Intuitionistic Fuzzy Jensen-Rényi Divergence: Applications to Multiple-Attribute Decision Making

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Vagueness in the scientific studies presents a challenging dimension. Intuitionistic fuzzy set theory has emerged as a tool for its characterization. There is need to associate measures which can measure vagueness and differences in the underlying characterizing IFSs. In the present paper we introduce an information theoretic divergence measure, called intuitionistic fuzzy Jensen-Rényi divergence. It is a difference measure in the setting of intuitionistic fuzzy set theory, involving parameters that provide flexibility and choice. The strength of the new measure lies in its properties and applications. An approach to multiple-attribute decision making based on intuitionistic fuzzy Jensen-Rényi divergence is proposed. A numerical example illustrates the application of the new measure and the role of various parameters therein to multipleattribute decision making problem formulated in terms of intuitionistic fuzzy sets.

Povzetek: Razvita je nova verzija intuitivne mehke logike za uporabo v procesu odločanja.

1 Introduction

In probability theory and statistics, divergence measures are commonly used for measuring the differences between two probability distributions [13 and 22]. Kullback-Leibler [13] divergence is the well known such information theoretic divergence. Another important information theoretic divergence measure is the Jensen-Shannon divergence (JSD) [22] which has attracted quite some attention. It has been shown that the square root of JSD turns out to be a metric [9], satisfying (i) non-negativity (ii) (minimal) zero value only for identical distributions (iii) symmetric and (iv) satisfying triangular inequality, i.e. it is bounded from below and from above in terms of the norms of the distributions. However it may be mentioned that JSD itself is not a metric. It satisfies the first three axioms, and not the triangular inequality. These divergence measures have been applied in several disciplines like signal processing, pattern recognition, finance, economics etc.

Some generalizations of Jensen-Shannon divergence measure have been studied in the last couple of years. For instance, He et al. [10] proposed a one parametric generalization of JSD based on Rényi's entropy function [21], called Jensen-Rényi divergence and used it in image registration.

Other than probabilistic, there are vague/fuzzy phenomena. These are best characterized in terms of 'fuzzy sets', and their generalizations. The theory of fuzzy sets proposed by Zadeh [32] in 1965 addresses these situations and has found applications in various fields. In fuzzy set theory, the membership of an element is a single value lying between zero and one, where the degree of non-membership is just automatically equal to one minus the degree of membership.

As a generalization of Zadeh's fuzzy sets, Atanassov [1, 2], introduced intuitionistic fuzzy sets. In their general setting, these involve three non-negative functions expressing the degree of membership, the degree of non-membership, and hesitancy, their sum being one. These considerations imbue IFSs with inbuilt structure to consider varieties of factors responsible of vagueness in the phenomena. IFSs have been applied in many practically uncertain/vague situations, such as decision making [3, 4, 8, 14, 16-18, 20, 25, 27-30 and 33] medical diagnosis [5, 24] and pattern recognition [6, 11, 12, 19 and 24] etc. Atanassov [2] and Szmidt and Kacprzyk [26] suggested some methods for measuring distance/difference between two intuitionistic fuzzy sets. Their measures are generalizations of the well known Hamming and Euclidean distances. Dengfeng and Chutian [6] and Dengfeng [7] proposed some other similarity and dissimilarity measures for measuring differences between pairs of intuitionistic fuzzy sets. In addition, Yanhong et al. [31] undertook a comparative analysis of these similarity measures. Recently, Verma and Sharma [25] proposed a generalized intuitionistic fuzzy divergence and studied its applications to multi criteria decision making.

In this paper, we extend the idea of Jensen-Rényi divergence to intuitionistic fuzzy sets and propose a new divergence measure, called *intuitionistic fuzzy Jensen-Rényi divergence* (IFJRD) to measure the difference between two IFSs. After studying its properties, we give an example of its applications in multiple-attribute decision making based on intuitionistic fuzzy information. The paper is organized as follows: In Section 2 some basic definitions related to probability theory, fuzzy set theory and intuitionistic fuzzy set theory are briefly given. In Section 3, the intuitionistic fuzzy Jensen-Rényi divergence (IFJRD) between two intuitionistic fuzzy sets is proposed. Some of its basic properties are analysed there, along with the limiting case. In Section 4 some more properties of the proposed measure are studied. In Section 5 application of proposed *intuitionistic fuzzy Jensen-Rényi divergence* measure to multiple-attribute decision making are illustrated and our conclusions are also presented here.

2 Preliminaries

We start with probabilistic background. We denote the set of n-complete $(n \geq 2)$ probability distributions by

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}. \quad (1)$$

For a probability distribution

$$P = (p_1, p_2, ..., p_n) \in \Gamma_n,$$

the well known Shannon's entropy [23], is defined as

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i. \quad (2)$$

Various generalized entropies have been introduced in the literature taking the Shannon entropy as basic and have found applications in various disciplines such as economics, statistics, information processing and computing etc.

A generalizations of Shannon's entropy introduced by Rényi's [21], Rényi's entropy of order α, is given by

$$H_\alpha(P) = \frac{1}{1-\alpha} \log(\sum_{i=1}^{n} p_i^\alpha), \quad \alpha \neq 1, \alpha > 0. \quad (3)$$

For $\alpha \in (0,1)$, it is easy to see that $H_\alpha(P)$ is a concave function of P, and in the limiting case $\alpha \to 1$, it tends to Shannon's entropy. It can also be easily verified that $H_\alpha(P)$ is a non-increasing function of $\alpha \in (0,1)$ and thus

$$H_\alpha(P) \geq H(P) \quad \forall \alpha \in (0,1) \quad (4)$$

In sequel, we will restrict $\alpha \in (0,1)$, unless otherwise specified and will use base 2 for the logarithm.

Next, we mention *Jensen-Shannon divergence* [15]. Let $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ be the weights of two probability distributions $P, Q \in \Gamma_n$, respectively. Then the Jensen-Shannon divergence, is defined as

$$JS_\lambda(P,Q) = H(\lambda_1 P + \lambda_2 Q) - \lambda_1 H(P) - \lambda_2 H(Q). \quad (5)$$

Since $H(P)$ is a concave function, according to Jensen's inequality, $JS_\lambda(P,Q)$ is nonnegative and vanishes when $P = Q$. One of the major features of the Jensen-Shannon divergence is that we can assign different weights to the probability distributions involved according to their importance. This is particularly useful in the study of decision problems.

