

**INVARIANCE OF  $g$ -NATURAL METRICS  
ON LINEAR FRAME BUNDLES**

OLDŘICH KOWALSKI AND MASAMI SEKIZAWA

ABSTRACT. In this paper we prove that each  $g$ -natural metric on a linear frame bundle  $LM$  over a Riemannian manifold  $(M, g)$  is invariant with respect to a lifted map of a (local) isometry of the base manifold. Then we define  $g$ -natural metrics on the orthonormal frame bundle  $OM$  and we prove the same invariance result as above for  $OM$ . Hence we see that, over a space  $(M, g)$  of constant sectional curvature, the bundle  $OM$  with an arbitrary  $g$ -natural metric  $\tilde{G}$  is locally homogeneous.

## INTRODUCTION

There are well-known classical examples of “lifted metrics” on the linear frame bundle  $LM$  over a Riemannian manifold  $(M, g)$ . Namely, these are the diagonal lift (which is also called the Sasaki-Mok metric), the horizontal lift and the vertical lift. As one can see, the classical constructions are examples of “natural transformations of second order”. In [10], the present authors have fully classified all (possibly degenerate) naturally lifted metrics “of second order” on  $LM$ . They have proved that the complete family of such natural metrics (for a fixed base metric) is a module over real functions generated by some generalizations of known classical lifts. Our idea of naturality is closely related to that of A. Nijenhuis, D. B. A. Epstein, P. Stredder and others (see [6] for the full references). We have used for our purposes the concepts and methods developed by D. Krupka [12, 13] and D. Krupka and V. Mikolášová [15]. See also I. Kolář, P. W. Michor and J. Slovák [6], and D. Krupka and J. Janyška [14] for the concept of naturality in general. We shall use further the name “ $g$ -natural metrics” for our metrics on  $LM$  after the name proposed by M. T. K. Abbassi in [1] for the metrics naturally lifted to tangent bundles. The Riemannian geometry of linear frame bundles (for a special class of  $g$ -natural metrics) has been studied by Mok [16], Cordero and de León [2, 3], Cordero, Dodson and de León [4], and by the present authors in [11].

One of the properties of  $g$ -natural metrics to be expected is the “invariance property” saying that the lifts of (local) isometries are again (local) isometries. In

---

2000 *Mathematics Subject Classification*: Primary: 53C07; Secondary: 53C20, 53C21, 53C40.

*Key words and phrases*: Riemannian manifold, linear frame bundle, orthonormal frame bundle,  $g$ -natural metrics, homogeneity.

The first author was supported by the grant GA ČR 201/05/2707 and by the project MSM 0021620839.

Received January 23, 2008. Editor I. Kolář.

fact, we have proved in [7] that this property holds for the  $g$ -natural metrics on the tangent bundles. After a short survey about our classification we present  $g$ -natural metrics on  $LM$  in the convenient setting. Then we prove that the invariance property really holds. Next we pass over to the orthonormal frame bundles.

We define  $g$ -natural metrics on orthonormal frame bundles  $OM$  as restrictions of  $g$ -natural metrics on  $LM$  to the submanifolds  $OM$  of  $LM$ , and we present an explicit formula. The Riemannian geometry of orthonormal frame bundles (for a special class of  $g$ -natural metrics) has been studied by Jensen [5], Mok [16], Cordero and de León [2, 3], Cordero, Dodson and de León [4], Zou [18], and by the present authors in [8] and [9]. We prove in this article that each  $g$ -natural metric on  $OM$  over  $(M, g)$  is invariant by a lifted map of a (local) isometry of  $(M, g)$ , and hence each orthonormal frame bundle  $OM$  equipped with a  $g$ -natural metric  $\tilde{G}$  over a space of constant sectional curvature is a locally homogeneous space.

### 1. LIFTS OF VECTORS

The linear frame bundle  $LM$  over a smooth manifold  $M$  consists of all pairs  $(x, u)$ , where  $x$  is a point of  $M$  and  $u$  is a basis for the tangent space  $M_x$  of  $M$  at  $x$ . We denote by  $p$  the natural projection of  $LM$  to  $M$  defined by  $p(x, u) = x$ . If  $(\mathcal{U}; x^1, x^2, \dots, x^n)$  is a system of local coordinates in  $M$ , then a basis  $u = (u_1, u_2, \dots, u_n)$  for  $M_x$  can be expressed in the unique way in the form

$$u_\lambda = \sum_{i=1}^n u_\lambda^i \left( \frac{\partial}{\partial x^i} \right)_x$$

for all indices  $\lambda = 1, 2, \dots, n$ , and hence  $(p^{-1}(\mathcal{U}); x^1, x^2, \dots, x^n, u_1^1, u_1^2, \dots, u_n^n)$  is a system of local coordinates in  $LM$ .

