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INVARIANCE OF THE GLOBAL MONODROMIES IN FAMILIES OF NONDEGENERATE POLYNOMIALS IN TWO VARIABLES

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Abstract

We are interested in a global version of Lê-Ramanujam μ-constant theorem for polynomials. We consider an analytic family $\{f_s\}$, $s \in [0,1]$, of complex polynomials in two variables, that are Newton non-degenerate. We suppose that the Euler characteristic of a generic fiber of f_s is constant, then we show that the global monodromy fibrations of f_s are all isomorphic, and that the degree of f_s is constant (up to an algebraic automorphism of $\mathbf{C}^2$).

1. Introduction

Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a complex polynomial function. It is well-known that there exists a (minimal) finite set $B(f)$ in $\mathbf{C}$, called the *bifurcation set of* f, such that the restriction:

$$f : \mathbf{C}^2 \backslash f^{-1}(B(f)) \to \mathbf{C} \backslash B(f)$$

is a C^∞-locally trivial fibration (see, for example, [28], [29], [17], [26], [7], [11]). The bifurcation set $B(f)$ contains the set $\Sigma_0(f)$ of critical values of f, but in general it is bigger.

The above fibration permits us to introduce the *global monodromy fibration* of f. Namely, for $r > \max\{|c| \mid c \in B(f)\}$ and $\mathbf{S}_r^1 := \{c \in \mathbf{C} \mid |c| = r\}$, this is the restriction

$$f : f^{-1}(\mathbf{S}_r^1) \to \mathbf{S}_r^1.$$

If $c \in \mathbf{S}_r^1$ then by translating the fiber $f^{-1}(c)$ along the circle $\mathbf{S}_r^1$ we obtain a homeomorphism of $f^{-1}(c)$ onto itself, and thus isomorphisms

$$m_q(f) : H_q(f^{-1}(c), \mathbf{Z}) \to H_q(f^{-1}(c), \mathbf{Z}), \quad q = 0, 1,$$

which we call the *global monodromy operators* of f.

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Let $\{f_s\}$, $s \in [0,1]$, be a family of complex polynomials in two variables, whose coefficients are analytic functions in s. We will be interested in families such that the Euler characteristic $\chi(f_s)$ of a generic fiber of f_s is constant. These families are interesting in the view of μ-constant type theorem, see [8], [10], [3], [5], [27]. We ask if for such families, the global monodromy fibrations are isomorphic. In general, the answer is negative, as the following example shows us:

Example 1.1. Let $f_s(x,y) = sx^2y^2 + xy$. Then $\chi(f_s) = 0$ for all s but the generic fibers of f_0 and f_s, $s \neq 0$, are isomorphic, respectively, to $\mathbf{C}^* := \mathbf{C}\setminus\{0\}$ and $\mathbf{C}^* \sqcup \mathbf{C}^*$ (disjoint union).

We shall prove that for the class of *Newton non-degenerate* polynomials, introduced in [14], the answer of our question is positive.

We will recall some basic facts about Newton polygons, see [14], [19], [25]. Let $f = \sum_{(p,q)\in \mathbf{Z}^2_{\geq 0}} a_{p,q}x^py^q$ be a given polynomial. We denote $\operatorname{supp}(f) = \{(p,q) \mid a_{p,q} \neq 0\}$, by abuse $\operatorname{supp}(f)$ will also denote the set of monomials $\{x^py^q \mid (p,q) \in \operatorname{supp}(f)\}$. The *Newton polygon* $\Gamma_-(f)$ is by definition the convex hull of the set $\{(0,0)\} \cup \operatorname{supp}(f)$. We denote $\Gamma(f)$ to be the union of closed faces of $\Gamma_-(f)$ which do not contain $(0,0)$. Zero dimensional faces are vertices of the polygon $\Gamma_-(f)$ and one dimensional faces are its edges. For a face γ, let $f_\gamma = \sum_{(p,q)\in\gamma} a_{p,q}x^py^q$. The polynomial f is (*Newton*) *non-degenerate* if for all faces γ of $\Gamma(f)$ the system

$$\frac{\partial f_\gamma}{\partial x}(x,y) = 0 \quad \text{and} \quad \frac{\partial f_\gamma}{\partial y}(x,y) = 0$$

has no solution in $\mathbf{C}^* \times \mathbf{C}^*$. Note that, by the definition, if $\dim \Gamma_-(f) = 1$ then the polynomial f is non-degenerate.

Our main result is the following μ-constant type theorem:

THEOREM 1.2. *Let* $\{f_s\}$, $s \in [0,1]$, *be a family of non-degenerate polynomials in two complex variables. If one of the two following conditions hold*:

(i) $\dim \Gamma_-(f_s) = 1$ *and* $\Gamma_-(f_s)$ *is constant for all* $s \in [0,1]$;

(ii) $\dim \Gamma_-(f_s) = 2$ *for all* $s \in (0,1]$, *and the Euler characteristic* $\chi(f_s)$ *is constant for all* $s \in [0,1]$;

then the global monodromy fibrations of f_s *are isomorphic*.

