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Introduction to the Maldacena Conjecture on AdS/CFT

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ABSTRACT

These lectures do not at all provide a general review of this rapidly growing field. Instead a rather detailed account is presented of a number of the most elementary aspects.

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1 Introduction

The Maldacena conjecture [1] is a conjecture concerning string theory or M theory on certain backgrounds of the form $AdS_d \times M_{D-d}$. Here AdS_d is an anti de Sitter space of space-time dimension d, and M_{D-d} is a certain compactification space of dimension $D-d$ with $D = 10$ for string theory and $D = 11$ for M theory. In addition, the background is specified by a statement about the flux of a certain field strength differential form. The conjecture asserts that the quantum string- or M-theory on this background is mathematically equivalent - or dual as the word goes - to an *ordinary* but conformally invariant quantum field theory in a space-time of dimension $d-1$, which in fact has the interpretation of "the boundary" of AdS_d. This seems to put the formulation of string/M-theory on a novel and rather unexpected footing. Also the relation between quantum and classical theory is illuminated in a surprising way by the conjecture. Several details in Maldacena's original formulation were left unspecified. Most of those were subsequently given a precise formulation by independent works of Gubser, Klebanov and Polyakov [2] and by Witten [3]. A priori it might seem very strange that quantum theories in different space-time dimensions could be equivalent. This possibility is related to the fact that the theory in the larger dimension is (among other things) a quantum theory of gravity. For such theories the concept of *holography* has been introduced as a generic property, and the Maldacena conjecture is an example of the realization of that (for discussion, see for example [4]).

In the meantime a large number of checks have been performed which we shall not attempt to review in these notes (for some recent reviews with many additional references, see for example [5, 6, 7, 8]). Supposing the conjecture is true, it remains somewhat unclear what the most significant consequence will be. On the one hand the conjecture allows one to obtain non perturbative information on ordinary, but mostly *conformally invariant* quantum field theories, especially at large N (of a gauge group $U(N)$), from classical string/M-theory or even classical supergravity. This is a remarkable unexpected development, and the one that has mostly been pursued until now. On the other hand it is conceivable that the conjecture will play an important role in the eventual non-perturbative formulation of M-theory, for which the matrix model of BFSS [9] was a first proposal. In a somewhat different line of development, Witten [10] showed how to apparently overcome the original restriction to conformally invariant (and mostly supersymmetric) quantum field theories, providing in fact an entirely new framework for studying large N "ordinary" QCD and similar theories. A rather new idea about how to achieve the same end in perhaps a more efficient way has recently appeared [11]. That approach, however will not be covered here at all (see also [5]). In any case, the AdS/CFT development attracts an enormous interest.

In these lectures we shall attempt a very elementary introduction to a somewhat restricted number of basic aspects. In sect. 2 we begin by reviewing properties of anti de Sitter spaces, their isometries, the fact that they may be associated with a "boundary" and the fact that the isometry group of anti de Sitter space becomes the conformal group on the boundary. It follows that *if* a quantum theory on anti de Sitter space is dual to another quantum theory on the boundary, then that second theory must necessarily be conformal.

In sect. 3 we expand on the discussion in [1] and provide a short review of classical supergravity solutions in the presence of branes. Both so called extremal (BPS) and non-extremal solutions will be considered for later reference. This subject has already been reviewed on numerous occasions (see for example [12, 13, 14, 15]). We describe how the so called near horizon approximation in some cases lead to geometries of the form $AdS_d \times S^{D-d}$. This fact has been known for several years by the experts, but its full significance was only realized by Maldacena.

In sec. 4 we follow rather closely [3] and describe in detail several instructive albeit rather trivial examples of how the duality between the bulk theory and the boundary theory works in the case of free theories. An important object which has a general significance is the generalized propagator describing propagation of certain modes from a space-time point in the bulk of anti de Sitter space to a "point" on the boundary. This propagator was the key object in the discussions in [2, 3] and will be constructed in a few of the simplest cases. At the same time the Maldacena conjecture will be made more precise.

In sect. 5 we follow [10] and describe how certain finite temperature scenarios may be used to provide a mechanism for breaking conformal invariance and supersymmetry, and thereby obtain a framework for studying large N QCD.

2 Elementary properties of anti de Sitter spaces

We begin by considering the Einstein-Hilbert action with a cosmological term.

$$S = -s\frac{1}{16\pi G_D}\int d^D x\sqrt{|g|}(R+\Lambda) \tag{1}$$

We consider (first) Minkowski metric with $s=-1$, and we take it to be "mostly plus". We shall also consider Euclidean signature, $s=+1$. Notice that the sign of the action flips if we go from a "mostly plus" to a "mostly minus" metric. Anti de Sitter space (AdS) as well as de Sitter space are solutions of the empty space Einstein equation:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}\Lambda g_{\mu\nu} \Rightarrow \\ R &= \frac{D}{2-D}\Lambda \Rightarrow \\ R_{\mu\nu} &= \frac{\Lambda}{2-D}g_{\mu\nu} \end{aligned} \tag{2}$$

So these spaces have the property that the Ricci tensor is proportional to the metric tensor: They are Einstein spaces. We shall be interested in various examples of such spaces, in particular in ones with maximal symmetry, for which in addition we have

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}) \tag{3}$$

Such spaces are (for $R \neq 0$): spheres, S^D, de Sitter spaces, dS_D, and anti de Sitter spaces, AdS_D. The difference between de Sitter space and anti de Sitter space is the sign of the cosmological constant. With the above conventions, AdS spaces have $\Lambda > 0$ (see below).

2.1 AdS_{n+1} by embedding

It is useful to consider an $(n+1)$-dim AdS_{n+1} as a submanifold of a pseudo-Euclidean $(n+2)$-dimensional embedding space with coordinates $(y^a) = (y^0, y^1, ..., y^n, y^{n+1})$ and metric

$$\eta_{ab} = \mathrm{diag}(+, -, -, ..., -, +)$$

with "length squared"

$$y^2 \equiv (y^0)^2 + (y^{n+1})^2 - \sum_{i=1}^{n} (y^i)^2$$

preserved by the "Lorentz-like" group $SO(2,n)$ (with "two times") acting as

$$y^a \to y'^a = \Lambda^a{}_b y^b, \quad \Lambda^a{}_b \in SO(2,n) \tag{4}$$

A possible definition of AdS_{n+1} is then as the locus of

$$y^2 = b^2 = \text{const.} \tag{5}$$

For de Sitter spaces we would use a "mostly plus" metric and the same definition (or equivalently, $b^2 \to -b^2$) and similarly for the spherical spaces, for which of course the metric is positive definite. We shall demonstrate below that this implies eq.(3).

If instead of AdS_{n+1} we consider the $n+1$ dimensional Minkowski space, we know that our theory should be invariant under the Poincaré group, which in $n+1$ dimensions has dimension $n+1$ (for the translations) plus $\frac{1}{2}n(n+1)$ (for the Lorenz transformations) in total $\frac{1}{2}(n+1)(n+2)$. In fact the Poincaré group is exactly the isometry group of flat space: invariant intervals, squared, are preserved by the Poincaré group. With the definition of AdS_{n+1} just given, it is obvious that the isometry group of that space instead is $SO(2,n)$. In fact, let $y_0^a, y_0^a + dy_{(1)}^a$ and $y_0^a + dy_{(2)}^a$ be 3 points lying in the submanifold given by eq.(5), and let $y'^a_0, y'^a_0 + dy'^a_{(1)}$ and $y'^a_0 + dy'^a_{(2)}$ be the corresponding images under the $SO(2,n)$ transformation eq.(4). Clearly these also lie in AdS_{n+1}, and in particular we have

$$dy_{(1)} \cdot dy_{(2)} = dy'_{(1)} \cdot dy'_{(2)} \tag{6}$$

where for any vectors in $(n+2)$-dimensional pseudo-Euclidean space

$$x \cdot y \equiv \eta_{ab} x^a y^b$$

Since the vectors $dy^a_{(1)}, dy^a_{(2)}, dy'^a_{(1)}, dy'^a_{(2)}$ are all vectors in AdS_{n+1}, this proves that the metric on AdS_{n+1} inherited from that of the embedding space, is $SO(2,n)$ invariant. It follows that quantum theories on AdS_{n+1} should have an $SO(2,n)$ invariance. Clearly the dimension of that group is (the same as the dimension of $SO(n+2)$) $\frac{1}{2}(n+1)(n+2)$. So the invariance group for theories on AdS is just a large as for theories on a flat space of the same dimensionality.

2.1.1 Polar/Stereographic coordinates

Introduce coordinates $(x^\mu) = (x^1, ..., x^{n+1})$ on AdS_{n+1} by

$$\begin{aligned} y^0 &= \rho \frac{1+x^2}{1-x^2} \\ y^\mu &= \rho \frac{2x^\mu}{1-x^2}, \quad \mu = 1, ..., n+1 \end{aligned} \tag{7}$$

where

$$x^2 \equiv (x^1)^2 + ... + (x^n)^2 - (x^{n+1})^2$$

We may think of the set (ρ, x^μ) as a possible set of coordinates on the $(n+2)$-dimensional embedding space. Clearly $y^2 = \rho^2$ and AdS is $\rho = b$. The metric in the embedding space is (convention: "mostly minus")

$$ds^2 = (dy^0)^2 + (dy^{n+1})^2 - d\vec{y}^2 \tag{8}$$

where $d\vec{y} = (dy^1, ..., dy^n)$. From this we may work out the metric in x coordinates. We get (Notation: $\mu = 1, ..., n+1$ and we raise and lower μ by the flat Minkowski metric diag$(+, +, ..., +, -)$)

$$\begin{aligned} dy^0 &= d\rho \frac{1+x^2}{1-x^2} + 4\rho \frac{x_\mu dx^\mu}{(1-x^2)^2} \\ dy^\mu &= d\rho \frac{2x^\mu}{1-x^2} + \frac{2\rho}{(1-x^2)^2} \{(1-x^2)\delta^\mu_\nu + 2x^\mu x_\nu\} dx^\nu \end{aligned} \tag{9}$$

Then work out

$$ds^2 = d\rho^2 - \frac{4\rho^2}{(1-x^2)^2} dx^2 \tag{10}$$

We see that in these coordinates the metric factorizes into a trivial "radial" part and an interesting ("angular") AdS part:

$$g_{\mu\nu} = +\frac{4b^2}{(1-x^2)^2} \eta_{\mu\nu} \tag{11}$$

(convention: "mostly plus").

We now want to verify that this metric indeed satisfies the Einstein equation in vacuum eq.(2) with a cosmological term:

$$R_{\mu\nu} \propto g_{\mu\nu}$$

and determine the constant in terms of the dimension $D = n+1$ and the AdS-scale, b. We shall do even more, and verify that the spaces also satisfy the maximal symmetry condition, eq.(3).

Let us in fact consider a general "conformally flat" metric of the form

$$g_{\mu\nu}(x) = e^{\phi(x)} \eta_{\mu\nu}$$

In our case

$$\phi(x) = \log 4b^2 - 2\log(1-x^2)$$

Then work out

$$
\begin{aligned}
\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}) \\
&= \frac{1}{2}(\partial_\nu \phi \delta^{\mu}_{\rho} + \partial_\rho \phi \delta^{\mu}_{\nu} - \partial^\mu \phi \eta_{\nu\rho}) \\
R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho} \\
\partial_\rho \Gamma^{\mu}_{\nu\sigma} - (\rho \leftrightarrow \sigma) &= \frac{1}{2}\,(\delta^{\mu}_{\sigma}\partial_\rho\partial_\nu\phi + \delta^{\mu}_{\nu}\partial_\rho\partial_\sigma\phi \\
&\quad -\eta_{\nu\sigma}\partial_\rho\partial^\mu\phi) - (\rho \leftrightarrow \sigma) \\
&= \frac{1}{2}\left(\delta^{\mu}_{\sigma}\partial_\rho\partial_\nu\phi - \delta^{\mu}_{\rho}\partial_\sigma\partial_\nu\phi - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi + \eta_{\nu\rho}\partial_\sigma\partial^\mu\phi\right) \\
\Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - (\rho \leftrightarrow \sigma) &= \frac{1}{4}\left(\delta^{\mu}_{\rho}\partial_\nu\phi\partial_\sigma\phi + \eta_{\sigma\nu}\partial^\mu\phi\partial_\rho\phi + \eta_{\nu\rho}\delta^{\mu}_{\sigma}(\partial\phi)^2\right) - (\rho \leftrightarrow \sigma) \\
\partial_\mu\phi\partial_\nu\phi &= \frac{16 x_\mu x_\nu}{(1-x^2)^2} \\
\partial_\mu\partial_\nu\phi &= \frac{4}{1-x^2}\eta_{\mu\nu} + \frac{8 x_\mu x_\nu}{(1-x^2)^2} \\
R^{\mu}{}_{\nu\rho\sigma} &= -\frac{4}{(1-x^2)^2}\left(\eta_{\nu\sigma}\delta^{\mu}_{\rho} - \eta_{\nu\rho}\delta^{\mu}_{\sigma}\right) \\
&= -\frac{1}{b^2}(-g_{\nu\rho}\delta^{\mu}_{\sigma} + g_{\nu\sigma}\delta^{\mu}_{\rho})
\end{aligned} \tag{12}
$$

the last equality being the statement of maximal symmetry. Then further

$$
R_{\nu\sigma} = -\frac{D-1}{b^2} g_{\nu\sigma}, \quad (D \equiv n+1) \tag{13}
$$

And we see that indeed eq.(2) is satisfied with

$$
\Lambda = \frac{n(n-1)}{b^2} = \frac{(D-1)(D-2)}{b^2} \tag{14}
$$

(Notice that under a shift in convention $g_{\mu\nu} \to -g_{\mu\nu}$, i.e. "mostly plus" $\to$ "mostly minus", $R_{\mu\nu}$ is unchanged, so the cosmological term will appear with the opposite sign.)

Exercise: Show for the conformally flat metric, that in general

$$
R_{\mu\nu} = \left(1 - \frac{D}{2}\right)(\partial_\nu\partial_\mu - \frac{1}{2}\partial_\nu\phi\partial_\mu\phi) + \frac{1}{2}\eta_{\mu\nu}\left(\left[1 - \frac{D}{2}\right](\partial\phi)^2 - \partial^2\phi\right) \tag{15}
$$

Use this to provide yet another derivation of 13.

We shall also need to consider the case of Euclidean signature or imaginary times. Then

$$
dx^2 = \sum_{\mu=1}^{n+1}(dx_\mu)^2
$$

In such coordinates the "Euclidean version" of AdS_{n+1} is topologically the ball B_{n+1}

$$\sum_{\mu=1}^{n+1}(x_\mu)^2 < 1$$

The "boundary" of the ball lies infinitely far away as measured in the AdS metric. We shall come back to that.

Exercise: In the Euclidean case, AdS_{n+1} may be viewed as the hyperbola

$$(y^0)^2 - r^2 = b^2$$

where

$$r^2 \equiv \sum_{\mu=1}^{n+1}(y^\mu)^2$$

Denoting the point $(y^0, r) = (-b, 0)$ as the "South Pole", show that the coordinates x^μ eq.(7) (Euclidean version) are the "stereographic" projections of AdS_{n+1} from the South Pole to the "equatorial plane" $y^0 = 0$ (in units of b).

Now define "light cone coordinates"

$$u = y^0 + iy^{n+1}, \quad v = y^0 - iy^{n+1} \tag{16}$$

or for Euclidean signature where the Euclidean AdS_{n+1} is the locus of

$$y_E^2 \equiv (y^0)^2 - (y^{n+1})^2 - \vec{y}^2 = b^2$$

with isometry group $SO(1, n+1)$,

$$u = y^0 + y^{n+1}, \quad v = y^0 - y^{n+1} \tag{17}$$

so that in both cases

$$y^2 = uv - \vec{y}^2 = b^2$$

We consider in turn various coordinates on AdS_{n+1} and the corresponding metrics.

The first set is the one used for example by Maldacena in [1]. Define

$$\begin{aligned} \xi^\alpha &\equiv \frac{y^\alpha}{u}, \quad \alpha = 1, ..., n \\ \vec{\xi}^2 &\equiv \sum_{\alpha=1}^{n}(\xi^\alpha)^2 \end{aligned} \tag{18}$$

then

$$\begin{aligned} y^2 &= uv - \vec{y}^2 = uv - u^2\vec{\xi}^2 = b^2 \\ \Rightarrow v &= \xi^2 u + \frac{b^2}{u} \end{aligned} \tag{19}$$

Use the set (u,ξ^α) on AdS_{n+1}:

$$
\begin{aligned}
dv &= 2\xi\cdot d\xi u + \xi^2 du - \frac{b^2 du}{u^2} \\
dy^\alpha &= u d\xi^\alpha + \xi^\alpha du \\
(ds^2)_{\text{embedding}} &= dudv - \vec{dy}^{\,2} \\
&= -\frac{b^2 du^2}{u^2} - u^2 d\vec{\xi}^2 \quad (y^2 \equiv b^2) \\
\Rightarrow (ds^2)_{AdS_{n+1}} &= +\frac{b^2 du^2}{u^2} + u^2 d\vec{\xi}^2 \quad (\text{“mostly plus”})
\end{aligned}
\tag{20}
$$

The second set is similar to Poincaré coordinates on the projective plane. For simplicity put $b \equiv 1$ and use the set

$$(\xi^0, \vec{\xi}) \equiv (u^{-1}, \vec{\xi})$$

Then $\log u = -\log \xi^0$ and $\frac{du}{u} = -\frac{d\xi^0}{\xi^0}$. Hence

$$ds^2 = \frac{(d\xi^0)^2}{(\xi^0)^2} + \frac{d\vec{\xi}^2}{(\xi^0)^2} = \frac{1}{(\xi^0)^2}\left((d\xi^0)^2 + d\vec{\xi}^2\right) \tag{21}$$

This is one of the forms used by Witten, [3].

2.2 The “boundary” of AdS_{n+1}

Anti de Sitter space has a kind of “projective boundary”. The idea is in embedding space to consider (y^0, y^μ) very large with $y \in AdS_{n+1}$. Hence define new variables

$$y^a = R\tilde{y}^a, \quad u = R\tilde{u}, \quad v = R\tilde{v} \tag{22}$$

and take $R \to \infty$. Then

$$y^2 = b^2 \Rightarrow \tilde{u}\tilde{v} - \vec{\tilde{y}}^2 = b^2/R^2 \to 0 \tag{23}$$

So the boundary is somehow the manifold

$$\tilde{u}\tilde{v} - \vec{\tilde{y}}^2 = 0 \tag{24}$$

But since tR is just as good as R for any $t \in \mathbb{R}$, we have to consider the boundary to be the projective equivalence classes

$$
\begin{aligned}
uv - \vec{y}^2 &= 0 \\
(u,v,\vec{y}) &\sim t(u,v,\vec{y})
\end{aligned}
\tag{25}
$$

so the boundary is n-dimensional – as it should be. Using the equivalence scaling, the boundary may be considered to be represented by (Minkowski signature)

$$(y^0)^2 + (y^{n+1})^2 = 1 = \vec{y}^2 \tag{26}$$

so that topologically the boundary is $S^1 \times S^{n-1}$. In another use of scaling, for points with $v \neq 0$ we may scale to $v = 1$. Then $u = \vec{y}^2$ and we may use $\vec{y}$ as coordinates on the boundary. Equivalently, if also $u \neq 0$ we may instead scale u to 1 and use coordinates $\vec{\tilde{y}}$ and have $v = \vec{\tilde{y}}^2$. Clearly the connection between the two sets is

$$\vec{\tilde{y}} = \frac{\vec{y}}{y^2} \tag{27}$$

When either $v = 0$ or $u = 0$, only one of the two sets may be used. For $v = 0$, $\vec{\tilde{y}} = \vec{0}$ whereas for $u = 0$, $\vec{y} = \vec{0}$. We may think of the (one) point $v = 0$ as "the point at infinity" in the $\vec{y}$ coordinates, and similarly for $u = 0$. So the boundary is automatically compactified. The situation is analogous to compactifying the Riemann sphere including the point $z = \infty$ with z a good coordinate in a neighbourhood of $z = 0$, and $\zeta = 1/z$ a good coordinate in a neighbourhood of $z = \infty$.

The above definition of AdS_{n+1} and its boundary in terms of the embedding space, implies that the isometry group $SO(2, n)$ ($SO(1, n+1)$ for Euclidean signature) acts in an obvious way on points of the boundary. The crucial result on which we would like to elaborate, is that *the isometry group $SO(2, n)$ ($SO(1, n+1)$) acts on the boundary as the conformal group acting on Minkowski (Euclidean) space.*

2.3 The conformal group

For definiteness, consider n-dimensional Euclidean space E^n. We first want to understand that the conformal group is $SO(1, n+1)$. Begin by counting the number of generators = number of generators in $SO(n+2)$ = number of linearly independent antisymmetric $(n+2) \times (n+2)$ matrices

$$\dim SO(1, n+1) = \frac{1}{2}(n+2)(n+1) \tag{28}$$

By comparison, the Poincaré group in n dimensions has n translation generators and $\frac{1}{2}n(n-1)$ rotation generators

$$\text{dim Poincaré}(E^n) = \frac{1}{2}n(n+1) \tag{29}$$

so the difference is $n+1$. This just fits with the following "extra" possible conformal transformations:

<u>Dilations</u>

$$\vec{x} \to \lambda \vec{x}, \quad \lambda \in \mathbb{R} \tag{30}$$

gives one generator and

<u>The Special Conformal Transformations</u>

$$\begin{aligned} \vec{x} &\to \vec{x}' \text{ such that} \\ \frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} + \alpha^\mu \end{aligned} \tag{31}$$

involve the n parameters $\alpha^\mu, \mu = 1, ..., n$ and give rise to the additional n generators. Equivalently we may write

$$x'^\mu = \frac{x^\mu + \alpha^\mu x^2}{1 + 2\vec{\alpha} \cdot \vec{x} + \alpha^2 x^2} \tag{32}$$

The equivalence between eq.(31) and eq.(32) follows after noting (from eq.(32)) that

$$x'^2 = \frac{x^2}{1 + 2\vec{\alpha}\cdot\vec{x} + \alpha^2 x^2} \tag{33}$$

To verify that these are really *conformal* transformations, consider 3 neighbouring points

$$\vec{x}, \vec{x} + d\vec{x}_1, \vec{x} + d\vec{x}_2$$

and their images

$$\vec{x}', \vec{x}' + d\vec{x}'_1, \vec{x}' + d\vec{x}'_2$$

The statement that the transformation is conformal, is the statement that the angles are preserved, or

$$\frac{d\vec{x}_1 \cdot d\vec{x}_2}{\sqrt{dx_1^2 dx_2^2}} = \frac{d\vec{x}'_1 \cdot d\vec{x}'_2}{\sqrt{(dx'_1)^2 (dx'_2)^2}} \tag{34}$$

But eq.(31) implies

$$\begin{aligned} \frac{x'^2 dx'^\mu - 2\vec{x}' \cdot d\vec{x}' x'^\mu}{(x'^2)^2} &= \frac{x^2 dx^\mu - 2\vec{x}\cdot d\vec{x} x^\mu}{(x^2)^2} \\ \Rightarrow \frac{d\vec{x}'_i \cdot d\vec{x}'_j}{x'^4} &= \frac{d\vec{x}_i \cdot d\vec{x}_j}{x^4}, \quad i,j = 1,2 \end{aligned} \tag{35}$$

and the claim follows.

Now we want to show that the action of $SO(1, n+1)$ on boundary points give conformal transformations. A point in AdS_{n+1}: $(u, v, \vec{y})$ with $uv - \vec{y}^2 = b^2$ is mapped by $SO(1, n+1)$ to $(u', v', \vec{y}')$ as

$$\Lambda \begin{pmatrix} u \\ v \\ \vec{y} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \vec{y}' \end{pmatrix}, \tag{36}$$

where $\Lambda \in SO(1, n+1)$ i.e. Λ preserves the norm $uv - \vec{y}^2$.

Similarly, a point on the boundary has coordinates $(u, v, \vec{y})$ subject to

$$\begin{aligned} (i) \qquad & uv - \vec{y}^2 = 0 \\ (ii) \qquad & (u, v, \vec{y}) \sim \lambda(u, v, \vec{y}) \end{aligned} \tag{37}$$

and is mapped by Λ to $(u', v', \vec{y}')$ as before. (Notice that of course $(u_2, v_2, \vec{y}_2) = \lambda(u_1, v_1, \vec{y}_1) \Rightarrow (u'_2, v'_2, \vec{y}'_2) = \lambda(u'_1, v'_1, \vec{y}'_1)$).