A generalization of the above concept is the *Jensen-Rényi divergence* proposed by He [10], given by

$$JR_{\lambda,\alpha}(P,Q) = H_\alpha(\lambda_1 P + \lambda_2 Q) - \lambda_1 H_\alpha(P) - \lambda_2 H_\alpha(Q), \quad \alpha \in (0,1) \quad (6)$$

where $H_\alpha(P)$ is Rényi's entropy, and $\lambda = (\lambda_1, \lambda_2)$ is the weight vector, with $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, as before.

***Properties of Jensen-Rényi Divergence*:** Briefly we note some simple properties:

i. $JR_{\lambda,\alpha}(P,Q)$ is nonnegative and is equal to zero when $P = Q$.

ii. For $\alpha \in (0,1)$, $JR_{\lambda,\alpha}(P,Q)$ is a convex function of P and Q.

iii. $JR_{\lambda,\alpha}(P,Q)$, achieves its maximum value when P and Q are degenerate distributions.

The Jensen-Shannon divergence (5) is a limiting case of $JR_{\lambda,\alpha}(P,Q)$ when $\alpha \to 1$.

Definition 1. *Fuzzy Set* [32]**:** A fuzzy set $\tilde{A}$ in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x) \rangle \mid x \in X \}, \quad (7)$$

where $\mu_{\tilde{A}}(x) : X \to [0,1]$ is measure of belongingness or degree of membership of an element $x \in X$ to $\tilde{A}$.

Thus, automatically the measure of non-belongingness of $x \in X$ to $\tilde{A}$ is $(1 - \mu_{\tilde{A}}(x))$.

Atanassov [1, 2] introduced following generalization of fuzzy sets, called intuitionistic fuzzy sets.

Definition 2. *Intuitionistic Fuzzy Set* [1, 2]**:** An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}, \quad (8)$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. For each $x \in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ denote the degree of membership and degree of non-membership of x to A respectively.

Further, we call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, the degree of hesitance or the intuitionistic index of $x \in X$ to A.

Obviously, when $\pi_A(x)=0$, i.e., $\nu_A(x)=1-\mu_A(x)$ for every $x\in X$, then the IFS A becomes a fuzzy set. Thus, FSs are the special cases of IFSs.

Definition 3: Let $IFS(X)$ denote the family of all IFSs defined in the universe X, and let $A,B\in IFS(X)$ be given by

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\},$$
$$B=\{\langle x,\mu_B(x),\nu_B(x)\rangle \mid x\in X\}.$$

These being sets, Atanassov further defined set operations on $IFS(X)$ as follows:

(i) $A\subseteq B$ iff $\mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\ge\nu_B(x)\ \ \forall\ x\in X$;

(ii) $A=B$ iff $A\subseteq B$ and $B\subseteq A$;

(iii) $A^C=\{\langle x,\nu_A(x),\mu_A(x)\rangle \mid x\in X\}$;

(iv) $A\bigcup B=\left\{\left\langle \begin{matrix} x,\max(\mu_A(x),\mu_B(x)), \\ \min(\nu_A(x),\nu_B(x)) \end{matrix}\right\rangle \mid x\in X\right\}$;

(v) $A\bigcap B=\left\{\left\langle \begin{matrix} x,\min(\mu_A(x),\mu_B(x)), \\ \max(\nu_A(x),\nu_B(x)) \end{matrix}\right\rangle \mid x\in X\right\}$.

Extending the idea from probabilistic to intuitionistic phenomena, in the next section, we propose a divergence measure called '*Intuitionistic Fuzzy Jensen-Rényi Divergence*' (IFJRD) on intuitionistic fuzzy sets to quantify the difference between two intuitionistic fuzzy sets and discuss its limiting case.

3 Intuitionistic Fuzzy Jensen-Rényi Divergence (IFJRD)

Single element universe: First, let A and B be two intuitionistic fuzzy sets defined on a single element universal set $X=\{x\}$.

Precisely speaking, we have:

$$A=(\mu_A(x),\nu_A(x),\pi_A(x)),$$

and
$$B=(\mu_B(x),\nu_B(x),\pi_B(x)),$$

where

$$\mu_A(x)+\nu_A(x)+\pi_A(x)=1,$$

and
$$\mu_B(x)+\nu_B(x)+\pi_B(x)=1,$$

with

$$0\le\mu_A(x),\nu_A(x),\pi_A(x),\ \mu_B(x),\nu_B(x),\pi_B(x)\le 1.$$

Regarding (μ_A,ν_A,π_A) and (μ_B,ν_B,π_B) as two probability distributions, in analogy of (6), we define the intuitionistic fuzzy Jensen-Rényi divergence measure between IFSs A and B, as

$$JR^*_{\lambda,\alpha}(A,B)=H_\alpha(\lambda_1A+\lambda_2B) -\lambda_1H_\alpha(A)-\lambda_2H_\alpha(B), \tag{9}$$

where $H_\alpha(\bullet)$ is Rényi's entropy for intuitionistic fuzzy set $(\bullet)$, $\alpha\in(0,1)$, $\lambda_1+\lambda_2=1$, $\lambda_1,\lambda_2\ge 0$, and

$$\lambda_1A+\lambda_2B=\begin{pmatrix}\lambda_1\mu_A(x)+\lambda_2\mu_B(x), \\ \lambda_1\nu_A(x)+\lambda_2\nu_B(x), \\ \lambda_1\pi_A(x)+\lambda_2\pi_B(x)\end{pmatrix}.$$

That is

$$JR^*_{\lambda,\alpha}(A,B) =\frac{1}{(1-\alpha)}\left[\begin{matrix}\log\left\{\begin{matrix}(\lambda_1\mu_A(x)+\lambda_2\mu_B(x))^\alpha \\ +(\lambda_1\nu_A(x)+\lambda_2\nu_B(x))^\alpha \\ +(\lambda_1\pi_A(x)+\lambda_2\pi_B(x))^\alpha\end{matrix}\right\} \\ -\lambda_1\log\left\{\begin{matrix}(\mu_A(x))^\alpha+(\nu_A(x))^\alpha \\ +(\pi_A(x))^\alpha\end{matrix}\right\} \\ -\lambda_2\log\left\{\begin{matrix}(\mu_B(x))^\alpha+(\nu_B(x))^\alpha \\ +(\pi_B(x))^\alpha\end{matrix}\right\}\end{matrix}\right], \tag{10}$$

where $\alpha\in(0,1)$.

Next, in theorem below we study properties of $JR^*_{\lambda,\alpha}(A,B)$ defined in (10).

