Intuitionistic fuzzy modal operators of second type over interval-valued intuitionistic fuzzy sets. Part 1

Krassimir Atanassov

Bioinformatics and Mathematical Modelling Dept., Institute of Biophysics
and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria,

and

Intelligent Systems Laboratory
Prof. Asen Zlatarov University, 8010 Burgas, Bulgaria
e-mail: `krat@bas.bg`

Abstract: The basic definitions of the concept of interval-valued intuitionistic fuzzy set and of the operations, relations and operators over it are given. Some of its most important applications are described. Ideas for future development of the theory of interval-valued intuitionistic fuzzy sets are proposed.
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In memory of my father Todor A. Atanassov (1928 – 1985)
who would have been celebrating his 90th birthday today.

1 Introduction

Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs; see, e.g., [2]) were introduced as extensions of Intuitionistic Fuzzy Sets (IFSs; see [1, 2]) by George Gargov (7.4.1947 – 9.11.1996) and the author in [4]. A lot of operations, relations and operators were defined over them.

In recent years, the interest in IVIFSs increased and now they are object of intensive research and applications.

A of 2018, the author has started working actively over the development of the IVIFSs theory. In [3], new operators from standard (first) modal type were introduced and some of their basic properties were discussed.

In the present paper, using the notations from paper [3], we introduce a series of modal type operators of second type over IVIFSs. They are analogues of the modal type operators of second type over IFSs, discussed in [2].

2 Preliminaries

An IVIFS A over universe E is an object of the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \le 1.$$

For any two IVIFSs A and B the following relations hold:

$$A \subset B \quad \text{iff} \quad (\forall x \in E)(\inf M_A(x) \le \inf M_B(x) \ \& \ \inf N_A(x) \ge \inf N_B(x)$$
$$\& \sup M_A(x) \le \sup M_B(x) \ \& \ \sup N_A(x) \ge \sup N_B(x)),$$

$$A = B \quad \text{iff} \quad A \subset B \ \& \ B \subset A.$$

For any two IVIFSs A and B the following operations hold:

$$\neg A \ = \ \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\},$$

$$A \cap B \ = \ \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))],$$
$$[\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\},$$

$$A \cup B \ = \ \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))],$$
$$[\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}$$

$$A + B \ = \ \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x).\inf M_B(x),$$
$$\sup M_A(x) + \sup M_B(x) - \sup M_A(x).\sup M_B(x)],$$
$$[\inf N_A(x).\inf N_B(x), \sup N_A(x).\sup N_B(x)]\rangle \mid x \in E\}$$

$$A.B \ = \ \{\langle x, [\inf M_A(x).\inf M_B(x), \sup M_A(x).\sup M_B(x)],$$
$$[\inf N_A(x) + \inf N_B(x) - \inf N_A(x).\inf N_B(x),$$
$$\sup N_A(x) + \sup N_B(x) - \sup N_A(x).\sup N_B(x)]\rangle \mid x \in E\}$$

$$A@B \ = \ \{\langle x, [(\inf M_A(x) + \inf M_B(x))/2, (\sup M_A(x) + \sup M_B(x))/2],$$
$$[(\inf N_A(x) + \inf N_B(x))/2, (\sup N_A(x)\rangle + \sup N_B(x))/2] \mid x \in E\}$$

Operators of modal type are defined similarly to those, defined for IFSs:

$$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\},$$

$$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\},$$

$$D_\alpha(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))],$$
$$[\inf N_A(x), \sup N_A(x) + (1 - \alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle$$
$$\mid x \in E\},$$

$$F_{\alpha,\beta}(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))],$$
$$[\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle$$
$$\mid x \in E\}, \text{ for } \alpha + \beta \le 1,$$

$$G_{\alpha,\beta}(A) = \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle$$
$$\mid x \in E\},$$

$$H_{\alpha,\beta}(A) = \{\langle x, [\alpha.\inf M(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x)$$
$$+\beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$

$$H^*_{\alpha,\beta}(A) = \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x)$$
$$+\beta.(1 - \alpha.\sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x)$$
$$- \sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},$$

$$J^*_{\alpha,\beta}(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x)$$
$$-\beta.\sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},$$

where $\alpha, \beta \in [0, 1]$.

