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Intuitionistic Fuzzifications of a-Ideals in BCI-Algebras

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Abstract

The notion of intuitionistic fuzzy a-ideals in BCI-algebras is introduced. Conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy a-ideal are provided. Using a collection of a-ideals, intuitionistic fuzzy a-ideals are established.

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1 Introduction

To develop the theory of BCI-algebras, the ideal theory plays an important role. Liu and Meng [6] introduced the notion of q-ideals and a-ideals in BCI-

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algebras. Liu and Zhang [7] discussed the fuzzification of a-ideals, gave relations between fuzzy ideals, fuzzy a-ideals and fuzzy p-ideals. They also considered characterizations of fuzzy a-ideals. Using the notion of fuzzy a-ideals, they provided characterization of associative BCI-algebras. After the introduction of fuzzy sets by Zadeh [9], there have been a number of generaizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Aranassov [1, 2] is one among them. In this paper, we apply the concept of an intuitionistic fuzzy set to a-ideals in BCI-algebras. We introduce the notion of an intuitionistic fuzzy a-ideal of a BCI-algebra, and investigate some related properties. We provide relations between an intuitionistic fuzzy ideal and an intuitionistic fuzzy a-ideal. We give characterizations of an intuitionistic fuzzy a-ideal. Using a collection of a-ideals, we establish intuitionistic fuzzy a-ideals.

2 Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

We can define a partial order '$\leq$' on X by $x \leq y$ if and only if $x * y = 0$. Any BCI-algebra X has the following properties:

(a1) $(\forall x \in X)\ (x * 0 = x)$.

(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

(a3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

A mapping $\mu : X \to [0, 1]$, where X is an arbitrary nonempty set, is called a *fuzzy set* in X. For any fuzzy set μ in X and any $t \in [0, 1]$ we define two sets

$$U(\mu; t) = \{x \in X \mid \mu(x) \geq t\} \text{ and } L(\mu; t) = \{x \in X \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

As an important generalization of the notion of fuzzy sets in X, Atanassov [1, 2] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a nonempty set X as objects having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for Example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [3]).

For the sake of simplicity, we shall use the symbol $A = \langle X, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$.

A nonempty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,

(I2) $(\forall x, y \in X)\ (\forall y \in A)\ (x * y \in A \Rightarrow x \in A)$.

A nonempty subset A of a BCI-algebra X is called an *a-ideal* of X if it satisfies (I1) and

(I3) $(\forall x, y \in X)\ (\forall z \in A)\ ((x * z) * (0 * y) \in A \Rightarrow y * x \in A)$.

Definition 2.1. *An IFS $A = \langle X, \mu_A, \gamma_A\rangle$ in a BCI-algebra X is called an intuitionistic fuzzy subalgebra of X if it satisfies:*

$$\begin{aligned} &(\forall x, y \in X)\,(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}), \\ &(\forall x, y \in X)\,(\gamma_A(x * y) \leq \max\{\gamma_A(x), \gamma_A(y)\}). \end{aligned} \tag{2.1}$$

Definition 2.2. *An IFS $A = \langle X, \mu_A, \gamma_A\rangle$ in a BCI-algebra X is called an intuitionistic fuzzy ideal of X if it satisfies:*

$$(\forall x \in X)\,(\mu_A(0) \geq \mu_A(x),\ \gamma_A(0) \leq \gamma_A(x)), \tag{2.2}$$

and

$$\begin{aligned} &(\forall x, y \in X)\,(\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}), \\ &(\forall x, y \in X)\,(\gamma_A(x) \leq \max\{\gamma_A(x * y), \gamma_A(y)\}). \end{aligned} \tag{2.3}$$

Definition 2.3. *An intuitionistic fuzzy ideal $A = \langle X, \mu_A, \gamma_A\rangle$ in a BCI-algebra X is said to be closed if it satisfies:*

$$(\forall x \in X)\,(\mu_A(0 * x) \geq \mu_A(x),\ \gamma_A(0 * x) \leq \gamma_A(x)). \tag{2.4}$$

3 Intuitionistic fuzzy a-ideals

In what follows, let X denotes a BCI-algebra unless otherwise specified. We first consider the intuitionistic fuzzification of the notion of a-ideals in a BCI-algebra as follows.