Theorem1: For $A,B\in IFS(X)$, $JR^*_{\lambda,\alpha}(A,B)$ satisfies the following properties:

i. $JR^*_{\lambda,\alpha}(A,B)\ge 0$, with equality if and only if $A=B$.

ii. $0\le JR^*_{\lambda,\alpha}(A,B)\le 1$.

iii. For three *IFSs* A,B,C in X and $A\subseteq B\subseteq C$,
$$JR^*_{\lambda,\alpha}(A,B)\le JR^*_{\lambda,\alpha}(A,C),$$
and
$$JR^*_{\lambda,\alpha}(B,C)\le JR^*_{\lambda,\alpha}(A,C).$$

Proof: (i) The result directly follows from Jensen's inequality.

(ii) Since $JR^*_{\lambda,\alpha}(A,B)$ is convex for $\alpha\in(0,1)$, refer Proposition 1 of He et al. [10], therefore, for $\alpha\in(0,1)$, $JR^*_{\lambda,\alpha}(A,B)$ increases as $\|A-B\|_1$ increases, where

$$\|A-B\|_1=|\mu_A(x)-\mu_B(x)|+|\nu_A(x)-\nu_B(x)| +|\pi_A(x)-\pi_B(x)|. \tag{11}$$

Thus, $JR^*_{\lambda,\alpha}(A,B)\ \forall\ \alpha\in(0,1)$, attains its maximum for following degenerate cases:

$$A=(1,0,0),B=(0,1,0) \text{ or } A=(0,1,0),B=(1,0,0) \text{ or } A=(0,0,1),B=(0,1,0).$$

This gives

$$0\le JR^*_{\lambda,\alpha}(A,B)\le 1.$$

(iii) For $A,B,C\in IFS(X)$,

$$\| A-B\|_1 \le \| A-C\|_1$$

and $\| B-C\|_1 \le \| A-C\|_1$, if $A\subseteq B\subseteq C$.

Thus,

$$JR^*_{\lambda,\alpha}(A,B)\le JR^*_{\lambda,\alpha}(A,C)$$

and $JR^*_{\lambda,\alpha}(B,C)\le JR^*_{\lambda,\alpha}(A,C)\ \ \forall\ \alpha\in(0,1)$.

(12)

This proves the theorem.

Limiting case: When $\alpha\to 1$ and $\lambda_1=\lambda_2=\frac{1}{2}$, then measure (10) reduces to *J*-divergence on intuitionistic fuzzy sets proposed by Hung and Yang [11] as

$$J(A,B) = \left[-\left(\begin{array}{l}\left(\frac{\mu_A(x)+\mu_B(x)}{2}\right)\log\left(\frac{\mu_A(x)+\mu_B(x)}{2}\right)\\ +\left(\frac{\nu_A(x)+\nu_B(x)}{2}\right)\log\left(\frac{\nu_A(x)+\nu_B(x)}{2}\right)\\ +\left(\frac{\pi_A(x)+\pi_B(x)}{2}\right)\log\left(\frac{\pi_A(x)+\pi_B(x)}{2}\right)\end{array}\right) + \left(\frac{\begin{pmatrix}\mu_A(x)\log\mu_A(x)\\ +\nu_A(x)\log\nu_A(x)\\ +\pi_A(x)\log\pi_A(x)\end{pmatrix}+\begin{pmatrix}\mu_B(x)\log\mu_B(x)\\ +\nu_B(x)\log\nu_B(x)\\ +\pi_B(x)\log\pi_B(x)\end{pmatrix}}{2}\right)\right]. \quad (13)$$

Definition 4: $JR_{\lambda,\alpha}(A,B)$ **on Finite Universe:** Previously, we considered single element universe set. The idea can be extended to any finite universe set. If A and B are two IFSs defined in finite universe of discourse $X=\{x_1,x_2,...,x_n\}$, then, we define, the associated intuitionistic fuzzy Jensen-Rényi divergence by

$$JR_{\lambda,\alpha}(A,B)=\frac{1}{n}\sum_{i=1}^{n} JR^*_{\lambda,\alpha}(A(x_i),B(x_i)) \quad (14)$$

where $A(x_i)=\{(x_i,\mu_A(x_i),\nu_A(x_i),\pi_A(x_i))\}$,

and $B(x_i)=\{(x_i,\mu_B(x_i),\nu_B(x_i),\pi_B(x_i))\}$.

In the next section, we study several properties of $JR_{\lambda,\alpha}(A,B)$. While proving these properties, we consider separation of X into two parts X_1 and X_2, such that

$$X_1=\{x_i \mid x_i\in X,\ A(x_i)\subseteq B(x_i)\}, \quad (15)$$

$$X_2=\{x_i \mid x_i\in X,\ A(x_i)\supseteq B(x_i)\}. \quad (16)$$

Further it may be noted that for all $x_i\in X_1$,

$$\mu_A(x_i)\le\mu_B(x_i) \text{ and } \nu_A(x_i)\ge\nu_B(x_i),$$

as also for $\forall\ x_i\in X_2$,

$$\mu_A(x_i)\ge\mu_B(x_i) \text{ and } \nu_A(x_i)\le\nu_B(x_i).$$

4 Properties of intuitionistic fuzzy Jensen- Rényi divergence measure

The measure $JR_{\lambda,\alpha}(A,B)$ defined in (10) has the following properties:

Theorem 2: For $A,B\in IFS(X)$,

(i) $JR_{\lambda,\alpha}(A\cup B,A\cap B)=JR_{\lambda,\alpha}(A,B)$,

(ii) $JR_{\lambda,\alpha}(A\cap B,A\cup B)=JR_{\lambda,\alpha}(B,A)$.

Proof: We prove (i) only, (ii) can be proved analogously.

(i) From definition in (10), we have:

$$JR_{\lambda,\alpha}(A\cup B,A\cap B)$$

$$=\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}\log\left\{\begin{array}{l}(\lambda_1\mu_{A\cup B}(x_i)+\lambda_2\mu_{A\cap B}(x_i))^\alpha\\ +(\lambda_1\nu_{A\cap B}(x_i)+\lambda_2\nu_{A\cup B}(x_i))^\alpha\\ +\begin{pmatrix}\lambda_1(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i))\\ +\lambda_2(1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i))\end{pmatrix}^\alpha\end{array}\right\}\\ -\lambda_1\log\left\{\begin{array}{l}(\mu_{A\cup B}(x_i))^\alpha+(\nu_{A\cap B}(x_i))^\alpha\\ +(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i))^\alpha\end{array}\right\}\\ -\lambda_2\log\left\{\begin{array}{l}(\mu_{A\cap B}(x_i))^\alpha+(\nu_{A\cup B}(x_i))^\alpha\\ +(1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i))^\alpha\end{array}\right\}\end{array}\right]$$