Now consider the infinitesimal transformation $\Lambda = \mathbf{1}_{n+2} + \omega$ with ω infinitesimal. In order for the relevant norm to be preserved, the $(n+2) \times (n+2)$ dimensional matrix, ω must be of the form

$$\omega = \begin{pmatrix} a & 0 & \vec{\alpha}^T \\ 0 & -a & \vec{\beta}^T \\ \frac{1}{2}\vec{\beta} & \frac{1}{2}\vec{\alpha} & \omega_n \end{pmatrix} \tag{38}$$

where $\vec{\alpha}, \vec{\beta}$ are n-vectors represented as columns and ω_n is an $n \times n$ antisymmetric matrix. (The strange looking factors $\frac{1}{2}$ are due to the fact that we have a non-trivial metric on E^{n+2} in the coordinates $(u, v, \vec{y})$, and that our matrices have indices like $\Lambda^a{}_b$ rather than two lower indices, say.)

Indeed

$$(1_{n+2} + \omega) \begin{pmatrix} u \\ v \\ \vec{y} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \vec{y}' \end{pmatrix} = \begin{pmatrix} u(1+a) + \vec{\alpha} \cdot \vec{y} \\ v(1-a) + \vec{\beta} \cdot \vec{y} \\ \left(\vec{y} + \frac{u}{2}\vec{\beta} + \frac{v}{2}\vec{\alpha}\right) + \omega_n \vec{y} \end{pmatrix} \tag{39}$$

and one checks that $u'v' - \vec{y'}^2 = uv - \vec{y}^2$ to first order in the infinitesimal quantities, $a, \vec{\alpha}, \vec{\beta}, \omega_n$.

Now, choose a representative $(u, v, \vec{y})$ for a boundary point with $v = 1, u = \vec{y}^2$. (This can always be done except for $v = 0$ corresponding to "a point at infinity" on the boundary.) We have seen that with this representation, $\vec{y}$ is a convenient representation of the boundary point. Now, the image point according to eq.(39) is not in the same convention: $v' \neq 1$ in general. But the image point is equivalent to $(u'/v', 1, \vec{y}'/v')$, which is in the same convention. Thus, the effect of the mapping is

$$\vec{y} \to \vec{y}'/v' = \vec{y}(1 + a - \vec{\beta} \cdot \vec{y}) + \frac{y^2}{2}\vec{\beta} + \frac{1}{2}\vec{\alpha} + \omega_n \vec{y} \tag{40}$$

Let us verify that this transformation is in fact a combination of infinitesimal (i) translations (ii) (Lorentz-) rotations (iii) dilations and (iv) special conformal transformations:

(i) Only $\vec{\alpha} \neq 0 \Rightarrow$

$$\vec{y} \to \vec{y} + \frac{1}{2}\vec{\alpha} \tag{41}$$

i.e. translations.

(ii) Only $\omega_n \neq 0 \Rightarrow$

$$\vec{y} \to \vec{y} + \omega_n \vec{y} \tag{42}$$

i.e. rotations.

(iii) Only $a \neq 0 \Rightarrow$

$$\vec{y} \to \vec{y}(1 + a) \tag{43}$$

i.e. dilation.

(iv) Only $\vec{\beta} \neq 0 \Rightarrow$

$$\vec{y} \to \vec{y}(1 - \vec{\beta} \cdot \vec{y}) + \frac{1}{2} y^2 \vec{\beta} \tag{44}$$

If we compare with eq.(32) and put $\vec{\alpha}$ in that equation equal to $\vec{\beta}/2$ we find

$$\vec{y} \to \frac{\vec{y} + \frac{1}{2}\vec{\beta} y^2}{1 + \vec{\beta} \cdot \vec{y} + \frac{1}{4}\beta^2 y^2} = \vec{y}(1 - \vec{\beta} \cdot \vec{y}) + + \frac{1}{2} y^2 \vec{\beta} + O(\beta^2) \tag{45}$$

in agreement with the above, i.e. indeed we find in this case the (infinitesimal) special conformal transformations.

This completes the main result in this section that $SO(1, n+1)$ (and $SO(2, n)$ in the Minkowski case) acts (i) as an isometry on AdS_{n+1} and (ii) as the conformal group on the boundary of AdS_{n+1}.

2.4 The conformal algebra

For completeness let us work out the Lie algebra of the conformal group. We choose the simplest possible representation, which is in terms of scalar fields $\phi(x)$ with $\vec{x}$ an n-tuple of Cartesian coordinates. It is trivial to check that the generators are represented as follows:
<u>Translations</u> $P_\mu = -i\partial_\mu$
<u>(Lorentz-)rotations</u> $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) = -(x_\mu P_\nu - x_\nu P_\mu)$
<u>Dilations</u> $D = -ix^\mu\partial_\mu$
<u>Special Conformal Transformations</u> $K_\mu = i(2x_\mu x\cdot\partial - x^2\partial_\mu) = -2x_\mu D + x^2 P_\mu$

One then easily finds:

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \\
[M_{\mu\nu}, M_{\rho\tau}] &= i\left(g_{\mu\tau}M_{\nu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau} - g_{\nu\tau}M_{\mu\rho}\right) \\
[M_{\mu\nu}, K_\rho] &= i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu) \\
[D, P_\mu] &= +iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[P_\mu, K_\nu] &= 2i(g_{\mu\nu}D + M_{\mu\nu}) \qquad (46)
\end{aligned}$$

all others zero.

3 The Maldacena Conjecture

3.1 On Dp-branes and other p-branes.

In this section we first briefly review the construction of solitonic p-branes in low energy effective supergravity. There are many excellent reviews available, for example. refs. [12, 13, 16, 14] to which we refer for more details and references to the extensive original literature. We begin by writing down the effective low energy string action for type II (A or B) strings in the string frame:

$$S_s = -s\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{|g|}\left(e^{-2\phi}(R+4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)-\frac{1}{2}\sum_n\frac{1}{n!}F_n^2+...\right) \tag{47}$$

($s=-1(+1)$ for Minkowski (Euclidean) signature, flipping to "mostly minus" signature introduces an additional sign of $(-)^n$ in front of F_n^2)). Here the dots represent fermionic terms as well as the NS-NS 3-form field strength term. ϕ is the dilaton, and the n-form field strengths F_n belong to the RR sector. For the Newton constant in D dimensions we write

$$16\pi G_D = 2\kappa_D^2$$

We shall only be concerned with the terms given. For IIA strings (IIB strings) we only have even (odd) values of n. For the IIB string the $n=5$ field strength tensor is self-dual (In Minkowski space, see below), and it is not strictly speaking possible to describe the theory by the simple action above. A more complicated formulation nonetheless exists [17]. However, it turns out to be suffcient to adopt the above action for deriving the equations of motions, and imposing self-duality a posteriori (making sure that the normalization of F_5^2 is unchanged). We shall therefore employ that procedure. It is convenient for various reasons to also represent the action for fields in the Einstein frame, obtained by a certain Weyl rescaling. In fact the following identity in D space-time dimensions may be verified [18]

$$\begin{aligned} g_{\mu\nu} &\rightarrow e^{2\sigma\phi}g_{\mu\nu} \Rightarrow \\ \sqrt{|g|}e^{-2\phi}R &\rightarrow \sqrt{|g|}e^{-\phi(\sigma(D-2)+2)}\{R+2\sigma(D-1)\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\partial^\mu\phi) \\ &\quad -\sigma^2(D-1)(D-2)(\partial\phi)^2\} \end{aligned} \tag{48}$$

We may therefore choose

$$\sigma = -\frac{2}{D-2}$$

and get rid of a total derivative, specifically in 10 dimensions:

$$g_{\mu\nu}(\text{Einstein}) = e^{-\frac{1}{2}\phi}g_{\mu\nu}(\text{string}) \tag{49}$$

so we obtain in the Einstein frame

$$S_E = -s\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{|g|}\left(R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\sum_n\frac{1}{n!}e^{a_n\phi}F_n^2+...\right) \tag{50}$$

with

$$a_n = -\frac{1}{2}(n-5)$$

We shall also be concerned with low-energy M-theory in the form of 11-dimensional super gravity. The bosonic fields of that theory are just the metric and a 3-form gauge potential C with a 4-form field strength tensor

$$K = dC$$

The bosonic part of the action is

$$S_{\text{bosonic}}(\text{11-dim SUGRA}) = -s\frac{1}{2\kappa_{11}^2}\left(\int d^{11}x\sqrt{|g|}\{R - \frac{1}{48}K^2\} - \frac{1}{6}\int C\wedge K\wedge K\right) \tag{51}$$

which only makes sense in the Einstein frame - there is no dilaton.

We shall be interested in classical solutions of the above theories, specifically in the ones describing Dp-branes. We shall consider static solutions corresponding to flat translationally invariant p-branes, isotropic in transverse directions. For such solutions the last term in eq.(51) will vanish. Hence one is able to cover all cases by considering the generic action

$$S = -s\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{g}\{R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sum_n \frac{1}{n!}e^{a_n\phi}F_n^2 + ..\} \tag{52}$$

in particular with $a = 0$ and $\phi \equiv 0$ for 11-dimensional supergravity. The p-brane is a source of charge for the $p+1$ form (RR-) gauge field and the $n = p+2$ form field strength. We write

$$D = (p+1) + d \tag{53}$$

where d is the number of dimensions transverse to the p-brane.

3.2 Summary on differential forms

An n-form F has components related to it by

$$F = \frac{1}{n!}F_{\mu_1\cdots\mu_n}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \tag{54}$$

We define the Levi-Civita symbol (unconventionally) as a non-tensor, so that simply

$$\epsilon_{01\ldots(D-1)} \equiv \epsilon^{01\ldots(D-1)} = 1 \tag{55}$$

From that we define the proper D-form, the volume form ω with tensor components

$$\omega_{\mu_1\ldots\mu_D} = \sqrt{g}\epsilon_{\mu_1\ldots\mu_D}, \quad \omega^{\mu_1\ldots\mu_D} = \frac{s}{\sqrt{g}}\epsilon^{\mu_1\ldots\mu_D} \tag{56}$$

Here $s = 1$ ($s = -1$) for Euclidean (Minkowski) metric (both mostly plus). We have dropped from now on the numerical signs around the determinant of the metric but they are always to be understood.

Integration of an n-form over a flat n-dimensional domain M becomes

$$\begin{aligned}
\int_M F_n &= \int_M F_{01\ldots(n-1)} dx^0 \wedge \ldots \wedge dx^{n-1} = \int d^n x F_{01\ldots(n-1)} \\
&= \frac{1}{n!}\int d^n x \epsilon^{\mu_1\ldots\mu_n} F_{\mu_1\ldots\mu_n}
\end{aligned} \tag{57}$$

Generally

$$V(M) = \int_M \omega = \int_M d^n x \sqrt{g_{\text{ind}}} \tag{58}$$

with $(g_{\text{ind}})_{ab}$ the induced metric on the sub manifold M. The Hodge dual satisfies

$$\begin{aligned}
(*F)^{\mu_{n+1}\ldots\mu_D} &= \frac{1}{n!}\omega^{\mu_1\ldots\mu_D} F_{\mu_1\ldots\mu_n} = \frac{1}{n!}\frac{s}{\sqrt{g}}\epsilon^{\mu_1\ldots\mu_D} F_{\mu_1\ldots\mu_n} \\
(*F)_{\mu_{n+1}\ldots\mu_D} &= \frac{1}{n!}\omega_{\mu_1\ldots\mu_D} F^{\mu_1\ldots\mu_n} = \frac{1}{n!}\sqrt{g}\epsilon_{\mu_1\ldots\mu_D} F^{\mu_1\ldots\mu_n} \\
(F\wedge *F)_{01\ldots(D-1)} &= \sqrt{g}\frac{1}{n!} F_{\mu_1\ldots\mu_n} F^{\mu_1\ldots\mu_n} \\
**F &= s(-1)^{n(D-n)} F \\
F\wedge *F &= s*F\wedge *(*F) \\
F^2 &\equiv F_{\mu_1\ldots\mu_n} F^{\mu_1\ldots\mu_n} \\
\frac{1}{n!}F^2 &= s\frac{1}{(D-n)!}(*F)^2
\end{aligned} \tag{59}$$

A self-dual tensor satisfies

$$*F = F \Rightarrow F = **F$$

which for a given dimension and rank is obviously impossible for both Minkowski signature and Euclidean signature. In particular, the 5-form field strength in 10 dimensional IIB string theory is self dual only for Minkowski signature. Even in Euclidean signature it continues to be true for the D3-brane solution considered below, that it will be of the form

$$F_5 = A_5 + *A_5 \tag{60}$$

Also

$$\begin{aligned}
*A &\equiv d*F \Rightarrow \\
A^{\mu_1\ldots\mu_{n-1}} &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu_1\ldots\mu_{n-1}\mu})
\end{aligned} \tag{61}$$

3.3 Equations of motion

The equations of motion for the generic problem (Einstein frame) eq.(52) are (often we shall write a for a_n):

$$R^\mu{}_\nu = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \frac{1}{2n!}e^{a\phi}\left(nF^{\mu\xi_2\ldots\xi_n}F_{\nu\xi_2\ldots\xi_n} - \frac{n-1}{D-2}\delta^\mu_\nu F_n^2\right)$$

$$
\begin{aligned}
\nabla^2\phi &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial_\nu\phi g^{\mu\nu}) = \frac{a}{2n!}F_n^2 \\
\partial_\mu(\sqrt{g}e^{a\phi}F^{\mu\nu_2\ldots\nu_n}) &= 0
\end{aligned}
\tag{62}
$$

where for simplicity we have considered the case with $F_n \neq 0$ only for one value of n, as will be the case. The Bianchi identity for F_n is

$$
\partial_{[\mu_1}F_{\mu_2\ldots\mu_{n+1}]} = 0 \tag{63}
$$

Our p-brane ansatz makes use of coordinates

$$
z^\mu = (t, x^i, y^a), \quad \mu = 0, ..., D-1; \quad i = 1, 2, ..., p; \quad a = 1, 2, ..., d; \quad D = p+1+d \tag{64}
$$

A metric respecting the symmetries is

$$
ds^2 = g_{\mu\nu}dz^\mu dz^\nu = sB^2dt^2 + C^2\sum_{i=1}^{p}(dx^i)^2 + F^2dr^2 + G^2r^2d\Omega_{d-1}^2 \tag{65}
$$

which is a diagonal metric, the components of which are all functions of the transverse "distance" coordinate,

$$
r^2 = \sum_{a=1}^{d}(y^a)^2
$$

only. Also $d\Omega_{d-1}^2$ is the metric on the unit sphere S^{d-1} in the transverse space. There is a gauge freedom which may be disposed of by putting $F = G$ or $G = 1$ or something else. We shall leave it for the time being in order to find a convenient form of the solutions in which we shall be interested. It is furthermore part of the p-brane ansatz to require, that the metric should tend to a flat value at $r \to \infty$, i.e. that all the coefficients B, C, F, G should tend to 1 in that limit.

A $p+1$ form gauge potential couples naturally to the world volume of the Dp brane. The resulting $p+2$ form field strength tensor is termed electric. However, we shall also need the magnetic possibility, which is a consequence of an electric/magnetic duality in the problem. In fact, defining

$$
\tilde{F}_{D-n} = e^{a\phi} * F_n \tag{66}
$$

and using the Hodge duality relations of the previous subsection, it is possible to verify that the equations of motions are invariant under the "duality transformations":

$$
a\phi \to -a\phi, \quad n \to D-n, \quad F_n \to \tilde{F}_{D-n} \tag{67}
$$

It is helpful first to establish

$$
\begin{aligned}
\sqrt{g}e^{a\phi}F^{\mu_1\ldots\mu_n} &= \frac{1}{(D-n)!}e^{\mu_1\ldots\mu_D}\tilde{F}_{\mu_{n+1}\ldots\mu_D} \\
\tilde{F}_{\mu_{n+1}\ldots\mu_D} &= \frac{1}{n!}e^{a\phi}F^{\mu_1\ldots\mu_n}\sqrt{g}\epsilon_{\mu_1\ldots\mu_D} \\
\frac{1}{n!}e^{a\phi}F_n^2 &= -\frac{1}{(D-n)}e^{-a\phi}\tilde{F}_{D-n}^2 \\
\frac{n}{n!}e^{a\phi}F^{\mu\xi_2\ldots\xi_n}F_{\nu\xi_2\ldots\xi_n} &= \frac{1}{(D-n)!}e^{-a\phi}\left((D-n)\tilde{F}^{\mu\xi_2\ldots\xi_{D-n}}\tilde{F}_{\nu\xi_2\ldots\xi_{D-n}} - \delta_\nu^\mu\tilde{F}_{D-n}^2\right)
\end{aligned}
\tag{68}
$$

The *electric ansatz* for the field strength is

$$F_{ti_1\ldots i_p r}(r) = \epsilon_{i_1\ldots i_p} k(r) \tag{69}$$

Since

$$\sqrt{g} = BC^p F (Gr)^{d-1} \sqrt{\gamma_{d-1}} \tag{70}$$

with $\gamma_{\alpha\beta}$ the metric on S^{d-1}, we find

$$F^{ti_1\ldots i_p r} = \frac{s}{B^2 C^{2p} F^2} \epsilon_{i_1\ldots i_p} k(r) \tag{71}$$

and the equation of motion for F_n becomes

$$\left(\frac{1}{BC^p F} (Gr)^{d-1} e^{a\phi} k(r) \right)' = 0 \tag{72}$$

with the result

$$\begin{aligned} k(r) &= e^{-a\phi} BC^p F \frac{Q}{(Gr)^{d-1}} \\ F_{ti_1\ldots i_p r} &= \epsilon_{i_1\ldots i_p} e^{-a\phi} BC^p F \frac{Q}{(Gr)^{d-1}} \end{aligned} \tag{73}$$

and Q a constant of integration.

$$\begin{aligned} \tilde{F}_{\alpha_1\ldots\alpha_{d-1}} &= \sqrt{\gamma_{d-1}} \epsilon_{\alpha_1\ldots\alpha_{d-1}} Q \\ \mu_p &= \frac{1}{\sqrt{16\pi G_D}} \int_{S^{d-1}} \tilde{F}_{d-1} = \frac{\Omega_{d-1} Q}{\sqrt{16\pi G_D}} \end{aligned} \tag{74}$$

where μ_p is the density of electric charge on the p-brane. The $\alpha_i = 1, ..., d-1$ are indices on the unit sphere in the transverse space, and Ω_{d-1} is the volume of S^{d-1}:

$$\begin{aligned} \Omega_d &= \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \\ \Omega_{2n-1} &= \frac{2\pi^n}{(n-1)!} \\ \Omega_{2n} &= \frac{2(2\pi)^n}{(2n-1)!!} \end{aligned} \tag{75}$$

For future reference we work out

$$\begin{aligned} \frac{1}{n!} F_n^2 &= F_{t12\ldots pr} F^{t12\ldots pr} = s e^{-2a\phi} \frac{Q^2}{(Gr)^{2(d-1)}} \\ \frac{1}{(n-1)!} F^{\mu\xi_2\ldots\xi_n} F_{\nu\xi_2\ldots\xi_n} &= \delta^\mu_\nu F_{t1\ldots pr} F^{t1\ldots pr} = s \delta^\mu_\nu e^{-2a\phi} \frac{Q^2}{(Gr)^{2(d-1)}} \end{aligned} \tag{76}$$

with $\mu, \nu \in \{t, 1, ..., p, r\}$, else 0.

The *magnetic ansatz* has $n = D-(p+2) = d-1$ with the non zero components of the field strength tensor given by

$$F_{\alpha_1\ldots\alpha_{d-1}} = \sqrt{\gamma_{d-1}}\epsilon_{\alpha_1\ldots\alpha_{d-1}}Q \tag{77}$$

and $Q = Q(r)$. But the equation of motion for F_n is trivially satisfied, whereas the Bianchi identity requires Q to be a constant. The magnetic charge (density) is

$$g_p = \frac{1}{\sqrt{16\pi G_D}}\int_{S^{d-1}} F_{d-1} = \frac{\Omega_{d-1}}{\sqrt{16\pi G_D}}Q \tag{78}$$

Now

$$F^{\alpha_1\ldots\alpha_{d-1}} = \frac{1}{\sqrt{\gamma_{d-1}}}\epsilon_{\alpha_1\ldots\alpha_{d-1}}\frac{Q}{(Gr)^{2(d-1)}} \tag{79}$$

from which

$$\begin{aligned} \frac{1}{n!}F_n^2 &= \frac{Q^2}{(Gr)^{2(d-1)}} \\ \frac{1}{(n-1)!}F^{\mu\xi_2\ldots\xi_n}F_{\nu\xi_2\ldots\xi_n} &= \delta^\mu_\nu\frac{Q^2}{(Gr)^{2(d-1)}} \end{aligned} \tag{80}$$

with δ^μ_ν non-vanishing only for $\mu\nu$ indices belonging to the sphere S^{d-1}. The similarity to the electric case will allow us to cover both possibilities at the same time.

To find the form of the equations of motion for our ansatz, we must work out the Riemann tensor for the metric. We choose to work via the spin connection, expressed in terms of the vielbein e^a_μ as (flat indices are either small latin letters from the beginning of the alphabet, or bar'ed Greek letters)

$$\begin{aligned} \omega_{\mu ab} &= \frac{1}{2}e^c_\mu(\Omega_{cab}+\Omega_{bac}+\Omega_{bca}) \\ \Omega_{abc} &= e^\mu_a e^\nu_b(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \\ \Omega_{bac} &= -\Omega_{abc}, \quad \omega_{\mu ba} = -\omega_{\mu ab} \\ R_{\mu\nu ab} &= S_{\mu\nu ab} + K_{\mu\nu ab} \\ S_{\mu\nu ab} &= \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} \\ K_{\mu\nu ab} &= \omega_{\mu a}{}^c\omega_{\nu cb} - \omega_{\nu a}{}^c\omega_{\mu cb} \end{aligned} \tag{81}$$

Our ansatz is of the "diagonal" type:

$$ds^2 = s(A_0)^2(dz^0)^2 + \sum_{\mu=1}^{D-1}(A_\mu)^2(dz^\mu)^2 \tag{82}$$

Then we may employ a diagonal vielbein

$$e^{\bar\mu}_\mu = A_\mu$$

and work out (no sums over μ and ν):

$$
\begin{aligned}
\Omega_{\bar\nu\bar\mu\bar\mu} &= -\Omega_{\bar\mu\bar\nu\bar\mu} = \eta_{\bar\mu\bar\mu}\frac{1}{A_\mu A_\nu}\partial_\nu A_\mu \\
\omega_{\mu\bar\mu\bar\nu} &= \eta_{\bar\mu\bar\mu}\frac{1}{A_\nu}\partial_\nu A_\mu \\
S_{\bar\mu\bar\nu\bar\mu\bar\nu} &= \eta_{\bar\nu\bar\nu}\frac{1}{A_\mu^2}\left(-\partial_\mu\partial_\mu \log A_\nu - (\partial_\mu \log A_\nu)^2 + (\partial_\mu \log A_\mu)(\partial_\mu \log A_\nu)\right) \\
&\quad +\eta_{\bar\mu\bar\mu}\frac{1}{A_\nu^2}\left(-\partial_\nu\partial_\nu \log A_\mu - (\partial_\nu \log A_\mu)^2 + (\partial_\nu \log A_\nu)(\partial_\nu \log A_\mu)\right) \\
K_{\bar\mu\bar\nu\bar\mu\bar\nu} &= -\sum_{\kappa\neq\mu,\nu}\eta_{\bar\mu\bar\mu}\eta_{\bar\nu\bar\nu}\eta_{\bar\kappa\bar\kappa}\frac{1}{A_\kappa^2}(\partial_\kappa \log A_\mu)(\partial_\kappa \log A_\nu)
\end{aligned} \tag{83}
$$

When the metric only depends on one coordinate, r, these are the only non-vanishing components (up to symmetries). Also notice that $S_{\bar\mu\bar\nu\bar\mu\bar\nu}$ and $K_{\bar\mu\bar\nu\bar\mu\bar\nu}$ are never simultaneously non-vanishing.