Definition 3.1. *An IFS $A = \langle X, \mu_A, \gamma_A \rangle$ in X is called an intuitionistic fuzzy a-ideal of X if it satisfies (2.2) and*

$$\begin{array}{l}(\forall x, y, z \in X)\,(\mu_A(y * x) \geq \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\}),\\ (\forall x, y, z \in X)\,(\gamma_A(y * x) \leq \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}).\end{array} \tag{3.1}$$

Example 3.2. *Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

We define an IFS $A = \langle X, \mu_A, \gamma_A \rangle$ in X by

$$A = \left\langle X, \left(\tfrac{0}{0.8}, \tfrac{a}{0.8}, \tfrac{b}{0.3}, \tfrac{c}{0.3}\right), \left(\tfrac{0}{0.1}, \tfrac{a}{0.1}, \tfrac{b}{0.4}, \tfrac{c}{0.4}\right)\right\rangle,$$

By routine calculations, we know that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X.

Theorem 3.3. *Every intuitionistic fuzzy a-ideal is both an intuitionistic fuzzy subalgebra and an intuitionistic fuzzy ideal.*

Proof. Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy a-ideal of X. Taking $y = z = 0$ in (3.1) and using (III) and (a1), we have

$$\mu_A(0 * x) \geq \mu_A(x),\ \gamma_A(0 * x) \leq \gamma_A(x). \tag{3.2}$$

Setting $x = z = 0$ in (3.1) induces from (a1), (III), (2.2) and (3.2) that

$$\mu_A(y) = \mu_A(y * 0) \geq \mu_A(0 * (0 * y)) \geq \mu_A(0 * y),$$

$$\gamma_A(y) = \gamma_A(y * 0) \leq \gamma_A(0 * (0 * y)) \leq \gamma_A(0 * y)$$

for all $y \in X$. It follows from (3.1) and (a1) that

$$\begin{array}{rcl}\mu_A(x) & \geq & \mu_A(0 * x) \geq \min\{\mu_A((x * z) * (0 * 0)), \mu_A(z)\}\\ & = & \min\{\mu_A(x * z), \mu_A(z)\},\end{array}$$

$$\begin{aligned}\gamma_A(x) &\leq \gamma_A(0 * x) \leq \max\{\gamma_A((x * z) * (0 * 0)), \gamma_A(z)\} \\ &= \max\{\gamma_A(x * z), \gamma_A(z)\}\end{aligned}$$

for all $x, z \in X$. Hence $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of X. Now, for any $x, y \in X$, we have

$$\begin{aligned}\mu_A(x * y) &\geq \min\{\mu_A((x * y) * x), \mu_A(x)\} \\ &= \min\{\mu_A(0 * y), \mu_A(x)\} \\ &\geq \min\{\mu_A(x), \mu_A(y)\},\end{aligned}$$

$$\begin{aligned}\gamma_A(x * y) &\leq \max\{\gamma_A((x * y) * x), \gamma_A(x)\} \\ &= \max\{\gamma_A(0 * y), \gamma_A(x)\} \\ &\leq \max\{\gamma_A(x), \gamma_A(y)\}.\end{aligned}$$

Therefore $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy subalgebra of X. □

The converse of Theorem 3.3 is not true in general as seen in the following Example.

Example 3.4. *Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:*

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in X defined by

$$A = \left\langle X, \left(\tfrac{0}{0.7}, \tfrac{a}{0.2}, \tfrac{b}{0.2}\right), \left(\tfrac{0}{0.2}, \tfrac{a}{0.5}, \tfrac{b}{0.5}\right)\right\rangle.$$

It is easy to verify that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal and an intuitionistic fuzzy subalgebra of X. But it is not an intuitionistic fuzzy a-ideal of X since

$$\mu_A(b * a) = \mu_A(a) = 0.2 < 0.7 = \min\{\mu_A((a * 0) * (0 * b)), \mu_A(0)\}$$

and/or

$$\gamma_A(b * a) = \gamma_A(a) = 0.5 > 0.2 = \max\{\gamma_A((a * 0) * (0 * b)), \gamma_A(0)\}.$$