Let  $g$  be a Riemannian metric on the manifold  $M$  and  $\nabla$  its Levi-Civita connection. Then the tangent space  $(LM)_{(x,u)}$  of  $LM$  at  $(x, u) \in LM$  splits into the horizontal and vertical subspace  $H_{(x,u)}$  and  $V_{(x,u)}$  with respect to  $\nabla$ :

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If a point  $(x, u) \in LM$  and a vector  $X \in M_x$  are given, then there exists a unique vector  $X^h \in H_{(x,u)}$  such that  $p_*(X^h) = X$ . We call  $X^h$  the *horizontal lift* of  $X$  to  $TM$  at  $(x, u)$ . We define naturally  $n$  different vertical lifts of  $X \in M_x$ . If  $\omega$  is a one-form on  $M$ , then  $\iota_\mu \omega$ ,  $\mu = 1, 2, \dots, n$ , are functions on  $LM$  defined by  $(\iota_\mu \omega)(x, u) = \omega(u_\mu)$  for all  $(x, u) = (x, u_1, u_2, \dots, u_n) \in LM$ . The *vertical lifts*  $X^{v,\lambda}$ ,  $\lambda = 1, 2, \dots, n$ , of  $X \in M_x$  to  $LM$  at  $(x, u)$  are the  $n$  vectors such that  $X^{v,\lambda}(\iota_\mu \omega) = \omega(X) \delta_\mu^\lambda$ ,  $\lambda, \mu = 1, 2, \dots, n$ , hold for all one-forms  $\omega$  on  $M$ , where  $\delta_\mu^\lambda$  denotes the Kronecker's delta. The  $n$  vertical lifts are always uniquely determined, and they are linearly independent if  $X \neq 0$ . They are expressed in a local coordinate system as

$$X_{(x,u)}^{v,\lambda} = \sum_{i=1}^n \xi^i \left( \frac{\partial}{\partial u_\lambda^i} \right)_{(x,u)}$$

for all indices  $\lambda = 1, 2, \dots, n$  (cf. [3]).

In an obvious way we can define horizontal and vertical lifts of vector fields on  $M$ . These are uniquely defined vector fields on  $TM$ . The *canonical vertical vector fields* on  $LM$  are vector fields  $U_\lambda^\mu$ ,  $\lambda, \mu = 1, 2, \dots, n$ , defined, in terms of local coordinates, by  $U_\lambda^\mu = \sum_{i=1}^n u_\lambda^i \partial / \partial u_\mu^i$ . Here  $U_\lambda^\mu$ 's do not depend on the choice of local coordinates and they are defined globally on  $LM$ .

For a vector  $u_\lambda = \sum_{i=1}^n u_\lambda^i (\partial / \partial x^i)_x \in M_x$ ,  $\lambda = 1, 2, \dots, n$ , we see that

$$(1.1) \quad (u_\lambda)_{(x,u)}^h = \sum_{i=1}^n u_\lambda^i \left( \frac{\partial}{\partial x^i} \right)_{(x,u)}^h$$

and

$$(1.2) \quad (u_\lambda)_{(x,u)}^{v,\mu} = \sum_{i=1}^n u_\lambda^i \left( \frac{\partial}{\partial x^i} \right)_{(x,u)}^{v,\mu} = \sum_{i=1}^n u_\lambda^i \left( \frac{\partial}{\partial u_\mu^i} \right)_{(x,u)} = U_\lambda^\mu(x, u).$$