$$=\frac{1}{n(1-\alpha)}\left[\sum_{x_i\in X_1}\left\{\begin{array}{l}\log\left\{\begin{array}{l}(\lambda_1\mu_B(x_i)+\lambda_2\mu_A(x_i))^\alpha\\ +(\lambda_1\nu_B(x_i)+\lambda_2\nu_A(x_i))^\alpha\\ +\begin{pmatrix}\lambda_1(1-\mu_B(x_i)-\nu_B(x_i))\\ +\lambda_2(1-\mu_A(x_i)-\nu_A(x_i))\end{pmatrix}^\alpha\end{array}\right\}\\ -\lambda_1\log\left\{\begin{array}{l}(\mu_B(x_i))^\alpha+(\nu_B(x_i))^\alpha\\ +(1-\mu_B(x_i)-\nu_B(x_i))^\alpha\end{array}\right\}\\ -\lambda_2\log\left\{\begin{array}{l}(\mu_A(x_i))^\alpha+(\nu_A(x_i))^\alpha\\ +(1-\mu_A(x_i)-\nu_A(x_i))^\alpha\end{array}\right\}\end{array}\right\} + \sum_{x_i\in X_2}\left\{\begin{array}{l}\log\left\{\begin{array}{l}(\lambda_1\mu_A(x_i)+\lambda_2\mu_B(x_i))^\alpha\\ +(\lambda_1\nu_A(x_i)+\lambda_2\nu_B(x_i))^\alpha\\ +\begin{pmatrix}\lambda_1(1-\mu_A(x_i)-\nu_A(x_i))\\ +\lambda_2(1-\mu_B(x_i)-\nu_B(x_i))\end{pmatrix}^\alpha\end{array}\right\}\\ -\lambda_1\log\left\{\begin{array}{l}(\mu_A(x_i))^\alpha+(\nu_A(x_i))^\alpha\\ +(1-\mu_A(x_i)-\nu_A(x_i))^\alpha\end{array}\right\}\\ -\lambda_2\log\left\{\begin{array}{l}(\mu_B(x_i))^\alpha+(\nu_B(x_i))^\alpha\\ +(1-\mu_B(x_i)-\nu_B(x_i))^\alpha\end{array}\right\}\end{array}\right\}\right]$$

$$=JR_{\lambda,\alpha}(A,B).$$

This proves the theorem.

Theorem 3: For $A, B \in IFS(X)$,

(i) $JR_{\lambda,\alpha}(A, A\cup B) + JR_{\lambda,\alpha}(A, A\cap B) = JR_{\lambda,\alpha}(A, B)$,

(ii) $JR_{\lambda,\alpha}(B, A\cup B) + JR_{\lambda,\alpha}(B, A\cap B) = JR_{\lambda,\alpha}(B, A)$.

Proof: In the following, we prove only (i), (ii) can be proved analogously.

(i) Using definition in (10), we first have

$$JR_{\lambda,\alpha}(A, A\cup B)$$

$$= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \left[\begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_{A\cup B}(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_{A\cap B}(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_{A\cup B}(x_i))^\alpha + (\nu_{A\cap B}(x_i))^\alpha \\ + (1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i))^\alpha \end{array} \right\} \end{array} \right]$$

$$= \frac{1}{n(1-\alpha)} \left[\sum_{x_i \in X_1} \left\{ \begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_B(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_B(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_B(x_i)-\nu_B(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_B(x_i))^\alpha + (\nu_B(x_i))^\alpha \\ + (1-\mu_B(x_i)-\nu_B(x_i))^\alpha \end{array} \right\} \end{array} \right\} \right.$$

$$\left. + \sum_{x_i \in X_2} \left\{ \begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_A(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_A(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_A(x_i)-\nu_A(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \end{array} \right\} \right]$$

$$= \frac{1}{n(1-\alpha)} \left[\sum_{x_i \in X_1} \left\{ \begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_B(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_B(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_B(x_i)-\nu_B(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_B(x_i))^\alpha + (\nu_B(x_i))^\alpha \\ + (1-\mu_B(x_i)-\nu_B(x_i))^\alpha \end{array} \right\} \end{array} \right\} \right] \quad (17)$$

Next, again from definition in (10), we have

$$JR_{\lambda,\alpha}(A, A\cap B)$$

$$= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \left[\begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_{A\cap B}(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_{A\cup B}(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_{A\cap B}(x_i))^\alpha + (\nu_{A\cup B}(x_i))^\alpha \\ + (1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i))^\alpha \end{array} \right\} \end{array} \right]$$

$$= \frac{1}{n(1-\alpha)} \left[\sum_{x_i \in X_1} \left\{ \begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_A(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_A(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_A(x_i)-\nu_A(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \end{array} \right\} \right.$$

$$\left. + \sum_{x_i \in X_2} \left\{ \begin{array}{l} \log\left\{ \begin{array}{l} (\lambda_1\mu_A(x_i) + \lambda_2\mu_B(x_i))^\alpha \\ + (\lambda_1\nu_A(x_i) + \lambda_2\nu_B(x_i))^\alpha \\ + \left(\begin{array}{l} \lambda_1(1-\mu_A(x_i)-\nu_A(x_i)) \\ + \lambda_2(1-\mu_B(x_i)-\nu_B(x_i)) \end{array} \right)^\alpha \end{array} \right\} \\ - \lambda_1 \log\left\{ \begin{array}{l} (\mu_A(x_i))^\alpha + (\nu_A(x_i))^\alpha \\ + (1-\mu_A(x_i)-\nu_A(x_i))^\alpha \end{array} \right\} \\ - \lambda_2 \log\left\{ \begin{array}{l} (\mu_B(x_i))^\alpha + (\nu_B(x_i))^\alpha \\ + (1-\mu_B(x_i)-\nu_B(x_i))^\alpha \end{array} \right\} \end{array} \right\} \right]$$

$$=\frac{1}{n(1-\alpha)}\left[\sum_{x_i\in X_2}\left\{\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_A(x_i)+\lambda_2\mu_B(x_i)\right)^\alpha\\+\left(\lambda_1\nu_A(x_i)+\lambda_2\nu_B(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^\alpha+\left(\nu_A(x_i)\right)^\alpha\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_B(x_i)\right)^\alpha+\left(\nu_B(x_i)\right)^\alpha\\+\left(1-\mu_B(x_i)-\nu_B(x_i)\right)^\alpha\end{array}\right\}\end{array}\right\}\right] \quad (18)$$

Adding (17) and (18), we get the result.