The level operators, defined over IVIFS A, are defined for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \le 1$ by:

$$P_{\alpha,\beta}(A) = \{\langle x, [\max(\alpha, \inf M_A(x)), \max(\alpha, \sup M_A(x))],$$
$$[\min(\beta, \inf N_A(x)), \min(\beta, \sup N_A(x))]\rangle \mid x \in E\},$$

$$Q_{\alpha,\beta}(A) = \{\langle x, [\min(\alpha, \inf M_A(x)), \min(\alpha, \sup M_A(x))],$$
$$[\max(\beta, \inf N_A(x)), \max(\beta, \sup N_A(x))]\rangle \mid x \in E\}.$$

The standard interval valued intuitionistic fuzzy topological operators are defined over each IVIFS A by:

$$C(A) = \{\langle x, [\sup_{x \in E} \inf M_A(x), \sup_{x \in E} \sup M_A(x)], [\inf_{x \in E} \inf N_A(x), \inf_{x \in E} \sup N_A(x)]\rangle \mid x \in E\},$$

$$I(A) = \{\langle x, [\inf_{x \in E} \inf M_A(x), \inf_{x \in E} \sup M_A(x)], [\sup_{x \in E} \inf N_A(x), \sup_{x \in E} \sup N_A(x)]\rangle \mid x \in E\}.$$

3   Main results

We describe standard interval-valued forms of some intuitionistic fuzzy modal-like operators of second type, following [2], where their IFS-forms are given. We study some basis properties of

the defined operators. Below, we formulate only these operator properties that are valid for the currently discussed type of operators and are not valid for the next operator extensions.

Let the IVIFS

$$A = \{\langle x, [\inf M_A(x), \sup M_A(x)], [\inf N_A(x), \sup N_A(x)]\rangle \mid x \in E\},$$

be given.

Now, we introduce the first two new IVIFS-operators of second type $\boxplus$ and $\boxtimes$ as follows.

$$\boxplus A = \left\{ \left\langle x, \left[\frac{\inf M_A(x)}{2}, \frac{\sup M_A(x)}{2}\right], \left[\frac{\inf N_A(x) + 1}{2}, \frac{\sup N_A(x) + 1}{2}\right] \right\rangle \Big| x \in E \right\},$$

$$\boxtimes A = \left\{ \left\langle x, \left[\frac{\inf M_A(x) + 1}{2}, \frac{\sup M_A(x) + 1}{2}\right], \left[\frac{\inf N_A(x)}{2}, \frac{\sup N_A(x)}{2}\right] \right\rangle \Big| x \in E \right\}.$$

All of their properties are valid for their immediare extensions, that for a given real number $\alpha \in [0, 1]$ and IVIFS A have the forms:

$$
\begin{aligned}
\boxplus_\alpha A &= \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], \\
&\quad [\alpha \inf N_A(x) + 1 - \alpha, \alpha \sup N_A(x) + 1 - \alpha]\rangle | x \in E\}, \\
\boxtimes_\alpha A &= \{\langle x, [\alpha \inf M_A(x) + 1 - \alpha, \alpha \sup M_A(x) + 1 - \alpha], \\
&\quad [\alpha \inf N_A(x), \alpha \sup N_A(x)]\rangle | x \in E\}.
\end{aligned}
$$

Obviously, for every IVIFS A:

$$\boxplus_{0.5} A = \boxplus A,$$

$$\boxtimes_{0.5} A = \boxtimes A.$$

Therefore, the new operators $\boxplus_\alpha$ and $\boxtimes_\alpha$ are generalizations of the first ones.

The following assertions hold for the first two types of the next operators. We give the proof of only one of them (Theorem 1 (b)), while the rest assertions are proved in the same manner.

Theorem 1. For every IVIFS A and for every $\alpha \in [0, 1]$:

(a) $\boxplus_\alpha A \subseteq A \subseteq \boxtimes_\alpha A$,

(b) $\neg \boxplus_\alpha \neg A = \boxtimes_\alpha A$,

(c) $\boxplus_\alpha \boxplus_\alpha A \subseteq \boxplus_\alpha A$,

(d) $\boxtimes_\alpha \boxtimes_\alpha A \supseteq \boxtimes_\alpha A$.