## 2. $g$ -NATURAL METRICS

We say that a bundle morphism of the form  $\zeta: LM \oplus TM \oplus TM \rightarrow M \times \mathbb{R}$  is an  $L$ -metric on  $M$  if it is linear in the second and the third argument (and smooth in the first argument). We also say that  $\zeta$  is symmetric or skew-symmetric if it is symmetric or skew-symmetric with respect to the second and third argument, respectively. Any Riemannian metric  $g$  on  $M$  is a symmetric  $L$ -metric which is independent on  $u$ . In our special case, letting  $g$  be a given Riemannian metric on  $M$ , we speak about *natural  $L$ -metrics derived from  $g$*  which are  $L$ -metrics  $\zeta$ , for a fixed  $u \in LM$ , whose components  $\zeta(u)_{ij} = \zeta(u, \partial / \partial x^i, \partial / \partial x^j)$  with respect to a system of local coordinates  $(x^1, x^2, \dots, x^n)$  in  $M$  are solutions of the system of differential equations

$$2 \sum_{a=1}^n g_{ap} \frac{\partial \zeta_{ij}}{\partial g_{aq}} - \sum_{\alpha=1}^n u_\alpha^q \frac{\partial \zeta_{ij}}{\partial u_\alpha^p} = \zeta_{ip} \delta_j^q + \zeta_{pj} \delta_i^q, \quad i, j, p, q = 1, 2, \dots, n.$$

We obtain

**Theorem 2.1** ([10]). *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Then all natural  $L$ -metrics  $\zeta$  on  $M$  derived from  $g$  are given by*

$$(2.1) \quad \zeta(u; X, Y) = \sum_{\alpha, \beta=1}^n \varphi_{\alpha\beta}(w_{\rho\sigma}) \omega^\alpha(X) \omega^\beta(Y),$$

where  $\{\omega^1, \omega^2, \dots, \omega^n\}$  are the dual frame to a linear frame  $u = \{u_1, u_2, \dots, u_n\}$  and  $\varphi_{\alpha\beta}$ ,  $\alpha, \beta = 1, 2, \dots, n$ , are arbitrary smooth functions of  $n(n+1)/2$  variables  $w_{\rho\sigma} = g(u_\rho, u_\sigma)$ ,  $1 \leq \rho \leq \sigma \leq n$ .

**Remark.** Let  $(\theta^1, \theta^2, \dots, \theta^n)$  be the frame of  $(M, g)$  metrically equivalent to a basis  $u = (u_1, u_2, \dots, u_n)$  for  $M_x$ , which is given by  $\theta^\alpha(u_\lambda) = g_x(u_\alpha, u_\lambda)$  for all indices  $\alpha, \lambda = 1, 2, \dots, n$ . Then, each  $\omega^\alpha$  of the frame  $(\omega^1, \omega^2, \dots, \omega^n)$  dual to  $u$  is a linear combination of  $\theta^\alpha$ 's whose factors are smooth functions of variables  $w_{\rho\sigma} = g(u_\rho, u_\sigma)$ ,  $\rho, \sigma = 1, 2, \dots, n$ . Because the factors  $\varphi_{\alpha\beta}(w_{\rho\sigma})$  in (2.1) are functions of the variables  $w_{\rho\sigma}$ , if we take  $(\omega^1, \omega^2, \dots, \omega^n)$  in Theorem 2.1 as the

frame metrically equivalent to  $u$ , then all natural L-metrics  $\zeta$  on  $M$  derived from  $g$  are still written in the form (2.1).

For a given Riemannian metric  $g$  on  $M$ , we define the *classical lifts of L-metrics from  $M$  to  $LM$  with respect to  $g$*  as symmetric  $(0, 2)$ -tensor fields on  $LM$  which are constructed as follows:

(a) Let  $\zeta$  be a symmetric L-metric and  $(\zeta^{\lambda\mu})$ ,  $1 \leq \lambda \leq \mu \leq n$ , a family of arbitrary L-metrics. The *diagonal lift*  $\xi^{d,g}$  of the family  $\xi = (\zeta, \zeta^{\lambda\mu})$  with respect to  $g$  is defined by

$$\begin{aligned} \xi_{(x,u)}^{d,g}(X^h, Y^h) &= \zeta_x(u; X, Y), \\ \xi_{(x,u)}^{d,g}(X^h, Y^{v,\mu}) &= 0, & \mu = 1, 2, \dots, n, \\ \xi_{(x,u)}^{d,g}(X^{v,\lambda}, Y^{v,\mu}) &= \zeta_x^{\lambda\mu}(u; X, Y), & 1 \leq \lambda \leq \mu \leq n, \end{aligned}$$

for all  $X, Y \in M_x$ .