We define $f(r)$ by

$$
f(r)r^{d-1} \equiv BC^pF^{-1}(Gr)^{d-1} \tag{84}
$$

Using the above formulas, we may then work out

$$
\begin{aligned}
R^{\bar t}{}_{\bar t} &= -\frac{1}{F^2}\left((\log B)'' + (\log B)'(\log(fr^{d-1}))'\right) \\
R^{\bar i}{}_{\bar i} &= -\frac{1}{F^2}\left((\log C)'' + (\log C)'(\log(fr^{d-1}))'\right) \\
R^{\bar r}{}_{\bar r} &= -\frac{1}{F^2}\left((\log(Ffr^{d-1}))'' - (\log F)'(\log(Ffr^{d-1}))' + ((\log B)')^2\right. \\
&\quad \left. +p((\log C)')^2 + (d-1)((\log Gr)')^2\right) \\
R^{\bar\alpha}{}_{\bar\alpha} &= -\frac{1}{F^2}\left((\log Gr)'' + (\log Gr)'(\log(fr^{d-1}))' - (d-2)\frac{F^2}{G^2r^2}\right)
\end{aligned} \tag{85}
$$

The equations of motion for the metric and the dilaton for the electric case then take the forms (no summation over indices):

$$
\begin{aligned}
R^{\bar t}{}_{\bar t} &= -\frac{(d-2)e^{-a_n\phi}n!}{2(D-2)}F_n^2 \equiv -(d-2)\frac{K^2}{F^2} \\
R^{\bar i}{}_{\bar i} &= -(d-2)\frac{K^2}{F^2} \\
R^{\bar r}{}_{\bar r} &= -(d-2)\frac{K^2}{F^2} + \frac{1}{2F^2}(\phi')^2 \\
R^{\bar\alpha}{}_{\bar\alpha} &= (p+1)\frac{K^2}{F^2} \\
\phi'' + \phi'(\log(fr^{d-1}))' &= a_n(D-2)K^2 \\
K^2 &\equiv \frac{1}{2(D-2)}e^{-a\phi}F^2\frac{Q^2}{(Gr)^{2(d-1)}}
\end{aligned} \tag{86}
$$

We see already here, that in order for the metric to reduce to a form similar to $AdS_q \times S^{D-q}$, the Riemann tensor in each sub space has to be proportional to the metric tensor, and a necessary condition therefore is that the dilaton decouples and becomes a constant (in particular, zero). This requires either $n = 5$ i.e. IIB string theory and $AdS_5 \times S^5$, or M-theory (or 11-dimensional supergravity) where 2-branes and 5-branes are possible corresponding to $AdS_4 \times S^7$ or $AdS_7 \times S^4$.

3.4 The extremal and non-extremal p-brane solutions

We begin by providing the final solution (in the Einstein frame)

$$\begin{aligned}
B &= f^{\frac{1}{2}} H^{-\frac{d-2}{\Delta}},\ C = H^{-\frac{d-2}{\Delta}},\ F = f^{-\frac{1}{2}} H^{\frac{p+1}{\Delta}},\ G = H^{\frac{p+1}{\Delta}},\ e^{\phi} = H^{a\frac{D-2}{\Delta}} \\
& \quad \text{i.e.} \\
ds^2 &= H^{-2\frac{d-2}{\Delta}} \left(s f dt^2 + \sum_{i=1}^{p} (dx^i)^2 \right) + H^{2\frac{p+1}{\Delta}} \left(f^{-1} dr^2 + r^2 (d\Omega_{d-1})^2 \right) \\
H &= 1 + \left(\frac{h}{r} \right)^{d-2},\ f = 1 - \left(\frac{r_0}{r} \right)^{d-2} \\
\Delta &= (p+1)(d-2) + \frac{1}{2} a^2 (D-2) \\
h^{2(d-2)} + r_0^{d-2} h^{d-2} &= \frac{\Delta Q^2}{2(d-2)(D-2)}
\end{aligned} \tag{87}$$

Notice that indeed the diagonal metric tensor components tend to 1 as $r \to \infty$. In the electric case

$$F_{ti_1 \ldots i_p r} = \epsilon_{i_1 \ldots i_p} H^{-2} \frac{Q}{r^{d-1}} \tag{88}$$

In the magnetic case the solutions are obtained by the above duality relations eq.(67). For the 5-form in IIB string theory we replace $F_5 \to F_5 + *F_5$.

The above solutions are not the most general ones, but represent a 2-parameter sub-family of solutions. For $r_0 = 0$ we have $f \equiv 1$ and we obtain the extremal solution, depending only on a single parameter, Q related to the common mass and charge density of the BPS D-brane. For $r_0 \neq 0$ a horizon develops at $r = r_0$.

Thus we are seeking a 2-parameter solution to be represented in a suitable form by some convenient gauge choice. Let us try the ansatz

$$\log \left(\frac{B}{C} \right) = c_B \log f,\ \log \left(\frac{F}{G} \right) = c_F \log f \tag{89}$$

with c_B and c_F constants to be sought for. Further define

$$g \equiv C^{p+1} G^{d-2} = f \frac{CG}{BF} = f^{1-(c_B - c_F)} \tag{90}$$

From the Einstein equations we derive

$$(\log B - \log C)'' + (\log B - \log C)' \left[(\log f)' + \frac{d-1}{r} \right] = 0 \tag{91}$$

or

$$(\log f)'' + ((\log f)')^2 + \frac{d-1}{r}(\log f)' = 0 \tag{92}$$

giving

$$\begin{aligned} f'' + \frac{d-1}{r}f' &= 0 \\ f &= 1 - \left(\frac{r_0}{r}\right)^{d-2} \end{aligned} \tag{93}$$

since we demand $f \to 1$ for $r \to \infty$, which is our first result for $f(r)$. From the equations of motion we further obtain

$$(\log g)'' + (\log g)' \left[(\log f r^{d-1})'\right] + \frac{d-1}{r}\left[(\log f)' + \frac{d-2}{r}\left(1 - \left(\frac{F}{G}\right)^2\right)\right] = 0 \tag{94}$$

This suggests trying to solve with $g \equiv 1$, hence $c_B - c_F = 1$. Then

$$(\log f)' + \frac{d-2}{r}(1 - f^{2c_F}) = 0 \tag{95}$$

or with the above solution for f

$$c_F = -\frac{1}{2},\ B = f^{\frac{1}{2}}C,\ F = f^{-\frac{1}{2}}G$$

With this the remaining equations of motion become

$$\begin{aligned} (\log G)'' + (\log G)'(\log f r^{d-1})' &= -(p+1)K^2 \\ (a\log G - \frac{p+1}{D-2}\phi)'' + (a\log G - \frac{p+1}{D-2}\phi)'(\log f r^{d-1})' &= 0 \Rightarrow \\ a\log G &= \frac{p+1}{D-2}\phi \end{aligned} \tag{96}$$

Exercise: Complete the calculation and derive the solution, eq.(87).

3.5 The extremal, non-dilatonic solutions. The near horizon approximation

The extremal solution is obtained by putting $r_0 = 0$. It corresponds to the brane being in the ground state in a quantum description. The non-extremal solution (presumably) represents excitations, corresponding to a definite temperature. In the extremal case we further consider the case with no dilaton coupling. We have already identified the relevant cases as

$$(D, n) = (10, 5),\ (11, 4),\ (11, 7)$$

For these values, fortuitously one has the "accidental" identity

$$\begin{aligned} \Delta &= (p+1)(d-2) = 2(D-2) \Rightarrow \\ h^{d-2} &= \frac{Q}{d-2} \end{aligned} \tag{97}$$

The solution then simplifies as follows:

$$
\begin{aligned}
f(r) &\equiv 1 \\
H &= 1 + \frac{Q}{(d-2)r^{d-2}} \\
ds^2 &= H^{-\frac{2}{p+1}}(sdt^2 + \sum_{i=1}^{p}(dx^i)^2) + H^{\frac{2}{d-2}} \sum_{a=1}^{d}(dy^a)^2 \\
\sum_{a=1}^{d}(dy^a)^2 &\equiv dr^2 + r^2(d\Omega_{d-1})^2
\end{aligned} \tag{98}
$$

We now want to consider N coincident branes. For a single extremal Dp brane, the flux as normalized in eq.(74) is given by [20, 16]

$$
\mu_p = T_p\sqrt{16\pi G_{10}} \tag{99}
$$

where the p-brane tension and Newton's constant are given by

$$
\begin{aligned}
T_p &= \frac{2\pi}{(2\pi\ell_s)^{p+1}g_s} \\
16\pi G_{10} &= \frac{(2\pi\ell_s)^8}{2\pi}g_s^2
\end{aligned} \tag{100}
$$

In fact in 11-dimensional supergravity, almost the same formulas apply, with an obvious change in dimensionality, but with the understanding that g_s is absent (say $g_s \equiv 1$ in 11 dimensions). With this somewhat vulgar notation, we may write

$$
\begin{aligned}
T_p &= \frac{2\pi}{(2\pi\ell)^{p+1}g_s} \\
16\pi G_D &= \frac{(2\pi\ell)^{D-2}}{2\pi}g_s^2
\end{aligned} \tag{101}
$$

The length ℓ is the string length $\ell_s^2 = \alpha'$ for $D = 10$ and the 11-dimensional Planck length for 11-dim. sugra. We therefore insist that we should choose Q so that

$$
\frac{\mu_p}{T_p\sqrt{16\pi G_D}} = N \tag{102}
$$

or from eq.(74)

$$
\frac{Q\Omega_{d-1}}{15\pi G_D}\frac{(2\pi\ell)^{p+1}g_s}{2\pi} = N \tag{103}
$$

or

$$
Q = Ng_s\frac{(2\pi\ell)^{d-2}}{\Omega_{d-1}}, \quad h_d^{d-2} = Ng_s\frac{(2\pi\ell)^{d-2}}{(d-2)\Omega_{d-1}} \tag{104}
$$

The Maldacena-conjecture arises by considering the so-called near horizon limit in which we consider the region very close to $r = 0$ and subsequently scale this region up in a singular way to be described. In this limit we simply have

$$
H \simeq \frac{h_d^{d-2}}{r^{d-2}} \tag{105}
$$

We see in particular that the r^2 in front of $d\Omega^2_{d-1}$ will get cancelled so that the metric becomes a direct product with an S^{d-1}. We now consider the various cases in turn.

$\underline{D = 10,\ p + 1 = 4,\ d = 6 \rightarrow AdS_5 \times S^5}$

This is the case of N $D3$ branes in IIB string theory. So according to the above prescription we take (cf. also eq.(75))

$$H = 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \tag{106}$$

Also define the scaled variable

$$U = r/\ell_s^2 \tag{107}$$

We consider the limit $\alpha' = \ell_s^2 \rightarrow 0$ and again also $r \rightarrow 0$ in such a way that U becomes the meaningful variable:

$$\begin{aligned} H &\simeq \frac{4\pi g_s N}{U^4 \ell_s^4} \\ ds^2 &= \ell_s^2 \left\{ \frac{U^2}{\sqrt{4\pi g_s N}} dx_4^2 + \sqrt{4\pi g_s N} \left(\frac{dU^2}{U^2} + d\Omega_5^2 \right) \right\} \\ &= \frac{U^2}{L^2} d\tilde{x}_4^2 + L^2 \frac{dU^2}{U^2} + L^2 d\Omega_5^2 \end{aligned} \tag{108}$$

with

$$dx_4^2 \equiv sdt^2 + \sum_{i=1}^{3} (dx^i)^2$$

and $\tilde{x}$ a suitably scaled version of the coordinate x. Comparing with eq.(20) we see that we exactly have the metric of $AdS_5(L) \times S^5(L)$ where we have indicated the length parameter, $L = b$, of AdS_5 and and the radius L of S^5. When $\ell_s \rightarrow 0$ the metric has to be rescaled to get a finite result - by removing the ℓ_s^2 overall factor. This is the singular blowing up alluded to above. The radius parameter is given by

$$L^4 = b^4 = 4\pi g_s N \ell_s^4 \tag{109}$$

Also, compared with eq.(20) $\xi^i = \frac{\alpha'}{b^2} x^i$.

$\underline{D = 11,\ p + 1 = 6,\ d = 5 \rightarrow AdS_7 \times S^4}$

This is the case of N $M5$-branes. We find

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}} dx_6^2 + H^{\frac{2}{3}} dy_5^2 = H^{-\frac{1}{3}} dx_6^2 + H^{\frac{2}{3}} (dr^2 + r^2 d\Omega_4^2) \\ H &= 1 + \frac{\pi N \ell_{11}^3}{r^3} \sim \frac{\pi N \ell_{11}^3}{r^3} \end{aligned} \tag{110}$$

This time define

$$\begin{aligned} U^2 &\equiv \frac{r}{\ell_{11}^3} \\ H &\simeq \frac{\pi N}{\ell_{11}^6} \frac{1}{U^6} \end{aligned} \tag{111}$$

Then

$$\begin{aligned}
dr &= \ell_{11} 2U dU, \quad r^2 = \ell_{11}^6 U^4 \\
ds^2 &= \ell_{11}^2 \left\{ \frac{U^2}{(\pi N)^{1/3}} dx_6^2 + 4(\pi N)^{2/3} \frac{dU^2}{U^2} + (\pi N)^{2/3} d\Omega_4^2 \right\} \\
&= \frac{U^2}{4L^2} d\tilde{x}_6^2 + 4L^2 \frac{dU^2}{U^2} + L^2 d\Omega_4^2 \qquad (112)
\end{aligned}$$

which this time is the metric of

$$AdS_7(2L) \times S^4(L)$$

with

$$L^2 = (\pi N)^{2/3} \ell_{11}^2 \qquad (113)$$

$\underline{D = 11,\ p+1 = 3,\ d = 8 \rightarrow AdS_4 \times S^7}$
This is the case of N $M2$-branes. Here

$$\begin{aligned}
ds^2 &= H^{-2/3} dx_3^2 + H^{1/3}(dr^2 + r^2 d\Omega_7^2) \\
H &= 1 + \frac{N \ell_{11}^6 2^5 \pi^2}{r^6} \simeq \frac{N \ell_{11}^6 2^5 \pi^2}{r^6} \qquad (114)
\end{aligned}$$

We introduce

$$\begin{aligned}
U^{\frac{1}{2}} &= \frac{r}{\ell_{11}^{3/2}} \\
H &\simeq \frac{2^5 \pi^2 N}{\ell_{11}^3 U^3} \\
ds^2 &= \ell_{11}^2 \left\{ \frac{U^2}{(2^5 \pi^2 N)^{2/3}} dx_3^2 + \left(\frac{1}{2} \pi^2 N \right)^{1/3} \frac{dU^2}{U^2} + 4 \left(\frac{1}{2} \pi^2 N \right)^{1/3} d\Omega_7^2 \right\} \\
&= \frac{4U^2}{L^2} d\tilde{x}_3^2 + \frac{L^2}{4} \frac{dU^2}{U^2} + L^2 d\Omega_7^2 \qquad (115)
\end{aligned}$$

corresponding to

$$AdS_4(\frac{1}{2}L) \times S^7(L)$$

with

$$L^2 = 4\ell_{11}^2 \left(\frac{1}{2} \pi^2 N \right)^{1/3} \qquad (116)$$

Exercise show that the spaces, $AdS_5(L) \times S^5(L)$, $AdS_7(2L) \times S^4(L)$ and $AdS_4(\frac{1}{2}L) \times S^7(L)$ are in fact *exact* solutions to the equations of motion of the appropriate low energy effective Lagrangians, in particular, that the last term in eq.(51) does not cause any modifications. Notice, however, that unlike the brane solutions, these solutions do not become asymptotically flat.

3.5.1 Non-extremal branes

For completeness and later reference we give her also the form of the non-extremal p-brane solutions eq.(87) in the near horizon approximation and with the same scaled U-variables as in the extremal case. We have used an independent scaling of the boundary coordinates $t, x^1, ..., x^{n-1}$ in some cases.

$\underline{D = 10,\ p + 1 = 4,\ d = 6 \to AdS_5 \times S^5}$

$$\begin{aligned} ds^2 &= \frac{U^2}{L^2}(sf(U)dt^2 + d\vec{x}_3^{\,2}) + L^2 \frac{dU^2}{f(U)U^2} + L^2 d\Omega_5^2 \\ L^2 &= \ell_s^2 \sqrt{4\pi g_s N} \\ f(U) &= 1 - \left(\frac{U_0}{U}\right)^4 \end{aligned} \tag{117}$$

$\underline{D = 11,\ p + 1 = 6,\ d = 5 \to AdS_7 \times S^4}$

$$\begin{aligned} ds^2 &= \frac{U^2}{4L^2}(sf(U)dt^2 + d\vec{x}_5^{\,2}) + 4L^2 \frac{dU^2}{f(U)U^2} + L^2 d\Omega_4^2 \\ L^2 &= \ell_{11}^2 (\pi N)^{2/3} \\ f(U) &= 1 - \left(\frac{U_0}{U}\right)^6 \end{aligned} \tag{118}$$

$\underline{D = 11,\ p + 1 = 3,\ d = 8 \to AdS_4 \times S^7}$

$$\begin{aligned} ds^2 &= \frac{4U^2}{L^2}(sf(U)dt^2 + d\vec{x}_2^{\,2}) + \frac{L^2}{4} \frac{dU^2}{f(U)U^2} + L^2 d\Omega_7^2 \\ L^2 &= 4\ell_{11}^2 \left(\frac{1}{2}\pi^2 N\right)^{1/3} \\ f(U) &= 1 - \left(\frac{U_0}{U}\right)^3 \end{aligned} \tag{119}$$

3.6 The Brane Theory and the Maldacena Conjecture

From the discussion in the previous section we know that IIB string theory on AdS_5 compactified on S^5 or M-theory on AdS_4 compactified on S^7 or finally on AdS_7 compactified on S^4, are all quantum theories with isometry groups (for Minkowski signature) $SO(2,4)$, $SO(2,3)$ and $SO(2,6)$ respectively. The remarkable Maldacena conjecture [1] is that these various quantum theories are exactly mathematically equivalent to (dual to) certain quantum theories on the boundary of the relevant AdS spaces, i.e. on the coincident branes.

What could these brane theories be? They would have to be conformally invariant quantum field theories according to the discussion in the previous section. And in particular for the case of $AdS_5 \times S^5$ the argumentation is perhaps not too far fetched. The important point is that we have seen that what we are looking at, is a small portion of space-time very close to the branes, and then subsequently blown up by formally letting $\alpha' \to 0$. But in precisely that limit we think we know what the effective quantum theory on the N coincident $D3$-branes should be [19]: It should be $N = 4$ super Yang Mills with gauge group $U(N)$. (For references on D-branes, see [20, 21]).

Let us try a heuristic argumentation: Excitations of D-branes may be thought of in terms of open strings with end-points on the D-branes. We are dealing with IIB string theory which is a theory of oriented strings, and there are N different branes for the strings to end on, so these open strings are automatically equipped with Chan-Paton labels relevant to $U(N)$. They interact with each other, but in the singular limit, $\alpha' \to 0$ we consider, only the zero-mass modes need be considered, and in 10-dimensions they form the $N = 1$ multiplet of pure Super Yang-Mills (for gauge group $U(N)$). Also, as has been known since the middle 1970'ies [22], they interact exactly according to that theory, the gauge coupling being related to the open string coupling g_s^o and α'. For a given Feynman-diagram, the corresponding open-string-diagram will have boundaries, which come with a natural orientation, because the string is oriented (the string connects two boundaries and is oriented, so the two boundaries are different: have opposite orientations). Further the different boundaries have labels $i = 1, ..., N$ equal to the label of the $D3$-brane. Thus the open string-diagram has an appearance identical to the (super) Yang-Mills Feynman-diagram in the 't Hooft double line representation for $U(N)$. In 10 dimensions such a string theory is anomalous, but here we are considering end points restricted to the D-branes and there is no such problem.

In our case, however, the zero-mass particles are confined strictly to the 4-dimensional world volume of the coincident $D3$ branes. Hence the theory is naturally 10-dim SYM dimensionally reduced to 4 dimensions, and that is exactly the $N = 4$ SYM theory mentioned above. The SYM-coupling g_{YM} is essentially the open string coupling constant which is itself the square root of the closed IIB string coupling constant. More precisely for N p-branes:

$$\frac{g_{YM}^2}{4\pi} = g_s(2\pi\ell_s)^{p-3} \tag{120}$$

(in a suitable normalization). The dimensionality is the well known one.

Exercise: Derive eq.(120) from the Born-Infeld-action

$$I_{BI} = T_p Tr \left\{ \int d^{p+1}x \sqrt{\det\left(G_{\mu\nu} + 2\pi\ell_2^2 F_{\mu\nu}\right)} \right\} \tag{121}$$

Here, for the gauge group $U(N)$, $F_{\mu\nu}$ is treated as an $N \times N$ matrix, but the "det" refers only to the $(p+1) \times (p+1)$ index structure of indices $\mu\nu$. The interpretation of the determinant in this case is via the symmetrized trace, denoted by Tr above. However, in the calculation needed here, only an expansion to 2nd order is required, and no ambiguity exists. So for a $D3$-brane

$$\frac{g_{YM}^2}{4\pi} = g_s, \quad D3\text{-brane} \tag{122}$$

From the perspective of the 4-dim. field theory, N is the N of the gauge group $U(N)$. From the perspective of IIB string theory on $AdS_5 \times S^5$, N is the flux (normalized carefully as in eq.(102)) through S^5.

So, the remarkable conjecture is that IIB quantum *string* theory on AdS_5 compactified on S^5 is identically equivalent to the quantum *field* theory in 4 dimensions. Let us collect a number of points in favour of this hypothesis. We shall try to compare properties of the two quantum theories: (i) IIB string theory on $AdS_5 \times S^5$ with N units of 5-form flux through S^5 - to be referred simply as IIB; and (ii) $N = 4$ super-Yang-Mills with gauge-group $U(N)$ in 4 dimensions - to be referred to simply as SYM.

First compare global symmetries. The IIB-theory has an isometry group $SO(2,4) \times SO(6)$ with the last $SO(6)$ being the isometry group of the 5-sphere. Actually, because spinors are involved the relevant groups for AdS_5 and the sphere, S^5 are the covering groups $SU(4)$ of $SO(6)$ and $SU(2,2)$ of $SO(2,4)$, so we have $SU(2,2) \times SU(4)$. But the 32 Majorana spinor supercharges of the IIB theory (which are all preserved in this background) transform under this symmetry in such a way that in fact the full invariance is given by the Lie-supergroup $SU(2,2|4)$.

Now we should try to understand that this is also the relevant invariance to consider for the SYM. We have already understood that the $SO(2,4)$ or $SU(2,2)$ part is realized as a conformal invariance. Indeed the SYM is known to have vanishing beta-function and be conformally invariant. How about $SO(6)$ (or $SU(4)$)? does the SYM theory know about the 5-sphere? Yes, indeed. That is the R-symmetry of SYM. In fact, consider briefly the field content of SYM:

In 10 dimensions, $N = 1$ pure Super Yang-Mills contains the gauge field potentials $A_\mu,\ \mu = 0, 1, ..., 9$ (the "gluons") giving $10-2 = 8$ bosonic physical degrees of freedom, all in the adjoint representation of the $U(N)$. Further we have the 8-dimensional Majorana-Weyl "gluinos" $\lambda_\alpha,\ \alpha = 1, ..., 8$, also all in the adjoint representation. The theory has 16 Majorana supercharges $Q_\alpha,\ \alpha = 1, ..., 16$. Under dimensional reduction, the gluon fields turn into $4 - 2 = 2$ gauge fields and a remaining 6 scalar fields, $\phi_1, ..., \phi_6$. The gluino fields turn into $4 \cdot 2$ Weyl spinors in 4 dimensions, $\lambda^A_\alpha,\ \alpha = 1, 2,\ A = 1, 2, 3, 4$. An $N = 1$ description in 4 dimensions put one of these spinors together with the gauge field in a gauge superfield, and the remaining 3 spinors each combine with a pair of scalars to give 3 scalar chiral superfields. The 16 supercharges turn into 4 sets of complex Majoranas $Q^A_\alpha, \bar{Q}^A_{\dot{\alpha}},\quad \alpha = 1, 2,\ A = 1, 2, 3, 4$ transforming as a $\{\mathbf{4}\}$ and a $\{\bar{\mathbf{4}}\}$ of the R-symmetry group $SU(4)$, and the ϕ_i transform as a $\{\mathbf{6}\}$ (the fundamental rep. of $SO(6)$ or the antisymmetric rank 2 tensor under $SU(4)$), so we see that $SO(6)$ (or $SU(4)$) is indeed present.