Lemma 3.5. *Let an IFS $A = \langle X, \mu_A, \gamma_A \rangle$ in X be an intuitionistic fuzzy ideal of X. If the inequlity $x * y \leq z$ holds in X, then*

$$\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}, \; \gamma_A(x) \leq \max\{\gamma_A(y), \gamma_A(z)\}. \tag{3.3}$$

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} \mu_A(x) &\geq \min\{\mu_A(x * y), \mu_A(y)\} \\ &\geq \min\{\min\{\mu_A((x * y) * z), \mu_A(z)\}, \mu_A(y)\} \\ &= \min\{\min\{\mu_A(0), \mu_A(z)\}, \mu_A(y)\} \\ &= \min\{\mu_A(y), \mu_A(z)\}, \end{aligned}$$

$$\begin{aligned} \gamma_A(x) &\leq \max\{\gamma_A(x * y), \gamma_A(y)\} \\ &\leq \max\{\max\{\gamma_A((x * y) * z), \gamma_A(z)\}, \gamma_A(y)\} \\ &= \max\{\max\{\gamma_A(0), \gamma_A(z)\}, \gamma_A(y)\} \\ &= \max\{\gamma_A(y), \gamma_A(z)\}. \end{aligned}$$

This completes the proof. □

Remark 3.6. *Example 3.4 shows that an intuitionistic fuzzy ideal is not an intuitionistic fuzzy a-ideal. So, we have a question: Under which condition(s), is every intuitionistic fuzzy ideal an intuitionistic fuzzy a-ideal? We give a solution for this question in the following Theorem.*

Theorem 3.7. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ideal of X. Then the following are equivalent:*

(i) $A = \langle X, \mu_A, \gamma_A \rangle$ *is an intuitionistic fuzzy a-ideal of X.*

(ii) $A = \langle X, \mu_A, \gamma_A \rangle$ *satisfies the following conditions:*

$$\begin{aligned} &(\forall x, y, z \in X)(\mu_A(y * (x * z)) \geq \mu_A((x * z) * (0 * y))) \\ &(\forall x, y, z \in X)(\gamma_A(y * (x * z)) \leq \gamma_A((x * z) * (0 * y))). \end{aligned} \tag{3.4}$$

(iii) $A = \langle X, \mu_A, \gamma_A \rangle$ *satisfies the following conditions:*

$$\begin{aligned} &(\forall x, y \in X)(\mu_A(y * x) \geq \mu_A(x * (0 * y))) \\ &(\forall x, y \in X)(\gamma_A(y * x) \leq \gamma_A(x * (0 * y))). \end{aligned} \tag{3.5}$$

Proof. (i) ⇒ (ii) Assume that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X and let $x, y, z \in X$. Using (3.1), (a1) and (2.2), we have

$$\begin{aligned} \mu_A(y * (x * z)) &\geq \min\{\mu_A(((x * z) * 0) * (0 * y)), \mu_A(0)\} \\ &= \mu_A((x * z) * (0 * y)), \end{aligned}$$

$$\begin{aligned} \gamma_A(y * (x * z)) &\leq \max\{\gamma_A(((x * z) * 0) * (0 * y)), \gamma_A(0)\} \\ &= \gamma_A((x * z) * (0 * y)). \end{aligned}$$

(ii) ⇒ (iii) Taking $z = 0$ in (ii) and using (a1) induce (iii).
(iii) ⇒ (i) Note that

$$(x * (0 * y)) * ((x * z) * (0 * y)) \leq x * (x * z) \leq z$$

for all $x, y, z \in X$. It follows from (3.5) and Lemma 3.5 that

$$\mu_A(y * x) \geq \mu_A(x * (0 * y)) \geq \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \gamma_A(x * (0 * y)) \leq \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

Hence $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X. □

Theorem 3.8. *If $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X, then the set*

$$H := \{x \in X \mid \mu_A(x) = \mu_A(0),\ \gamma_A(x) = \gamma_A(0)\}$$

is an a-ideal of X.