(b) The *horizontal lift*  $\xi^{h,g}$  of an  $n$ -tuple  $\xi = (\zeta^\mu)$  of L-metrics with respect to  $g$  is defined by

$$\begin{aligned} \xi_{(x,u)}^{h,g}(X^h, Y^h) &= 0, \\ \xi_{(x,u)}^{h,g}(X^h, Y^{v,\mu}) &= \zeta_x^\mu(u; Y, X), & \mu = 1, 2, \dots, n, \\ \xi_{(x,u)}^{h,g}(X^{v,\lambda}, Y^{v,\mu}) &= 0, & \lambda, \mu = 1, 2, \dots, n, \end{aligned}$$

for all  $X, Y \in M_x$ .

If we take  $\zeta = g$  and  $\zeta^{\lambda\mu} = g \delta^{\lambda\mu}$  in (a), and  $\zeta^\mu = g$  in (b), then  $\xi^{d,g}$  and  $\xi^{h,g}$  are just the diagonal lift  $g^d$  and the horizontal lift  $g^h$ , respectively. Also, if we take  $\zeta = g$  and  $\zeta^{\lambda\mu} = 0$  in (a), then  $\xi^{d,g}$  is the vertical lift  $g^v$ .

Thus we have all metrics on  $LM$  which come from a second order natural transformation of a given Riemannian metric on  $M$ :

**Theorem 2.2** ([10]). *Let  $g$  be a Riemannian metric on an  $n$ -dimensional smooth manifold  $M$ ,  $n \geq 2$ , and let  $G$  be a (possibly degenerate) pseudo-Riemannian metric on the linear frame bundle  $LM$  which comes from a second order natural transformation of  $g$ . Then there are families  $\xi_1 = (\zeta, \zeta^{\lambda\mu})$  and  $\xi_2 = (\zeta^\nu)$  of natural L-metrics derived from  $g$ , where  $1 \leq \lambda \leq \mu \leq n$ ,  $\nu = 1, 2, \dots, n$  and  $\zeta$  is symmetric, such that*

$$G = \xi_1^{d,g} + \xi_2^{h,g}.$$

Moreover, all natural L-metrics derived from  $g$  are given by Theorem 2.1.

The family of all natural metrics  $G$  on  $LM$  over an  $n$ -dimensional Riemannian manifold  $(M, g)$  depends on  $n(n^3 + 3n^2 + n + 1)/2$  arbitrary functions of  $n(n + 1)/2$  variables.

For any point  $(x, u) \in LM$ , if we take the frame  $(\omega^1, \omega^2, \dots, \omega^n)$  in Theorem 2.1 as the frame metrically equivalent to  $u = (u_1, u_2, \dots, u_n)$ , then the metrics  $G$  in

Theorem 2.2 is expressed in the following form:

$$(2.2) \quad \begin{aligned} G_{(x,u)}(X^h, Y^h) &= \sum_{\alpha,\beta=1}^n \varphi_{\alpha\beta}(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y), \\ G_{(x,u)}(X^h, Y^{v,\mu}) &= \sum_{\alpha,\beta=1}^n \varphi_{\alpha\beta}^\mu(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y), \\ G_{(x,u)}(X^{v,\lambda}, Y^{v,\mu}) &= \sum_{\alpha,\beta=1}^n \varphi_{\alpha\beta}^{\lambda\mu}(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y) \end{aligned}$$

for all  $X, Y \in M_x$ , where  $\varphi_{\alpha\beta}$ ,  $\varphi_{\alpha\beta}^\mu$  and  $\varphi_{\alpha\beta}^{\lambda\mu}$ ,  $\alpha, \beta, \lambda, \mu = 1, 2, \dots, n$ , are arbitrary smooth functions of  $n(n+1)/2$  variables  $w_{\rho\sigma} = g(u_\rho, u_\sigma)$ ,  $1 \leq \rho \leq \sigma \leq n$ .

In the following we shall call  $G$  a  $g$ -natural metric on  $LM$ .

### 3. INVARIANCE OF $g$ -NATURAL METRICS

Let  $\phi$  be a (local) transformation of a manifold  $M$ . Then we define a transformation  $\Phi$  of  $LM$  by

$$(3.1) \quad \Phi(x, u) = (\phi x, \phi_{*x} u_1, \phi_{*x} u_2, \dots, \phi_{*x} u_n)$$

for all  $(x, u) = (x, u_1, u_2, \dots, u_n) \in LM$ .