Remark 1.3. (i) In fact, in Section 3, we shall prove a stronger form of Theorem 1.2(i): *Assume that* $\dim \Gamma_-(f_s) = 1$ *for all* $s \in [0,1]$. *Then* $\Gamma_-(f_s)$ *is constant if and only if the global monodromy fibrations of* f_s *are isomorphic*.

(ii) For non-degenerate polynomial functions with constant Newton polygon, Theorem 1.2 was obtained in [27], for *any* number of variables. However, the hypothesis that the Newton polygon $\Gamma_-(f_s)$ of f_s does not change is a *non-topological hypothesis*. What is new here is the improvement in the result when

$\Gamma_-(f_s)$ is not constant, and the method of proof is a thorough analysis of the change of the Newton polygon $\Gamma_-(f_s)$.

Example 1.4. Let us consider $f_s(x, y) := sx^4 + x^2y$. An easy calculation shows that the polynomial f_s is non-degenerate and $\chi(f_s) = 0$ for all $s \in [0, 1]$. By Theorem 1.2, the global monodromy fibrations of f_0 and f_1 are isomorphic. Namely, the following diagram commutes:

$$\begin{array}{ccc} f_0^{-1}(\mathbf{S}_r^1) & \xrightarrow{f_0} & \mathbf{S}_r^1 \\ \Phi \downarrow & & \downarrow \mathrm{id} \\ f_1^{-1}(\mathbf{S}_r^1) & \xrightarrow{f_1} & \mathbf{S}_r^1 \end{array}$$

where $r > 0$ and $\Phi(x, y) := (x, y - x^2)$ is a homeomorphism. We notice that the Newton polygon of f_s is not constant and that f_s has non-isolated critical points, $\Sigma_0(f_s) = \{0\}$. Moreover, it follows from Proposition 2.2 below that $B(f_s) = \{0\}$ for all $s \in [0, 1]$.

As a corollary of Theorem 1.2, we obtain the following result (see also [10, Theorem 1.3]).

Corollary 1.5. *With the hypotheses of Theorem* 1.2. *Then the global monodromy operators of* f_0 *and* f_1 *are conjugate.*

We are now interested in the constancy of the degree. It is well known that the degree of a polynomial depends on the coordinate system of $\mathbf{C}^2$. Also in families of non-degenerate polynomial functions with constant Euler characteristic it can happen that the degree changes (see Example 1.4). On the other hand, as a by-product of Theorem 1.2, we obtain the following result (see also [4, Theorem 3]):

Corollary 1.6. *With the hypotheses of Theorem* 1.2. *Then the family* f_s *is of constant degree up to an algebraic automorphism of* $\mathbf{C}^2$.

Remark 1.7. In the above results, the polynomials f_s can have *non-isolated* singularities. Moreover, the Newton polygon $\Gamma_-(f_0)$ may be of one dimension.

The paper is organized as follows. In Section 2 we recall some useful notations and results. The proofs are given in Section 3.

2. Tools

2.1. Fibrations. We will denote $\mathbf{B}_R^2 := \{(x, y) \in \mathbf{C}^2 \mid \|(x, y)\| < R\}$, $\mathbf{S}_R^3 := \{(x, y) \in \mathbf{C}^2 \mid \|(x, y)\| = R\}$ and $D_r := \{c \in \mathbf{C} \mid |c| < r\}$.

Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial function. Let us choose $r > 0$ such that the bifurcation set $B(f)$ of f is contained in the open disc D_r. The following lemma is a consequence of transversality properties.

LEMMA 2.1. *Let R_0 be a positive number such that for all $c \in \mathbf{S}_r^1$ and for all $R \geq R_0$, the fiber $f^{-1}(c)$ intersects the sphere $\mathbf{S}_R^3$ transversally. Then the global monodromy fibration $f : f^{-1}(\mathbf{S}_r^1) \to \mathbf{S}_r^1$ is isomorphic to the fibration $f : f^{-1}(\mathbf{S}_r^1) \cap \mathbf{B}_R^2 \to \mathbf{S}_r^1$ for all $R \geq R_0$.*

Proof. See [10] or [27, Lemma 3.1]. □

2.2. Bifurcation set. We recall the result of Némethi A. and Zaharia A. [19] on how to estimate the bifurcation set. A polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ is *convenient for the x-axis* if there exists a monomial x^a in $\mathrm{supp}(f)$ $(a > 0)$; f is *convenient for the y-axis* if there exists a monomial y^b in $\mathrm{supp}(f)$ $(b > 0)$; f is *convenient* if it is convenient for the x-axis and the y-axis. Let γ_x and γ_y be the two faces of $\Gamma_-(f)$ that contain the origin. If f is convenient for the x-axis then we set $\mathfrak{C}_x(f) = \emptyset$, otherwise γ_x is not included in the x-axis and we set

$$\mathfrak{C}_x(f) := \left\{ f_{\gamma_x}(x, y) \mid (x, y) \in \mathbf{C}^* \times \mathbf{C}^* \text{ and } \frac{\partial f_{\gamma_x}}{\partial x}(x, y) = \frac{\partial f_{\gamma_x}}{\partial y}(x, y) = 0 \right\}.$$

In a similar way we define $\mathfrak{C}_y(f)$. Let $\Sigma_\infty(f) := \mathfrak{C}_x(f) \cup \mathfrak{C}_y(f)$.