Proof. Assume that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X and let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in H$ and $z \in H$. Then

$$\mu_A(0) \geq \mu_A(y * x) \geq \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\} = \mu_A(0),$$

$$\gamma_A(0) \leq \gamma_A(y * x) \leq \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\} = \gamma_A(0)$$

by using (2.2) and (3.1). Hence $\mu_A(y * x) = \mu_A(0)$ and $\gamma_A(y * x) = \gamma_A(0)$, which imply that $y * x \in H$. Obviously, $0 \in H$. Therefore H is an a-ideal of X. □

We give another condition for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy a-ideal.

Theorem 3.9. *If X is associative, i.e., X satisfies the identity $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$, then every intuitionistic fuzzy ideal of X is an intuitionistic fuzzy a-ideal of X.*

Proof. Suppose that X is an associative BCI-algebra. Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ideal of X. Note that $0 * x = x$ for all $x \in X$. Hence

$$y * x = (0 * y) * x = (0 * x) * y = x * y = x * (0 * y)$$

for all $x, y \in X$, and so

$$\mu_A(y * x) = \mu_A(x * (0 * y)),\ \gamma_A(y * x) = \gamma_A(x * (0 * y)).$$

It follows from Theorem 3.7 that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X. □

Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in a set X and let $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$. Then the set

$$X_A^{(\alpha,\beta)} := \{x \in X \mid \mu_A(x) \geq \alpha,\ \gamma_A(x) \leq \beta\}$$

is called an (α, β)-*level subset* of $A = \langle X; \mu_A, \gamma_A \rangle$.

Theorem 3.10. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy a-ideal of X. Then $X_A^{(\alpha,\beta)}$ is an a-ideal of X for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha + \beta \leq 1$.*

Proof. Obviously $0 \in X_A^{(\alpha,\beta)}$. Let $x, y, z \in X$ be such that $(x*z)*(0*y) \in X_A^{(\alpha,\beta)}$ and $z \in X_A^{(\alpha,\beta)}$. Then $\mu_A((x*z)*(0*y)) \geq \alpha$, $\gamma_A((x*z)*(0*y)) \leq \beta$, $\mu_A(z) \geq \alpha$, and $\gamma_A(z) \leq \beta$. It follows from (3.1) that

$$\mu_A(y * x) \geq \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\} \geq \alpha,$$

$$\gamma_A(y * x) \leq \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\} \leq \beta$$

so that $y * x \in X_A^{(\alpha,\beta)}$. Hence $X_A^{(\alpha,\beta)}$ is an a-ideal of X. □

Theorem 3.11. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in X such that $X_A^{(\alpha,\beta)}$ is an a-ideal of X for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha + \beta \leq 1$. Then $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X.*

Proof. Since $0 \in X_A^{(\alpha,\beta)}$, we have $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $A((x * z) * (0 * y)) = (\alpha_1, \beta_1)$ and $A(z) = (\alpha_2, \beta_2)$, that is, $\mu_A((x * z) * (0 * y)) = \alpha_1$, $\gamma_A((x * z) * (0 * y)) = \beta_1$, $\mu_A(z) = \alpha_2$, and $\gamma_A(z) = \beta_2$. Then $(x * z) * (0 * y) \in X_A^{(\alpha_1,\beta_1)}$ and $z \in X_A^{(\alpha_2,\beta_2)}$. We may assume that $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, i.e., $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, without loss of generality. It follows that $X_A^{(\alpha_2,\beta_2)} \subseteq X_A^{(\alpha_1,\beta_1)}$ so that $(x*z)*(0*y) \in X_A^{(\alpha_1,\beta_1)}$ and $z \in X_A^{(\alpha_1,\beta_1)}$. Since $X_A^{(\alpha_1,\beta_1)}$ is an a-ideal of X, we have $y * x \in X_A^{(\alpha_1,\beta_1)}$ by (I3). Thus

$$\mu_A(y * x) \geq \alpha_1 = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \beta_1 = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

Consequently, $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X. □

Note that

$$\begin{aligned} X_A^{(\alpha,\beta)} &= \{x \in X \mid \mu_A(x) \geq \alpha,\ \gamma_A(x) \leq \beta\} \\ &= \{x \in X \mid \mu_A(x) \geq \alpha\} \cap \{x \in X \mid \gamma_A(x) \leq \beta\} \\ &= U(\mu_A; \alpha) \cap L(\gamma_A; \beta). \end{aligned}$$

Hence we have the following Corollary.