But the IIB theory had 32 fermionic supercharges, the SYM only 16. Indeed from the perspective of the N coincident BPS $D3$-branes, half the IIB supersymmetries are broken. In any case, where are the remaining 16 fermionic generators? The answer is that they arise as part of an extension of the conformal group that takes place when supersymmetry is present as described in the famous paper by Haag, Lopuzanski and Sohnius [23]. For completeness let us give here their form of the superconformal algebra:

Define first

$$P_{\alpha\dot{\alpha}} \equiv P_\mu \sigma^\mu_{\alpha\dot{\alpha}}, \quad K_{\alpha\dot{\alpha}} \equiv K_\mu \sigma^\mu_{\alpha\dot{\alpha}} \tag{123}$$

and

$$\sigma^\mu_{\ \alpha_1\dot{\beta}_1} \sigma^\nu_{\ \alpha_2\dot{\beta}_2} M_{\mu\nu} \equiv M_{\alpha_1\alpha_2} \epsilon_{\dot{\beta}_1\dot{\beta}_2} + \bar{M}_{\dot{\beta}_1\dot{\beta}_2} \epsilon_{\alpha_1\alpha_2} \tag{124}$$

In addition to the generators of the conformal algebra with commutation relations, eq.(46), one now has the *16 new fermionic generators*

$$Q^{(1)A}_{\beta}, \bar{Q}^{(1)A}_{\dot{\beta}}$$

obtained as

$$[K_{\alpha\dot{\beta}}, Q^A_\gamma] = 2i\epsilon_{\alpha\gamma}\bar{Q}^{(1)A}_{\dot{\beta}} \tag{125}$$

The R-symmetry $SU(4)$ generators, T^{AB}, commute with the supersymmetry generators in a way dictated by the fact that these transform as a fundamental and and anti-fundamental 4-dimensional multiplet under R. Some of the remaining commutation relations are

$$\begin{aligned}
[Q^A_\alpha, D] &= \frac{i}{2}Q^A_\alpha \\
[P_{\alpha\dot{\beta}}, \bar{Q}^{(1)A}_{\dot{\gamma}}] &= 2i\epsilon_{\dot{\beta}\dot{\gamma}}Q^A_\alpha \\
\{Q^A_\alpha, \bar{Q}^{(1)B}_{\dot{\beta}}\} &= \delta^{AB}K_{\alpha\dot{\beta}} \\
\{Q^A_\alpha, Q^{(1)B}_{\beta}\} &= \delta^{AB}(\epsilon_{\alpha\beta}D + M_{\alpha\beta}) + i\epsilon_{\alpha\beta}T^{AB} \\
[\bar{Q}^{(1)A}_{\dot{\alpha}}, T^{BC}] &= 2\delta^{AB}\bar{Q}^{(1)C}_{\dot{\alpha}} - \frac{1}{2}\delta^{BC}\bar{Q}^{(1)A}_{\dot{\alpha}}
\end{aligned} \tag{126}$$

Indeed the combination of eq.(46) and eq.(126) together with the standard $N=4$ supersymmetry algebra constitute the Lie super-algebra $SU(2,2|4)$.

It is well known that the IIB theory (almost certainly) contains a (non-perturbative) $SL(2,\mathbb{Z})$ invariance [24]. It is best viewed as arising from compactification of M-theory on a 2-torus with modular parameter

$$\tau = \chi + ie^{-\phi}$$

with χ the RR-scalar of IIB (the "axion"). In $N=4$ Super Yang-Mills there is a corresponding $SL(2,\mathbb{Z})$ invariance of the theory with modular parameter in this case

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}$$

In this latter case the symmetry is represented by an $SL(2,\mathbb{Z})$ transformation of the lattice of electric and magnetic charges in that theory: $q + ig = g_{YM}(n_e + \tau n_m)$ by treating (n_e, n_m) as a doublet.

This finishes our very brief comparison of symmetries of the two theories.

3.7 Implication of the Maldacena conjecture

We continue for definiteness to focus on the case of $AdS_5 \times S^5$. As we have seen in eq.(109) the common "radius-" or length-parameter is given be

$$b^4 = \ell_s^4 4\pi g_s N = \ell_s^4 g^2_{YM} N = \ell_s^4 \lambda \tag{127}$$

with

$$\lambda \equiv g^2_{YM} N$$

the 't Hooft coupling relevant to large N Yang-Mills theory. Thus, it is tempting in particular to consider the limit λ fixed while $N \to \infty$. We see that in this limit the string coupling tends to zero, so that we may perform calculations on the string theory side, simply by restricting ourselves to string tree-diagrams, the *classical* limit of string theory! The full *quantum* non-perturbative description of $N = 4$ Super Yang-Mills would be obtained from this classical theory - in the large N limit. This seems like a program which might have some success eventually, even though the NS-R formulation of IIB string theory on the $AdS_5 \times S^5$ background with N units of RR 5-form flux is unknown (for a preliminary attempt, see [25]). In the Green-Schwarz-formulation there is a proposal [26], [27], but non-trivial calculations remain to be performed. In any case, this looks like a concrete proposal for the so called Master-field idea of Witten, that the large-N $U(N)$ Yang-Mills theory path integral could be described by a single field configuration, reminiscent indeed of classical theory.

The situation becomes even more astonishing if we furthermore consider the strong coupling limit, i.e. large λ limit of SYM. If we keep the AdS/sphere radii fixed, we are therefore dealing with the $\ell_s^2 = \alpha' \to 0$ limit of string theory, or in other words, with the limit in which classical string theory simply becomes classical supergravity! This is the limit mostly considered in concrete calculations so far. Even in that extreme limit the conjecture has dramatic predictions: It predicts how the SYM theory at large N behaves in the extreme non-perturbative, strong coupling regime. String excitations become infinitely heavy and decouple in that limit, but since we kept the radius of the sphere S^5 fixed, we cannot at all neglect the Kaluza-Klein states associated with the compactification on that sphere. (Strictly speaking one cannot send the dimensionful string length to zero. What one means by this is to consider energy scales for which string excitations may be neglected. If we consider the string length fixed and still take λ large, the radius of S^5 tends to infinity. People often phrase the situation that way. Then all curvatures are "small" and quantum gravity corrections may be neglected: classical supergravity is adequate. KK-masses now become very small. Obviously the situation is entirely equivalent to our formulation, but as usual one properly has to consider dimensionless ratios for such arguments to make sense.)

4 A Detailed Specification of the Conjecture. Sample Calculations.

4.1 Presentation of the idea

Maldacena conjectured an equivalence - a duality - between two theories: (i) String/M-theory on a manifold of the form $AdS_d \times \mathcal{M}$, with $\mathcal{M}$ being a compactification manifold, and (ii) an appropriate conformal field theory on the boundary of AdS_d. But his conjecture did not specify the precise way in which these two theories should be mapped onto each other. Subsequently a detailed proposal was made independently by Gubser, Klebanov and Polyakov [2] and by Witten [3]. Here we shall describe that and carry out several sample calculations considered in ref. [3] filling in a few details (while leaving several aspects of [2, 10] untreated). For definiteness of presentation we would think of the canonical $AdS_5 \times S^5$ example, but in fact the discussion will be much more general, ignoring mostly the details of the compactification manifold.

On the boundary theory of n-dimensional Minkowski (or Euclidean) space-time, we should typically like to understand a general correlator of the the form

$$\langle \mathcal{O}_1(x_1) ... \mathcal{O}_q(x_q) \rangle$$

One wants to know how to obtain the same object in terms of the quantum theory on AdS_{n+1}. The proposal of [2] and [3] is to identify this object with the result of a path integral in the AdS_{n+1} theory, with certain fields attaining specific boundary properties. The fields in question would be related to the corresponding boundary theory operators $\mathcal{O}_i(x_i)$ by the requirement that their boundary values - to be identified in a non-trivial way - should couple to the operators in a way consistent with the symmetries of the problem. In practice the resulting path integral might be evaluated by means of generalized Feynman diagrams with q "external" propagators "ending" on the boundary at points x_i. In particular in the large N limit, as we have seen, the Feynman diagrams would be tree diagrams only, albeit string tree diagrams in general. If furthermore we consider the limit of large 't Hooft coupling, the string tree diagrams become tree diagrams of supergravity. We shall need to understand how to evaluate these generalized propagators. A particularly neat formulation is possible if we can construct a standard generating functional for the correlators on the boundary:

$$Z(\{\phi_i\}) = \sum_q \frac{1}{q!} \int \prod_{k=1}^{q} d^n x_k \langle \mathcal{O}_1(x_1) ... \mathcal{O}_q(x_q) \rangle \phi_1(x_1) ... \phi_q(x_q) = \langle exp\{ \int dx \sum_i \phi_i(x) \mathcal{O}_i(x) \} \rangle \tag{128}$$

This requires short distance singularities of the correlators to be integrable, or else the introduction of some device to render the expression meaningful. In any case we might always go back to considering individual correlators. Obviously, if the operators $\mathcal{O}_i$ on the boundary CFT have conformal dimension Δ_i then the currents ϕ_i should have conformal dimension $n - \Delta_i$. Similarly any other quantum number the operators may have (say as multiplets of $SU(2,2|4)$ in the case of $AdS_5 \times S^5$) would have to be supplemented by conjugate quantum numbers of the currents so that singlets may be formed. Supposing the generating functional of the "currents" $\{\phi_i(x)\}$ ($x \in \partial \Big(AdS_{n+1} \Big) \sim E^n$) makes sense, the description of this object in AdS_{n+1} should be by a path integral with fields $\phi_i(y)$ in that theory (i.e. with $y \in AdS_{n+1}$) tending to the boundary currents $\{\phi_i(x)\}$ in a certain prescribed way, that we shall have to infer. In the large N limit we would just have to work out the classical action, and in the large N strong coupling limit, just the classical supergravity action on fields satisfying the equations of motion and tending to the prescribed boundary "currents" in some particular way.

4.2 Free scalar fields on AdS_{n+1}

By a scalar field we mean one which transforms as a scalar under the AdS isometry group, hence it would tend to a boundary value with conformal dimension zero, and couple to operators with conformal dimension n. In that case it turns out to be meaningful to simply require the path integral on AdS_{n+1} with the scalar field in question tending to a definite *value* on the boundary

$$\phi(x) \to \phi_0(x')$$

for $x \in AdS_{n+1}$ and $x' \in E^n$ and (somehow) $x \to x'$. Let us study the case of free scalar fields. Here the path integral becomes trivial, and is simply equal to the exponential of (minus or i times) the classical action, up to a normalization constant. Thus in this case the classical approximation is the exact result. The action on AdS_{n+1} is

$$I(\phi) = \frac{1}{2} \int_{AdS_{n+1}} d^{n+1}x \sqrt{g} \partial_\mu \phi \partial^\mu \phi \tag{129}$$

So we seek a classical field, satisfying the equation of motion

$$D_\mu D^\mu \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu \Big(\sqrt{g} \partial^\mu \phi(x) \Big) = 0 \tag{130}$$

throughout AdS_{n+1}, but such that $\phi(x) \to \phi_0(x')$ whenever the point x in AdS_{n+1} runs away to ∞ in the particular way that defines the boundary point $x' \in E^n$ of the boundary.

It is plausible that the classical solution to this problem is unique. Indeed, imagine $\phi_1(x)$ and $\phi_2(x)$ both being solutions of the equation of motion with the same boundary value. Then $\delta\phi(x) \equiv \phi_1(x) - \phi_2(x)$ has boundary value zero and also satisfies eq.(130). We should show that $\delta\phi$ vanishes identically. Since it vanishes at infinity we take it to be square integrable. Then

$$0 = -\int d^{n+1}x \sqrt{g} \delta\phi D_\mu D^\mu \delta\phi = \int d^{n+1}x \sqrt{g} \partial_\mu \delta\phi \partial^\mu \delta\phi \tag{131}$$

Since $\partial_\mu \delta\phi \partial^\mu \delta\phi$ is positive (semi) definite (in the Euclidean case), we find

$$\partial_\mu \delta\phi \equiv 0 \tag{132}$$

or $\delta\phi$ constant. But as it tends to zero, it must vanish everywhere.

We may solve the problem in terms of a Greens function, the generalized propagator,

$$K(x, x')$$

with $x \in AdS_{n+1}$ and $x' \in E^n$ (or $E^{1,n-1}$ in the Minkowski case), or since we have seen the boundary to be compactified, S^n, which we will take to denote the boundary. Thus we seek a solution to the problem

$$\frac{1}{\sqrt{g}}\partial^x_\mu\Big(\sqrt{g}\partial^\mu_x K(x,x')\Big) = 0 \tag{133}$$

and somehow $K(x,x')$ a delta function when x is on the boundary. Then we may construct the sought for classical solution as

$$\phi(x) = \int_{S^n} d^n x' K(x,x')\phi_0(x'), \quad x \in AdS_{n+1} \tag{134}$$

Following Witten [3], we construct this propagator in the coordinates eq.(21), taking for definiteness Euclidean signature:

$$ds^2 = \frac{1}{(x^0)^2}\sum_{\mu=0}^{n}(dx^\mu)^2 \tag{135}$$

Now AdS_{n+1} is described by the upper half space $x^0 > 0$, and $x^\mu \equiv x_\mu$, $\mu = 1, ..., n$ are coordinates in E^n, the coordinates of the boundary. However, we know that we must be dealing with a compactified boundary S^n, and there is an extra "point at infinity" described at some length around eq.(27). The boundary here is $x^0 = 0$; the single point at infinity is $x^0 = \infty$, a single point indeed, no matter what the values of the remaining coordinates are, since the metric tensor vanishes there.

In these coordinates then

$$g_{\mu\nu} = \frac{1}{(x^0)^2}\delta_{\mu\nu}, \ \sqrt{g} = \frac{1}{(x^0)^{n+1}}, \ g^{\mu\nu} = (x^0)^2\delta^{\mu\nu} \tag{136}$$

Exercise: Verify by explicit calculation, that

$$K(x^0,\vec{x}\,;\vec{x}\,') = c\frac{(x^0)^n}{((x^0)^2 + (\vec{x}\, - \vec{x}\,')^2)^n} \tag{137}$$

satisfies Laplace's equation eq.(133) for $x^0 \neq 0$ and $\vec{x}\, \neq \vec{x}\,'$, and becomes the desired delta function in the limit $x_0 \to 0$.

We may also infer the above more elegantly from Witten's trick [3], useful in the sequel. First notice, that just as the scalar propagator in flat space is Poincaré invariant so the propagator in our case is invariant under the AdS isometry group. Then let the boundary point $\vec{x}\,'$ represent the point, P at infinity:

$$K(x;P) = K(x^0,\vec{x}\,;P) \tag{138}$$

This cannot depend on $\vec{x}$ due to translation invariance, so in that particular case, K is a function of x^0 only, and Laplace's equation becomes

$$\partial_0\Big(\sqrt{g}\partial^0 K(x^0)\Big) = 0$$

Here

$$\partial_0 = \frac{\partial}{\partial x^0}; \ \partial^0 = g^{00}\partial_0 = (x^0)^2\partial_0$$

so the equation becomes

$$\frac{d}{dx^0}\Big((x^0)^{-n+1}\frac{d}{dx^0}K(x^0)\Big) = 0 \tag{139}$$

If we try to solve with $K(x^0) = c(x^0)^p$ we find

$$p(-n+p) = 0$$

The solution with $p = 0$ cannot describe something with delta function support "at infinity", $x^0 = \infty$, since such a delta function should vanish on the boundary $x^0 = 0$ for any $\vec{x}$. Hence

$$K(x^0, \vec{x}\,; P) = c(x^0)^n$$

Next apply a transformation to map $P \to \vec{x}\,' = \vec{0}$:

$$x^\mu \to z^\mu \equiv \frac{x^\mu}{(x^0)^2 + \vec{x}^{\,2}},\ \ \mu = 0, ..., n \tag{140}$$

This transformation is indeed an $SO(1, n+1)$ isometry of AdS_{n+1} (in this case with Euclidean signature). In fact we find with $x^2 \equiv (x^0)^2 + \vec{x}^{\,2}$

$$dz^\mu = \frac{x^2 dx^\mu - x^\mu 2x \cdot dx}{(x^2)^2},\ \ (x \cdot dx \equiv \sum_{\nu=0}^{n} x^\nu dx^\nu) \tag{141}$$

It follows that

$$dz^2 = \frac{dx^2}{(x^2)^2} \Rightarrow \frac{dz^2}{(z^0)^2} = \frac{dx^2}{(x^0)^2} \tag{142}$$

so that the transformation is an isometry and the parametrization space is preserved. Under this mapping

$$\begin{aligned} x^0 &\to \frac{x^0}{(x^0)^2 + \vec{x}^{\,2}} \\ K(x^0, \vec{x}\,; P) &\to K(x^0, \vec{x}\,; \vec{0}) = c\frac{(x^0)^n}{((x^0)^2 + \vec{x}^{\,2})^n} \end{aligned} \tag{143}$$

as claimed. Finally from translational invariance in the boundary we find

$$K(x^0, \vec{x}\,; \vec{x}\,') = c\frac{(x^0)^n}{((x^0)^2 + (\vec{x}\, - \vec{x}\,')^2)^n} \tag{144}$$

For $x^0 \to 0^+$ this becomes proportional to a $\delta^n(\vec{x}\, - \vec{x}\,')$. Indeed, clearly for $\vec{x}\,' \neq \vec{x}$, $K \to 0$ for $x^0 \to 0$. Also

$$\int d^n x \frac{(x^0)^n}{((x^0)^2 + \vec{x}^{\,2})^n}$$

is independent of x^0 and convergent, as seen by scaling to the new integration variable x^i/x^0, so that a limit is obtained for $x^0 \to 0$, and we may adjust c to get proper normalization if desired.

So we have obtained the sought for classical solution

$$\phi(x^0,\vec{x}\,) = c\int d^nx' \frac{(x^0)^n}{((x^0)^2+(\vec{x}\,-\vec{x}\,')^2)^n}\phi_0(\vec{x}\,') \tag{145}$$

We want to evaluate the classical action on that field, remembering that

$$\Delta\phi = \frac{1}{\sqrt{g}}\partial_\mu\Big(\sqrt{g}\partial^\mu\phi\Big) = 0 \tag{146}$$

or (as we have seen, since $\partial^\mu = (x^0)^2\partial_\mu$)

$$\sum_{\mu=0}^{n}\partial_\mu\Big((x^0)^{-n+1}\partial_\mu\phi\Big) = 0 \tag{147}$$

Now

$$\begin{aligned} I(\phi) &= \frac{1}{2}\int d^{n+1}x\sqrt{g}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}\int d^{n+1}x(x^0)^{-n+1}\partial_\mu\phi\partial_\mu\phi \\ &= \frac{1}{2}\int d^{n+1}x\partial_\mu\left((x^0)^{-n+1}\phi\partial_\mu\phi\right) - \frac{1}{2}\int d^{n+1}x\phi\left\{\partial_\mu\Big((x^0)^{-n+1}\partial_\mu\phi\Big)\right\} \end{aligned} \tag{148}$$

The last term vanishes by the equation of motion, and the total derivative term vanishes in all directions except in the x^0 direction where we have a boundary. To avoid the divergence for $x^0=0$ we put first $x^0=\epsilon$. Then

$$I(\phi) = -\frac{1}{2}\int_{x^0=\epsilon} d^nx(x^0)^{-n+1}\phi(x^0,\vec{x}\,)\partial_0\phi(x_0,\vec{x}\,) \tag{149}$$

In the limit $x^0\to 0$ we may put $\phi(x^0,\vec{x}\,)=\phi_0(\vec{x}\,)$. We then evaluate

$$\begin{aligned} \partial_0\phi(x^0,\vec{x}\,) &= c\frac{\partial}{\partial x^0}\int d^nx'\frac{(x^0)^n}{((x^0)^2+(\vec{x}\,-\vec{x}\,')^2)^n}\phi_0(\vec{x}\,') \\ &= cn(x^0)^{n-1}\int d^nx'\frac{1}{(\vec{x}\,-\vec{x}\,')^{2n}}\phi_0(\vec{x}\,') + \mathcal{O}((x^0)^{n+1}),\ x^0\to 0 \end{aligned} \tag{150}$$

Inserting into eq.(149) we see that the singular x^0 behaviour drops out and we get

$$I(\phi) = -\frac{cn}{2}\int d^nxd^nx'\frac{\phi_0(\vec{x}\,')\phi_0(\vec{x}\,)}{(\vec{x}\,-\vec{x}\,')^{2n}} \tag{151}$$

In the classical (super)gravity limit the generating function for operators $\mathcal{O}(\vec{x}\,)$ in the boundary theory coupling to the "source" $\phi_0(\vec{x}\,)$, is then given by the exponential of (minus) that. It follows that in this (trivial) example, there are only connected 2-point functions

$$\langle\mathcal{O}(\vec{x}\,)\mathcal{O}(\vec{x}\,')\rangle \sim \frac{1}{(\vec{x}\,-\vec{x}\,')^{2n}} \tag{152}$$

This is the expected result: The $SO(1, n+1)$-scalar, $\phi(x)$ will couple to conformal operators of dimension n on the boundary in order for the coupling

$$\int d^n x \mathcal{O}(\vec{x}\,)\phi_0(\vec{x}\,)$$

to be invariant. And conformal invariance (indeed dilatation invariance) suffices then to fix the form of the 2-point function to eq.(152). In the case of $AdS_5 \times S^5$ this operator will turn out to be the $Tr(F^2)$ of the YM field strength (sect. 4.5).

Notice that the 2-point function eq.(152) is characteristic of a *quantum* theory with a non trivial short distance singularity. This is so even though it was derived from a classical calculation in the bulk of AdS_{n+1}.

4.3 Massless abelian gauge field on AdS_{n+1}

We continue to follow ref.[3]. A gauge field $A_\mu(x)$ in AdS_{n+1} with $\mu = 0, 1, ..., n$ gives rise to a field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

satisfying the free equation of motion (no currents)

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0 \tag{153}$$

Now again we seek to construct, first a propagator with one delta function source on the boundary. With that we then build any gauge field $A_\mu(\vec{x}\,)$ in the bulk with the property that the components $A_i(x_0, \vec{x}\,),\ i \geq 1$ tend to prescribed functions on the boundary, corresponding a certain 1-form on the boundary

$$A_0(\vec{x}\,) = a_i(\vec{x}\,)dx^i \tag{154}$$

As before, we use the trick of first working out the propagator when the point on the boundary is the point P "at infinity". Again, in that case we expect the propagator to be independent of any $\vec{x}$. Further, the propagator should be a 1-form, but in that case, one with no 0-component. The remaining components are all treated the same way. We treat the components in the boundary one by one.