Theorem 4: For $A,B,C\in IFS(X)$,

(i) $JR_{\lambda,\alpha}(A\cup B,C)\le JR_{\lambda,\alpha}(A,C)+JR_{\lambda,\alpha}(B,C)$;

(ii) $JR_{\lambda,\alpha}(A\cap B,C)\le JR_{\lambda,\alpha}(A,C)+JR_{\lambda,\alpha}(B,C)$;

Proof: We prove (i) only, (ii) can be proved analogously.

(i) Let us consider the expression

$$JR_{\lambda,\alpha}(A,C)+JR_{\lambda,\alpha}(B,C)-JR_{\lambda,\alpha}(A\cup B,C) \quad (19)$$

$$=\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_A(x_i)+\lambda_2\mu_C(x_i)\right)^\alpha\\+\left(\lambda_1\nu_A(x_i)+\lambda_2\nu_C(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^\alpha+\left(\nu_A(x_i)\right)^\alpha\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^\alpha+\left(\nu_C(x_i)\right)^\alpha\\+\left(\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\right)^\alpha\end{array}\right\}\end{array}\right]$$

$$+\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_B(x_i)+\lambda_2\mu_C(x_i)\right)^\alpha\\+\left(\lambda_1\nu_B(x_i)+\lambda_2\nu_C(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_B(x_i)\right)^\alpha+\left(\nu_B(x_i)\right)^\alpha\\+\left(1-\mu_B(x_i)-\nu_B(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^\alpha+\left(\nu_C(x_i)\right)^\alpha\\+\left(\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\right)^\alpha\end{array}\right\}\end{array}\right]$$

$$-\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_{A\cup B}(x_i)+\lambda_2\mu_C(x_i)\right)^\alpha\\+\left(\lambda_1\nu_{A\cap B}(x_i)+\lambda_2\nu_C(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_{A\cup B}(x_i)\right)^\alpha+\left(\nu_{A\cap B}(x_i)\right)^\alpha\\+\left(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^\alpha+\left(\nu_C(x_i)\right)^\alpha\\+\left(\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\right)^\alpha\end{array}\right\}\end{array}\right]$$

$$=\frac{1}{n(1-\alpha)}\left[\sum_{x_i\in X_2}\left\{\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_B(x_i)+\lambda_2\mu_C(x_i)\right)^\alpha\\+\left(\lambda_1\nu_B(x_i)+\lambda_2\nu_C(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_B(x_i)\right)^\alpha+\left(\nu_B(x_i)\right)^\alpha\\+\left(1-\mu_B(x_i)-\nu_B(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^\alpha+\left(\nu_C(x_i)\right)^\alpha\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^\alpha\end{array}\right\}\end{array}\right\}\right]$$

$$+\frac{1}{n(1-\alpha)}\left[\sum_{x_i\in X_1}\left\{\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1\mu_A(x_i)+\lambda_2\mu_C(x_i)\right)^\alpha\\+\left(\lambda_1\nu_A(x_i)+\lambda_2\nu_C(x_i)\right)^\alpha\\+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^\alpha+\left(\nu_A(x_i)\right)^\alpha\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^\alpha+\left(\nu_C(x_i)\right)^\alpha\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^\alpha\end{array}\right\}\end{array}\right\}\right]$$

≥ 0

This proves the theorem.

Theorem 5: For $A,B,C\in IFS(X)$,

$$JR_{\lambda,\alpha}(A\cup B,C)+JR_{\lambda,\alpha}(A\cap B,C)=JR_{\lambda,\alpha}(A,C)+JR_{\lambda,\alpha}(B,C).$$

Proof: Using definition in (10), we first have:

$$JR_{\lambda,\alpha}(A\cup B,\,C)$$

$$
=\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_{A\cup B}(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_{A\cap B}(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_{A\cup B}(x_i)\right)^{\alpha}+\left(\nu_{A\cap B}(x_i)\right)^{\alpha}\\+\left(1-\mu_{A\cup B}(x_i)-\nu_{A\cap B}(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right]
$$

$$
=\frac{1}{n(1-\alpha)}\left[\begin{array}{l}
\sum_{x_i\in X_1}\left\{\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_B(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_B(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_B(x_i)\right)^{\alpha}+\left(\nu_B(x_i)\right)^{\alpha}\\+\left(1-\mu_B(x_i)-\nu_B(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right\}\\
+\sum_{x_i\in X_2}\left\{\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_A(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_A(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^{\alpha}+\left(\nu_A(x_i)\right)^{\alpha}\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right\}
\end{array}\right]. \quad (20)
$$

Next, again using definition in (10), we have

$JR_{\lambda,\alpha}(A\cap B, C)$

$$
=\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_{A\cap B}(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_{A\cup B}(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_{A\cap B}(x_i)\right)^{\alpha}+\left(\nu_{A\cup B}(x_i)\right)^{\alpha}\\+\left(1-\mu_{A\cap B}(x_i)-\nu_{A\cup B}(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right]
$$

$$
=\frac{1}{n(1-\alpha)}\left[\begin{array}{l}
\sum_{x_i\in X_1}\left\{\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_A(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_A(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^{\alpha}+\left(\nu_A(x_i)\right)^{\alpha}\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right\}\\
+\sum_{x_i\in X_2}\left\{\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_B(x_i)+\lambda_2\mu_C(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_B(x_i)+\lambda_2\nu_C(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\\+\lambda_2\left(1-\mu_C(x_i)-\nu_C(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_B(x_i)\right)^{\alpha}+\left(\nu_B(x_i)\right)^{\alpha}\\+\left(1-\mu_B(x_i)-\nu_B(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\mu_C(x_i)\right)^{\alpha}+\left(\nu_C(x_i)\right)^{\alpha}\\+\left(1-\mu_C(x_i)-\nu_C(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right\}
\end{array}\right]. \quad (21)
$$

Adding (20) and (21), we get the result.

Theorem 6: For $A, B\in IFS(X)$,

(a) $JR_{\lambda,\alpha}(A,B)=JR_{\lambda,\alpha}(A^C,B^C)$

(b) $JR_{\lambda,\alpha}(A,B^C)=JR_{\lambda,\alpha}(A^C,B)$;

(c) $JR_{\lambda,\alpha}(A,B)+JR_{\lambda,\alpha}(A^C,B)$
$=JR_{\lambda,\alpha}(A^C,B^C)+JR_{\lambda,\alpha}(A,B^C)$.

where A^C and B^C represents the complement of intuitionistic fuzzy sets A and B respectively.

Proof: (a) The proof simply follows from the relation of membership and non-membership functions of an element in a set and its complement.