Corollary 3.12. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in X. Then $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X if and only if $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are a-ideals of X for every $\alpha \in [0, \mu_A(0)]$ and $\beta \in [\gamma_A(0), 1]$ with $\alpha + \beta \leq 1$.*

Corollary 3.13. *Let I be an a-ideal of X and let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in X defined by*

$$\mu_A(x) := \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1 & \text{otherwise,} \end{cases} \qquad \gamma_A(x) := \begin{cases} \beta_0 & \text{if } x \in I, \\ \beta_1 & \text{otherwise,} \end{cases}$$

for all $x \in X$ where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X.

Proposition 3.14. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy a-ideal of X and (α_1, β_1), $(\alpha_2, \beta_2) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $X_A^{(\alpha_1, \beta_1)} = X_A^{(\alpha_2, \beta_2)}$ if and only if $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.*

Proof. If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, then clearly $X_A^{(\alpha_1, \beta_1)} = X_A^{(\alpha_2, \beta_2)}$. Assume that $X_A^{(\alpha_1, \beta_1)} = X_A^{(\alpha_2, \beta_2)}$. Since $(\alpha_1, \beta_1) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$, there exists $x \in X$ such that $\mu_A(x) = \alpha_1$ and $\gamma_A(x) = \beta_1$. It follows that $x \in X_A^{(\alpha_1, \beta_1)} = X_A^{(\alpha_2, \beta_2)}$ so that $\alpha_1 = \mu_A(x) \geq \alpha_2$ and $\beta_1 = \gamma_A(x) \leq \beta_2$. Similarly we have $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Hence $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. □

Theorem 3.15. *Let $A = \langle X, \mu_A, \gamma_A \rangle$ be an IFS in X and*

$$\mathrm{Im}(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \cdots, (\alpha_k, \beta_k)\}$$

where $(\alpha_i, \beta_i) < (\alpha_j, \beta_j)$ whenever $i > j$. Let $\{G_n \mid n = 0, 1, \cdots, k\}$ be a family of a-ideals of X such that

(i) $G_0 \subset G_1 \subset \cdots \subset G_k = X$,

(ii) $A(G_n^*) = (\alpha_n, \beta_n)$, *i.e.,* $\mu_A(G_n^*) = \alpha_n$ *and* $\gamma_A(G_n^*) = \beta_n$, *where* $G_n^* = G_n \setminus G_{n-1}$, $G_{-1} = \emptyset$ *for* $n = 0, 1, \cdots, k$.

Then $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X.

Proof. Since $0 \in G_0$, we have $\mu_A(0) = \alpha_0 \geq \mu_A(x)$ and $\gamma_A(0) = \beta_0 \leq \gamma_A(x)$ for all $x \in X$. Let $x, y, z \in X$. To prove that $A = \langle X, \mu_A, \gamma_A \rangle$ satisfies condition (3.1), we discuss the following cases: If $(x * z) * (0 * y) \in G_n^*$ and $z \in G_n^*$, then $y * x \in G_n$ because G_n is an a-ideal of X. Thus

$$\mu_A(y * x) \geq \alpha_n = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \beta_n = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

If $(x * z) * (0 * y) \notin G_n^*$ and $z \notin G_n^*$, then the following four cases arise:

1. $(x*z)*(0*y) \in X \setminus G_n$ and $z \in X \setminus G_n$,
2. $(x*z)*(0*y) \in G_{n-1}$ and $z \in G_{n-1}$,
3. $(x*z)*(0*y) \in X \setminus G_n$ and $z \in G_{n-1}$,
4. $(x*z)*(0*y) \in G_{n-1}$ and $z \in X \setminus G_n$.