The following result gives an estimation for the bifurcation set $B(f)$ of f in terms of its Newton boundary at infinity.

PROPOSITION 2.2 [14], [6], [19] (see also, [30], [12], [4]). *Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a non-degenerate polynomial function. Then the following statements hold*

(i) *If f is convenient, then $B(f) = \Sigma_0(f)$.*

(ii) *If f is not convenient, then $B(f) \subset \Sigma_0(f) \cup \Sigma_\infty(f) \cup \{f(0)\}$.*

2.3. Euler characteristic. Let us recall the definition of the Newton number ν, see [14]. Let T be a compact polytope $T \subset \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. The *Newton number* of T is defined as follows

$$\nu(T) := 2S - a - b + 1,$$

where S is the area of T, a is the length of the intersection of T with the x-axis, and b is the length of the intersection of T with the y-axis.

The following formula gives an explicit expression for the Euler characteristics $\chi(f)$ in terms of the Newton number of $\Gamma_-(f)$ (see [2], [13], [21], [22], [23], [24], [25], [1]):

PROPOSITION 2.3. *Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a complex polynomial function. If f is non-degenerate then*

$$\chi(f) = 1 - \nu(\Gamma_-(f)).$$

2.4. Additivity and positivity. We need a variation of the Newton number ν, see [4]. Let T be a compact polytope whose vertices are in $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. We define

$$\tau(T) = \nu(T) - 1.$$

It is clear that τ is additive: $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2) - \tau(T_1 \cap T_2)$, and in particular if $T_1 \cap T_2$ has null area then $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2)$. This formula enables us to argue on triangles only (after a triangulation of T).

We denote $\mathfrak{A}$ to be the set of triangles T such that T has two edges contained in the x-axis and the y-axis, and the length of one of these edges is 1. Then $\tau(T) = -1$ for every triangle $T \in \mathfrak{A}$. Moreover, we have the following facts

- $\nu(T) \geq 0$; and
- $\nu(T) = 0$ if and only if $T \in \mathfrak{A}$.

2.5. Families of polytopes. We consider a family $\{f_s\}$, $s \in [0, 1]$, of complex polynomials in two variables. We will always assume that the only critical parameter is $s = 0$. We will say that a monomial $x^p y^q$ *disappears* if $(p, q) \in \operatorname{supp}(f_s) \backslash \operatorname{supp}(f_0)$ for $s \neq 0$. By extension a triangle of $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ disappears if one of its vertices does. We triangulate $\Gamma(f_s)$ such that a finite number of triangles T disappear (see Figure 1, on pictures of the Newton polygon, a plain circle is drawn for a monomial that does not disappear and an empty circle for monomials that disappear).

We have the following simple results (see also [4, Lemma 9]).

LEMMA 2.4. *With the hypotheses of Theorem* 1.2(ii). *Suppose that there exists a triangulation of* $\Gamma(f_s)$, $s \neq 0$, *with a triangle* $T \in \mathfrak{A}$ *that disappears. Then either* $\deg_x(f_s) = 1$ *or* $\deg_y(f_s) = 1$ *for all* $s \in [0, 1]$, *where* $\deg_x(f_s)$ (*resp.*, $\deg_y(f_s)$) *is the degree of* f *in* x (*resp.*, y).

Proof. By assumption, it is not hard to see that $\Gamma(f_s)$ coincides with T for $s \in (0, 1]$. Then either $\deg_x(f_s) = 1$ or $\deg_y(f_s) = 1$ for $s \in (0, 1]$. Moreover, $\chi(f_s) = -\tau(T) = 1$. As the Euler characteristic $\chi(f_s)$ is constant, we must have either $\deg_x(f_0) = 1$ or $\deg_y(f_0) = 1$. $\square$

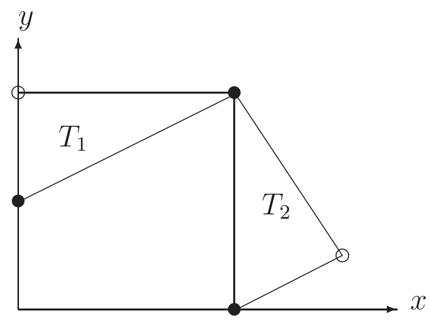

FIGURE 1. Triangles that disappear.