Thus we look for a 1-form in the bulk of AdS_{n+1}, only depending on x^0, and with a single component only, say the i'th

$$A^{(i)}(x) = f(x^0)dx^i$$

It should satisfy the equation of motion, eq.(153). We have

$$\begin{aligned} A_i^{(i)} &= f(x^0),\ \ A_\mu^{(i)} \equiv 0,\ \mu \neq i \\ F_{0i} &= f'(x^0) = -F_{i0},\ \text{all other } F_{\mu\nu} \equiv 0 \\ F^{0i} &= (x^0)^4 f'(x^0) \\ \sqrt{g}F^{0i} &= (x^0)^{-n+3} f'(x^0) \end{aligned} \tag{155}$$

Then the equations of motion give

$$
\begin{aligned}
\frac{d}{dx^0}(\sqrt{g}F^{0i}) &= \frac{d}{dx^0}\left((x^0)^{-n+3}f'(x^0)\right) = 0 \Rightarrow \\
f'(x^0) &\propto (x^0)^{n-3} \Rightarrow f(x^0) \propto (x^0)^{n-2} \Rightarrow \\
A^{(i)} &= \frac{n-1}{n-2}(x^0)^{n-2}dx^i \text{ (fixed } i\text{)}
\end{aligned} \tag{156}
$$

(the normalization is for later convenience). We hope this 1-form will have a delta-function singularity at P. We exhibit this as before using the $SO(1,n+1)$ isometry (inversion)

$$
\begin{aligned}
x^\mu &\rightarrow \frac{x^\mu}{(x^0)^2+\vec{x}^{\,2}} \Rightarrow \\
A^{(i)} &\rightarrow \frac{n-1}{n-2}\left(\frac{x^0}{(x^0)^2+\vec{x}^{\,2}}\right)^{n-2} d\left(\frac{x^i}{(x^0)^2+\vec{x}^{\,2}}\right)
\end{aligned} \tag{157}
$$

This new propagator represents propagation from $\vec{x}^{\,\prime} = \vec{0}$ on the boundary to $(x^0, \vec{x}\,)$ in the bulk. When we work out the derivatives we see that this new 1-form propagator will have components along all the different dx^μ's. We may simplify, using the fact that the propagator is unique only up to gauge transformation. We shall in fact get a simpler expression if we subtract the "pure gauge"

$$
\frac{1}{n-2}d\left(\frac{(x^0)^{n-2}x^i}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\right)
$$

Then

$$
\begin{aligned}
A^{(i)} &= \frac{n-1}{n-2}\left(\frac{x^0}{(x^0)^2+\vec{x}^{\,2}}\right)^{n-2} d\left(\frac{x^i}{(x^0)^2+\vec{x}^{\,2}}\right) - \frac{1}{n-2}d\left(\frac{(x^0)^{n-2}x^i}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\right) \\
&= \frac{1}{n-2}\left\{(n-1)(x^0)^{n-2}\frac{x^i}{((x^0)^2+\vec{x}^{\,2})^{n-2}}d\left(\frac{1}{(x^0)^2+\vec{x}^{\,2}}\right) + (n-1)\frac{(x^0)^{n-2}}{((x^0)^2+\vec{x}^{\,2})^{n-1}}dx^i\right. \\
&- \left. d\left(\frac{(x^0)^{n-2}x^i}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\right)\right\} \\
&= \frac{1}{n-2}\left\{(x^0)^{n-2}x^i d\left(\frac{1}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\right) + (n-1)\frac{(x^0)^{n-2}}{((x^0)^2+\vec{x}^{\,2})^{n-1}}dx^i\right. \\
&- \left. d\left(\frac{(x^0)^{n-2}x^i}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\right)\right\} \\
&= \frac{1}{n-2}\left\{-\frac{1}{((x^0)^2+\vec{x}^{\,2})^{n-1}}d\left((x^0)^{n-2}x^i\right) + (n-1)\frac{(x^0)^{n-2}}{((x^0)^2+\vec{x}^{\,2})^{n-1}}dx^i\right\} \\
&= \frac{1}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\left\{-(x^0)^{n-3}dx^0x^i + (x^0)^{n-2}dx^i\right\}
\end{aligned} \tag{158}
$$

Or,

$$
A^{(i)}_\mu = \frac{1}{((x^0)^2+\vec{x}^{\,2})^{n-1}}\begin{cases} -(x^0)^{n-3}x^i & \mu = 0 \\ +(x^0)^{n-2} & \mu = i \\ 0 & \text{otherwise} \end{cases} \tag{159}
$$

We may now collect results and obtain for the general classical solution 1-form field

$$\begin{aligned} A^{(i)}(x^0, \vec{x}\,) &= \int d^n x' \sum_{i=1}^{n} A^{(i)}(x^0, \vec{x}\,; \vec{x}\,')a_i(\vec{x}\,') \\ &= \int d^n x' \left\{ \frac{(x^0)^{n-2}}{((x^0)^2 + (\vec{x}\, - \vec{x}\,')^2)^{n-1}} a_i(\vec{x}\,')dx^i \right. \\ &\quad \left. -(x^0)^{n-3} dx^0 \frac{(x' - x)^i a_i(\vec{x}\,')}{((x^0)^2 + (\vec{x}\,' - \vec{x}\,)^2)^{n-1}} \right\} \end{aligned} \tag{160}$$

Notice that only the first term acts as a delta-function for $x^0 \to 0$. In fact, a function of the form

$$\frac{\epsilon^\beta}{(\epsilon^2 + \vec{x}\,^2)^\alpha} \tag{161}$$

is a model of $\delta^n(\vec{x}\,)$ only if $0 < \beta = 2\alpha - n$. Then, namely

$$\int d^n x \frac{\epsilon^\beta}{(\epsilon^2 + \vec{x}\,^2)^\alpha} = \epsilon^{\beta + n - 2\alpha} \int d^n \left(\frac{x}{\epsilon}\right) \frac{1}{\left(1 + \left(\frac{\vec{x}}{\epsilon}\right)^2\right)^\alpha}$$

is a constant, independent of ϵ. For $x^0 = \epsilon$, the first term in 160 is of that form. The last term in eq.(160) has an extra power of ϵ and vanishes for $\epsilon \to 0$. This is perhaps not completely obvious. In fact it looks like there is a power *less* of ϵ. But first we must remember to scale also the factor $(x' - x)^i$, and second we see by expanding $a_i(\vec{x}\,')$ around $\vec{x}$ that the leading term vanishes since the integrand is odd, and the following terms *do* have extra powers of ϵ. Hence, clearly eq.(160) will tend to (up to a constant normalization)

$$A_0(\vec{x}\,) = \sum_{i=1}^{n} a_i(\vec{x}\,)dx^i, \text{ for } x^0 \to 0 \tag{162}$$

We are now instructed to evaluate the classical action for the classical solution eq.(160).

In form language

$$I(A) = \frac{1}{2} \int_{AdS_{n+1}} F \wedge *F \tag{163}$$

where $F = dA$, and the equation of motion is $*d*F = 0$ i.e. $d*F = 0$. Then

$$I(A) = \frac{1}{2} \int_{AdS_{n+1}} dA \wedge *F = \frac{1}{2} \int_{AdS_{n+1}} d(A \wedge *F) = \frac{1}{2} \int_{\text{boundary}(\epsilon)} A \wedge *F \tag{164}$$

where again we take the boundary to be $x^0 = \epsilon$ at first, and only let $\epsilon \to 0$ at the end. On the boundary, we only need the $i = 1, ..., n$ components of $A \wedge *F$. Thus a component of A, A_i has $i = 1, ..., n$. Also $*F$ has components $j_1, ..., j_{n-1} = 1, ..., n$, but never 0 or i. Therefore we need exactly the components F_{0i}. In coordinates on the boundary

$$I \sim \int d^n x \sqrt{h} A^\ell n^0 F_{0\ell} \tag{165}$$

Near the boundary the metric is

$$h_{ij} = \frac{1}{(x^0)^2}\delta_{ij}, \quad i,j = 1,...,n$$

The fact that the metric *on the boundary* is not uniquely obtained (seems singular) is related to the fact that the boundary theory as a conformal theory knows of no metric - only of a conformal class. We shall come back to that. n^μ is an outward pointing unit vector normal to the boundary. We may take

$$n_\mu = (-\frac{1}{x^0}, 0, ..., 0); \quad n^\mu = (-x^0, 0, ..., 0)$$

and $\sqrt{h} = (x^0)^{-n}$. We must find $F_{0\ell}$. $F = dA$ is obtained hitting A with $d = dx^0\partial_0 + dx^i\partial_i$, but in the result we only need bother about terms with a dx^0. Introducing

$$D \equiv (x^0)^2 + (\vec{x} - \vec{x}\,')^2$$

we find

$$\begin{aligned} F &= (n-2)(x^0)^{n-3}dx^0 \wedge \int d^nx' \frac{a_i(x')dx^i}{D^{n-1}} \\ &-2(n-1)(x^0)^{n-1}dx^0 \wedge \int d^nx' \frac{a_i(x')dx^i}{D^n} \\ &+2(n-1)(x^0)^{n-3}dx^\ell \wedge dx^0 \int d^nx'(x^\ell - (x')^\ell)a_i(x')(x^i - (x')^i)\frac{1}{D^n} \\ &-(x^0)^{n-3}dx^i \wedge dx^0 \int d^nx' \frac{a_i(x')}{D^{n-1}} + \text{ terms with no } dx^0 \end{aligned} \tag{166}$$

Using $dx^0 \wedge dx^i = -dx^i \wedge dx^0$ we get further

$$\begin{aligned} F &= (n-1)(x^0)^{n-3}dx^0 \wedge \int d^nx' \frac{a_i(x')dx^i}{D^{n-1}} \\ &-2(n-1)(x^0)^{n-1}dx^0 \wedge \int d^nx' \frac{a_i(x')dx^i}{D^n} \\ &-2(n-1)(x^0)^{n-3}dx^0 \wedge \int d^nx' \frac{(\vec{x} - \vec{x}\,')\cdot d\vec{x}\, a_i(x')(x^i - (x')^i)}{D^n} + ... \end{aligned} \tag{167}$$

Now

$$I = \int d^nx'(x^0)^{-n+3}A_i(x^0, \vec{x}\,')F_{0i}(x^0, \vec{x}\,') \tag{168}$$

Here, on the boundary $A_i \to a_i(\vec{x}\,')$ and from eq.(167)

$$\begin{aligned} F_{0i}(x^0, \vec{x}\,) &= (x^0)^{n-3}\Big\{(n-1)\int d^nx' \frac{a_i(x')}{D^{n-1}} \\ &\quad -2(n-1)\int d^nx'(x_i - x'_i)\frac{a_k(x')(x^k - (x')^k)}{D^n}\Big\} \\ &\quad +\mathcal{O}((x^0)^{n-1}) \end{aligned} \tag{169}$$

We use a notation with $x_i \equiv x^i$. Only the term with $(x^0)^{n-3}$ survives for $x^0 \to 0$, and we find

$$I(a) = \int d^n x d^n x' a_i(\vec{x}\,) a_j(\vec{x}\,') \left(\frac{\delta^{ij}}{(\vec{x}\, - \vec{x}\,')^{2n-2}} - \frac{2(x-x')^i (x-x')^j}{(\vec{x}\, - \vec{x}\,')^{2n}} \right) \tag{170}$$

This is the final result. To see that this is in accord with the conjecture, notice that a gauge-field 1-form in AdS_{n+1} is a scalar under $SO(1, n+1)$. Hence, the *components* have conformal dimension $+1$ on the boundary, so they couple to operators, J_i, in the conformal field theory on the boundary with conformal dimension $n-1$, but these "currents" must be conserved by virtue of gauge invariance in the bulk. According to the conjecture we have calculated the generating function for these operators in eq.(170). We see that they have only non vanishing 2-point functions:

$$\langle J_i(\vec{x}\,) J_j(\vec{x}\,') \rangle \sim \frac{1}{(\vec{x}\, - \vec{x}\,')^{2(n-1)}} \left\{ \delta_{ij} - \frac{2(x_i - x'_i)(x_j - x'_j)}{(\vec{x}\, - \vec{x}\,')^2} \right\} \tag{171}$$

The last term ensures current conservation:

$$\begin{aligned} & \partial_i^x \langle J_i(\vec{x}\,) J_j(\vec{x}\,') \rangle \\ = \; & \partial_i \left\{ \frac{\delta_{ij}}{(\vec{x}\, - \vec{x}\,')^{2(n-1)}} - \frac{2(x_i - x'_i)(x_j - x'_j)}{(\vec{x}\, - \vec{x}\,')^{2n}} \right\} \\ = \; & 0 \end{aligned} \tag{172}$$

We conclude that the conjecture also works for free massless gauge fields. Furthermore, we have constructed a propagator also in that case (in a particular gauge).

The case of massless gravitons in the bulk is a little more complicated [2, 3]. They couple to the energy momentum tensor on the boundary.

4.4 Free massive fields on AdS_{n+1}

Following Witten again[3] we shall argue that a massive scalar with mass m in AdS_{n+1} must couple to operators $\mathcal{O}_\Delta$ with conformal dimension Δ in the boundary theory, given by the largest root of

$$\Delta(\Delta - n) = m^2 \tag{173}$$

Of course we have already checked the case of $m = 0$. In the massive case it turns out that we have to reinterpret the idea that the field ϕ "should tend to" a definite (current) field on the boundary.

We take the free massive theory in the bulk to be described by

$$I(\phi) = \frac{1}{2} \int d^{n+1} x \sqrt{g} (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) \tag{174}$$

Consider coordinates x^μ, $\mu = 0, ..., n$ with metric eq.(11)

$$ds^2 = \frac{4}{(1-x^2)^2} \sum_0^n (dx^\mu)^2 \tag{175}$$

with

$$x^2 \equiv \sum_0^n (x^\mu)^2 \equiv r^2, \ 0 \leq r < 1 \tag{176}$$

Then change variable to

$$\begin{aligned} r &= \tanh \frac{y}{2}, \quad 0 \leq y < \infty \\ dr &= \frac{dy}{2\cosh^2 y/2}, \ 1 - r^2 = \frac{1}{\cosh^2 y/2} \\ \frac{r^2}{(1-r^2)^2} &= \sinh^2 \frac{y}{2} \cosh^2 \frac{y}{2} = \frac{1}{4} \sinh^2 y \end{aligned} \tag{177}$$

Next write

$$\sum_0^n (dx^\mu)^2 = dr^2 + r^2 d\Omega_n^2$$

with $d\Omega_n$ the metric of the unit S^n. Thus the metric on AdS_{n+1} may be expressed as

$$ds^2 = dy^2 + \sinh^2 y d\Omega_n^2 \tag{178}$$

In these coordinates

$$\det g = \sinh^{2n} y \det \gamma$$

with $\gamma_{\alpha\beta}$ the metric tensor on S^n. Now the Laplacian on scalars becomes

$$\begin{aligned} \Delta &= \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \\ &= \frac{1}{\sinh^n y} \frac{d}{dy} \sinh^n y \frac{d}{dy} + \frac{1}{\sqrt{\gamma}} \partial_\alpha \sqrt{\gamma} \partial^\alpha \\ &= \frac{1}{\sinh^n y} \frac{d}{dy} \sinh^n y \frac{d}{dy} - \frac{L^2}{\sinh^2 y} \end{aligned} \tag{179}$$

where

$$-L^2 = \frac{1}{\sqrt{\gamma}} \partial_\alpha \sqrt{\gamma} \partial^\alpha \tag{180}$$

is the Laplacian on the sphere, the "angular momentum" or centrifugal contribution. (The notation is perhaps slightly confusing here: In 179

$$\partial^\alpha = g^{\alpha\nu} \partial_\nu = g^{\alpha\beta} \partial_\beta = \frac{1}{\sinh^2 y} \gamma^{\alpha\beta} \partial_\beta$$

In 180 instead, ∂^α is just $\gamma^{\alpha\beta}\partial_\beta$). One might imagine expanding ϕ in eigenmodes of L^2 which indeed as wee shall see in sect. 4.5, is the Casimir of $SO(n+1)$. We want to understand the behaviour of $\phi(y, \theta^\alpha)$ for $y \to \infty$ (θ^α are coordinates on S^n). The Klein-Gordon equation in these coordinates becomes

$$\frac{1}{\sinh^n y} \frac{d}{dy} (\sinh^n y \frac{d}{dy} \phi) - \frac{L^2}{\sinh^2 y} \phi = m^2 \phi$$

For large y, the centrifugal term is negligible and we get approximately

$$e^{-ny}\frac{d}{dy}(e^{ny}\frac{d}{dy}\phi) = m^2\phi$$

The solution is an exponential $e^{\lambda y}$ provided

$$\lambda(n+\lambda) = m^2 \tag{181}$$

Thus there are 2 linearly independent solutions which asymptotically behave as

$$e^{\lambda_+ y} \text{ and } e^{\lambda_- y}$$

with λ_+ and λ_- the larger and the smaller solutions for λ respectively. Only one particular linear combination is an allowed solution free of singularities in the interior of AdS_{n+1}, and hence it's asymptotic behaviour is dominated by

$$e^{\lambda_+ y}$$

It follows that we cannot assume that $\phi(x)$ tends to a definite value on the boundary!

We might try to assume (with $\vec{x}$ a coordinate on the boundary)

$$\phi(y,\vec{x}\,) \sim (e^y)^{\lambda_+}\phi_0(\vec{x}\,) \tag{182}$$

near the boundary. But the form e^y is rather arbitrary. In fact, the function

$$e^{-y} \equiv f(y,\vec{x}\,)$$

has a 1st order zero on the boundary (taking into account the metric on AdS_{n+1}). It is therefore just of the form needed to build a finite metric from the divergent AdS one. But this construction has a degree of arbitrariness about it. Thus, if we transform

$$f(y,\vec{x}\,) = e^{-y}$$

into

$$e^{w(\vec{x}\,)}e^{-y} \equiv \tilde{f}(y,\vec{x}\,)$$

that new function has an equally good 1st order zero. Thus, demanding an asymptotic behaviour

$$\phi(y,\vec{x}\,) \sim \Big(f(y,\vec{x}\,)\Big)^{-\lambda_+}\phi_0(\vec{x}\,)$$

this freedom in f implies a freedom in ϕ_0:

$$f \to e^w f \Rightarrow \phi_0 \to e^{w\lambda_+}\phi_0 \tag{183}$$

So, the arbitrariness, $f \to e^w f$, shows that the metric on AdS_{n+1} in a natural way only defines a *conformal class* of metrics on the boundary: if h_{ij} is a metric on the boundary defined by means of f, then $e^{2w}h_{ij}$ is a conformally transformed metric obtained from $e^w f$.

This is all consistent with the field theory on the boundary being conformally invariant. Then namely, the field theory is insensitive to the conformal rescalings of the metric. The field theory can only conceive of a conformal class.

But then the behaviour

$$h_{ij} \to e^{2w} h_{ij} \;\Rightarrow\; \phi_0 \to e^{w\lambda_+}\phi_0$$

is the statement that ϕ_0 is not a function, but rather a *density* with conformal weight $-\lambda_+$. Indeed, a density of weight d would have the property that for a small length $\Delta\ell$

$$(\Delta\ell)^d \phi_0$$

should be invariant under conformal scaling. That works, since the above is transformed into

$$(\Delta\ell e^w)^d e^{w\lambda_+}\phi_0$$

which is invariant for $d = -\lambda_+$.

So the only natural procedure is to require that massive fields should tend to *densities* ϕ_0, and therefore should couple to operators $\mathcal{O}_\Delta$ with conformal dimension $n + \lambda_+$ so that

$$\mathcal{O}_\Delta \phi_0$$

is a density with weight n.

We now verify that this state of affairs is in full accordance with the general prescription.

As before we want to solve for a propagator. We use again coordinates with metric (in units where the "radius" b of AdS_{n+1} has been put equal to 1)

$$ds^2 = \frac{1}{(x^0)^2}\sum_0^n (dx^\mu)^2$$

Again we begin with a propagator vanishing on the boundary $x^0 = 0$, but developing a delta-function at P: $x^0 = \infty$, and thus being independent of $\vec{x}$. Denoting again the propagator as $K(x^0, \vec{x}\,; \vec{x}\,')$ and in particular $K(x^0, \vec{x}\,; P) = K(x^0)$, the equation of motion is

$$\Big(-(x^0)^{n+1}\frac{d}{dx^0}(x^0)^{-n+1}\frac{d}{dx^0} + m^2\Big) K(x^0) = 0$$

We may find a solution of the form $K(x^0) \propto (x^0)^{\lambda+n}$ provided

$$-(\lambda + n)\lambda + m^2 = 0 \tag{184}$$

with the solutions $\lambda = \lambda_\pm$

$$\begin{aligned} \lambda_+ &= \frac{1}{2}(-n + \sqrt{n^2 + 4m^2}) \\ \lambda_- &= \frac{1}{2}(-n - \sqrt{n^2 + 4m^2}) \end{aligned} \tag{185}$$

Only the solution $K(x^0) = (x^0)^{n+\lambda_+}$ will vanish for $x^0 = 0$. As before the propagator $K(x^0, \vec{x}\,; \vec{0})$ is found by the inversion

$$x^\mu \to \frac{x^\mu}{(x^0)^2 + \vec{x}^{\,2}}$$

giving

$$\begin{aligned} K(x^0, \vec{x}\,; \vec{0}) &= \frac{(x^0)^{n+\lambda_+}}{((x^0)^2 + \vec{x}^{\,2})^{n+\lambda_+}} \\ K(x^0, \vec{x}\,; \vec{x}\,') &= \frac{(x^0)^{n+\lambda_+}}{((x^0)^2 + (\vec{x} - \vec{x}\,')^2)^{n+\lambda_+}} \end{aligned} \tag{186}$$

Notice that according to the rule eq.(161), this does *not* tend to a delta-function when $x^0 \to 0$, rather it is

$$\frac{(x^0)^{n+2\lambda_+}}{((x^0)^2 + (\vec{x} - \vec{x}\,')^2)^{n+\lambda_+}}$$

which tends to $\delta^n(\vec{x} - \vec{x}\,')$ for $x^0 \to 0$. Hence, when we build the classical field as

$$\begin{aligned} \phi(x^0, \vec{x}\,) &= c \int d^n x' \frac{(x^0)^{n+\lambda_+}}{((x^0)^2 + (\vec{x} - \vec{x}\,')^2)^{n+\lambda_+}} \phi_0(\vec{x}\,') \\ &= (x^0)^{-\lambda_+} c \int d^n x' \frac{(x^0)^{n+2\lambda_+}}{((x^0)^2 + (\vec{x} - \vec{x}\,')^2)^{n+\lambda_+}} \phi_0(\vec{x}\,') \end{aligned} \tag{187}$$

we see that for $x^0 \to 0$, this behaves as

$$(x^0)^{-\lambda_+} \phi_0(\vec{x}\,) \sim e^{\lambda_+ y} \phi_0(\vec{x}\,)$$

as anticipated.

As explained, $\phi_0(\vec{x}\,)$ has conformal dimension $-\lambda_+$ and couples on the boundary to operators $\mathcal{O}_\Delta$ with conformal dimension $\Delta = n + \lambda_+$. Therefore we would expect to find a 2-point function

$$\langle \mathcal{O}_\Delta(\vec{x}\,) \mathcal{O}_\Delta(\vec{x}\,') \rangle = \frac{1}{(\vec{x} - \vec{x}\,')^{2n+2\lambda_+}} \tag{188}$$

We now verify that this is indeed what is obtained, using the by now well established prescription.

Namely, we evaluate the classical free action on the classical field as follows:

$$\begin{aligned} I(\phi) &= \frac{1}{2} \int d^{n+1}x \sqrt{g} (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) \\ &= \frac{1}{2} \int d^{n+1}x \sqrt{g} \left\{ \frac{1}{\sqrt{g}} \partial_\mu \Big(\sqrt{g} \phi \partial^\mu \phi \Big) - \phi \Big(\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \phi - m^2 \phi \Big) \right\} \end{aligned} \tag{189}$$

the last term vanishes, and the first term is evaluated as in the massless case

$$I(\phi) = -\frac{1}{2} \int_{x^0 = \epsilon} d^n x (x^0)^{-n+1} \phi(x^0, \vec{x}\,) \partial_0 \phi(x^0, \vec{x}\,) \tag{190}$$

Now,

$$\begin{aligned}
\partial_0\phi(x^0,\vec{x}\,) &= c(n+\lambda_+)(x^0)^{n+\lambda_+-1}\int d^nx'\frac{\phi_0(\vec{x}\,')}{((x^0)^2+(\vec{x}\,-\vec{x}\,')^2)^{n+\lambda_+}} \\
&\quad + \text{non leading terms as } x^0\to 0 \qquad (191)
\end{aligned}$$

and as we have seen, $\phi(x^0,\vec{x}\,)\sim(x^0)^{-\lambda_+}\phi_0(\vec{x}\,),\ x^0\to 0$. Hence

$$I_{Cl}(\phi_0)\propto\int d^nxd^nx'\frac{\phi_0(\vec{x}\,)\phi_0(\vec{x}\,')}{(\vec{x}\,-\vec{x}\,')^{2(n+\lambda_+)}} \qquad (192)$$

in complete agreement with the expectation eq.(188). Since λ_+ is the larger root of

$$\lambda(\lambda+n)=m^2,$$

Δ is the larger root of $(\Delta=n+\lambda,\lambda=\Delta-n)$

$$\begin{aligned}
(\Delta-n)\Delta &= m^2 \\
\Delta &= \frac{1}{2}\Big(n+\sqrt{n^2+4m^2}\Big) \qquad (193)
\end{aligned}$$

It is intuitively plausible how to generalize to fields other than scalars. We saw that the massless 1-form gauge-field $A=A_\mu dx^\mu$ restricting to A_idx^i on the boundary, naturally had component fields of dimension 1. Likewise a massless p-form field C_p has component fields with dimension p and couples to operators on the boundary with dimension $n-p$.

A *massive* p-form field would couple to operators which would have dimensions shifted as in the scalar case to

$$\Delta=n+\lambda_+-p \qquad (194)$$

or

$$\begin{aligned}
(\Delta-n+p)(\Delta+p) &= m^2 \\
\Delta &= \frac{1}{2}\Big(n+\sqrt{n^2+4m^2-4np}\Big) \qquad (195)
\end{aligned}$$

4.5 Comparison of multiplet data in the bulk and on the boundary

In ref.[3] Witten makes a check on the Maldacena conjecture in the case of $AdS_5\times S^5$. Similar checks may be performed in more complicated situations. The check is restricted to the case of the strong coupling (in the boundary theory) and large N approximation, which may be treated by classical supergravity. Even though we can neglect string excitations in that limit, the compactification on S^5 gives rise to Kaluza-Klein excitations with massive modes. We may use the inverse radius of of S^5 as our unit of mass (as in the preceding subsections), i.e. continue to put that equal to 1. Thus we should at first analyze the spectrum of KK excitations (for a general discussion, see [28]). This was done some time ago, in [29] by studying small fluctuations of the supergravity fields around the $AdS_5\times S^5$ background, and in [30] by applying the powerful technique of (super) group representation theory. These analyses lead to several infinite families of massive field modes with definite masses and transformation properties under $SO(1,5) \times SO(6)$ or even better, under $SU(1,3|4)$ ($SU(2,2|4)$ for Minkowski signature). The representation theory of that supergroup has been considered for example in [30, 31, 32]. According to the Maldacena conjecture, these give rise to predictions concerning the spectrum of conformal operators in the $N = 4$ $U(\infty)$ boundary theory. For a given set of quantum numbers (conjugate to the ones for the modes in the bulk theory) we may predict conformal dimensions using the result of the previous subsection. We must ask whether in fact quantum corrections would upset the result of such a simple analysis. However, it turns out that both in the bulk theory and in the boundary theory, there exist large classes of *small* representations with properties similar to properties of BPS states and for which such quantum corrections cannot occur. This makes it possible to perform meaningful checks. In the boundary theory these are the so called chiral primaries (see for example [33]).