(b) Let us consider the expression

$$JR_{\lambda,\alpha}(A,B^C)-JR_{\lambda,\alpha}(A^C,B) \quad (22)$$

$$
=\frac{1}{n(1-\alpha)}\left[\begin{array}{l}
\log\left\{\begin{array}{l}
\left(\lambda_1\mu_A(x_i)+\lambda_2\nu_B(x_i)\right)^{\alpha}\\
+\left(\lambda_1\nu_A(x_i)+\lambda_2\mu_B(x_i)\right)^{\alpha}\\
+\left(\begin{array}{l}\lambda_1\left(1-\mu_A(x_i)-\nu_A(x_i)\right)\\+\lambda_2\left(1-\mu_B(x_i)-\nu_B(x_i)\right)\end{array}\right)^{\alpha}
\end{array}\right\}\\
-\lambda_1\log\left\{\begin{array}{l}\left(\mu_A(x_i)\right)^{\alpha}+\left(\nu_A(x_i)\right)^{\alpha}\\+\left(1-\mu_A(x_i)-\nu_A(x_i)\right)^{\alpha}\end{array}\right\}\\
-\lambda_2\log\left\{\begin{array}{l}\left(\nu_B(x_i)\right)^{\alpha}+\left(\mu_B(x_i)\right)^{\alpha}\\+\left(1-\nu_B(x_i)-\mu_B(x_i)\right)^{\alpha}\end{array}\right\}
\end{array}\right]
$$

$$-\frac{1}{n(1-\alpha)}\left[\begin{array}{l}\log\left\{\begin{array}{l}(\lambda_1\nu_A(x_i)+\lambda_2\mu_B(x_i))^\alpha\\+(\lambda_1\mu_A(x_i)+\lambda_2\nu_B(x_i))^\alpha\\+\left(\begin{array}{l}\lambda_1(1-\nu_A(x_i)-\mu_A(x_i))\\+\lambda_2(1-\mu_B(x_i)-\nu_B(x_i))\end{array}\right)^\alpha\end{array}\right\}\\-\lambda_1\log\left\{\begin{array}{l}(\nu_A(x_i))^\alpha+(\mu_A(x_i))^\alpha\\+(1-\nu_A(x_i)-\mu_A(x_i))^\alpha\end{array}\right\}\\-\lambda_2\log\left\{\begin{array}{l}(\mu_B(x_i))^\alpha+(\nu_B(x_i))^\alpha\\+(1-\mu_B(x_i)-\nu_B(x_i))^\alpha\end{array}\right\}\end{array}\right]$$

$= 0.$

(c) It immediately follows **(a)** and **(b)**.

This completes proof the theorem.

In the next section, we suggest an application of the measure proposed to multiple-attribute decision making problem and give an illustrative example.

5 Applications of intuitionistic fuzzy Jensen-Rényi divergence to multiple-attribute decision making

Vagueness is a fact of life and needs attention in matters of management. It can have several forms, for example, imperfectly defined facts, indirect data, or imprecise knowledge. For mathematical study, vague phenomena have got to be first suitably represented. IFSs are found to be suitable tools for this purpose. In this section, we present a method based on our proposed *intuitionistic fuzzy Jensen-Rényi divergence* defined over IFSs, to solve multiple-attribute decision making problems. It may be remarked that for a deterministic or probabilistic phenomenon where patterns show *stability* of the form, parameters have perhaps limited rule, but in vague phenomena, parameters provide a class of measures and choice for making appropriate selection by testing further. Intuitionistic fuzzy Jensen-Rényi divergence defined has parameters of two categories- the averaging parameters, $\lambda's$, and an extraneous parameter α, each serving a different purpose. In the example below, we bring out their role in multiple-attribute decision making.

Multiple-attribute decision making problems are defined on a set of alternatives, from which the decision maker has to select the best alternative according to some attributes. Suppose that there exists an alternative set $A=\{A_1,A_2,...,A_m\}$ which consists of m alternatives, the decision maker will choose the best alternative from the set A according to a set of n attributes $G=\{G_1,G_2,...,G_n\}$. Further let $D=(d_{ij})_{n\times m}$ be the intuitionistic fuzzy decision matrix, where $d_{ij}=(\mu_{ij},\nu_{ij},\pi_{ij})$ is an attribute value provided by the decision maker, such that μ_{ij} indicates the degree with which the alternative A_j satisfies the attribute G_i, ν_{ij} indicates the degree with which the alternative A_j does not satisfies the attribute G_i, and π_{ij} indicates the indeterminacy degree of alternative A_j to the attribute G_i, such that:
$\mu_{ij}\in[0,1]$, $\nu_{ij}\in[0,1]$, $\mu_{ij}+\nu_{ij}\le\pi_{ij}=1$, $\pi_{ij}=1-\mu_{ij}-\nu_{ij}$ $i=1,2,...,n$ and $j=1,2,...,m$.

To harmonize the data, first step is to look at the attributes. These, in general, can be of different types. If all the attributes $G=\{G_1,G_2,...,G_n\}$ are of the same type, then the attribute values do not need harmonization. However if these involve different scales and/or units, there is need to convert them all to the same scale and/or unit. Just to make this point clear, let us consider two types of attributes, namely, (i) cost type and the (ii) benefit type. Considering their natures, a benefit attribute (the bigger the values better is it) and cost attribute (the smaller the values the better) are of rather opposite type. In such cases, we need to first transform the attribute values of cost type into the attribute values of benefit type. So, we transform the intuitionistic fuzzy decision matrix $D=(d_{ij})_{n\times m}$ into the normalized intuitionistic fuzzy decision matrix $R=(r_{ij})_{m\times n}$ by the method given by Xu and Hu [30], where

$$r_{ij}=(\mu_{ij},\nu_{ij},\pi_{ij})=\begin{cases}d_{ij}, & \text{for benefit attribute } G_i\\ (d_{ij})^C, & \text{for cost attribute } G_i\end{cases}, \quad (23)$$

$$i=1,2,...,n;\ \ j=1,2,...,m$$

where $(d_{ij})^C$ is the complement of d_{ij}, such that $(d_{ij})^C=(\nu_{ij},\mu_{ij},\pi_{ij})$.