But, in either case, we know that

$$\mu_A(y*x) \geq \min\{\mu_A((x*z)*(0*y)), \mu_A(z)\},$$

$$\gamma_A(y*x) \leq \max\{\gamma_A((x*z)*(0*y)), \gamma_A(z)\}.$$

If $(x*z)*(0*y) \in G_n^*$ and $z \notin G_n^*$, then either $z \in G_{n-1}$ or $z \in X \setminus G_n$. It follows that either $y*x \in G_n$ or $y*x \in X \setminus G_n$. Thus

$$\mu_A(y*x) \geq \min\{\mu_A((x*z)*(0*y)), \mu_A(z)\},$$

$$\gamma_A(y*x) \leq \max\{\gamma_A((x*z)*(0*y)), \gamma_A(z)\}.$$

If $(x*z)*(0*y) \notin G_n^*$ and $z \in G_n^*$, then by similar process we have

$$\mu_A(y*x) \geq \min\{\mu_A((x*z)*(0*y)), \mu_A(z)\},$$

$$\gamma_A(y*x) \leq \max\{\gamma_A((x*z)*(0*y)), \gamma_A(z)\}.$$

This completes the proof. □

Theorem 3.16. *Let $\{G_\alpha \mid \alpha \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a finite collection of a-ideals of X such that $X = \bigcup_{\alpha \in \Lambda} G_\alpha$, and for every $\alpha, \beta \in \Lambda$, $\alpha < \beta$ if and only if $G_\beta \subset G_\alpha$. Then an IFS $A = \langle X, \mu_A, \gamma_A \rangle$ in X defined by*

$$\mu_A(x) = \sup\{\alpha \in \Lambda \mid x \in G_\alpha\} \text{ and } \gamma_A(x) = \inf\{\alpha \in \Lambda \mid x \in G_\alpha\}$$

for all $x \in X$ is an intuitionistic fuzzy a-ideal of X.

Proof. According to Corollary 3.12, it is sufficient to show that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are a-ideals of X for every $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. In order to show that $U(\mu_A; \alpha)$ is an a-ideal, we divide into the following two cases:

$$\text{(i) } \alpha = \sup\{\delta \in \Lambda \mid \delta < \alpha\} \text{ and (ii) } \alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}.$$

Case (i) implies that

$$x \in U(\mu_A;\alpha) \Leftrightarrow x \in G_\delta \text{ for all } \delta < \alpha \\ \Leftrightarrow x \in \bigcap_{\delta<\alpha} G_\delta,$$

so that $U(\mu_A;\alpha) = \bigcap_{\delta<\alpha} G_\delta$, which is an a-ideal of X. For the case (ii), we claim that $U(\mu_A;\alpha) = \bigcup_{\delta\geq\alpha} G_\delta$. If $x \in \bigcup_{\delta\geq\alpha} G_\delta$, then $x \in G_\delta$ for some $\delta \geq \alpha$. It follows that $\mu_A(x) \geq \delta \geq \alpha$ so that $x \in U(\mu_A;\alpha)$. This proves that $\bigcup_{\delta\geq\alpha} G_\delta \subset U(\mu_A;\alpha)$. Now assume that $x \notin \bigcup_{\delta\geq\alpha} G_\delta$. Then $x \notin G_\delta$ for all $\delta \geq \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha-\varepsilon,\alpha)\cap\Lambda = \emptyset$. Hence $x \notin G_\delta$ for all $\delta > \alpha-\varepsilon$, which means that if $x \in G_\delta$ then $\delta \leq \alpha-\varepsilon$. Thus $\mu_A(x) \leq \alpha-\varepsilon < \alpha$, and so $x \notin U(\mu_A;\alpha)$. Therefore $U(\mu_A;\alpha) = \bigcup_{\delta\geq\alpha} G_\delta$. Next we show that $L(\gamma_A;\beta)$ is an a-ideal of X for all $\beta \in [\gamma_A(0),1]$. We consider the following two cases:

$$\text{(iii) } \beta = \inf\{\delta \in \Lambda \mid \beta < \delta\} \text{ and (iv) } \beta \neq \inf\{\delta \in \Lambda \mid \beta < \delta\}.$$