LEMMA 2.5. *With the hypotheses of Theorem* 1.2(ii). *Let* $T \notin \mathfrak{A}$ *be a triangle that disappears then* $\tau(T) = 0$.

Proof. The proof is analogous to that of [4, Lemma 9]. In fact, we suppose that $\tau(T) > 0$. By the additivity and positivity of $\tau(T)$ we have for $s \in (0, 1]$:

$$\nu(\Gamma_-(f_s)) \geq \nu(\Gamma_-(f_0)) + \tau(T) > \nu(\Gamma_-(f_0)).$$

By Proposition 2.3, then

$$\chi(f_s) = 1 - \nu(\Gamma_-(f_s)) < 1 - \nu(\Gamma_-(f_0)) = \chi(f_0).$$

This gives a contradiction with $\chi(f_s) = \chi(f_0)$. □

We will widely use the following observation.

LEMMA 2.6. *Under the hypotheses of Theorem* 1.2(ii), *we have*
(i) *A vertex* $x^p y^q$, $p > 0$, $q > 0$, *of* $\Gamma(f_s)$ *cannot disappear*.
(ii) *If a vertex* x^a (*resp.*, y^b) *of* $\Gamma(f_s)$ *disappears, then there exists a monomial* $x^p y$ (*resp.*, xy^q) *of* $\mathrm{supp}(f_s)$.

Proof. We will adapt the proof of [4, Section 3].

(i) We suppose that a vertex $x^p y^q$, $p > 0$, $q > 0$, of $\Gamma(f_s)$ disappears. Let T be a triangle that contains $x^p y^q$. Then T disappears and $T \notin \mathfrak{A}$. By Lemma 2.5, $\tau(T) = 0$. Hence, T has an edge contained in either the x-axis or the y-axis, but not both, and the height of T (with respect to this edge) is 1 (see Figure 2). Then certainly we have $\Gamma(f_s)$ coincides with T for $s \in (0, 1]$, otherwise there exists a region T' that disappears with $\tau(T') > 0$, which contradicts Lemmas 2.4 and 2.5. Now an easy calculation shows that $\chi(f_s) = 0 < \chi(f_0)$ for $s \in (0, 1]$, which is a contradiction.

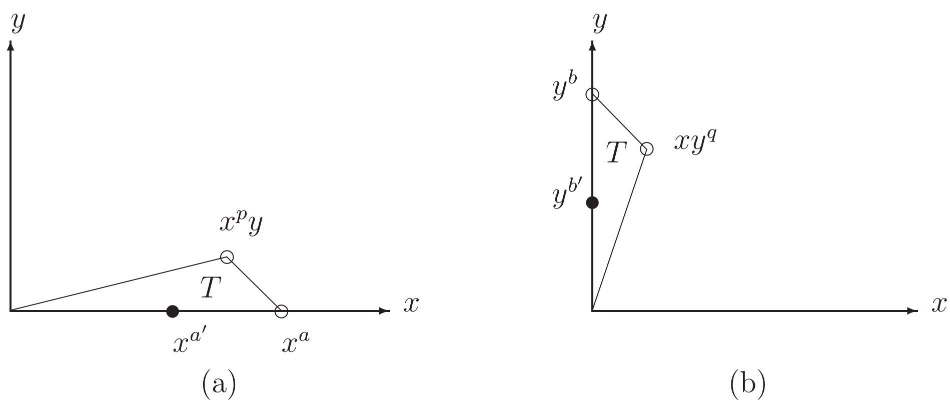

FIGURE 2. Case where a vertex $x^p y^q$ of $\Gamma(f_s)$ disappears: (a) $q = 1$; (b) $p = 1$.

(ii) Suppose that a vertex x^a of $\Gamma(f_s)$ disappears (a similar proof holds for y^b). Let $x^p y^q$, $q > 0$, be a monomial of $\mathrm{supp}(f_s)$ with q minimal. Since $\dim \Gamma(f_s) = 2$ for all $s \in (0, 1]$, such a monomial exists. Then certainly we have $q = 1$, otherwise there exists a region T that disappears with $\tau(T) > 0$, which contradicts Lemmas 2.4 and 2.5 (see Figure 3). □

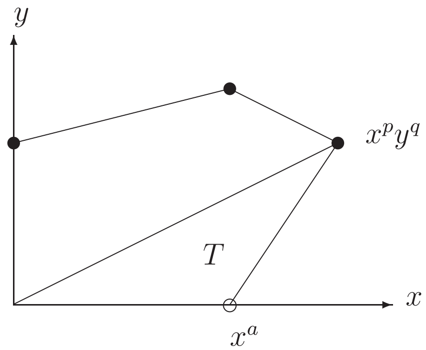

FIGURE 3. Case where a monomial x^a of $\Gamma(f_s)$ disappears: no monomial $x^p y^q$ in $\Gamma(f_s)$ with $p \geq 0$ and $q = 1$.