**Proposition 3.1.** *Let  $\phi$  be a (local) affine transformation of a manifold  $M$  with an affine connection  $\nabla$  and let  $\Phi$  be the lift of  $\phi$  to  $LM$  defined as above. Then we have*

$$(3.2) \quad \Phi_*(X^h) = (\phi_* X)^h, \quad \Phi_*(X^{v,\lambda}) = (\phi_* X)^{v,\lambda}$$

for all  $X \in \mathfrak{X}(M)$  and all indices  $\lambda = 1, 2, \dots, n$ . In particular, for the canonical vertical vector fields, we have

$$(3.3) \quad \Phi_*(U_\lambda^\mu) = U_\lambda^\mu$$

for all indices  $\lambda, \mu = 1, 2, \dots, n$ .

**Proof.** We use the formula  $p \circ \Phi = \phi \circ p$ . For all vector fields  $X$  and functions  $f$  on  $M$  we calculate at  $(x, u) \in LM$ :

$$\begin{aligned} (p_{*\Phi(x,u)}(\Phi_{*(x,u)}(X_{(x,u)}^h)))f &= X_{(x,u)}^h(f \circ p \circ \Phi) = X_{(x,u)}^h(f \circ \phi \circ p) \\ &= (p_{*(x,u)}(X_{(x,u)}^h))(f \circ \phi) = X_x(f \circ \phi) = (\phi_{*x} X_x)f. \end{aligned}$$

Since  $\Phi$  preserves the horizontal distribution, we have

$$\Phi_{*(x,u)}(X_{(x,u)}^h) = (\phi_{*x} X_x)_{\Phi(x,u)}^h.$$

Next, using the formula  $\iota_\nu \omega \circ \Phi = \iota_\nu(\phi^* \omega)$  for all one-forms  $\omega$  on  $M$  and all indices  $\nu = 1, 2, \dots, n$ , we calculate at  $(x, u) \in LM$  for all  $X \in \mathfrak{X}(M)$ :

$$\begin{aligned} (\Phi_{*(x,u)}(X_{(x,u)}^{v,\mu}))(\iota_\nu \omega) &= (X_{(x,u)}^{v,\mu})(\iota_\nu \omega \circ \Phi) = (X_{(x,u)}^{v,\mu})(\iota_\nu(\phi^* \omega)) \\ &= (\phi^* \omega)_x(X_x) \delta_\nu^\mu = \omega_{\phi(x)}(\phi_{*x} X_x) \delta_\nu^\mu = (\phi_{*x} X_x)^{v,\mu}(\iota_\nu \omega). \end{aligned}$$

Since  $\Phi$  preserves the vertical distribution, we have  $\Phi_{*(x,u)}(X_{(x,u)}^{v,\mu}) = (\phi_{*x}X)_{\Phi(x,u)}^{v,\mu}$ . In particular, we have  $\Phi_{*(x,u)}(u_\lambda^{v,\mu}) = (\phi_{*x}u_\lambda)^{v,\mu}$ , that is, by (1.2), we have (3.3).  $\square$

**Theorem 3.2.** *Let  $\phi$  be a (local) isometry of a Riemannian manifold  $(M, g)$ . Then every  $g$ -natural metric  $G$  on the linear frame bundle  $LM$  over  $(M, g)$  is invariant by the lift  $\Phi$  of  $\phi$ . In other words,  $\Phi$  is a (local) isometry of  $(LM, G)$  whose projection on  $(M, g)$  is  $\phi$ .*

**Proof.** For any basis  $u = (u_1, u_2, \dots, u_n)$  for  $M_x$ ,  $x \in M$ , we assume that the frame  $(\omega^1, \omega^2, \dots, \omega^n)$  in Theorem 2.2 is metrically equivalent to  $u$ . We abbreviate  $(\phi_{*x}u_1, \phi_{*x}u_2, \dots, \phi_{*x}u_n)$  as  $\phi_{*x}u$ . Let  $X$  and  $Y$  be vectors from  $M_x$ . Then, by Proposition 3.1 and the first formula of (2.2), we have at any point  $(x, u) \in LM$  that

$$\begin{aligned} (\Phi^*G)_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h) &= G_{\Phi(x,u)}(\Phi_{*(x,u)}X_{(x,u)}^h, \Phi_{*(x,u)}Y_{(x,u)}^h) \\ &= G_{(\phi_x, \phi_{*x}u)}((\phi_{*x}X)_{(\phi_x, \phi_{*x}u)}^h, (\phi_{*x}Y)_{(\phi_x, \phi_{*x}u)}^h) \\ &= \sum_{\alpha, \beta=1}^n \varphi_{\alpha\beta}(g_{\phi_x}(\phi_{*x}u_\rho, \phi_{*x}u_\sigma))g_{\phi_x}(\phi_{*x}u_\alpha, \phi_{*x}X)g_{\phi_x}(\phi_{*x}u_\beta, \phi_{*x}Y). \end{aligned}$$