Proof. Let A be an IVIFS and $\alpha \in [0, 1]$. Then

$$\neg \boxplus_\alpha \neg A$$

$$= \neg \boxplus_\alpha \neg \{\langle x, [\inf M_A(x), \sup M_A(x)], [\inf N_A(x), \sup N_A(x)]\rangle \mid x \in E\}$$

$$= \neg \boxplus_\alpha \{\langle x, [\inf N_A(x), \sup N_A(x)], [\inf M_A(x), \sup M_A(x)]\rangle \mid x \in E\}$$

$$= \neg \{\langle x, [\alpha \inf N_A(x), \alpha \sup N_A(x)], [\alpha \inf M_A(x) + 1 - \alpha, \alpha \sup M_A(x) + 1 - \alpha]\rangle | x \in E\},$$

$$= \{\langle x, [\alpha \inf M_A(x) + 1 - \alpha, \alpha \sup M_A(x) + 1 - \alpha], [\alpha \inf N_A(x), \alpha \sup N_A(x)]\rangle | x \in E\}$$

$$= \boxtimes_\alpha A. \qquad \square$$

Theorem 2. For every two IVIFSs A and B:

(a) $\boxplus_\alpha(A + B) \subseteq \boxplus_\alpha A + \boxplus_\alpha B$,

(b) $\boxtimes_\alpha(A + B) \supseteq \boxtimes_\alpha A + \boxtimes_\alpha B$,

(c) $\boxplus_\alpha(A.B) \supseteq \boxplus_\alpha A.\boxplus_\alpha B$,

(d) $\boxtimes_\alpha(A.B) \subseteq \boxtimes_\alpha A.\boxtimes_\alpha B$.

Moreover, the following assertions are also true.

Theorem 3. For every IVIFS A and for every two real numbers $\alpha, \beta \in [0, 1]$:

(a) $\boxplus_\alpha \boxplus_\beta A = \boxplus_\beta \boxplus_\alpha A$,

(b) $\boxtimes_\alpha \boxtimes_\beta A = \boxtimes_\beta \boxtimes_\alpha A$,

(c) $\boxtimes_\alpha \boxplus_\beta A \supseteq \boxplus_\beta \boxtimes_\alpha A$.

Theorem 4. For every IVIFS A and for every three real numbers $\alpha, \beta, \gamma \in [0, 1]$:

(a) $\boxplus_\alpha D_\beta(A) = D_\beta(\boxplus_\alpha A)$,

(b) $\boxplus_\alpha F_{\beta,\gamma}(A) = F_{\beta,\gamma}(\boxplus_\alpha A)$, where $\beta + \gamma \leq 1$,

(c) $\boxplus_\alpha G_{\beta,\gamma}(A) \subseteq G_{\beta,\gamma}(\boxplus_\alpha A)$,

(d) $\boxplus_\alpha H_{\beta,\gamma}(A) = H_{\beta,\gamma}(\boxplus_\alpha A)$,

(e) $\boxplus_\alpha H^*_{\beta,\gamma}(A) = H^*_{\beta,\gamma}(\boxplus_\alpha A)$,

(f) $\boxplus_\alpha J_{\beta,\gamma}(A) = J_{\beta,\gamma}(\boxplus_\alpha A)$,

(g) $\boxplus_\alpha J^*_{\beta,\gamma}(A) = J^*_{\beta,\gamma}(\boxplus_\alpha A)$,

(h) $\boxtimes_\alpha D_\beta(A) = D_\beta(\boxtimes_\alpha A)$,

(i) $\boxtimes_\alpha F_{\beta,\gamma}(A) = F_{\beta,\gamma}(\boxtimes_\alpha A)$, where $\beta + \gamma \leq 1$,

(j) $\boxtimes_\alpha G_{\beta,\gamma}(A) \subseteq G_{\beta,\gamma}(\boxtimes_\alpha A)$,

(k) $\boxtimes_\alpha H_{\beta,\gamma}(A) = H_{\beta,\gamma}(\boxtimes_\alpha A)$,

(l) $\boxtimes_\alpha H^*_{\beta,\gamma}(A) = H^*_{\beta,\gamma}(\boxtimes_\alpha A)$,

(m) $\boxtimes_\alpha J_{\beta,\gamma}(A) = J_{\beta,\gamma}(\boxtimes_\alpha A)$,

(n) $\boxtimes_\alpha J^*_{\beta,\gamma}(A) = J^*_{\beta,\gamma}(\boxtimes_\alpha A)$.