Here we shall not attempt an account of this which is anywhere near complete. Instead we shall restrict ourselves to analyzing certain aspects of one family, the one corresponding to KK-excitations of the dilaton field. We shall show, that the masses of these excitations obey the rule

$$m^2 = k(k+4), \quad k = 0, 1, 2, ... \tag{196}$$

In the boundary conformal field theory Witten pointed out that the corresponding operators are of the form

$$\mathcal{O}_{(i_1,...,i_k)}(x) = Tr(\phi_{(i_1} \cdots \phi_{i_k)} F_{\mu\nu} F^{\mu\nu})(x) \tag{197}$$

But only when the *symmetrized* tensor in $\{i_1, ..., i_k\}$ is taken, do these fields belong to a multiplet of chiral primaries. Precisely then do the fields transform as an irreducible representation of $SO(6)$ (see below). The trace is over the adjoint of the $U(N)$ gauge group. The scalar ($N \times N$ matrix valued) fields $\phi_i(x)$, $i = 1, ..., 6$ have been mentioned before. They transform in the $\{\mathbf{6}\}$ vector representation of $SO(6)$. The $F_{\mu\nu}$ is the $U(N)$ field strength matrix written as an $N \times N$ matrix. It is trivial to count the conformal dimension in *the weak coupling limit* where free field dimensions apply. There the scalars have dimension 1, and the field strength tensor dimension 2, so $\mathcal{O}_{1_1,...,i_k}(x)$ has dimension Δ_k

$$\Delta_k = k + 4 \tag{198}$$

This fits with the formula of the previous subsection eq.(193) (for $k = \lambda$) in the case of the mass values eq.(196). In general the check would not be convincing since we used a strong coupling argument in the bulk and a weak coupling one on the boundary. But because it may be shown that we are dealing with the above mentioned small representations, the result will survive quantum corrections. As emphasized we shall not go into these crucial matters, but here restrict ourselves to an elementary account of KK-modes of the dilaton field.

Exercise: Consider the case of $AdS_5 \times S^5$ and consider the KK-mode of the dilaton field which is independent of the coordinates on S^5, the "s-wave". Supposing it couples indeed to $Tr(F^2)$ as described, work out the 2-point function of that operator in the supergravity picture, using the result of sect. 4.2. In particular work out the coefficient in the value of the classical action, left out in the calculation there, taking into account the integration over S^5 and the scale b. Verify that the coefficient is a numerical constant times N^2. Argue that the form of the 2-point function is exactly the expected one for the operator $Tr\Big(F^2\Big)$ in the large N limit (cf. [2]).

As we have seen before, the dilaton decouples in the background $AdS_5 \times S^5$ and satisfies the free 10−dimensional equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu\Big(\sqrt{g}\partial_\nu\phi g^{\mu\nu}\Big) = 0 \tag{199}$$

We may use a splitting of this into the 5 components of the AdS_5 and the 5 components of the S^5. Thus, if we expand the dilaton field on S^5 in eigen modes of the Laplacian on S^5, we see that these eigenvalues will play the role of (minus) m^2-values in AdS_5.

Our first task will be to understand the connection between the Laplacian on S^5 (more generally S^{n+1}) and the quadratic Casimir of $SO(6)$ ($SO(n+2)$).

We may think of S^{n+1} as imbedded in R^{n+2} in close analogy to the case of AdS_{n+1} (indeed many of the results below carry over to results for $SO(2,n)$, but subtle important differences exist). Thus define S^{n+1} by the condition

$$y_0^2 + y_1^2 + ... + y_n^2 + y_{n+1}^2 = 1 \tag{200}$$

We are interested in scalar fields defined on S^{n+1}, but we may trivially extend those to scalar fields on $\mathbb{R}^{n+2}$. In fact, define

$$\rho^2 = \sum_{\mu=0}^{n+1}(y_\mu)^2$$

Then a scalar field, ϕ on S^{n+1} is only defined for $\rho = 1$, but we may extend the definition to $\mathbb{R}^{n+2}$ by demanding ϕ independent of ρ along fixed directions in $\mathbb{R}^{n+2}$. More concretely, introduce coordinates (ρ, x_μ) similar to eq.(7) by

$$\begin{aligned} y_0 &= \rho\frac{1-x^2}{1+x^2},\ y_\mu = \rho\frac{2x_\mu}{1+x^2},\ \mu = 1, ..., n+1 \\ y^2 &= \rho^2 \end{aligned} \tag{201}$$

We then extend a field $\phi(x)$ into

$$\phi(\rho, x) \equiv \phi(1, x) \equiv \phi(x)$$

On any scalar field $\phi(y)$, the generators of $SO(n+2)$ are simply (for generalized treatments along these lines, see for example [32])

$$L_{mn} = i(y_m\partial_n - y_n\partial_m) \tag{202}$$

with the standard algebra:

$$[L_{mn}, L_{pq}] = i\{\delta_{np}L_{mq} + \delta_{mq}L_{np} - \delta_{nq}L_{mp} - \delta_{mp}L_{nq}\} \tag{203}$$

Further we have the vielbeins and their relation to the metric on S^{n+1} (the $y^m \equiv y_m$'s are the flat coordinates, the set $(\rho, x^\mu) \equiv (x^0, x^\mu)$ are the curvilinear ones).

$$
\begin{aligned}
e^0_m &= \frac{\partial\rho}{\partial y^m} = \frac{y_m}{\rho},\ e^\mu_m = \frac{\partial x^\mu}{\partial y^m} \\
e^m_0 &= \frac{\partial y^m}{\partial\rho} = \frac{y^m}{\rho},\ e^m_\mu = \frac{\partial y^m}{\partial x^\mu} \\
y_m e^{m\mu} &= 0,\quad y_m e^{m0} = \rho^2,\quad e^\mu_n e^n_0 = 0 \\
e^\mu_m e^{\nu m} &= g^{\mu\nu} \\
e^0_n e^n_0 &= \frac{y_n}{\rho}\cdot\frac{y^n}{\rho} = 1
\end{aligned} \tag{204}
$$

We now want to show that

$$
L_{mn}\phi(y)L^{mn}\phi(y) = -2\rho^2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{205}
$$

It then follows that the dilaton action on S^{n+1} may be formulated as

$$
\begin{aligned}
S(S^{n+1}) &= -\frac{1}{4}\int d^{n+2}y\delta(y^2-1)L_{mn}\phi(y)L^{mn}\phi(y) \\
&= \frac{1}{4}\int d^{n+2}y\delta(y^2-1)\phi(y)L_{mn}L^{mn}\phi(y)
\end{aligned} \tag{206}
$$

(use: $d^{n+2}y\delta(y^2-1) = d^{n+1}xd\rho\sqrt{g}\delta(\rho-1)$) and the statement about the connection between the Laplacian and the quadratic Casimir

$$
C_{n+2} \equiv \frac{1}{2}L_{mn}L^{mn} \tag{207}
$$

follows. Hence we work out (the distinction between lower and upper indices has no significance)

$$
\begin{aligned}
L_{mn}\phi L^{mn}\phi &= -2\Big(y_m\partial_n\phi y^m\partial^n\phi - y_m\partial_n\phi y^n\partial^m\phi\Big) \\
&= -2\left\{y_m\Big(\frac{\partial x^\mu}{\partial y^n}\frac{\partial}{\partial x^\mu} + \frac{\partial\rho}{\partial y^n}\frac{\partial}{\partial\rho}\Big)\phi\right. \\
&\qquad \left.\cdot\left[y^m\Big(\frac{\partial x^\nu}{\partial y_n}\frac{\partial}{\partial x^\nu} + \frac{\partial\rho}{\partial y_n}\frac{\partial}{\partial\rho}\Big)\phi - (m\leftrightarrow n)\right]\right\} \\
&= -2\left\{y_m(e^\mu_n\partial_\mu + e^0_n\partial_\rho)\phi(y)\right. \\
&\qquad \left.\cdot\left[y^m(e^{\nu n}\partial_\nu + e^{0n}\partial_\rho)\phi(y) - (m\leftrightarrow n)\right]\right\} \\
&= -2\left\{y^2(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + e^0_n e^{n0}(\partial_\rho\phi)^2) - y_m e^{0m}\partial_\rho\phi e^0_n y^n\partial_\rho\phi\right\}
\end{aligned} \tag{208}
$$

But here we have made the choice that $\partial_\rho\phi \equiv 0$, so we simply get

$$
L_{mn}\phi L^{mn}\phi = -2\rho^2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{209}
$$

as we wanted[1].

[1] It is easy to verify, that putting instead $\phi(\rho,x) = \rho^N\phi(1,x)$ for any N, would yield exactly the same result.

Thus we are let to perform an analysis of the dilaton field on S^{n+1} similar to the analysis of scalars on S^2 according to spherical harmonics $Y_\ell^m(\theta,\phi)$. In this latter case the Laplacian is well known to be identified with (minus) the square of the angular momentum with eigenvalues $\ell(\ell+1)$ for integer ℓ. Of course the square of the angular momentum is just the Casimir of $SO(3)$. We want to arrive at a similar understanding in general.

Let us single out as special coordinate

$$u = y_0 + iy_{n+1} \equiv Ye^{i\phi},\ Y,\phi \in \mathbb{R},\quad Y = \sqrt{(y^0)^2 + (y^{n+1})^2} \tag{210}$$

On S^{n+1} we have

$$y_0^2 + y_{n+1}^2 + \sum_1^n y_i^2 \equiv Y^2 + z^2 = 1$$

so that we may define

$$z^2 = \sum_1^n y_i^2 = \cos^2\theta,\ Y^2 = \sin^2\theta \tag{211}$$

In general a representation of $SO(n+2)$ would be characterized by a highest weight state, $|\vec{\Lambda}_{n+2}\rangle$ with a certain highest weight, $\vec{\Lambda}_{n+2}$ of $SO(n+2)$, a vector in weight space, the components of which are eigenvalues of a mutually commuting set of Cartan generators of the algebra $so(n+2)$. Let us take one of these to be

$$H \equiv L_{0,n+1} \tag{212}$$

corresponding to rotations in the complex u-plane, and generating an $SO(2)$ subgroup. The remaining Cartan generators pertain to an $SO(n)$ subgroup commuting with that $SO(2)$. With this choice our weights are then automatically labelled by (i) the eigenvalue of H and (ii) by a highest weight of that $SO(n)$, i.e. we classify irreducible representations according to the

$$SO(2) \times SO(n)$$

subgroup. And we write

$$\vec{\Lambda}_{n+2} = (k, \vec{\Lambda}_n) \tag{213}$$

There are n raising and lowering operators relative to H. Indeed, define

$$J_i^+ \equiv L_{i,n+1} + iL_{i0} \tag{214}$$

Then work out

$$\begin{aligned}
[H, J_i^+] &= [L_{0,n+1}, L_{i,n+1} + iL_{i0}] = [L_{0,n+1}, L_{i,n+1}] + i[L_{0,n+1}, L_{i0}] \\
&= -iL_{0i} + i^2 L_{n+1,i} = J_i^+
\end{aligned} \tag{215}$$

Thus we may write

$$\begin{aligned}
C_{n+2} &= \frac{1}{2}L_{mn}L_{mn} = \sum_{1\le i<j\le n} L_{ij}L_{ij} + L_{0i}L_{0i} + L_{n+1,i}L_{n+1,i} + L_{0,n+1}L_{0,n+1} \\
&= C_n + (L_{i,n+1} - iL_{i0})(L_{i,n+1} + iL_{i0}) - iL_{i,n+1}L_{i0} + iL_{i0}L_{i,n+1} + H^2 \\
&= C_n + J_i^- J_i^+ - i[L_{i,n+1}, L_{i0}] + H^2 \\
&= C_n + J_i^- J_i^+ - nL_{n+1,0} + H^2 = C_n + J_i^- J_i^+ + nH + H^2
\end{aligned} \tag{216}$$

So acting on a highest weight state, J_i^+ will vanish, and we get

$$C_{n+1} = C_n + k(k+n) \tag{217}$$

where k denotes the eigenvalue of the $SO(2)$ generator H.

Let us first consider the $(n+2)$-dimensional *vector* representation of $SO(n+2)$. The L_{mn}'s are represented by $(n+2) \times (n+2)$ matrices

$$(L_{mn})_{ab} = i(\delta_{ma}\delta_{nb} - \delta_{na}\delta_{mb}) \tag{218}$$

And H gives eigenvalue 1 for the "highest weight state"

$$(y_0, y_1, ..., y_n, y_{n+1}) = \frac{1}{\sqrt{2}}(1, 0, ..., 0, -i)$$

(interpreted as a column). The raising operators J_i^+ give zero on the same "state", and so do the $SO(n)$ generators $L_{ij},\ i,j = 1,...,n$. It follows that the highest weight of the vector representation is

$$\vec{\Lambda}_{n+2}(\text{vector}) = (1, \vec{0}) \tag{219}$$

and the Casimir has the value above with $k = 1$ and $C_n = 0$. Next consider the k-fold tensor product of the vector representation with k a positive integer. The highest weight is trivially $(k, \vec{0})$ but the representation is highly reducible. The unique irreducible representation with the same highest weight is the symmetrized tensor product. This is the one we encountered in eq.(197), and we see now that the Casimir of the representation is given by $k(k+n), k = 0, 1, 2, ...$ (generalizing the result $k(k+1)$ for $n+2 = 3$).

We now consider the generalized spherical harmonics of the scalar (dilaton) field on S^{n+1}. In the case of $SO(3)$ these are the usual spherical harmonics $Y_\ell^m(\theta, \phi)$ with ℓ integer. They constitute a complete set of scalar functions on S^2. They carry irreducible representations of $SO(3)$ with the highest weight member

$$Y_\ell^\ell(\theta, \phi) = N_\ell \sin^\ell \theta e^{i\ell\phi} \tag{220}$$

The normalization is not interesting here, but is easily evaluated to

$$N_\ell = \sqrt{\frac{2\ell+1}{4\pi}}\frac{\sqrt{(2\ell)!}}{2^\ell \ell!}$$

All the other spherical harmonics are obtained from this one by applying lowering operators.

In general for S^{n+1} and $SO(n+2)$, we may construct a similar highest weight scalar function for any fixed positive integer k

$$Y_k(\hat{y}) \equiv N_k u^k = N_k(y_0 + iy_{n+1})^k = N_k \sin^k \theta e^{ik\phi} \tag{221}$$

where we used the definitions in eq.(210) and eq.(211). This is a regular well defined function on S^{n+1} only for positive integer values of k. Of course the similarity with the elementary case is strong (the normalization would depend on n). It is trivial to verify with the form of the generators we have given, that this field is indeed a highest weight "state" of $SO(n+2)$ with highest weight

$$\vec{\Lambda}_{n+1} = (k, \vec{\Lambda}_n = \vec{0}) \tag{222}$$

just as for the symmetrized k-fold tensor representation. Therefore, constructing a representation of $SO(n+2)$ by rotating this $Y_k(\hat{y})$ in all possible ways, or, equivalently by forming all possible linear combinations of fields obtained from it by applying lowering operators, we shall get a finite dimensional irreducible representation of $SO(n+2)$, in fact precisely the one we met above by considering symmetrized tensor products.

In the present case of $SO(n+2)$ there are of course very many more irreducible representations to worry about than for the case of $SO(3)$, and we might wonder if the generalized scalar spherical harmonics defined by eq.(221) (together with all of its multiplet members), would really form a *complete* set of functions on S^{n+1} for $n > 2$. This will be so, if we are able to construct arbitrarily good approximations to delta functions with support at any point on S^{n+1}, using (linear combinations of) these functions. It is actually intuitively obvious that this should be possible. In fact, consider what we may do with the highest weight functions $Y_k(\hat{y})$ themselves, directly. For very high values of k it follows from eq.(221) and the sphere condition

$$y_0^2 + y_{n+1}^2 = 1 - \vec{y}^2 \leq 1$$

that these functions only have appreciable support in the neighbourhood of the unit circle

$$y_0^2 + y_{n+1}^2 = 1, \;\; \vec{y} = \vec{0} \tag{223}$$

Along this circle the coordinate is ϕ and

$$Y_k \sim e^{ik\phi}$$

essentially a 1-dim. plane wave. It follows that we may construct arbitrarily good approximations to delta functions with support at any point along the circle eq.(223). Since our representation space also contains all possible rotations of the functions Y_k we may form delta functions with support at any point we please on S^{n+1}, and thus indeed we have found the complete set of generalized spherical harmonics.

The Casimir for this representation is therefore $k(k+n)$ or $k(k+4)$ in the case of S^5. But these are just the squared mass values we needed for the KK-excitations, eq.(196), in order for the Maldacena conjecture to be checked in this particular instance.

5 Breaking SUSY and conformal invariance in the boundary theory. A possible new approach to (large N) QCD

The Maldacena conjecture suggests a mathematical equivalence between string/M theory in certain backgrounds, and a conformally invariant ordinary quantum field theory on "the boundary". Field theories, such as the Standard Model are not conformally invariant: They typically posses a mass gap, there is a lightest *massive* meson for example. Although the Maldacena conjecture would seem to throw extremely interesting light on non-perturbative aspects of quantum field theory, it would therefore also seem that we are restricted to theories rather far removed from reality. However, Witten [10] proposed a scheme whereby it seems possible to overcome these difficulties. Perhaps the most interesting case is that of $AdS_7 \times S^4$. The boundary theory here is a certain so called (2,0) exotic 6-dimensional conformally invariant theory for which no action seems available [34] (see also lectures by E. Bergshoeff and by P.C. West, this school). By compactifying everything on a 2-torus, T^2, however, the boundary theory becomes 4-dimensional, and provided the fermions in the theory are taken anti-periodic around a cycle on the T^2, supersymmetry and conformal invariance are broken at low energies, and the set up provides a novel way of treating (regularized) large N ordinary QCD in 4 dimensions (albeit without quarks)! For a long time it has been a dream for theoretical particle physics [35] to be able to do something non trivial with that theory. Glueballs would be stable in that limit so that perhaps the theory could be sufficiently tractable to furnish an analytic understanding of confinement. Despite intense interest in this large N limit, however, very little has been achieved in the way of concrete results. The Maldacena conjecture combined with Witten's proposal seems to introduce a truly novel approach.

Here, at first we shall be more general, following the discussion of [10], and in the end we shall mostly restrict to the somewhat simpler case of $AdS_5 \times S^5$ for which we introduce a single $T^1 = S^1$ compactification, thereby rendering a framework for studying large N QCD in 3 space-time dimensions. At the end we shall indicate some of the first steps that have to be taken in order to treat also QCD_4.

To see how a suitable spin structure can break supersymmetry and conformal invariance at low energies, we go to Euclidean time and take it periodic on a circle of radius R, corresponding to an inverse temperature of $2\pi R$. A bosonic degree of freedom has to be periodic along this time t:

$$q(t) = \sum_{n \in \mathbb{Z}} \left(a_n e^{-int/R} \right) \tag{224}$$

There are of course then (KK-like) excitations with masses n/R. For fermions, we have the option of considering a non trivial spin structure. Clearly, if we also take the fermions periodic, they will be quantized with the same integer modes as the bosons, and supersymmetry will be preserved. But if we take fermions to be *anti*-periodic around Euclidean time, they have modes

$$(n + \frac{1}{2})/R$$

In particular, the lowest mode $n = 0$ is very different for bosons and for fermions. For bosons we have a massless mode, but for fermions the lowest mode can be considered to decouple for high enough temperatures. So supersymmetry is broken. If we investigate the theory with very high frequencies, much higher than the temperature, these details are irrelevant and we expect to regain a supersymmetric situation.

The scalar supersymmetric partners of fermions will get masses due to renormalization. In the supersymmetric theory the masses are protected from being divergent. Thus in the effective low energy theory, we shall have divergencies cut off by the "cut off" of the effective theory, which is the temperature. Thus we also expect scalar super partners of fermions to become massive and therefore to decouple at low energies. Finally, the effective theory will have a non-vanishing beta function, because some of the field modes which make the beta function vanish in the full theory are absent in the low energy effective theory. Thus, the effective low energy theory is no longer conformal.

It follows, that apparently we have a scheme for dealing with a realistic (QCD like) theory. In that theory, the coupling constant should run at high energies to a small value, since the theory is asymptotically free. The smallest value is the one attained at the cut off, the temperature, and it would be given by the fixed coupling constant of the unbroken theory. Therefore, in order to use this scheme in a fully realistic way, we should arrange for the coupling constant of the boundary theory to be *small*. We have previously seen, that the very simple supergravity approximation is obtained when the coupling constant of the boundary theory is *large*. Therefore, a straight forward study of large N QCD in 3 or 4 dimensions based on supergravity, cannot be hoped to be realistic. It is perhaps similar to a strong coupling analysis of lattice gauge theory, known to be in a "wrong phase". What we would rather like to do, would be to study not supergravity, but the full string theory in the appropriate background. At large N it should even be enough to study *classical* string theory - or the tree diagram limit only, in order to have a realistic framework for large N QCD. This goal has not yet been achieved, although preliminary proposals have been given [26, 27, 25].

Instead, a large number of studies have been performed in the supergravity approximation. Such studies can at best be considered to have an exploratory nature, but it is very instructive to see how several expectations from confinement are brought out in a very simple way (see refs. [36, 10, 37, 38, 39, 40, 41, 42, 43, 5], to name but a few).

5.1 Classical, finite temperature versions of AdS

According to the previous discussion, we are led to consider "finite temperature" versions of Anti de Sitter spaces. As shown by Hawking and Page [44] and generalized by Witten [10] there are two relevant manifolds denoted X_1 and X_2 which we must consider.

5.1.1 The manifold X_1

The first one is the simplest. In the standard case of Poincaré isometry, we may arrange for a finite temperature by taking Euclidean signature, and by compactifying the time on S^1. That compactification may be thought of as a periodic identification of Euclidean time. All bosonic fields would have to be periodic under: $t \to t + 2\pi R$. This mapping is a translation in time, $\tau : t \mapsto \tau(t)$, and the set of all translations, τ^n, form a group isomorphic to $\mathbb{Z}$. Thus in the case of AdS^+_{n+1} we may try to do something similar. Consider the embedding condition:

$$uv - \sum_{i=1}^{n} x_i^2 = b^2 \tag{225}$$

with AdS^+_{n+1} being the branch $u, v > 0$. Now introduce a real, positive parameter λ (analogous to R and to be related to the temperature), and define the mapping

$$u \to \lambda^{-1} u, \quad v \to \lambda v, \quad x_i \to x_i \tag{226}$$

This is a mapping, f_λ of AdS_{n+1} onto itself. The set of all repeated applications of this mapping (an the inverses) $\{f_\lambda^n | n \in \mathbb{Z}\}$ constitute a group isomorphic to $\mathbb{Z}$. Generalizing the case of Poincaré isometry we may consider the manifold

$$X_1 \equiv AdS^+_{n+1}/\mathbb{Z}$$

So X_1 is the set of "equivalence classes" where two points P, P' in AdS_{n+1} are equivalent if they are related by one of these mappings. We see that a fundamental domain for v is

$$1 \le v/b \le \lambda \tag{227}$$

namely $v/b = 1$ and $v/b = \lambda$ is the same point in X_1. These points parametrize indeed a circle

$$\begin{aligned} v/b &= \lambda^{\theta/2\pi} \\ \theta &= \frac{2\pi \ln v/b}{\ln \lambda} \end{aligned} \tag{228}$$

($\theta = 0 \Rightarrow v/b = 1, \theta = 2\pi \Rightarrow v/b = \lambda \sim 1$). For any v in the fundamental domain, we may solve for

$$u = \Big(b^2 + \sum_{n=1}^{n} x_i^2\Big)/v$$

and use for X_1 the coordinates $(x_1, ..., x_n)$ together with the angular coordinate θ. It follows that topologically

$$X_1 \sim \mathbb{R}^n \times S^1$$

As for the boundary of X_1, we know it is obtained by scaling u, v, x_i by $s \to \infty$, equivalent to the condition

$$uv - \sum_{i=1}^{n} x_i^2 = 0$$

subject to projective equivalence. Thus we may fix a scale so that

$$\sum_{i=1}^{n} \Big(x_i/b\Big)^2 = 1 \tag{229}$$

defining S^{n-1}. We see that the boundary of X_1 has topology

$$\partial X_1 \sim S^{n-1} \times S^1$$

These "spheres" have two different radii, but only the ratio is relevant for the conformal structure. We shall come back to that.