Now, since  $\phi$  is a (local) isometry of  $(M, g)$ , the right-hand side of this formula is just  $G_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h)$ . Hence we have

$$(\Phi^*G)_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h) = G_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h)$$

for all  $X, Y \in M_x$ .

The rest of the assertion is proved by similar calculations.  $\square$

Let us remark that the natural projection  $p$  of  $(LM, G)$  onto  $(M, g)$  is *not* a Riemannian submersion, in general.

#### 4. THE ORTHONORMAL FRAME BUNDLE

The orthonormal frame bundle

$$OM = \{(x, u) \in LM \mid g_x(u_\lambda, u_\mu) = \delta_{\lambda\mu}, \lambda, \mu = 1, 2, \dots, n\}$$

over a Riemannian manifold  $(M, g)$  is an  $n(n+1)/2$ -dimensional subbundle of  $LM$  with the structure group  $O(n)$ .

The vector fields  $T_{\lambda\mu}$  defined on  $LM$  by

$$(4.1) \quad T_{\lambda\mu} = U_\lambda^\mu - U_\mu^\lambda$$

for all indices  $\lambda, \mu = 1, 2, \dots, n$  are tangent to  $OM$  at every point  $(x, u) \in OM$ . Here  $T_{\lambda\mu}$  is skew-symmetric with respect to all indices  $\lambda, \mu = 1, 2, \dots, n$ . In particular  $T_{\lambda\lambda} = 0$  for all  $\lambda = 1, 2, \dots, n$ . At every point  $(x, u) \in OM$ , a collection  $\{T_{\lambda\mu}(x, u) \mid 1 \leq \lambda < \mu \leq n\}$  forms a basis for the vertical space  $V_{(x,u)} \cap (OM)_{(x,u)}$ .

**Definition 4.1.** Consider a smooth Riemannian manifold  $(M, g)$ . A  $g$ -natural metric on the orthonormal frame bundle  $OM$  over  $(M, g)$  is restriction  $\tilde{G}$  of any  $g$ -natural metric  $G$  given by (2.2) on the linear frame bundle  $LM$  to the submanifold  $OM \subset LM$ .

We say that a natural L-metric on  $M$  derived from  $g$  is of constant type if all factors  $\varphi_{\alpha\beta}(w_{\rho\sigma})$  in (2.1) are constants for all indices  $\alpha, \beta = 1, 2, \dots, n$ .

**Theorem 4.2.** Let  $g$  be a Riemannian metric on an  $n$ -dimensional smooth manifold  $M$ ,  $n \geq 2$ , and let  $\tilde{G}$  be a  $g$ -natural metrics on the orthonormal frame bundle  $OM$ . Then there are families  $\xi_1 = (\zeta, \zeta^{\lambda\mu})$  and  $\xi_2 = (\zeta^\nu)$  of natural L-metrics of constant type derived from  $g$ , where  $1 \leq \lambda \leq \mu \leq n$ ,  $\nu = 1, 2, \dots, n$  and  $\zeta$  is symmetric, such that

$$\tilde{G} = \xi_1^{d,g} + \xi_2^{h,g}.$$

Equivalently, we have

**Corollary 4.3.** The  $g$ -natural metrics  $\tilde{G}$  in Theorem 4.2 can be expressed at any point  $(x, u) \in OM$  in the following form:

$$\begin{aligned} \tilde{G}_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h) &= \sum_{\alpha,\beta=1}^n c_{\alpha\beta} g_x(u_\alpha, X) g_x(u_\beta, Y), \\ (4.2) \quad \tilde{G}_{(x,u)}(X_{(x,u)}^h, T_{\lambda\mu}(x, u)) &= \sum_{\alpha=1}^n (c_{\alpha\lambda}^\mu - c_{\alpha\mu}^\lambda) g_x(u_\alpha, X), \\ \tilde{G}_{(x,u)}(T_{\lambda\mu}(x, u), T_{\nu\omega}(x, u)) &= c_{\lambda\nu}^{\mu\omega} - c_{\lambda\omega}^{\mu\nu} - c_{\mu\nu}^{\lambda\omega} + c_{\mu\omega}^{\lambda\nu} \end{aligned}$$

