beta^\ell\gamma^m\rangle = 0 \text{ unless both } k=0 \text{ and } \ell=m. \tag{6.12}$$

A similar argument gives the result

$$\langle\delta^k\beta^\ell\gamma^m\rangle = 0 \text{ unless } k=0 \text{ and } \ell=m. \tag{6.13}$$

To obtain the remaining nonvanishing average values $\langle\beta^k\gamma^k\rangle$, compute

$$\begin{aligned}
&\langle y_+\alpha\ \gamma^k\beta^{k-1}\rangle\\
&\qquad = \lambda q^{-1}[k]\alpha\delta\gamma^{k-1}\beta^{k-1} + q^{-2}\lambda\gamma^k\beta^k,
\end{aligned} \tag{6.14}$$

where we have used the notation of (5.28). The average of the left hand side of (6.14) vanishes. Using (6.5) we obtain the recursion relation

$$\langle\beta^k\gamma^k\rangle = -q\frac{[k]}{[k+1]}\langle\beta^{k-1}\gamma^{k-1}\rangle. \tag{6.15}$$

Choosing the normalization

$$\langle 1\rangle = 1, \tag{6.16}$$

we obtain finally

$$\langle\beta^k\gamma^k\rangle = \frac{(-q)^k}{[k+1]}. \tag{6.17}$$

With the unitarity conditions (6.3) and (6.4) the above results agree with those obtained by Woronowicz [8] with a different method. In particular (6.18) becomes

$$\langle\bar{\gamma}^k\gamma^k\rangle = \frac{1}{[k+1]}. \tag{6.18}$$

It is remarkable that the results (6.12), (6.13) and (6.17) make sense even without the unitarity conditions, although one loses the positivity property of (6.18). So, it seems possible to define a left and right invariant average for polynomials on $SL_q(2)$. As $q \to 1$ this average still makes sense and is invariant, but apparently cannot be defined in terms of an integral over the group, even though a left and right invariant volume element exists on $SL(2)$.

7. Conclusion

The methods and results described above are examples of differential and intergal calculus on non commutative spaces. Other related examples can be found in [10] to [13].

The way seems to be open for the construction of consistent deformations of quantum mechanics and quantum field theory. Perhaps these deformations will provide a form of realistic regularization (too many as yet inconclusive papers to cite on this). In any case, it is remarkable that our present physical laws seem to allow consistent deformations, which are not required or suggested by experiment.

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