With attributes harmonized, using the measure defined in (10), we now stipulate following steps to solve our multiple-attribute intuitionistic fuzzy decision making problem:

Step 1: Based on the matrix $R=(r_{ij})_{m\times n}$, specify the options A_j ($j=1,2,...,m$) by the characteristic sets:

$$A_j=\{\langle G_i,\mu_{ij},\nu_{ij},\pi_{ij}\rangle \mid G_i\in G\},\quad j=1,2,...,m \text{ and } i=1,2,...,n.$$

Step 2: Find the ideal solution A^*, given by:

$$A^*=\left\{\begin{array}{l}\langle\mu_{1*},\nu_{1*},\pi_{1*}\rangle,\langle\mu_{2*},\nu_{2*},\pi_{2*}\rangle,\\ \qquad ...,\langle\mu_{n*},\nu_{n*},\pi_{n*}\rangle\end{array}\right\}, \quad (24)$$

where, for each $i=1,2,...,n$,

$$(\mu_{i*},\nu_{i*},\pi_{i*})=\left(\begin{array}{l}\max_j\mu_{ij},\min_j\nu_{ij},\\ \qquad 1-\max_j\mu_{ij}-\min_j\nu_{ij}\end{array}\right). \quad (25)$$

Step 3: Calculate $JR_{\lambda',\alpha}(A_j,A^*)$ using the following expression for it:

$$= \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\left[\begin{array}{l}\log\left\{\begin{array}{l}\left(\lambda_1^j\mu_{A_j}(x_i)+\lambda_2^j\mu_{A^*}(x_i)\right)^\alpha\\ +\left(\lambda_1^j\nu_{A_j}(x_i)+\lambda_2^j\nu_{A^*}(x_i)\right)^\alpha\\ +\left(\lambda_1^j\pi_{A_j}(x_i)+\lambda_2^j\pi_{A^*}(x_i)\right)^\alpha\end{array}\right\}\\ -\lambda_1^j\log\left\{\begin{array}{l}\left(\mu_{A_j}(x_i)\right)^\alpha+\left(\nu_{A_j}(x_i)\right)^\alpha\\ +\left(\pi_{A_j}(x_i)\right)^\alpha\end{array}\right\}\\ -\lambda_2^j\log\left\{\begin{array}{l}\left(\mu_{A^*}(x_i)\right)^\alpha+\left(\nu_{A^*}(x_i)\right)^\alpha\\ +\left(\pi_{A^*}(x_i)\right)^\alpha\end{array}\right\}\end{array}\right] \quad (26)$$

where $\lambda_1^j,\lambda_2^j\in[0,1]$, and $\lambda_1^j+\lambda_2^j=1\ \ \forall\ j=1,2,\ldots,m$.

Step 4: Rank the alternatives A_j, $j=1,2,\ldots,m$, in accordance with the values $JR_{\lambda^j,\alpha}(A_j,A^*)$, $j=1,2,\ldots,m$, and select the best one alternative, denoted by A_k with smallest $JR_{\lambda^j,\alpha}(A_j,A^*)$. Then A_k is the best choice.

In order to demonstrate the application of the above proposed method to a real multiple attribute decision making, we consider below a numerical example.

Example: Consider a customer who wants to buy a car. Let five types of cars (alternatives) A_j $(j=1,2,3,4,5)$ be available. The customer takes into account six attributes to decide which car to buy: (1) G_1: fuel economy, (2) G_2: aerodynamic degree, (3) G_3: price, (4) G_4: comfort, (5) G_5: design and (6) G_6: safety. We note that G_3 is a cost attribute while other five are benefit attributes. Next let us assume that the characteristics of the alternatives A_j $(j=1,2,3,4,5)$ are represented by the intuitionistic fuzzy decision matrix $D=(d_{ij})_{6\times5}$ shown in the following table:

Table I: Intuitionistic fuzzy decision matrix D

	A_1	A_2	A_3	A_4	A_5
G_1	(0.5,0.4, 0.1)	(0.4,0.3, 0.3)	(0.5,0.2, 0.3)	(0.4,0.2, 0.4)	(0.6,0.4, 0.0)
G_2	(0.7,0.2, 0.1)	(0.8,0.2, 0.0)	(0.9,0.1, 0.0)	(0.8,0.0, 0.2)	(0.5,0.2, 0.3)
G_3	(0.4,0.3, 0.3)	(0.5,0.2, 0.3)	(0.6,0.1, 0.3)	(0.7,0.3, 0.0)	(0.8,0.1, 0.1)
G_4	(0.6,0.2, 0.2)	(0.6,0.3, 0.1)	(0.8,0.1, 0.1)	(0.9,0.1, 0.0)	(0.4,0.2, 0.4)
G_5	(0.4,0.5, 0.1)	(0.6,0.4, 0.0)	(0.3,0.5, 0.2)	(0.5,0.3, 0.2)	(0.9,0.0, 0.1)
G_6	(0.3,0.1, 0.6)	(0.7,0.1, 0.2)	(0.6,0.2, 0.2)	(0.6,0.1, 0.3)	(0.4,0.3, 0.3)

First, we transform the attribute values of cost type (G_3) into the attribute values of benefit type (G_3') by using Eq. (23):

$$G_3'=(G_3)^C=\left\{\begin{array}{l}(0.3,0.4,0.3),(0.2,0.5,0.3),(0.1,0.6,0.3),\\(0.3,0.7,0.0),(0.1,0.8,0.1)\end{array}\right\},$$

and then $D=(d_{ij})_{6\times5}$ is transformed into $R=(r_{ij})_{6\times5}$, we get the following table:

Table II: Normalized intuitionistic fuzzy decision matrix R

	A_1	A_2	A_3	A_4	A_5
G_1	(0.5,0.4, 0.1)	(0.4,0.3, 0.3)	(0.5,0.2, 0.3)	(0.4,0.2, 0.4)	(0.6,0.4, 0.0)
G_2	(0.7,0.2, 0.1)	(0.8,0.2, 0.0)	(0.9,0.1, 0.0)	(0.8,0.0, 0.2)	(0.5,0.2, 0.3)
G_3'	(0.3,0.4, 0.3)	(0.2,0.5, 0.3)	(0.1,0.6, 0.3)	(0.3,0.7, 0.0)	(0.1,0.8, 0.1)
G_4	(0.6,0.2, 0.2)	(0.6,0.3, 0.1)	(0.8,0.1, 0.1)	(0.9,0.1, 0.0)	(0.4,0.2, 0.4)
G_5	(0.4,0.5, 0.1)	(0.6,0.4, 0.0)	(0.3,0.5, 0.2)	(0.5,0.3, 0.2)	(0.9,0.0, 0.1)
G_6	(0.3,0.1, 0.6)	(0.7,0.1, 0.2)	(0.6,0.2, 0.2)	(0.6,0.1, 0.3)	(0.4,0.3, 0.3)

The step-wise procedure now goes as follows.