for all  $X, Y \in M_x$  and all indices  $\lambda, \mu, \nu, \omega = 1, 2, \dots, n$ , where  $c_{\alpha\beta}$ ,  $c_{\alpha\beta}^\mu$  and  $c_{\alpha\beta}^{\lambda\mu}$ ,  $\alpha, \beta, \lambda, \mu = 1, 2, \dots, n$ , are constants.

**Proof.** Let  $(x, u) = (x, u_1, u_2, \dots, u_n)$  be any point of  $OM$ . Then we have  $w_{\rho\sigma} = g_x(u_\rho, u_\sigma) = \delta_{\rho\sigma}$  for all indices  $\rho, \sigma = 1, 2, \dots, n$ , where  $\delta_{\rho\sigma}$ 's are the Kronecker's deltas. Hence all factors  $\varphi_{\alpha\beta}$ ,  $\varphi_{\alpha\beta}^\mu$  and  $\varphi_{\alpha\beta}^{\lambda\mu}$ ,  $\alpha, \beta, \lambda, \mu = 1, 2, \dots, n$ , are arbitrary constants. By (4.1) and (1.2), we have (4.2).  $\square$

Let  $\phi$  be a (local) transformation. We denote by  $\tilde{\Phi}$  restriction of  $\Phi$  defined by (3.1) to  $OM$ . Then, using the fact that  $\tilde{\Phi}_{*(x,u)}\mathbf{X} = \Phi_{*(x,u)}\mathbf{X}$  for all vectors  $\mathbf{X} \in (OM)_{(x,u)}$ , from (3.2), (3.1) and (4.1) we have

$$\begin{aligned} \tilde{\Phi}_{*(x,u)}(X_{(x,u)}^h) &= (\phi_{*x}X)_{\tilde{\Phi}(x,u)}^h, \\ (4.3) \quad \tilde{\Phi}_{*(x,u)}(T_{\lambda\mu}(x, u)) &= T_{\lambda\mu}(\tilde{\Phi}(x, u)) \end{aligned}$$

at  $(x, u) \in OM$  for all vectors  $X \in M_x$  and all indices  $\lambda, \mu = 1, 2, \dots, n$ .

**Theorem 4.4.** Let  $\phi$  be a (local) isometry of a Riemannian manifold  $(M, g)$ . Then every  $g$ -natural metric  $\tilde{G}$  on the orthonormal frame bundle  $OM$  over  $(M, g)$  is invariant by the lift  $\tilde{\Phi}$  of  $\phi$ . In other words,  $\tilde{\Phi}$  is a (local) isometry of  $(OM, \tilde{G})$  whose projection on  $(M, g)$  is  $\phi$ .

**Proof.** We use Corollary 4.3. Let  $(x, u)$  be any point of  $OM$ . By the first formula of (4.3), we have

$$\begin{aligned} (\tilde{\Phi}^* \tilde{G})_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h) &= \tilde{G}_{\tilde{\Phi}(x,u)}(\tilde{\Phi}_{*(x,u)}(X_{(x,u)}^h), \tilde{\Phi}_{*(x,u)}(Y_{(x,u)}^h)) \\ &= \sum_{\alpha, \beta=1}^n c_{\alpha\beta} g_{\phi x}(\phi_{*x} u_\alpha, \phi_{*x} X) g_{\phi x}(\phi_{*x} u_\beta, \phi_{*x} Y) \\ &= \tilde{G}_{(x,u)}(X_{(x,u)}^h, Y_{(x,u)}^h) \end{aligned}$$

for all  $X, Y \in M_x$ . Next, by (4.3), we have

$$\begin{aligned} (\tilde{\Phi}^* \tilde{G})_{(x,u)}(X_{(x,u)}^h, T_{\lambda\mu}(x, u)) &= \tilde{G}_{\tilde{\Phi}(x,u)}(\tilde{\Phi}_{*(x,u)}(X_{(x,u)}^h), \tilde{\Phi}_{*(x,u)}(T_{\lambda\mu}(x, u))) \\ &= \sum_{\alpha=1}^n (c_{\alpha\lambda}^\mu - c_{\alpha\mu}^\lambda) g_{\phi x}(\phi_{*x} u_\alpha, \phi_{*x} X) \\ &= \tilde{G}_{(x,u)}(X_{(x,u)}^h, T_{\lambda\mu}(x, u)) \end{aligned}$$