3. Proofs of the results

Proof of Theorem 1.2. We will always suppose that $s = 0$ is the only problematic parameter. In particular $\Gamma(f_s)$ is constant for all $s \in (0, 1]$.

(i) We assume that $\dim \Gamma_-(f_s) = 1$ for all $s \in [0, 1]$. Then $\Gamma(f_s)$ is a single point. Hence, there exist integers p, q and $d \geq 1$ such that $\Gamma(f_0) = \{(p, q)\}$ and $\Gamma(f_s) = \{(dp, dq)\}$, $s \neq 0$, (see Figure 4). By [27, Theorem 1], the global monodromy fibrations of f_0 and f_s, $s \neq 0$, are isomorphic, respectively, to ones of the polynomials $x^p y^q$ and $x^{dp} y^{dq}$. On the other hand, it is not hard to see that the global monodromy fibrations of the polynomials $x^p y^q$ and $x^{dp} y^{dq}$ are isomorphic if and only if $d = 1$. Therefore, the global monodromy fibrations of f_s are

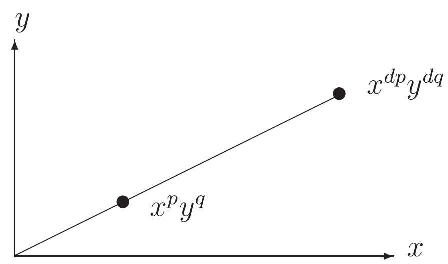

FIGURE 4. Case where $\dim \Gamma_-(f_s) = 1$.

isomorphic if and only if $d = 1$, that means that the Newton polygon $\Gamma_-(f_s)$ is constant.

(ii) Assume that we have proved the following claims:

- There exists a positive constant r such that

$$\Sigma_0(f_s) \cup \Sigma_\infty(f_s) \cup \{f_s(0)\} \subset D_r \quad \text{for all } s \in [0,1].$$

- There exists a positive number R_0 such that for all $R \geq R_0$, for all $s \in [0,1]$, and all $c \in \mathbf{S}^1_r$, the fiber $f_s^{-1}(c)$ intersects the sphere $\mathbf{S}^3_R$ transversally.

Then it follows from Proposition 2.2 that

$$B(f_s) \subset \Sigma_0(f_s) \cup \Sigma_\infty(f_s) \cup \{f_s(0)\} \subset D_r \quad \text{for all } s \in [0,1].$$

Hence, by Lemma 2.1, the global monodromy fibration of the polynomial function f_s:

$$f_s : f_s^{-1}(\mathbf{S}^1_r) \to \mathbf{S}^1_r$$

is isomorphic to the following fibration

$$f_s : f_s^{-1}(\mathbf{S}^1_r) \cap \mathbf{B}^2_R \to \mathbf{S}^1_r.$$

Now, with arguments similar to the ones used in the proof of the classical Lê D. T. and Ramanujam C. P. theorem (see [15], [10, Lemma 2.1] or [3, Lemma 12]), we have that the fibrations $f_s : f_s^{-1}(\mathbf{S}^1_r) \cap \mathbf{B}^2_R \to \mathbf{S}^1_r$, $s \in [0,1]$, are isomorphic. As a conclusion, the global monodromy fibrations of the polynomials f_s are isomorphic. Consequently, the statement (ii) is proved. □

So we are left with proving the above claims. Firstly, we have the following observation.

Remark 3.1. We suppose that a vertex x^a of $\Gamma(f_s)$ disappears. By Lemma 2.6(ii), there exists a monomial $x^p y \in \operatorname{supp}(f_s)$. We choose $x^p y$ in $\operatorname{supp}(f_s)$ with maximal p. We assume that $p = 0$. Then $\deg_y(f_s) = 1$ for $s \in (0,1]$. An easy calculation shows that $\chi(f_s) = 1$. As the Euler characteristic $\chi(f_s)$ is constant, we must have either $\deg_y f_0 = 1$ or $\deg_x f_0 = 1$. Therefore the polynomials f_s are all topologically equivalent. In particular, the conclusion of Theorem 1.2(ii) holds. We exclude this case for the end of the proof.

3.1. Boundedness of affine singularities. The following result says that the set $\Sigma_0(f_s)$ of critical values of f_s is contained in some open disc of radius independent of s.

LEMMA 3.2. *There exists a positive number r such that*

$$\Sigma_0(f_s) \subset D_r \quad \textit{for all } s \in [0,1].$$

Proof. It is enough to prove the lemma on an interval $[0, s_0]$ with a small $s_0 > 0$. Assume the contrary. Then by the Curve Selection Lemma [18], [20], there exist an analytic curve $(x(s), y(s))$ and an analytic function $\lambda(s)$, $s \in (0, \varepsilon)$, such that:

(a1) $\lim_{s\to 0}\|(x(s), y(s))\| = \infty$;

(a2) $\lim_{s\to 0} f_s(x(s), y(s)) = \infty$;

(a3) $\dfrac{\partial f_s}{\partial x}(x(s), y(s)) \equiv 0$; and

(a4) $\dfrac{\partial f_s}{\partial y}(x(s), y(s)) \equiv 0$.

