Contents

Part I Foundations of the Renormalization Group

1 Phase Transitions and the Scaling Hypothesis 5
1.1 Classification of Phase Transitions 5
1.2 Critical Exponents and Universality of Continuous Phase Transitions 9
1.3 The Scaling Hypothesis 12
1.4 Scaling in the Vicinity of Quantum Critical Points 15
References 21

2 Mean-Field Theory and the Gaussian Approximation 23
2.1 Mean-Field Theory 24
2.1.1 Landau Function and Free Energy 24
2.1.2 Thermodynamic Critical Exponents 26
2.2 Ginzburg–Landau Theory 30
2.2.1 Exact Effective Field Theory 30
2.2.2 Truncated Effective Action: φ^4-Theory 34
2.3 The Gaussian Approximation 39
2.3.1 Gaussian Effective Action 39
2.3.2 Gaussian Corrections to the Specific Heat Exponent 41
2.3.3 Correlation Function 44
2.3.4 Failure of the Gaussian Approximation in $D < 4$ 47
References 52

3 Wilsonian Renormalization Group 53
3.1 The Basic Idea 54
3.2 Real-Space RG for the One-Dimensional Ising Model 61
3.2.1 Exact Decimation 61
3.2.2 Iteration and Fixed Points of the RG 66
3.2.3 Infinitesimal Form of RG Recursion Relations 69
3.3 General Properties of RG Flows 72
3.3.1 RG Fixed Points and the Critical Surface 72

3.3.2 Local RG Flow Close to a Fixed Point: Classification of Couplings and Justification of the Scaling Hypothesis . 75
3.3.3 Global Properties of RG Flows and Classification of Fixed Points 81
References 87

4 Critical Behavior of the Ising Model Close to Four Dimensions 91
4.1 Diagrammatic Perturbation Theory 91
4.1.1 Wick Theorem 93
4.1.2 Feynman Diagrams and Linked Cluster Theorem 95
4.1.3 Diagrams for Correlation Functions and the Irreducible Self-Energy 98
4.2 One-Loop Momentum Shell RG 101
4.2.1 Derivation of the RG Flow Equations 102
4.2.2 The Wilson–Fisher Fixed Point 108
4.2.3 Wave Function Renormalization and Anomalous Dimension 113
References 120

5 Field-Theoretical Renormalization Group 123
5.1 Divergencies and Their Regularization in Field Theory 123
5.2 Perturbative Renormalization 128
5.2.1 The Renormalized Lagrangian 128
5.2.2 Perturbative Calculation of Renormalization Factors 130
5.2.3 Relating Renormalized Perturbation Theory to Experiments 133
5.3 Callan–Symanzik Equation 135
References 139

Part II Introduction to the Functional Renormalization Group

6 Functional Methods 147
6.1 Generating Functionals for Green Functions 150
6.1.1 Disconnected Green Functions 150
6.1.2 Connected Green Functions 151
6.1.3 Amputated Connected Green Functions 156
6.2 One-Line Irreducible Vertices 159
6.2.1 Generating Functional of the Irreducible Vertices 160
6.2.2 Tree Expansion 163
6.3 Symmetries 167
6.3.1 General Properties 169
6.3.2 Nonrelativistic Particles 171
6.3.3 Dyson–Schwinger and Skeleton Equations 176
References 179

7 Exact FRG Flow Equations 181
7.1 Cutoffs 181
7.2 Exact FRG Flow Equations for Generating Functionals 186
7.2.1 Disconnected Green Functions 187
7.2.2 Connected Green Functions 188
7.2.3 Amputated Connected Green Functions 189
7.2.4 One-Line Irreducible Vertices 190
7.3 Exact FRG Equations for the Irreducible Vertices 194
7.4 Spontaneous Symmetry Breaking: The Vertex Expansion with Vacuum Expectation Values 200
References 207

8 Vertex Expansion 209
8.1 Vertex Expansion for Classical φ^4-Theory 210
8.1.1 Exact FRG Flow Equations 212
8.1.2 Rescaled Flow Equations 213
8.1.3 FRG Flow Equations for a Sharp Momentum Cutoff 217
8.2 Recovering the Momentum Shell Results from the FRG 219
8.3 Momentum-Dependent Self-Energy in the Symmetric Phase 221
8.3.1 Scaling Functions 221
8.3.2 Truncation Strategy Based on Relevance 222
8.3.3 FRG Results for the Self-Energy Scaling Function 228
8.4 Momentum-Dependent Self-Energy in the Symmetry-Broken Phase 229
References 232

9 Derivative Expansion 233
9.1 Derivative Expansion for the $O(N)$-Symmetric Classical φ^4-Theory 233
9.2 Local Potential Approximation 237
9.2.1 RG Equation for the Effective Potential 237
9.2.2 Fixed Points and Critical Exponents 239
9.3 Beyond the Local Potential Approximation 242
References 247

Part III Functional Renormalization Group Approach to Fermions

10 Fermionic Functional Renormalization Group 255
10.1 Symmetries of the Two-Fermion Interaction 255
10.2 Exact FRG Flow Equations for the Irreducible Vertices 259
10.2.1 From Superfield to Partially Symmetrized Notation 259
10.2.2 Exact FRG Flow Equations 262
10.2.3 $SU(2)$-Invariant Flow Equations 268
10.3 Single-Channel Truncations 269

10.3.1 Ladder Approximation in the Particle–Particle Channel . . 269
10.3.2 Random Phase Approximation in the Forward Scattering Channel 273
10.4 Rescaled Flow Equations and Definition of the Fermi Surface 274
10.4.1 Scaling Toward the Fermi Surface 275
10.4.2 Classification of Couplings 282
10.4.3 Exact Integral Equation for the Fermi Surface 284
10.5 One-Loop Patching Approximations 285
10.5.1 Flow of Marginal Couplings 287
10.5.2 Spinless Fermions 290
10.5.3 One-Dimensional g-ology for $SU(2)$-Invariant Models . . . 292
10.5.4 Many-Patch RG for the Square-Lattice Hubbard Model . . 297
References 302

11 Normal Fermions: Partial Bosonization in the Forward Scattering Channel 305
11.1 Hubbard–Stratonovich Transformation in the Forward Scattering Channel 307
11.2 Exact FRG Flow Equations 310
11.3 Dyson–Schwinger and Skeleton Equations 315
11.4 Ward Identities 317
11.5 Exact Solution of the FRG Flow Equations for Fermions with Linear Dispersion via Ward Identities 322
References 324

12 Superfluid Fermions: Partial Bosonization in the Particle–Particle Channel 327
12.1 Hubbard–Stratonovich Transformation in the Particle–Particle Channel 328
12.2 Mean-field Approximation and BCS–BEC Crossover 330
12.3 Gaussian Fluctuations 335
12.3.1 Gaussian Effective Action 335
12.3.2 Bosonic Propagators and Order-Parameter Correlations . . 339
12.4 Dyson–Schwinger Equations and Ward Identities 342
12.4.1 Dyson–Schwinger and Skeleton Equations 342
12.4.2 Ward Identities 345
12.5 FRG Approach with Total Momentum Cutoff 346
12.5.1 Superfield Notation 347
12.5.2 Truncation of the Vertex Expansion 351
12.5.3 Truncation with Momentum-Independent Self-Energies . . 357
12.6 Outlook 362
References 365

Index 369