The second extension has the forms:

$$\boxplus_{\alpha,\beta} A = \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], [\alpha \inf N_A(x) + \beta, \alpha \sup N_A(x) + \beta]\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, [\alpha \inf M_A(x) + \beta, \alpha \sup M_A(x) + \beta], [\alpha \inf N_A(x), \alpha \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

Obviously, for every IVIFS A:

$$\boxplus A = \boxplus A_{0.5, 0.5},$$

$$\boxtimes A = \boxtimes A_{0.5, 0.5},$$

$$\boxplus A_\alpha = \boxplus A_{\alpha, 1-\alpha},$$

$$\boxtimes A_\alpha = \boxtimes A_{\alpha, 1-\alpha}.$$

The following assertions hold for the new operators.

Theorem 5. For every IVIFS A and for every $\alpha, \beta, \alpha + \beta \in [0, 1]$:

(a) $\neg \boxplus_{\alpha,\beta} \neg A = \boxtimes_{\alpha,\beta} A,$

(b) $\neg \boxtimes_{\alpha,\beta} \neg A = \boxplus_{\alpha,\beta} A.$

Theorem 6. For every IFS A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta, \gamma + \delta \in [0, 1]$:

$$\boxplus_{\alpha,\beta} \boxtimes_{\gamma,\delta} A \subseteq \boxtimes_{\gamma,\delta} \boxplus_{\alpha,\beta} A.$$

Now, we introduce the third extension of the above operators. They have the forms:

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], [\beta \inf N_A(x) + \gamma, \beta \sup N_A(x) + \gamma]\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, [\alpha \inf M_A(x) + \gamma, \alpha \sup M_A(x) + \gamma], [\beta \inf N_A(x), \beta \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \le 1$.

Obviously, for every IVIFS A:

$$\boxplus A = \boxplus A_{0.5, 0.5, 0.5},$$

$$\boxtimes A = \boxtimes A_{0.5, 0.5, 0.5},$$

$$\boxplus A_\alpha = \boxplus A_{\alpha, \alpha, 1-\alpha},$$

$$\boxtimes A_\alpha = \boxtimes A_{\alpha, 1-\alpha},$$

$$\boxplus A_{\alpha,\beta} = \boxplus A_{\alpha, \alpha, \beta},$$

$$\boxtimes A_{\alpha,\beta} = \boxtimes A_{\alpha, \alpha, \beta}.$$

The following assertions hold for the new operators.

Theorem 7. For every IVIFS A and for every $\alpha, \beta, \gamma \in [0, 1]$, for which $\max(\alpha, \beta) + \gamma \le 1$:

(a) $\neg \boxplus_{\alpha,\beta,\gamma} \neg A = \boxtimes_{\beta,\alpha,\gamma} A,$

(b) $\neg \boxtimes_{\alpha,\beta,\gamma} \neg A = \boxplus_{\beta,\alpha,\gamma} A.$

Theorem 8. For every two IVIFSs A and B and for every $\alpha, \beta, \gamma \in [0, 1]$, for which $\max(\alpha, \beta) + \gamma \le 1$:

(a) $\boxplus_{\alpha,\beta,\gamma}(A \cap B) = \boxplus_{\alpha,\beta,\gamma} A \cap \boxplus_{\alpha,\beta,\gamma} B$,

(b) $\boxtimes_{\alpha,\beta,\gamma}(A \cap B) = \boxtimes_{\alpha,\beta,\gamma} A \cap \boxtimes_{\alpha,\beta,\gamma} B$,

(c) $\boxplus_{\alpha,\beta,\gamma}(A \cup B) = \boxplus_{\alpha,\beta,\gamma} A \cup \boxplus_{\alpha,\beta,\gamma} B$,

(d) $\boxtimes_{\alpha,\beta,\gamma}(A \cup B) = \boxtimes_{\alpha,\beta,\gamma} A \cup \boxtimes_{\alpha,\beta,\gamma} B$,

(e) $\boxplus_{\alpha,\beta,\gamma}(A@B) = \boxplus_{\alpha,\beta,\gamma} A @ \boxplus_{\alpha,\beta,\gamma} B$,

(f) $\boxtimes_{\alpha,\beta,\gamma}(A@B) = \boxtimes_{\alpha,\beta,\gamma} A @ \boxtimes_{\alpha,\beta,\gamma} B$.