If $x(s) \equiv 0$ (resp., $y(s) \equiv 0$) we let $m := 0$ (resp., $n := 0$), otherwise, we put $m := \mathrm{val}(x(s))$ (resp., $n := \mathrm{val}(y(s))$), here val $\lambda(s)$ for $\lambda(s) = \sum_{i\geq k}^{\infty} a_i s^i$, $a_k \neq 0$, meromorphic at infinity is defined as follows: $\mathrm{val}\,\lambda := k$. By Condition (a1), $\min\{m, n\} < 0$. Let γ be the face of $\Gamma(f_s)$, $s \neq 0$, where the linear function $mp + nq$ defined on γ takes its minimal value. If the face γ does not disappear, then we obtain a contradiction as in the proof of [27, Lemma 3.2]. So we suppose that the face γ disappears, i.e., at least one vertex of the boundary of γ disappears. By Lemma 2.6(i), we may assume without loss of generality that a monomial x^a of γ disappears (a similar proof holds for y^b). Then it follows from Lemma 2.6(ii) that there exists a monomial $x^p y \in \mathrm{supp}(f_s)$. We choose $x^p y$ in $\mathrm{supp}(f_s)$ with maximal p. Remark 3.1 now yields $p > 0$ (see Figure 5).

Then we conclude from Lemma 2.6(i) that the monomial $x^p y$ of f_s cannot disappear, and hence that

$$0 \equiv \frac{\partial f_s}{\partial y}(x(s), y(s)) = cs^{mp} + \text{higher order terms in } s,$$

for some $c \neq 0$, which is impossible. □

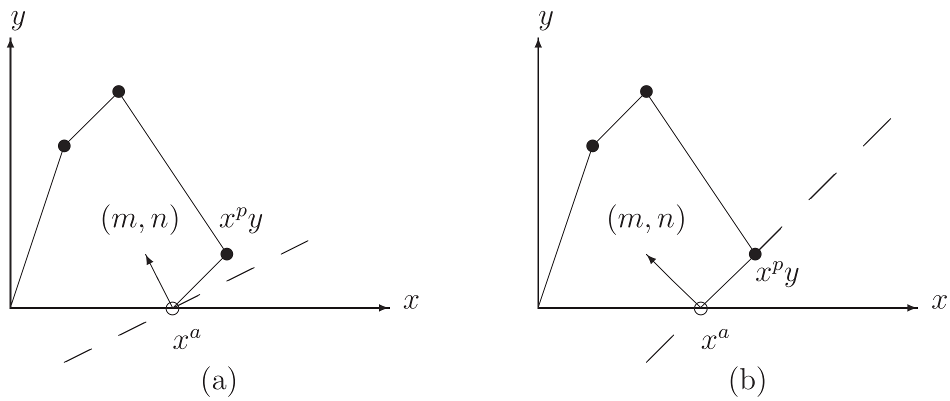

FIGURE 5. Case where a monomial x^a of $\Gamma(f_s)$ disappears: (a) $\gamma = \{x^a\}$; (b) γ joints the vertices x^a and $x^p y$.

3.2. Boundedness of singularities at infinity. The following lemmas show that the sets $\Sigma_\infty(f_s)$ and $\{f_s(0,0)\}$ are contained in some open disc of radius independent of s.

LEMMA 3.3. *There exists a positive number r such that*

$$\Sigma_\infty(f_s) = \mathfrak{C}_x(f_s) \cup \mathfrak{C}_y(f_s) \subset D_r \quad \textit{for all} \;\; s \in [0,1].$$

Proof. Let $\gamma_x(s)$ and $\gamma_y(s)$ be the two faces of $\Gamma_-(f_s)$, $s \geq 0$, that contain the origin. We will prove that there exists $r > 0$ such that the following inclusion holds

$$\mathfrak{C}_x(f_s) \subset D_r \quad \text{for all} \;\; s \in [0,1].$$

(A similar proof holds for $\mathfrak{C}_y(f_s)$.) If $\gamma_x(s)$ is constant, then with arguments similar to the ones used in the proof of [27, Lemma 3.2] we obtain the desired conclusion.