For the case (iii) we have

$$x \in L(\gamma_A;\beta) \Leftrightarrow x \in G_\delta \text{ for all } \beta < \delta \\ \Leftrightarrow x \in \bigcap_{\beta<\delta} G_\delta,$$

and hence $L(\gamma_A;\beta) = \bigcap_{\beta<\delta} G_\delta$, which is an a-ideal of X. For the case (iv), we will show that $L(\gamma_A;\beta) = \bigcup_{\beta\geq\delta} G_\delta$. If $x \in \bigcup_{\beta\geq\delta} G_\delta$, then $x \in G_\delta$ for some $\beta \geq \delta$. It follows that $\gamma_A(x) \leq \delta \leq \beta$ so that $x \in L(\gamma_A;\beta)$. Hence $\bigcup_{\beta\geq\delta} G_\delta \subset L(\gamma_A;\beta)$. Conversely, if $x \notin \bigcup_{\beta\geq\delta} G_\delta$ then $x \notin G_\delta$ for all $\delta \leq \beta$. Since $\beta \neq \inf\{\delta \in \Lambda \mid \beta < \delta\}$, there exists $\varepsilon > 0$ such that $(\beta,\beta+\varepsilon)\cap\Lambda = \emptyset$, which implies that $x \notin G_\delta$ for all $\delta < \beta+\varepsilon$, that is, if $x \in G_\delta$ then $\delta \geq \beta+\varepsilon$. Thus $\gamma_A(x) \geq \beta+\varepsilon > \beta$, that is, $x \notin L(\gamma_A;\beta)$. Therefore $L(\gamma_A;\beta) \subset \bigcup_{\beta\geq\delta} G_\delta$ and consequently $L(\gamma_A;\beta) = \bigcup_{\beta\geq\delta} G_\delta$. This completes the proof. □

Theorem 3.17. *Let $A = \langle X,\mu_A,\gamma_A\rangle$ be an intuitionistic fuzzy a-ideal of X with the finite image. Then every descending chain of a-ideals of X terminates at finite step.*

Proof. Suppose that there exists a strictly descending chain $G_0 \supset G_1 \supset G_2 \supset \cdots$ of a-ideals of X which does not terminate at finite step. Define an IFS $A = \langle X, \mu_A, \gamma_A \rangle$ in X by

$$\mu_A(x) := \begin{cases} \frac{n}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

$$\gamma_A(x) := \begin{cases} \frac{1}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 0 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

where G_0 stands for X. We prove that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X. Clearly $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(x) \leq \gamma_A(0)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $(x * z) * (0 * y) \in G_n \setminus G_{n+1}$ and $z \in G_k \setminus G_{k+1}$ for $n = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $z \in G_n$, and so $y * x \in G_n$ because G_n is an a-ideal of X. Hence

$$\mu_A(y * x) \geq \tfrac{n}{n+1} = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \tfrac{1}{n+1} = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

If $(x * z) * (0 * y), z \in \bigcap_{n=0}^{\infty} G_n$, then $y * x \in \bigcap_{n=0}^{\infty} G_n$. Thus

$$\mu_A(y * x) = 1 = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) = 0 = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

If $(x * z) * (0 * y) \notin \bigcap_{n=0}^{\infty} G_n$ and $z \in \bigcap_{n=0}^{\infty} G_n$, then there exists $k \in \mathbb{N}$ such that $(x * z) * (0 * y) \in G_k \setminus G_{k+1}$. It follows that $y * x \in G_k$ so that

$$\mu_A(y * x) \geq \tfrac{k}{k+1} = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \tfrac{1}{k+1} = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