for all  $X \in M_x$  and all indices  $\lambda, \mu = 1, 2, \dots, n$ . Finally, by the second formula of (4.3), we have easily that

$$(\tilde{\Phi}^* \tilde{G})_{(x,u)}(T_{\lambda\mu}(x, u), T_{\nu\omega}(x, u)) = \tilde{G}_{(x,u)}(T_{\lambda\mu}(x, u), T_{\nu\omega}(x, u))$$

for all  $\lambda, \mu, \nu, \omega = 1, 2, \dots, n$ . □

**Corollary 4.5.** *If  $(M, g)$  is a space of constant sectional curvature, then its orthonormal frame bundle  $(M, \tilde{G})$  with a  $g$ -natural metric is always locally homogeneous.*

## REFERENCES

- [1] Abbassi, M. T. K., *Note on the classification theorems of  $g$ -natural metrics on the tangent bundle of a Riemannian manifolds  $(M, g)$* , Comment. Math. Univ. Carolin. **45** (2004), 591–596.
- [2] Cordero, L. A., de León, M., *Lifts of tensor fields to the frame bundle*, Rend. Circ. Mat. Palermo **32** (1983), 236–271.
- [3] Cordero, L. A., de León, M., *On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold*, J. Math. Pures Appl. **65** (1986), 81–91.
- [4] Cordero, L. A., Dodson, C. T. J., de León, M., *Differential Geometry of Frame Bundles*, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1989.
- [5] Jensen, G., *The scalar curvature of left invariant Riemannian metrics*, Indiana Univ. Math. J. **20** (1971), 1125–1143.
- [6] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1993.
- [7] Kowalski, O., Sekizawa, M., *Invariance of  $g$ -natural metrics on tangent bundles*, to appear in Proceedings of 10th International Conference on Differential Geometry and Its Applications, World Scientific.
- [8] Kowalski, O., Sekizawa, M., *On the geometry of orthonormal frame bundles*, to appear in Math. Nachr.
- [9] Kowalski, O., Sekizawa, M., *On the geometry of orthonormal frame bundles II*, to appear in Ann. Global Anal. Geom.



- [10] Kowalski, O., Sekizawa, M., *Natural transformations of Riemannian metrics on manifolds to metrics on linear frame bundles—a classification*, Differential Geometry and its Applications, Proceeding of the Conference, August 24–30, 1986, Brno, Czechoslovakia, D. Reidel Publ. Comp., pp. 149–178, 1987.
- [11] Kowalski, O., Sekizawa, M., *On curvatures of linear frame bundles with naturally lifted metrics*, Rend. Sem. Mat. Univ. Politec. Torino **63** (2005), 283–295.
- [12] Krupka, D., *Elementary theory of differential invariants*, Arch. Math. (Brno) **4** (1978), 207–214.
- [13] Krupka, D., *Differential invariants*, Lecture Notes, Faculty of Science, Purkyně University, Brno (1979).
- [14] Krupka, D., Janyška, J., *Lectures on Differential Invariants*, University J. E. Purkyně in Brno, 1990.
- [15] Krupka, D., V. Mikolášová, *On the uniqueness of some differential invariants:  $d, [ , ], \nabla$* , Czechoslovak Math. J. **34** (1984), 588–597.
- [16] Mok, K. P., *On the differential geometry of frame bundles of Riemannian manifolds*, J. Reine Angew. Math. **302** (1978), 16–31.
- [17] O’Neill, B., *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
- [18] Zou, X., *A new type of homogeneous spaces and the Einstein metrics on  $O(n + 1)$* , Nanjing Univ. J. Math. Biquarterly **23** (2006), 70–78.

FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY IN PRAGUE

SOKOLOVSKÁ 83, 186 75 PRAHA 8, CZECH REPUBLIC

*E-mail:* kowalski@karlin.mff.cuni.cz

DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY

KOGANEI-SHI NUKUIKITA-MACHI 4-1-1, TOKYO 184-8501, JAPAN

*E-mail:* sekizawa@u-gakugei.ac.jp