So we suppose that the face $\gamma_x(s)$ is not constant. We also assume that the only critical parameter is $s = 0$. By Lemma 2.6(i), there exists a monomial x^a $(a > 0)$ of $\gamma_x(s)$ that disappears. Then for $s \in (0,1]$ the monomial x^a is in $\Gamma(f_s)$, so $\mathfrak{C}_x(f_s) = \emptyset$. If $\Gamma(f_0)$ contains a monomial $x^{a'}$ $(a' > 0)$, then $\mathfrak{C}_x(f_0) = \emptyset$. So we suppose that all monomials x^k disappear. It follows from Lemma 2.6(ii) that there exists a monomial $x^p y \in \operatorname{supp}(f_s)$. We can suppose that $p \geq 0$ is maximal among monomials $x^k y \in \operatorname{supp}(f_s)$. By Remark 3.1, $p > 0$. Note that the monomial $x^p y$ does not disappear by Lemma 2.6(i). Now the edge of $\Gamma_-(f_0)$ that contains the origin and the monomial $x^p y$ begins at the origin and ends at $x^p y$. Then it is easy to check that $\mathfrak{C}_x(f_0) = \emptyset$. So in case where $\gamma_x(s)$ changes, we have for all $s \in [0,1]$, $\mathfrak{C}_x(f_s) = \emptyset$. □

LEMMA 3.4. *There exists a positive number r such that*

$$\{f_s(0)\} \subset D_r \quad \textit{for all} \;\; s \in [0,1].$$

Proof. The claim follows easily from the continuity of the family $f_s(x,y)$. □

3.3. Transversality in the neighbourhood of infinity. Let us make the following observation.

Remark 3.5. We suppose that a monomial x^a of $\Gamma(f_s)$ disappears. It follows from Lemma 2.6(ii) that there exists a monomial $x^p y \in \operatorname{supp}(f_s)$. We also suppose that $p \geq 0$ is maximal among monomials $x^k y \in \operatorname{supp}(f_s)$. Remark 3.1 now gives $p > 0$. Then we can further assume, for the end of the proof of Theorem 1.2, that all monomials x^k disappear.

LEMMA 3.6. *Let r be a positive number such that the conclusions of Lemmas* 3.2, 3.3 *and* 3.4 *are fulfilled. Then there exists R_0 sufficiently large such that for* *all $R \geq R_0$ and for all $c \in \mathbf{S}^1_r$ we have that the fiber $f_s^{-1}(c)$ meets transversally the sphere $\mathbf{S}^3_R$ for each $s \in [0, 1]$.*

Proof. It is sufficient to prove the lemma for a family $\{f_s\}$ parameterized by s in an interval $[0, s_0]$ for a small $s_0 > 0$. Assume the contrary. Then by the Curve Selection Lemma [18], [20] there exist an analytic curve $(x(s), y(s))$ and an analytic function $\lambda(s)$, $s \in (0, \varepsilon)$, such that:

(b1) $\lim_{s\to 0}\|(x(s), y(s))\| = \infty$;

(b2) $\lim_{s\to 0} f_s(x(s), y(s)) = c$;

(b3) $\dfrac{\partial f_s}{\partial x}(x(s), y(s)) = \lambda(s)\overline{x(s)}$; and

(b4) $\dfrac{\partial f_s}{\partial y}(x(s), y(s)) = \lambda(s)\overline{y(s)}$.

By Lemma 3.2, $\lambda(s) \not\equiv 0$. Thus we can write

$$\lambda(s) = \lambda^0 s^\delta + \text{higher order terms in } s,$$

here $\lambda^0 \neq 0$ and $\delta \in \mathbf{Q}$.

We first suppose that $y(s) \equiv 0$ (a similar proof holds for $x(s) \equiv 0$). Then we may write

$$x(s) = x_0 s^m + \text{higher order terms in } s,$$

where $x_0 \neq 0$ and $m < 0$. Since Condition (b2), there exists a monomial x^a $(a > 0)$ in $\mathrm{supp}(f_s)$, $s \neq 0$. We also suppose that a is maximal among monomials $x^k \in \mathrm{supp}(f_s)$. Let $u(s)$ be the coefficient of the monomial x^a in f_s. If the monomial x^a does not disappear, then $u(0) \neq 0$ and we have that

$$\lim_{s\to 0} f_s(x(s), y(s)) = \lim_{s\to 0}[u(0)x_0^a s^{ma} + \text{higher order terms in } s] = \infty,$$

which contradicts Condition (b2).

So we suppose that the monomial x^a disappears. By Lemma 2.6(ii), there exists a monomial $x^p y \in \mathrm{supp}(f_s)$. We choose $x^p y$ in $\mathrm{supp}(f_s)$ with maximal p. Remark 3.1 now leads to $p > 0$. It follows from Lemma 2.6(i) that the monomial $x^p y$ of f_s cannot disappear. Let $v(s)$ be the coefficient of the monomial $x^p y$ in f_s. Then $v(0) \neq 0$. By Condition (b4), therefore

$$0 \equiv \frac{\partial f_s}{\partial y}(x(s), y(s)) = v(0)x_0^p s^{mp} + \text{higher order terms in } s,$$

which is impossible.

We now suppose that $x(s) \not\equiv 0$ and $y(s) \not\equiv 0$. Let us write

$$x(s) = x_0 s^m + \text{higher order terms in } s,$$

$$y(s) = y_0 s^n + \text{higher order terms in } s,$$

where $x_0 \neq 0$, $y_0 \neq 0$, and $\min\{m, n\} < 0$.