Let us introduce a convenient metric on X_1. Write

$$\begin{aligned} r^2 &= \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} dx_i^2 &= dr^2 + r^2 d\Omega_{n-1}^2 \end{aligned} \tag{230}$$

and define

$$t = \ln v/b - \frac{1}{2}\ln\Big(1 + (r/b)^2\Big) = \ln\lambda \cdot \frac{\theta}{2\pi} - \frac{1}{2}\ln\Big(1+(r/b)^2\Big) \tag{231}$$

Then work out

$$\begin{aligned} ds^2 &= dudv - \sum dx_i^2 \\ u &= (b^2 + \sum x_i^2)/v \Rightarrow du = -\frac{dv}{v^2}\Big(b^2 + r^2\Big) + \frac{2rdr}{v} \\ dt &= \frac{dv}{v} - \frac{\frac{r}{b^2}dr}{1 + \Big(r/b\Big)^2} \end{aligned} \tag{232}$$

Then find

$$\begin{aligned} dudv &= -dt^2(b^2 + r^2) + \frac{r^2 dr^2}{b^2 + r^2} \\ ds^2 &= dt^2(b^2 + r^2) + \frac{b^2 dr^2}{b^2 + r^2} + r^2 d\Omega_{n-1}^2, \text{ or} \\ d\tilde{s}^2 &= ds^2/b^2 = dt^2\Big(1 + (r/b)^2\Big) + \frac{dr^2}{1 + (r/b)^2} + (r/b)^2 d\Omega_{n-1}^2 \end{aligned} \tag{233}$$

(we performed as usual the "mostly minus" to "mostly plus" operation). This is our final metric on X_1.

5.1.2 The manifold X_2

This is the case where a temperature is introduced by in fact inserting a black hole into AdS_{n+1}, the Schwarzschild metric generalized to AdS_{n+1}. Thus we seek a static, spherically symmetric metric of the general form

$$ds^2 = A(r)dt^2 + B(r)dr^2 + r^2 d\Omega_{n-1}^2 \tag{234}$$

with $r = 0$ being "the position of the black hole". Outside the black hole we have Einstein's empty space equation (with a cosmological constant)

$$R_{\mu\nu} = -\frac{n}{b^2} g_{\mu\nu}, \quad D = n + 1 \tag{235}$$

Thus the black hole solution X_2 will also be an example of an Einstein space, but one less symmetric than AdS_{n+1}, even though the two are "asymptotically similar".

We find the following non zero Christoffel symbols (γ_{ij} denotes the metric on S^{n-1}):

$$\begin{aligned}\Gamma^t_{rt} &= \frac{1}{2}A^{-1}A',\ \Gamma^r_{tt} = -\frac{1}{2}B^{-1}A',\ \Gamma^r_{rr} = \frac{1}{2}B^{-1}B' \\ \Gamma^r_{ij} &= -B^{-1}r\gamma_{ij},\ \Gamma^i_{jk},\ \Gamma^i_{rj} = \frac{1}{r}\delta^i_j \end{aligned} \tag{236}$$

We then find the non vanishing Riemann tensor components contributing to R_{tt}:

$$\begin{aligned} R^r_{trt} &= \frac{1}{4}(AB)^{-1}(A')^2 - \frac{1}{4}B^{-2}A'B' - \frac{1}{2}(B^{-1}A')' \\ R^i_{tit} &= -\frac{n-1}{2r}B^{-1}A' \end{aligned} \tag{237}$$

and the first Einstein equation

$$R_{tt} = \frac{(A')^2}{4AB} - \frac{n-1}{2r}\frac{A'}{B} - \frac{A'B'}{4B^2} - \frac{1}{2}\Big(\frac{A'}{B}\Big)' = -\frac{n}{b^2}A \tag{238}$$

Further find

$$\begin{aligned} R^t_{rtr} &= -\frac{1}{2}(A^{-1}A')' - \frac{1}{4}A^{-2}(A')^2 + \frac{1}{4}(AB)^{-1}A'B' \\ R^i_{rir} &= \frac{n-1}{2r}B^{-1}B' \end{aligned} \tag{239}$$

and the second Einstein equation

$$R_{rr} = \frac{A'B'}{4AB} + \frac{n-1}{2r}\frac{B'}{B} - \frac{(A')^2}{4A^2} - \frac{1}{2}\Big(\frac{A'}{A}\Big)' = -\frac{n}{b^2}B \tag{240}$$

Put

$$B \equiv A^{-1}f;\quad B' = -A^{-2}A'f + A^{-1}f'$$

Then the two equations become

$$\begin{aligned} -\frac{1}{2}A\left\{f^{-1}\Big(\frac{n-1}{r}A' + A''\Big) + (f^{-1})'\frac{1}{2}A'\right\} &= -\frac{n}{b^2}A \\ -\frac{1}{2}A^{-1}\Big(A'' + \frac{n-1}{r}A'\Big) + f^{-1}f'\Big(\frac{1}{4}A^{-1}A' + \frac{n-1}{2r}\Big) &= -\frac{n}{b^2}A^{-1}f \end{aligned} \tag{241}$$

or

$$\begin{aligned} f^{-1}\Big(\frac{n-1}{r}A' + A''\Big) + (f^{-1})'\frac{1}{2}A' &= \frac{2n}{b^2} \\ f^{-1}\Big(\frac{n-1}{r}A' + A''\Big) + (f^{-1})'\Big(\frac{1}{2}A' + \frac{n-1}{r}A\Big) &= \frac{2n}{b^2} \end{aligned} \tag{242}$$

Subtracting, we find

$$(f^{-1})' = 0$$

so f is a constant, and

$$A'' + \frac{n-1}{r}A' = 2n\frac{f}{b^2} \tag{243}$$

This is solved into

$$\begin{aligned} A &= \frac{f}{b^2}r^2 + \frac{c_1}{r^{n-2}} + c_2 \\ B &= \frac{f}{A} \end{aligned} \tag{244}$$

Redefining the scale of t and the meaning of c_1, c_2, we may put $f = 1$. Also, to get the empty solution for X_1 as a special case, we put the new $c_2 = 1$. Thus we finally have the Schwarzschild solution in AdS_{n+1} for X_2:

$$ds^2 = \left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right)} + r^2 d\Omega_{n-1}^2 \tag{245}$$

We have renamed a constant, putting

$$w_n \equiv \frac{16\pi G_N}{(n-1)\Omega_{n-1}} \tag{246}$$

This will turn out to mean that M becomes the "mass" of the black hole.

Notice, that this of course generalizes the metric of the standard $b = \infty$, 4-dimensional black hole (Euclidean, $n = 3$, $r_g = 2G_N M = w_3 M$):

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 d\Omega_{n-1}^2$$

5.1.3 Temperature of the black hole

We expect that a black hole will have a (Beckenstein-Hawking) temperature, and that of course was our motivation for picking this metric. Let us work out what the temperature is. The metric eq.(245) is of the form

$$ds^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-1}^2 \tag{247}$$

where

$$V(r) = \frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} \tag{248}$$

and vanishes at various values of r, the largest of which we shall denote, r_+. This is characteristic of a metric with a horizon: g_{00} vanishes and g_{rr} has a pole at the horizon, $r = r_+$. Ordinary physical space is the region $r \geq r_+$. In our case of a diagonal metric, $g^{rr} = 1/g_{rr}$. It is rather easy to establish the general formula [41] for the temperature, T of the black hole in the case of such a diagonal metric:

$$2\pi T = \sqrt{g^{rr}}\frac{d}{dr}\sqrt{g_{00}}|_{r=r_+} \tag{249}$$

(other equivalent formulas are easily obtained). To see this, notice that the horizon property implies that near $r = r_+$

$$g_{00} \sim A(r - r_+), \quad g^{rr} \sim B(r - r_+) \tag{250}$$

and the metric takes the following form in the vicinity of the horizon

$$ds^2 \sim \frac{dr^2}{B(r - r_+)} + dt^2 A(r - r_+) + r^2 d\Omega^2_{n-1} \tag{251}$$

This metric has a coordinate singularity at the horizon, which we may remove by a suitable choice of coordinates, but only if the Euclidean time is periodic with a particular period, which one then identifies with the inverse temperature. In fact [44] the coordinate singularity may be made analogous to a standard 2-dimensional polar coordinate singularity

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 \tag{252}$$

where ρ is the polar distance and θ is the polar angle. At $\rho = 0$ the metric is singular, but we know very well that the geometry is regular *provided* the polar angle has period 2π. If the periodicity is anything else than 2π there is a genuine geometrical conical singularity.

We may arrange for the metric eq.(251) to look similar to eq.(252) if we put

$$d\rho^2 = \frac{dr^2}{B(r - r_+)}, \; \frac{d\rho}{dr} = \frac{1}{\sqrt{B(r - r_+)}}, \quad \rho = \frac{2}{\sqrt{B}}\sqrt{r - r_+} \tag{253}$$

where we have chosen ρ to vanish at the horizon. Then the first two terms of eq.(251) become

$$ds^2 \sim d\rho^2 + \rho^2 \frac{AB}{4} dt^2 \tag{254}$$

We see that this metric describes a regular geometry provided the Euclidean time t is periodic with period

$$\beta = \frac{1}{T} = \frac{4\pi}{\sqrt{AB}}$$

From this eq.(249) is easily obtained. We also see that only $r \leq r_+$ is relevant.

In our case, we have near $r = r_+$

$$\begin{aligned} V(r) &= V'(r_+) \cdot (r - r_+), \quad V'(r_+) = \frac{2r_+}{b^2} + \frac{(n-2)w_n M}{r_+^{n-1}} \\ 0 &= 1 + \frac{r_+^2}{b^2} - \frac{w_n M}{r_+^{n-2}} \;\Rightarrow\; V'(r_+) = \frac{(n-2)b^2 + nr_+^2}{r_+ b^2} \end{aligned} \tag{255}$$

It follows that our black hole has inverse temperature

$$\beta_0(r_+) = \frac{4\pi r_+ b^2}{nr_+^2 + (n-2)b^2} \tag{256}$$

Notice that $\beta_0(r_+)$ vanishes at $r_+ = 0$ and for $r_+ \to \infty$. Also it attains a maximum (of $4\pi b/\sqrt{n(n-2)}$, at $r_+ = \sqrt{\frac{n-2}{n}b}$). It follows, that unlike the manifold X_1, which may be constructed for any temperature, the black hole metric, X_2 only exists for "small" values of β or "large" values of the temperature. It will turn out in fact, that X_1 dominates the dynamics at low temperatures, and X_2 at high temperatures. We see that to get a high temperature requires either $r_+ \to 0$ or $r_+ \to \infty$. We shall see below, that the thermodynamics is dominated by $r_+ \to \infty$. In that case the $V = 0$ equation for r_+ implies that also $M \to \infty$, indeed that

$$0 \;=\; 1 + \frac{r_+^2}{b^2} - \frac{w_n M}{r_+^{n-2}} \simeq \frac{r_+^2}{b^2} - \frac{w_n M}{r_+^{n-2}} \;\Rightarrow\; r_+ = (Mb^2 w_N)^{1/n} \tag{257}$$

Then approximately in the large mass limit

$$ds^2 = \Big(\frac{r^2}{b^2} - \frac{w_n M}{r^{n-2}}\Big)dt^2 + \frac{dr^2}{\Big(\frac{r^2}{b^2} - \frac{w_n M}{r^{n-2}}\Big)} + r^2 d\Omega_{n-1}^2 \tag{258}$$

Remarkably, this form of the X_2 metric makes it completely equivalent to the non-extremal brane solutions eq.(117), eq.(118), eq.(119). The relation between the two forms is in all cases $U = r$ and $d\Omega_{n-1}^2 \sim \sum_1^{n-1} d\vec{x}^{\,2}$ after a suitable scaling (see below). The relations between the pairs of parameters (L, U_0) and (b, M) are in the relevant cases:

$$\begin{aligned} &D = 10,\ n = 4 \qquad L = b, \quad \frac{U_0^4}{L^2} = w_4 M \\ &D = 11,\ n = 6 \qquad L = \frac{1}{2}b, \quad \frac{U_0^6}{4L^2} = w_6 M \\ &D = 11,\ n = 3 \qquad L = 2b, \quad \frac{4U_0^3}{L^2} = w_3 M \end{aligned} \tag{259}$$

Although the solutions are expressed in terms of two parameters, one may in fact bee scaled away (see below). Notice that the p-brane solutions and the black hole solution have totally different symmetries and asymptotic behaviours. Only in the limits considered here (near horizon and large mass) do they agree.

In the same large M limit we have

$$\beta_0 \sim \frac{4\pi b^2}{n r_+} = \frac{4\pi b^2}{n}(w_n b^2 M)^{-1/n} \tag{260}$$

The topology of the solution is

$$X_2 \sim S^2 \times S^{n-1}$$

since the (t, r) space is topologically similar to the 2-dimensional plane (described by polar coordinates), which compactifies to S^2. On this space there are no non-contracitible loops, in contrast to the case of X_1.

The topology becomes $S^1 \times S^{n-1}$ on the boundary. However, we would like to consider limits where the boundary looks more like $S^1 \times \mathbb{R}^{n-1}$, corresponding to the radius of S^{n-1} being "much larger" than the radius of the S^1.

Near the boundary, i.e. at $r \to \infty$ the metric becomes

$$ds^2 \sim \frac{r^2}{b^2}dt^2 + \frac{b^2}{r^2}dr^2 + r^2 d\Omega_{n-1}^2 \tag{261}$$

At these asymptotically large values of r the metric on S^1 is

$$\frac{r^2}{b^2}dt^2$$

corresponding to a radius of

$$\frac{r}{b} \cdot \frac{\beta_0}{2\pi}$$

On the other hand, the radius of the S^{n-1} (the $r^2 d\Omega_{n-1}^2$ term in the metric) is simply r. The ratio of the two is therefore

$$\frac{\beta_0}{2\pi b}$$

If we want this ratio to be small, we see that we need β_0 small.

We now scale as follows

$$\begin{aligned}
r &= \left(\frac{w_n M}{b^{n-2}}\right)^{1/n} \rho, \quad t = \left(\frac{w_n M}{b^{n-2}}\right)^{-1/n} \tau \\
V &\sim \left(\frac{w_n M}{b^{n-2}}\right)^{2/n} \left\{\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right\} \Rightarrow \\
ds^2 &= \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)^{-1} d\rho^2 + \left(\frac{w_n M}{b^{n-2}}\right)^{2/n} \rho^2 d\Omega^2
\end{aligned} \tag{262}$$

Notice that now the radius of S^{n-1} is of order $M^{2/n} \to \infty$, so that indeed we have managed to replace S^{n-1} effectively by $\mathbb{R}^{n-1}$ as far as coordinates are concerned. The period of τ is

$$\begin{aligned}
\left(\frac{w_n M}{b^{n-2}}\right)^{1/n} \beta_0 &= \left(\frac{w_n M}{b^{n-2}}\right)^{1/n} \frac{4\pi b^2}{n(w_n b^2 M)^{1/n}} \\
&= \frac{4\pi b}{n} \equiv \beta_1
\end{aligned} \tag{263}$$

and we may write

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 dx_{n-1}^2 \tag{264}$$

with the boundary being $S^1(\beta_1) \times \mathbb{R}^{n-1}$. Notice that in these scaled coordinates, the metric is characterized by just one parameter, b; any explicit reference to M has disappeared.

5.1.4 Thermodynamics of the black hole/non-extremal brane solution

In the classical supergravity limit, the CFT partition function should be evaluated as

$$\begin{aligned}
e^{-I} = e^{-I(X_1)} + e^{-I(X_2)} &= e^{-I(X_1)}\left(1 + e^{-\Delta I}\right) \\
&= e^{-I(X_2)}\left(e^{\Delta I} + 1\right) \\
\Delta I &\equiv I(X_2) - I(X_1)
\end{aligned} \tag{265}$$

We shall find

$$\Delta I = \frac{\Omega_{n+1}}{4G_N} \cdot \frac{b^2 r_+^{n-1} - r_+^{n+1}}{n r_+^2 + (n-2)b^2} \tag{266}$$

We see then that for small r_+, $\Delta I > 0$ and tends to zero when r_+ tends to zero. For large r_+ on the other hand $\Delta I < 0$. Thus we have the following situation

$$\begin{aligned} \text{small } r_+ &: e^{-I} \sim 2e^{-I(X_1)} \\ \text{large } r_+ &: e^{-I} \sim e^{-I(X_2)} \end{aligned} \tag{267}$$

For low temperatures, X_2 cannot come into play at all and

$$e^{-I_{CFT}} \sim e^{-I(X_1)}$$

For high temperatures, however, we have both manifolds, and either $r_+ \to 0$ or $r_+ \to \infty$. In the first case, $I(X_2) \to I(X_1)$. In the latter case (the relevant one as it turns out), $I(X_2) < I(X_1)$:

$$\begin{aligned} I(X_2) &= I(X_1) + \Delta I = I(X_1) + \frac{\Omega_{n-1}}{4G_N} \frac{b^2 r_+^{n-1} - r_+^{n+1}}{n r_+^2 + (n-2)b^2} \\ &\sim I(X_1) - \frac{\Omega_{n-1}}{4G_N} \frac{r_+^{n-1}}{n} \ll I(X_1) \end{aligned} \tag{268}$$

It follows that

$$e^{-I(X_1)} + e^{-I(X_2)}|_{r_+ \to 0} \ll e^{-I(X_1)} + e^{-I(X_2)}|_{r_+ \to \infty} \tag{269}$$

All of that is a consequence of the formula for ΔI eq.(266). We now justify that formula.

The relevant part of the action is

$$I = -\frac{1}{16\pi G_N} \int d^{n+1}x \sqrt{g} \Big(R + \frac{n(n-1)}{b^2} \Big) \tag{270}$$

This is clearly badly divergent. In fact the equations of motion give

$$R_{\mu\nu} = -\frac{n}{b^2} g_{\mu\nu} \Rightarrow R = -\frac{n(n+1)}{b^2} \tag{271}$$

so that

$$I = \frac{1}{8\pi G_N} \frac{n}{b^2} \int d^{n+1} \sqrt{g} = \frac{n}{8\pi G_N b^2} V_{n+1} \tag{272}$$

i.e. a volume divergence. We may hope however, to make sense of the *difference,* ΔI, between I evaluated for the two manifolds X_1 and X_2. This requires that we arrange for the two manifolds somehow to be "asymptotically identical" in their geometrical respect. Now we have for both

$$ds^2 = V dt^2 + V^{-1} dr^2 + r^2 d\Omega^2 \Rightarrow \sqrt{g} = r^{n-1} \sqrt{\gamma} \tag{273}$$

γ being the determinant of the metric on the sphere S^{n-1}. V is given by eq.(248) for the geometry of X_2 and by $V = 1 + (r/b)^2$ for the geometry of X_1, eq.(233) To regulate the integrals, we terminate the integral over r at some large R:

$$\begin{aligned} \text{Volume}(X_1) \equiv V(X_1) &= \int_0^{\beta(X_1)} dt \int_0^R dr r^{n-1} \Omega_{n-1}, \quad 0 \le r \le R \\ V(X_2) &= \int_0^{\beta_0} dt \int_{r_+}^R dr r^{n-1} \Omega_{n-1}, \quad r_+ \le r \le R \end{aligned} \tag{274}$$

So we must choose $\beta(X_1)$ such that the geometry at the R-surface is the same in the two cases, in particular that the circumference of the S^1 is the same for the two manifolds:

$$\begin{aligned}
\text{for } X_1 &: \quad \sqrt{1+\frac{R^2}{b^2}}\beta(X_1) \\
\text{for } X_2 &: \quad \sqrt{1+\frac{R^2}{b^2}-\frac{w_n M}{R^{n-2}}}\beta_0 \Rightarrow \\
V(X_1) &= \left[\left(1+\frac{R^2}{b^2}-\frac{w_n M}{R^{n-2}}\right)/\left(1+\frac{R^2}{b^2}\right)\right]^{\frac{1}{2}}\beta_0\frac{R^n}{n}\Omega_{n-1} \\
&\simeq \left(1-\frac{w_n M}{2R^n}\right)\frac{R^n\beta_0}{n}\Omega_{n-1}, \quad R\to\infty \\
V(X_2) &= \beta_0\frac{R^n-r_+^n}{n}\Omega_{n-1} \\
\Delta I = I(X_2)-I(X_1) &\simeq \frac{\Omega_{n-1}\beta_0}{8\pi G_N}\left\{\frac{w_n M}{2}-\frac{r_+^n}{b^2}\right\}
\end{aligned} \tag{275}$$

independent of R. But

$$\frac{w_n M}{r_+^{n-2}} = \frac{r_+^2}{b^2}+1$$

so

$$\Delta I = \frac{\Omega_{n+1}}{4G_N}\frac{b^2 r_+^{n-1}-r_+^{n+1}}{nr_+^2+(n-2)b^2}$$

as promised eq.(266). This completes the discussion.

We now want to interpret $e^{-\Delta I}$ as a statistical average $e^{-\beta_0 E}$ where

$$E = \frac{\partial \Delta I}{\partial \beta_0} = \frac{\partial \Delta I}{\partial r_+}\frac{\partial r_+}{\partial \beta_0} \tag{276}$$

A slightly long but straight forward calculation gives the result

$$E = \frac{(n-1)\Omega_{n-1}}{16\pi G_N}\left(\frac{r_+^n}{b^2}+r_+^{n-2}\right) = M$$

This in fact justifies the long used notation for M.

More interestingly the Beckenstein-Hawking entropy is found from identifying ΔI with the free energy F:

$$\begin{aligned}
e^{-F} &= \sum_{\text{“states”}} e^{-\beta_0 H} = \langle e^{-\beta_0 H}\rangle \cdot \Big(\text{effective no. of degrees of freedom}\Big) \\
&= e^{-\beta_0 E}e^{S}
\end{aligned} \tag{277}$$

with $S=\beta_0 E-\Delta I$ the entropy. A very simple calculation now gives

$$S = \frac{\Omega_{n-1}r_+^{n-1}}{4G_N} = \frac{\text{“area of horizon in } n-1 \text{ dimensions”}}{4G_N} \tag{278}$$

which of course is the very famous result generalized from the flat case of $b \to \infty$ to the present case of AdS_{n+1}. Notice incidentally that this kind of calculation does not work in flat space directly since the classical action vanishes there.

Exercise: Verify the expressions given for the mass and entropy of the black hole.

5.2 On hadrons and confinement in QCD at large N. The case of QCD_3

We have emphasized that the supergravity approximation is not satisfactory for a serious study of large N QCD, even supposing the Maldacena conjecture works, and that no unpleasant phenomena corrupt the set up proposed by Witten. Nevertheless it is very instructive to see how this approximation very naturally gives a picture in qualitative agreement with expectations in hadronic physics. Thus the area law for Wilson loops, associated with confinement, comes out and goes away appropriately at high (physical) temperatures. Several other aspects have been treated [36, 40, 43]. Here, however, we shall concentrate on just one of these aspects, generation of a *mass gap* [3, 10, 39, 37, 45, 38, 42, 46, 47].

When we use the scheme described above in the case of $AdS_5 \times S^5$, compactifying Euclidean time, we expect (as argued) in the boundary theory to obtain an effective (Euclidean) version of QCD_3 at energies much below the temperature cut-off. In the next subsection we shall briefly indicate the framework for also dealing with QCD_4. In both cases, the hadrons we expect to find will be stable glueballs, in particular with a minimal positive mass greater than zero: a mass gap. This then is a non-trivial confinement effect, not visible in classical YM-theory, nor in perturbation theory.

How would we identify such a mass gap? We would need in the boundary theory to consider a gauge invariant operator which could create a glueball, the simplest example perhaps being

$$\mathcal{O}(\vec{x}\,) = trF^2(\vec{x}\,) \equiv \sum_{i,j=1}^{N} \Big(F_{\mu\nu}\Big)_{ij} \Big(F^{\mu\nu}\Big)_{ji}(\vec{x}\,) \quad \vec{x}\, \in \mathbb{R}^3 \tag{279}$$

with $(F_{\mu\nu})_{ij} = \sum_{a=1}^{N^2} F^a_{\mu\nu} T^a_{ij}$ being the $U(N)$ Yang Mills field strength expressed as an $N \times N$ matrix. We should then work out a 2-point function

$$\langle \mathcal{O}(\vec{x}\,)\mathcal{O}(\vec{0})\rangle \sim \exp\{-|\vec{x}\,|m\}, \quad |\vec{x}\,| \to \infty \tag{280}$$

and look for the indicated exponential decay at large distances, dominated by the minimal mass, $m > 0$, the mass gap. Non leading exponential terms would correspond to gluon excitations.

To perform the calculation in the *bulk theory*, we would have to identify the field, $\Phi(y)$, $y \in$ bulk, to which this operator couples. But we have already indicated in sect. 4.5.1 that the expected candidate is the dilaton field with no KK-excitations on S^5, the S^5 s-wave, or $k = 0$ mode [10]. We might then perform a classical supergravity calculation of the action in the AdS_5 background with the black hole and with prescribed boundary conditions on the dilaton field.