Finally suppose that $(x * z) * (0 * y) \in \bigcap_{n=0}^{\infty} G_n$ and $z \notin \bigcap_{n=0}^{\infty} G_n$. Then $z \in G_r \setminus G_{r+1}$ for some $r \in \mathbb{N}$. Hence $y * x \in G_r$, and so

$$\mu_A(y * x) \geq \tfrac{r}{r+1} = \min\{\mu_A((x * z) * (0 * y)), \mu_A(z)\},$$

$$\gamma_A(y * x) \leq \tfrac{1}{r+1} = \max\{\gamma_A((x * z) * (0 * y)), \gamma_A(z)\}.$$

Consequently, we conclude that $A = \langle X, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy a-ideal of X and $A = \langle X, \mu_A, \gamma_A \rangle$ has infinite number of different values. This is a contradiction, and the proof is complete. □

Finally, we consider the converse of Theorem 3.17.

Theorem 3.18. *Let X be a BCI-algebra in which every descending chain of a-ideals terminates at finite step. For an intuitionistic fuzzy a-ideal $A = \langle X, \mu_A, \gamma_A \rangle$ of X, if a sequence of elements of* $\mathrm{Im}(A)$ *is strictly intuitionistic increasing, that is, a sequence of elements of* $\mathrm{Im}(\mu_A)$ *is strictly increasing and a sequence of elements of* $\mathrm{Im}(\gamma_A)$ *is strictly decreasing, then $A = \langle X, \mu_A, \gamma_A \rangle$ has finite number of intuitionistic values, that is, μ_A and γ_A have finite number of values.*

Proof. Suppose that $\mathrm{Im}(\mu_A)$ is not finite. Let $\{\alpha_n\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu_A)$. Then $0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1$. Define

$$U(\mu_A; t) := \{x \in X \mid \mu_A(x) \geq \alpha_t\}$$

for $t = 2, 3, \cdots$. Then $U(\mu_A; t)$ is an a-ideal of X. Let $x \in U(\mu_A; t)$. Then $\mu_A(x) \geq \alpha_t > \alpha_{t-1}$, which implies that $x \in U(\mu_A; t-1)$. Hence $U(\mu_A; t) \subseteq U(\mu_A; t-1)$. Since $\alpha_{t-1} \in \mathrm{Im}(\mu_A)$, there exists $x_{t-1} \in X$ such that $\mu_A(x_{t-1}) = \alpha_{t-1}$. It follows that $x_{t-1} \in U(\mu_A; t-1)$, but $x_{t-1} \notin U(\mu_A; t)$. Thus $U(\mu_A; t)$ is a proper subset of $U(\mu_A; t-1)$, and so we obtain a strictly descending chain

$$U(\mu_A; 1) \supset U(\mu_A; 2) \supset U(\mu_A; 3) \supset \cdots$$

of a-ideals of X which is not terminating. This is a contradiction. Now assume that $\mathrm{Im}(\gamma_A)$ is not finite. Let $\{\beta_n\}$ be a strictly decreasing sequence of elements of $\mathrm{Im}(\gamma_A)$. Then $1 \geq \beta_1 > \beta_2 > \beta_3 > \cdots \geq 0$. Note that

$$L(\gamma_A; k) := \{x \in X \mid \gamma_A(x) \leq \beta_k\}$$

is an a-ideal of X for $k = 2, 3, \cdots$. If $z \in L(\gamma_A; k)$, then $\gamma_A(z) \leq \beta_k < \beta_{k-1}$ and so $z \in L(\gamma_A; k-1)$. This shows that $L(\gamma_A; k) \subseteq L(\gamma_A; k-1)$. Since $\beta_{k-1} \in \mathrm{Im}(\gamma_A)$, we have $\gamma_A(y_{k-1}) = \beta_{k-1}$ for some $y_{k-1} \in X$. Hence $y_{k-1} \in L(\gamma_A; k-1)$, but $y_{k-1} \notin L(\gamma_A; k)$. Therefore $L(\gamma_A; k)$ is a proper subset of $L(\gamma_A; k-1)$, and thus we get a strictly descending chain

$$L(\gamma_A; 1) \supset L(\gamma_A; 2) \supset L(\gamma_A; 3) \supset \cdots$$

of a-ideals of X which is not terminating. This is impossible, and the proof is complete. □

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