Theorem 9. For every IVIFS A and for every $\alpha, \beta, \gamma \in [0, 1]$, for which $\max(\alpha, \beta) + \gamma \le 1$:

(a) $\boxplus_{\alpha,\beta,\gamma} C(A) = C(\boxplus_{\alpha,\beta,\gamma} A)$,

(b) $\boxtimes_{\alpha,\beta,\gamma} C(A) = C(\boxtimes_{\alpha,\beta,\gamma} A)$,

(c) $\boxplus_{\alpha,\beta,\gamma} I(A) = I(\boxplus_{\alpha,\beta,\gamma} A)$,

(d) $\boxtimes_{\alpha,\beta,\gamma} I(A) = I(\boxtimes_{\alpha,\beta,\gamma} A)$.

A natural extension of the last two operators ($\boxplus_{\alpha,\beta,\gamma}$ and $\boxtimes_{\alpha,\beta,\gamma}$) is the operator

$$\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A = \{\langle x, [\alpha \inf M_A(x) + \gamma, \alpha \sup M_A(x) + \gamma],$$

$$[\beta \inf N_A(x) + \delta, \beta \sup N_A(x) + \delta]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha, \beta) + \gamma + \delta \le 1$.

It is the fourth type of operators from the current type.

Obviously, for every IVIFS A:

$$\boxplus A = \boxed{\bullet} A_{0.5,0.5,0,0.5},$$

$$\boxtimes A = \boxed{\bullet} A_{0.5,0.5,0.5,0},$$

$$\boxplus A_\alpha = \boxed{\bullet} A_{\alpha,\alpha,0,1-\alpha},$$

$$\boxtimes A_\alpha = \boxed{\bullet} A_{\alpha,\alpha,1-\alpha,0},$$

$$\boxplus A_{\alpha,\beta} = \boxed{\bullet} A_{\alpha,\alpha,0,\beta},$$

$$\boxtimes A_{\alpha,\beta} = \boxed{\bullet} A_{\alpha,\alpha,\beta,0}.$$

$$\boxplus A_{\alpha,\beta,\gamma} = \boxed{\bullet} A_{\alpha,\beta,0,\gamma},$$

$$\boxtimes A_{\alpha,\beta,\gamma} = \boxed{\bullet} A_{\alpha,\beta,\gamma,0}.$$

The following assertions hold for the new operator.

Theorem 10. For every IFS A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$, for which

$$\max(\alpha, \beta) + \gamma + \delta \le 1 :$$

(a) $\neg\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}\neg A = \boxed{\bullet}_{\beta,\alpha,\delta,\gamma}A,$

(b) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(\boxed{\bullet}_{\varepsilon,\zeta,\eta,\theta}A) = \boxed{\bullet}_{\alpha\varepsilon,\beta\zeta,\alpha\eta+\gamma,\beta\theta+\delta}A,$

(c) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}\,\square\,A \supseteq \square\,\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A,$

(d) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}\Diamond A \subseteq \Diamond\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A.$

Theorem 11. For every two IFSs A and B and for every $\alpha,\beta,\gamma,\delta \in [0,1]$, for which

$$\max(\alpha,\beta) + \gamma + \delta \leq 1:$$

(a) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(A \cap B) = \boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A \cap \boxed{\bullet}_{\alpha,\beta,\gamma,\delta}B,$

(b) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(A \cup B) = \boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A \cup \boxed{\bullet}_{\alpha,\beta,\gamma,\delta}B,$

(c) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(A@B) = \boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A@\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}B.$

In a series of papers (see, e.g., [5,6]), Gökhan Çuvalcioğlu introduced other operators from the second type, but having different forms. The author supposes that he will give for them interval-valued interpretations and for this reason, here the author will not discuss them.