However, [3, 10] it seems technically simpler to adopt a slightly different point of view. To see how, begin noticing that (according to the Maldacena conjecture) the two quantum theories, the bulk theory and the boundary theory, are entirely equivalent. They have the same Hilbert space, the same operators and correlators, only the physical interpretation of operators differ drastically in the two theories. First consider the boundary theory. At $N \to \infty$, glueballs are free particles. When moving according to plane waves

$$e^{i\vec{k}\cdot\vec{x}}$$

their mass is given by

$$m^2 = -\vec{k}^2, \tag{281}$$

(so only imaginary momenta correspond to being on the mass shell, as is usual in Euclidean space-time). In the Hilbert space there is an operator describing translations in the boundary theory, and $\vec{k}$ is the eigenvalue of that.

In the bulk theory, the very same operators occur, just with different interpretation. We need to think of the Hilbert space of the dilaton theory. First recall the situation for a free scalar in flat space-time. It satisfies the KG equation

$$(\partial^2 - m^2)\phi = 0$$

To build the Hilbert space (in this case the Fock space) one finds "modes", solutions of the KG equation, such as e^{ipx}, and build field operators as sums over modes, each mode being multiplied by a creation or annihilation operator. States in the Hilbert space are spanned by multi-particle states created by the creation operators. Actually, although we often consider plane waves, we really need to put them in a quantization volume, or better in fact, consider wave packets, *square integrable modes* for which

$$\int_{\text{space}} (\text{mode})^2 < \infty$$

We wish to imitate this procedure in the present case of a modified AdS_5 background with a black hole in it, the manifold X_2. Thus we must

1. Formulate the equation of motion on X_2,
2. find the square integrable modes,
3. interpret those in terms of possible m^2(glueball) values.

We follow here the treatments in [10, 37, 45, 39]. In the case of $AdS_5 \times S^5$ we have

$$b^2 = \ell_s^2\sqrt{4\pi g_s N} = \ell_s^2\sqrt{g_{YM}^2 N} \tag{282}$$

Putting $n = 4$ in the black hole metric eq.(264) we find

$$\frac{ds^2}{\ell_s^2\sqrt{4\pi g_s N}} = \left(\frac{\rho^2}{b^4} - \frac{1}{\rho^2}\right)d\tau^2 + \frac{d\rho^2}{\left(\rho^2 - \frac{b^4}{\rho^2}\right)} + \frac{\rho^2}{b^2}dx_3^2 + d\Omega_5^2 \tag{283}$$

(with a *unit radius* S^5). Now put $\tau = b^2\tilde{\tau}$ and get (scaling also x_i)

$$\frac{ds^2}{b^2} = \Big(\rho^2 - \frac{b^4}{\rho^2}\Big)d\tilde{\tau}^2 + \frac{d\rho^2}{\Big(\rho^2 - \frac{b^4}{\rho^2}\Big)} + \rho^2 dx_3^2 + d\Omega_5^2 \tag{284}$$

Here the period of τ is $\beta_1 = 4\pi b/n = \pi b$ and the period of $\tilde{\tau}$ is

$$\beta(\tilde{\tau}) \equiv 2\pi R(\tilde{\tau}) = \beta_1/b^2 = \pi/b \tag{285}$$

while the horizon is still at $\rho = b$.

Now, the dilaton is massless in 10 dimensions, and the $k = 0$ mode on S^5 which we should consider, is still massless in 5 dimensions. This mode does not depend on coordinates of S^5 and it satisfies the equation of motion on X_2

$$\partial_\mu \left[\sqrt{g}\partial_\nu \Phi g^{\mu\nu}\right] = 0 \tag{286}$$

We now wish to look for modes that are (i) square integrable on X_2, and (ii) correspond to a definite momentum in the boundary theory. Thus we consider the ansatz

$$\Phi(\rho, \vec{x}\,) = f(\rho)e^{i\vec{k}\cdot\vec{x}} \tag{287}$$

Also our Euclidean time $\tilde{\tau}$ is compactified corresponding to a "high" temperature, and we do not want the mode to depend on $\tilde{\tau}$ in the "low energy approximation". Clearly we may then identify $\vec{k}^2 = -m^2$ as the glueball mass in the boundary theory. Now we have

$$g^{00} = \Big(\rho^2 - \frac{b^4}{\rho^2}\Big)^{-1},\ g^{\rho\rho} = \Big(\rho^2 - \frac{b^4}{\rho^2}\Big),\ g^{ij} = \delta^{ij}\rho^{-2},\ \sqrt{g} = \rho^3 \tag{288}$$

The equation of motion then becomes

$$\begin{aligned} \partial_\rho\left[\rho^3\partial_\rho\Phi\Big(\rho^2 - \frac{b^4}{\rho^2}\Big)\right] + \partial_i\left[\rho^3\partial_i\Phi\rho^{-2}\right] &= 0 \ \text{ or} \\ \partial_\rho\left[\Big(\rho^5 - b^4\rho\Big)f'(\rho)\right] - k^2\rho f(\rho) &= 0 \text{ or} \\ \rho^{-1}\frac{d}{d\rho}\left[\rho\Big(\rho^4 - b^4\Big)f'(\rho)\right] &= -m^2 f(\rho) \end{aligned} \tag{289}$$

Put

$$x = \rho^2,\ \frac{d}{d\rho} = \frac{dx}{d\rho}\frac{d}{dx} = 2\rho\frac{d}{dx}$$

and obtain

$$4x(x^2 - b^4)\frac{d^2 f}{dx^2} + 4(3x^2 - b^4)\frac{df}{dx} - k^2 f = 0 \tag{290}$$

and of course we may scale b away and replace it by 1. This ordinary differential equation is the equation of motion. It has solutions for any values of $\vec{k}^2$, but now we must understand the additional information coming from boundary conditions and square integrability. These will imply that for generic $\vec{k}^2$ there is no acceptable solution, only for a particular *spectrum* of *strictly positive* $-\vec{k}^2$ values do such solutions exist.

5.2.1 Boundary condition

Putting $b = 1$ the metric eq.(284) becomes (we write τ for $\tilde{\tau}$ in the following):

$$ds^2 \sim \left(x - \frac{1}{x}\right)^{-1} d\rho^2 + \left(x - \frac{1}{x}\right) d\tau^2 + ... = \frac{dx^2}{4(x^2-1)} + \left(x - \frac{1}{x}\right) d\tau^2 + ... \tag{291}$$

As before there is a coordinate singularity at the horizon $x = 1$ which we wish to cast into the form of a 2-dimensional polar coordinate singularity

$$\begin{aligned} dz^2 &= \frac{dx^2}{4(x^2-1)} \quad (z \text{ is polar radius}) \\ \frac{dz}{dx} &= \frac{1}{2\sqrt{x^2-1}}, \quad z = \frac{1}{2}\cosh^{-1} x, \quad x = \cosh 2z \end{aligned} \tag{292}$$

with $z = 0$ at the horizon $x = 1$, and we find

$$\left(x - \frac{1}{x}\right) = \frac{\sinh^2 2z}{\cosh 2z} \simeq 4z^2 \text{ near } z = 0 \tag{293}$$

and

$$ds^2 \simeq dz^2 + 4z^2 d\tau^2 + ...$$

near the horizon, showing that we have "polar-like" coordinates with polar distance $\propto z$. The function $f(\rho)$ above is then a function of the "polar distance" only, not of the "angle" τ. The proper boundary condition for such a function to be smooth at the origin is therefore

$$\frac{df}{dz} = 0$$

But

$$\frac{df}{dz} = \frac{dx}{dz}\frac{df}{dx} = 2\sinh 2z \frac{df}{dx} \simeq 4z\frac{df}{dx}$$

near $z \simeq 0$. So we merely want to ensure that f is *regular* at $x = 1$.

5.2.2 Square integrability

We want to demand that

$$\int \sqrt{g} d\rho |f(\rho)|^2 < \infty$$

But

$$\sqrt{g} d\rho = \rho^3 d\rho = \rho^2 \rho d\rho = \frac{1}{2} x dx$$

Hence we should demand, that if f is inverse power bounded, by $f \sim \mathcal{O}(x^{-a})$, then $a > 1$.

5.2.3 Determination of the spectrum

The differential equation eq.(290) divided by $4x(x^2-1)$ becomes

$$y'' + \Big(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\Big)y' - \frac{p}{x(x^2-1)}y = 0 \tag{294}$$

$p \equiv \vec{k}^2/4 = -m^2/4$, and $y \equiv f$. This homogeneous, linear, 2. order, ordinary differential equation has a 2-dimensional space of solutions for any value of the mass parameter, p. We follow now the treatment of [45]. These solutions may be expressed as linear combinations of any 2 linearly independent solutions. Generically these would be analytic functions of x with singularities at $x = 0, 1, \infty$. Therefore these solutions cannot be represented by series expansions, convergent throughout the physical region $1 \leq x < \infty$. Instead, however, we may consider expansions convergent either in

$$I(\infty) \equiv \{x \in \mathbb{C} | 1 < |x| < \infty\}$$

or in

$$I(1) \equiv \{x \in \mathbb{C} | |x-1| < 1\}$$

respectively. Thus for the case of $I(\infty)$ we use the ansätze

$$\begin{aligned} y_1^\infty(x) &= \frac{1}{x^2} + \sum_{n=1}^{\infty} a_n^\infty \frac{1}{x^{n+2}} \\ y_2^\infty(x) &= \frac{p^2}{2} y_1^\infty(x) \log x + \sum_{n=1}^{\infty} b_n^\infty \frac{1}{x^n} \end{aligned} \tag{295}$$

whereas for the case of $I(1)$ we may use

$$\begin{aligned} y_1^1(x) &= 1 + \sum_{n=1}^{\infty} a_n^1 (x-1)^n \\ y_2^1(x) &= y_1^1(x) \log(x-1) + \sum_{n=1}^{\infty} b_n^1 (x-1)^n \end{aligned} \tag{296}$$

Inserting these expansions into the differential equation, one finds recursion relations for the unknown coefficients with simple unique solutions.

Any solution of the differential equation may then be represented, *either* as a linear combination of $y_1^\infty(x)$ and $y_2^\infty(x)$, *or* a linear combination of $y_1^1(x)$ and $y_2^1(x)$. The series expansions only converge in $I(\infty)$ or $I(1)$ respectively, but the solutions may be uniquely analytically continued to the entire complex plane (with the exception of the singularities at $x = 0, 1$). In the overlap (containing $1 < x < 2$), they may furthermore be directly compared. It is now clear that $y_1^\infty(x)$ but not $y_2^\infty(x)$ satisfies the square integrability condition. Also, only $y_1^1(x)$, but not $y_2^1(x)$ satisfies the boundary condition at $x = 1$.

We may therefore assert, that an acceptable solution is expressed in the y_i^∞ basis simply as

$$c \cdot y_1^\infty(x)$$

and such a solution may be built for any p. However, when analytically continued to $I(1)$, it will generically be a expressed as a linear combination

$$\alpha_1 y_1^1(x) + \alpha_2 y_2^1(x)$$

with $\alpha_2 \neq 0$, and therefore be unacceptable from the point of view of the behaviour near $x = 1$. Similarly, we may assert, that any acceptable solution is expressed in the y_i^1 basis simply as

$$c' \cdot y_1^1(x)$$

and of course also such a solution may be built for any p, but when analytically continued to $I(\infty)$ it will generically be expressed as a linear combination

$$\beta_1 y_1^\infty(x) + \beta_2 y_2^\infty(x)$$

with $\beta_2 \neq 0$, and therefore be unacceptable from the point of view of square integrability.

It follows, that if we restrict ourselves to the overlap $1 < x < 2$, we are able to require the condition

$$y_1^\infty(x) \propto y_1^1(x) \tag{297}$$

and this condition is only satisfied for certain discrete values of p. These represent the sought for spectrum of mass values. More concretely, we may pick a value such as $x = x_0 = 3/2 \in I(1) \cap I(\infty)$, and require the existence of a common constant, c such that

$$\begin{aligned} y_1^\infty(x_0) &= c y_1^1(x_0) \\ (y_1^\infty)'(x_0) &= c (y_1^1)'(x_0) \end{aligned} \tag{298}$$

This will ensure that eq.(297) is satisfied throughout, since we are dealing with a 2-dimensional set of solutions. These conditions are of course just equivalent to the Wronski condition

$$\left| \begin{array}{cc} y_1^\infty(x_0) & y_1^1(x_0) \\ (y_1^\infty)'(x_0) & (y_1^1)'(x_0) \end{array} \right| = 0 \tag{299}$$

Here we should think of $y_i^a(x_0)$ and $(y_i^a)'(x_0)$ $(a = \infty, 1)$ as functions of p (for fixed x_0). Indeed the coefficients a_n^a are determined uniquely as polynomials in p of order n. Truncating the series expansions at some high cut-off, eq.(299) becomes the condition for zeros of a certain high order polynomial in p. As the summation cut-off is taken higher and higher, the numerically smaller zeros of the polynomial rather quickly converge, whereas the larger zeros take more terms to stabilize.

Concretely and by way of illustration, it is easy from the differential equation to establish the following recursion relations (using the notation $a_{n<0} \equiv 0, a_0^\infty = 1 = a_0^1$):

$$\begin{aligned} a_{n+1}^\infty &= \frac{1}{(n+2)^2 - 1}\Big(a_{n-1}^\infty (n+1)^2 + p a_n^\infty\Big) \\ a_{n+1}^1 &= \frac{1}{2(n+1)^2}\Big(a_{n-1}^1 (n^2 - 1) + (3n(n+1) - p) a_n^1\Big) \end{aligned} \tag{300}$$

Thus

$$a_1^\infty = \frac{p}{3},\ a_2^\infty = \frac{p^2+12}{24}, ...$$
$$a_1^1 = -\frac{p}{2},\ a_2^1 = \frac{p(p-6)}{16}, ...$$

The glueball masses thus determined may be shown to pertain to $J^{PC} = 0^{++}$ states. Numerically one finds (in units of $1/b$) the following strictly positive mass values

$$11.6,\ 34.5,\ 69.0,\ 114.9, ..$$

(close, but not equal to the values, $6n(n+1)$ [37, 45]). Similar studies have been made for other J^{PC} quantum numbers. Comparisons with results from lattice calculations indicate reasonable agreement for the mass *ratios.*

5.3 On QCD in 4 dimensions

Witten [10] has also explained how to obtain 4-dimensional QCD based on the $AdS_7 \times S^4$ compactification of 11-dimensional M-theory with an AdS_7 scale eq.(113)

$$b = 2\ell_{11}(\pi N)^{1/3}$$

In that case the boundary theory is the conformally invariant 6-dimensional so called (2,0) theory[34]. Excitations of this 5-brane cannot be understood in terms of open strings ending on it, but in fact in terms of 2-branes ending on it. Upon compactification on a circle of radius R_1, we obtain type IIA string theory, and for small R_1, IIA string theory in the perturbative regime. There the 2-branes of M-theory turn into strings and we *can* use string theory to describe excitations of that once compactified theory. Now the effective boundary theory is 5-dimensional. We still have to compactify once more, on a circle of radius R_2 before we obtain the desired 4-dimensional theory. In this second step we must take fermions anti periodic around the second S^1 in order to break supersymmetry and conformal invariance. It is seen that we assumed it made sense to perform the compactification in two steps, corresponding to $R_1 \ll R_2$.

In the first step, we take SUSY preserving boundary conditions for the fermions, and we get a string theory 4-brane with an effective 5-dimensional SYM theory with YM coupling eq.(120)

$$g_5^2 = 8\pi^2 \ell_s g_s = 8\pi^2 R_1^{(10)} \tag{301}$$

where the last step is the standard identification between M-theory and IIA string theory upon compactification on a circle of 10-dimensional radius $R_1^{(10)}$ [48]. In the second compactification on $S^1(R_2)$, we may write

$$\int d^5x \frac{1}{g_5^2} \to \int d^4x \frac{2\pi R_2}{g_5^2} \Rightarrow \frac{1}{g_4^2} = \frac{2\pi R_2}{g_5^2}$$

So we get for the 4-dimensional YM coupling

$$\frac{g_{YM}^2}{4\pi} = \frac{R_1}{R_2} \tag{302}$$

At low energies this effective D3 brane theory is large N QCD with a running coupling constant, which at large energies becomes smaller and smaller (asymptotic freedom), until the theory changes over into a 5-dimensional one at the "cut-off" R_2 (in string units). Thus the YM coupling eq.(302) is the value of α_s at that cut-off, and ideally we want that to be small in accord with the requirement $R_1 \ll R_2$. If the second compactification also preserves SUSY, we get the $N = 4$ superconformal theory, but if we take fermions anti periodic around $S^1(R_2)$ we expect to obtain ordinary large N QCD (with no quarks).

Let $\eta \equiv \frac{g_{YM}^2 N}{4\pi}$ denote the 't Hooft coupling, so

$$R_1 = \frac{\eta R_2}{N} \tag{303}$$

with fixed (preferably small) η at $N \to \infty$. Let us in fact begin with the second compactification first. The AdS_7 metric with a black hole is given by eq.(264)

$$ds^2 = \Big(\frac{\rho^2}{b^2} - \frac{b^4}{\rho^4}\Big)d\tau^2 + \frac{d\rho^2}{\Big(\frac{\rho^2}{b^2} - \frac{b^4}{\rho^4}\Big)} + \rho^2 dx_5^2 \tag{304}$$

and $b = 2\ell_{11}(\pi N)^{1/3}$. Here τ has period ($n = 6$)

$$\beta_1 = \frac{4\pi b}{n} = \frac{2\pi b}{3} = \ell_{11}\frac{4\pi}{3}(\pi N)^{1/3}$$

Then put

$$\tau = \frac{b}{3}\theta, \quad \rho = b\lambda$$

with θ an angular variable $\in [0, 2\pi]$. Then (with a trivial rescaling of the x_i)

$$ds^2 = \Big(\lambda^2 - \frac{1}{\lambda^4}\Big)\frac{1}{9}b^2 d\theta^2 + \frac{b^2 d\lambda^2}{\Big(\lambda^2 - \frac{1}{\lambda^4}\Big)} + \frac{b^2\lambda^2}{9}dx_5^2 \tag{305}$$

This metric of course is in 11-dimensional units. But we should also compactify on a circle of radius R_1, say use the coordinate x_5 for that, with that radius being η/N times the radius of the first compactification eq.(303), at least at the boundary, i.e. for very large values of λ. For large λ, the circle with coordinate θ (the R_2 circle) has metric

$$\frac{1}{9}\lambda^2 b^2 d\theta^2$$

and therefore radius $b\lambda/3$. For large λ the x_5 coordinate has metric

$$\frac{1}{9}b^2\lambda^2 dx_5^2$$

(before, dx_5^2 meant the flat metric in 5 dimensions, now it refers just to the 5'th coordinate) so to compactify with the correct radius, we want to write

$$x_5 = \frac{\eta}{N}\psi$$

with ψ an angular variable. The full 11-dimensional metric, including the S^4 part is now

$$\begin{aligned} ds^2 &= \frac{b^2}{9}\Big(\lambda^2 - \frac{1}{\lambda^4}\Big)d\theta^2 + \frac{b^2\lambda^2\eta^2}{9N^2}d\psi^2 \\ &+ \frac{b^2 d\lambda^2}{\Big(\lambda^2 - \frac{1}{\lambda^4}\Big)} + \frac{b^2\lambda^2}{9}\sum_1^4 dx_i^2 + \frac{b^2}{4}d\Omega_4^2 \end{aligned} \tag{306}$$

It is the circle parametrized by ψ which gives us IIA compactification. It is seen to have a radius of

$$\frac{1}{3}b\lambda\frac{\eta}{N}$$

in 11-dimensional units. The general rule to go between 11-dimensional and 10-dimensional units is [48]

$$\begin{aligned} L^{(11)} &= g_s^{-1/3}L^{(10)}; \quad R_1^{(11)} = g_s^{2/3}\ell_{11}, \quad R_1^{(10)} = g_s\ell_s \\ ds^2_{(11)} &= g_s^{-2/3}ds^2_{(10)} + (R_1^{11})^2 d\psi^2 \Rightarrow ds^2_{(10)} = \frac{R_1^{(11)}}{\ell_{11}}ds^2_{(11)} - (x_5 - \text{part}) \\ g_s &= e^{\phi}, \quad \phi = \text{ dilaton field} \end{aligned} \tag{307}$$

So the final 10-dimensional metric in the string frame becomes independent of N (!)

$$\begin{aligned} 3\frac{ds^2_{(10)}}{8\pi\eta\ell^3} &= \frac{1}{9}\Big(\lambda^3 - \frac{1}{\lambda^3}\Big)d\theta^2 + \frac{\lambda^2 d\lambda^2}{\Big(\lambda^3 - \frac{1}{\lambda^3}\Big)} \\ &+ \frac{1}{9}\lambda^3\sum_1^4 dx_i^2 + \frac{1}{4}\lambda d\Omega_4^2 \end{aligned} \tag{308}$$

This is the form given in [10]. From the radius of the circle parametrized by ψ we find

$$\frac{b\lambda\eta}{3N} = \frac{2\pi^{1/3}\lambda\eta\ell_{11}}{3N^{2/3}} = g_s^{2/3}\ell_{11}$$

and

$$g_s = e^{\phi} = \frac{1}{N}\sqrt{\pi\Big(\frac{2\lambda\eta}{3}\Big)^3} \tag{309}$$

So this time we see that we have a non-trivial dilaton field depending on the coordinate λ. This fact, actually makes the analysis considerably more complicated. We must set up fluctuation equations for dilatons around the background provided by the above solution [37]. But a number of subtleties exist [38, 41, 42] which we do not want to discuss here.

It is instructive to derive an equivalent version of the above, starting from the non-extremal near horizon limit of the 11-dimensional 5-brane solution eq.(118) in 11 dimensional units

$$ds^2 = \frac{U^2}{4L^2}\left\{\Big(1 - \Big(\frac{U_0}{U}\Big)^6\Big)dt^2 + d\vec{x}^{\,2}\right\} + 4L^2\frac{dU^2}{U^2\Big(1 - \Big(\frac{U_0}{U}\Big)^6\Big)} + L^2 d\Omega_4^2 \tag{310}$$

with $L^2 = \ell_{11}^2 (\pi N)^{2/3}$. Now put

$$L^2 r = \frac{U^2}{4L^2}$$

and recast this into

$$\frac{ds^2}{L^2} = r\left\{\left(1 - \left(\frac{r_0}{r}\right)^3\right)dt^2 + d\vec{x}^{\,2}\right\} + \frac{dr^2}{r^2\left(1 - \left(\frac{r_0}{r}\right)^3\right)} + d\Omega_4^2 \tag{311}$$

From the discussion above we have seen that when compactifying on a circle to get to the 10-dimensional IIA metric, we must in fact write

$$ds^2_{(11)} = \left(R_1^{(11)}\right)^2 d\psi^2 + g_s^{-2/3} ds^2_{(10)} = \ell_{11}^2 e^{4\phi/3} d\psi^2 + e^{-2\phi/3} ds^2_{(10)} \tag{312}$$

Thus we find (in suitable units)

$$\begin{aligned} r &= e^{4\phi/3} \Rightarrow e^{\phi} = r^{3/4} \\ ds^2_{(10)} &= r^{3/2}\left\{\left(1 - \left(\frac{r_0}{r}\right)^3\right)dt^2 + \frac{dr^2}{r^3 - r_0^3} + \sum_1^4 dx_i^2\right\} + r^{\frac{1}{2}} d\Omega_4^2 \end{aligned} \tag{313}$$

This is the form given in [38, 41]. The transformation

$$r \propto \lambda^2$$

makes this form in agreement with eq.(308). It is instructive to see that it also agrees with the non-extremal D4-brane solution. From eq.(87) we find with

$$D = 10,\ p = 4,\ d = 5,\ a = -\frac{1}{2},\ \Delta = 16$$

$$\begin{aligned} H &= 1 + \left(\frac{h}{r}\right)^3 \sim \left(\frac{h}{r}\right)^3 \quad \text{in the near horizon approximation} \\ f &= 1 - \left(\frac{r_0}{r}\right)^3 \end{aligned} \tag{314}$$

Then (in un units of h)

$$\begin{aligned} ds^2_{(10)}(\text{Einstein}) &= r^{9/8}\left\{\left(1 - \left(\frac{r_0}{r}\right)^3\right)dt^2 + \sum_1^4 dx_i^2\right\} \\ &\quad + r^{-15/8}\left\{\frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^3} + r^2 d\Omega_4^2\right\} \\ e^{\phi} &= r^{3/4} \end{aligned} \tag{315}$$

The dilaton agrees with above, but the metric being in the Einstein frame does not yet. However, we may put it in the appropriate string frame using eq.(49), by multiplying the Einstein frame metric by

$$e^{\frac{1}{2}\phi} = r^{3/8}$$

This precisely reproduces eq.(313).

In the supergravity approximation we should now set up fluctuation equations of motion for dilatons and other fields in the backgrounds given. However, [38] it becomes important to diagonalize these fluctuations appropriately in a non trivial way, since the dilaton background itself is non trivial. We shall not, however, pursue these finer points.

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