Step 1: Based on $R=(r_{ij})_{6\times5}$, we have characteristic sets of the alternatives A_j $(j=1,2,\ldots,5)$ by

$$A_1=\left\{\begin{array}{l}(0.5,0.4,0.1),(0.7,0.2,0.1),(0.3,0.4,0.3),\\(0.6,0.2,0.2),(0.4,0.5,0.1),(0.3,0.1,0.6)\end{array}\right\},$$

$$A_2=\left\{\begin{array}{l}(0.4,0.3,0.3),(0.8,0.2,0.0),(0.2,0.5,0.3),\\(0.6,0.3,0.1),(0.6,0.4,0.0),(0.7,0.1,0.2)\end{array}\right\},$$

$$A_3=\left\{\begin{array}{l}(0.5,0.2,0.3),(0.9,0.1,0.0),(0.1,0.6,0.3),\\(0.8,0.1,0.1),(0.3,0.5,0.2),(0.6,0.2,0.2)\end{array}\right\},$$

$$A_4=\left\{\begin{array}{l}(0.4,0.2,0.4),(0.8,0.0,0.2),(0.3,0.7,0.0),\\(0.9,0.1,0.0),(0.5,0.3,0.2),(0.6,0.1,0.3)\end{array}\right\},$$

$$A_5=\left\{\begin{array}{l}(0.6,0.4,0.0),(0.5,0.2,0.3),(0.1,0.8,0.1),\\(0.4,0.2,0.4),(0.9,0.0,0.1),(0.4,0.3,0.3)\end{array}\right\}.$$

Step 2: Using (24) and (25), we obtain A^*:

$$A^*=\left\{\begin{array}{l}(0.6,0.2,0.2),(0.9,0.0,0.1),(0.3,0.4,0.3),\\(0.9,0.1,0.0),(0.9,0.0,0.1),(0.7,0.1,0.2)\end{array}\right\}.$$

Step3: We use formula (26) to measure $JR_{\lambda^j,\alpha}(A_j,A^*)$, choosing the various values of parameter. First we take $\lambda_1^j=\lambda_2^j=0.5\ \ \forall\ j=1,2,\ldots,5$; and $\alpha=0.3$, $\alpha=0.5$ and $\alpha=0.7$ respectively, we get the following table:

Table III: Values of $JR_{\lambda^j,\alpha}(A_j,A^*)$ for $\alpha=0.3,0.5,0.7$

	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$
$JR_{\lambda^j,\alpha}(A_1,A^*)$	0.1453	0.1409	0.1345
$JR_{\lambda^j,\alpha}(A_2,A^*)$	0.1908	0.1584	0.1299
$JR_{\lambda^j,\alpha}(A_3,A^*)$	0.1617	0.1400	0.1214
$JR_{\lambda^j,\alpha}(A_4,A^*)$	0.0946	0.0905	0.0849
$JR_{\lambda^j,\alpha}(A_5,A^*)$	0.1483	0.1467	0.1424

Based on the calculated values of $JR_{\lambda^j,\alpha}(A_j,A^*)$ in table III, we get the following orderings of ranks of the alternatives A_j $(j=1,2,3,4,5)$:

For $\alpha=0.3$, $\quad A_4\succ A_1\succ A_5\succ A_3\succ A_2$.

For $\alpha = 0.5$, $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$.

For $\alpha = 0.7$, $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$.

Since $JR_{\lambda^j,\alpha}(A_4, A^*)$ is smallest among the values of $JR_{\lambda^j,\alpha}(A_j, A^*)$ $\{j = 1,2,...,5\}$ for $\alpha = 0.3$, $\alpha = 0.5$ and $\alpha = 0.7$, so A_4 is the most preferable alternative. Thus here we find that variation in values of α brings about change in ranking, but leaves the best choice unchanged.

Change in Consideration: In the above consideration, same values of λ_i^j were taken. But in a realistic situation these can also be different for different alternatives. The value of λ_i^j may then depend on an un-explicit (like past experience or pressures) on the decision maker.

Let us next consider intuitionistic fuzzy Jensen-Rényi divergence measures $JR_{\lambda^j,\alpha}(A_j, A^*)$, taking different values of λ_i^j:

We take $\lambda_1^1 = 0.5$, $\lambda_2^1 = 0.5$; $\lambda_1^2 = 0.4$, $\lambda_2^2 = 0.6$; $\lambda_1^3 = 0.8$, $\lambda_2^3 = 0.2$; $\lambda_1^4 = 0.5$, $\lambda_2^4 = 0.5$; $\lambda_1^5 = 0.3$, $\lambda_2^5 = 0.7$ and $\alpha = 0.5$.

Calculating $JR_{\lambda^j,\alpha}(A_j, A^*)$, we get the following table:

Table IV: Values of $JR_{\lambda^j,\alpha}(A_j, A^*)$ for $\alpha = 0.5$

$JR_{\lambda^1,\alpha}(A_1, A^*)$	0.0965
$JR_{\lambda^2,\alpha}(A_2, A^*)$	0.1644
$JR_{\lambda^3,\alpha}(A_3, A^*)$	0.0856
$JR_{\lambda^4,\alpha}(A_4, A^*)$	0.1178
$JR_{\lambda^5,\alpha}(A_5, A^*)$	0.1479

The resulting order of rankings then is

$$A_3 \succ A_1 \succ A_4 \succ A_5 \succ A_2.$$

Thus A_3 is the most preferable alternative.

If we take

$\lambda_1^1 = 0.5$, $\lambda_2^1 = 0.5$; $\lambda_1^2 = 0.7$, $\lambda_2^2 = 0.3$; $\lambda_1^3 = 0.3$, $\lambda_2^3 = 0.7$; $\lambda_1^4 = 0.4$, $\lambda_2^4 = 0.6$; $\lambda_1^5 = 0.8$, $\lambda_2^5 = 0.2$ and $\alpha = 0.5$, calculating $JR_{\lambda^j,\alpha}(A_j, A^*)$, we get the following table:

Table V: Values of $JR_{\lambda^j,\alpha}(A_j, A^*)$ for $\alpha = 0.5$

$JR_{\lambda^1,\alpha}(A_1, A^*)$	0.1409
$JR_{\lambda^2,\alpha}(A_2, A^*)$	0.1296
$JR_{\lambda^3,\alpha}(A_3, A^*)$	0.1493
$JR_{\lambda^4,\alpha}(A_4, A^*)$	0.1268
$JR_{\lambda^5,\alpha}(A_5, A^*)$	0.0965

The resulting order of rankings then is

$$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3.$$

Resulting in A_5 as the most preferable option. Thus for a given value of parameter α, averaging parameters λ's can effect the choice.

The numerical example shows that change in order of the rankings results by change in parameters λ & α establishing the significance of these parameters in multi-attribute sensitive decision making problems.

6 Conclusions

The paper provides a measure and application in multiple-attribute decision making problem under intuitionistic fuzzy environment. This study can lead to symmetric measure and resulting other insight into studying IFSs.

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