Theorem 12: For every IVIFS A and for every $\alpha,\beta,\gamma,\delta \in [0,1]$, for which

$$\max(\alpha,\beta) + \gamma + \delta \leq 1:$$

(a) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}C(A) = C(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A),$

(b) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}I(A) = I(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A),$

(c) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}P_{\varepsilon,\zeta}((A)) = P_{\alpha\varepsilon+\gamma,\beta\zeta+\delta}(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A),$

(d) $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}P_{\varepsilon,\zeta}((A)) = Q_{\alpha\varepsilon+\gamma,\beta\zeta+\delta}(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}A).$

The extended form of all above operators is the operator

$$\boxed{\odot}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A = \{\langle x, [\alpha\inf M_A(x) - \varepsilon\inf N_A(x) + \gamma, \alpha\sup M_A(x) - \varepsilon\inf N_A(x) + \gamma],$$

$$[\beta\inf N_A(x) - \zeta\inf M_A(x) + \delta, \beta\sup N_A(x) - \zeta\inf M_A(x) + \delta]\rangle | x \in E\},$$

where $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta \in [0,1]$, and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1, \tag{1}$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0. \tag{2}$$

Obviously, for every IVIFS A:

$$\boxplus A = \boxed{\odot}_{0.5,0.5,0,0.5,0,0}A,$$

$$\boxtimes A = \boxed{\odot}_{0.5,0.5,0.5,0,0,0}A,$$

$$\boxplus_\alpha A = \boxed{\odot}_{\alpha,\alpha,0,1-\alpha,0,0}A,$$

$$\boxtimes_\alpha A = \boxdot_{\alpha,\alpha,1-\alpha,0,0,0}A,$$

$$\boxplus_{\alpha,\beta}A = \boxdot_{\alpha,\alpha,0,\beta,0,0}A,$$

$$\boxtimes_{\alpha,\beta}A = \boxdot_{\alpha,\alpha,\beta,0,0,0}A,$$

$$\boxplus_{\alpha,\beta,\gamma}A = \boxdot_{\alpha,\beta,0,\gamma,0,0}A,$$

$$\boxtimes_{\alpha,\beta,\gamma}A = \boxdot_{\alpha,\beta,\gamma,0,0,0}A,$$

$$\boxdot_{\alpha,\beta,\gamma,\delta}A = \boxdot_{\alpha,\beta,\gamma,\delta,0,0}A.$$

The following assertions hold for the operator $\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$.

Theorem 13. For every IVIFS A and for every $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$, for which (1) and (2) are valid, the equality

$$\neg\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}\neg A = \boxdot_{\beta,\alpha,\delta,\gamma,\zeta,\varepsilon}A$$

holds.

Theorem 14. For every two IVIFSs A and B and for every $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$, for which (1) and (2) are valid, the equality

$$\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A@B) = \boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A@\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B$$

holds.

We must note that equalities

$$\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A\cap B) = \boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \cap \boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B,$$

$$\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A\cup B) = \boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \cup \boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B,$$

which are valid for operator $\boxdot_{\alpha,\beta,\gamma,\delta}$, are not always valid now.

Theorem 15. For every IVIFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0,1]$ for which conditions that are similar to (1) and (2) are valid, the equality

$$\boxdot_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}\left(\boxdot_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A\right)$$

$$= \boxdot_{\alpha_1.\alpha_2+\varepsilon_1.\zeta_2,\ \beta_1.\beta_2+\zeta_1.\varepsilon_2,\ \alpha_1.\gamma_2-\varepsilon_1.\delta_2+\gamma_1,\ \beta_1.\delta_2-\zeta_1.\gamma_2+\delta_1,\ \alpha_1.\varepsilon_2+\varepsilon_1.\beta_2,\ \beta_1.\zeta_2+\zeta_1.\alpha_2}A$$

holds.

4 Conclusion

In a next paper, the author will introduce generalizations of all the operators, introduced here for a first time. Here, inf- and sup-components have equal parameters (real numbers). In the next research, these parameters for the inf- and sup-components will be different and their specific properties will be studied.

In near future, the author plans to extend the results from the present paper to a book.

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