Let γ be the face of $\Gamma(f_s)$, $s \neq 0$, where the linear function $mp + nq$ defined on γ takes its minimal value. If the face γ does not disappear, then we obtain a contradiction as in the proof of [27, Lemma 3.5]. So we suppose that the face γ disappears, i.e., at least one vertex of the boundary of γ disappears. By Lemma 2.6(i), we may assume without loss of generality that a monomial x^a of γ disappears (a similar proof holds for y^b). We also suppose that a is maximal among monomials $x^a \in \operatorname{supp}(f_s)$, $s \neq 0$. Again by Lemma 2.6(ii), there exists a monomial $x^p y \in \operatorname{supp}(f_s)$. We choose $x^p y$ in $\operatorname{supp}(f_s)$ with maximal p. According to Remark 3.1, we have $p > 0$. Then by a simple Plane Geometry argument we would have (see Figure 5)

$$m < 0.$$

Let $u(s)$ (resp., $v(s)$) be the coefficient of the monomial x^a (resp., $x^p y$) in f_s. As the monomial x^a disappears and $x^p y$ does not, we find that

$$u(s) = u_0 s^\kappa + \text{higher order terms in } s,$$

$$v(s) = v_0 + v_1 s + \text{higher order terms in } s,$$

where $u_0 \neq 0$, $v_0 \neq 0$, and $\kappa > 0$.

Let us note that all monomials x^k disappear (see Remark 3.5). There are three cases to be considered.

Case 1: $\kappa + ma < mp + n$. We have

$$f_s(x(s), y(s)) = u_0 x_0^a s^{\kappa + ma} + \text{higher order terms in } s,$$

$$\frac{\partial f_s}{\partial x}(x(s), y(s)) = a u_0 x_0^{a-1} s^{\kappa + m(a-1)} + \text{higher order terms in } s,$$

$$\frac{\partial f_s}{\partial y}(x(s), y(s)) = v_0 x_0^p s^{mp} + \text{higher order terms in } s.$$

Then we conclude from Conditions (b2)–(b4) that

$$\kappa + ma = 0,$$

$$\kappa + m(a-1) = \delta + m,$$

$$mp = \delta + n,$$

hence that $\delta = -2m$, and finally that $n = m(p+2) < 0$. This gives a contradiction with

$$0 = \kappa + ma < mp + n.$$

CASE 2: $\kappa + ma > mp + n$. We have

$$f_s(x(s), y(s)) = v_0 x_0^p y_0 s^{mp+n} + \text{higher order terms in } s,$$
$$\frac{\partial f_s}{\partial x}(x(s), y(s)) = p v_0 x_0^{p-1} y_0 s^{m(p-1)+n} + \text{higher order terms in } s,$$
$$\frac{\partial f_s}{\partial y}(x(s), y(s)) = v_0 x_0^p s^{mp} + \text{higher order terms in } s.$$

By Conditions (b2)–(b4), we get

$$mp + n = 0,$$
$$m(p-1) + n = \delta + m,$$
$$mp = \delta + n.$$

Hence $\delta = -2m$, and so that $n = m(p+2) < 0$, which contradicts the equation $mp + n = 0$.

CASE 3: $\kappa + ma = mp + n$. We have

$$f_s(x(s), y(s)) = (u_0 x_0^a + v_0 x_0^p y_0) s^{\kappa+ma} + \text{higher order terms in } s,$$
$$\frac{\partial f_s}{\partial x}(x(s), y(s)) = (a u_0 x_0^{a-1} + p v_0 x_0^{p-1} y_0) s^{\kappa+m(a-1)} + \text{higher order terms in } s,$$
$$\frac{\partial f_s}{\partial y}(x(s), y(s)) = v_0 x_0^p s^{mp} + \text{higher order terms in } s.$$

CASE 3.1: $u_0 x_0^a + v_0 x_0^p y_0 = 0$. We first suppose that

$$a u_0 x_0^{a-1} + p v_0 x_0^{p-1} y_0 = 0.$$

Then we must have $a = p$, and hence $\kappa = n$. It follows from Conditions (b3)–(b4) that

$$\kappa + m(a-1) < \delta + m,$$
$$mp = \delta + n.$$

Therefore $n < m$. Consequently, $mp = ma < \kappa + ma = mp + n < mp + m$. Thus $0 < m$. This gives a contradiction.

We now suppose that

$$a u_0 x_0^{a-1} + p v_0 x_0^{p-1} y_0 \neq 0.$$

Observe that

$$\kappa + m(a-1) = \delta + m,$$
$$mp = \delta + n.$$

These constraints, together with the equation $\kappa + ma = mp + n$, imply that $n = m < 0$. Hence

$$ma < \kappa + ma = mp + n = m(p+1).$$

Therefore

$$a > p + 1.$$

On the other hand, it is easy to see that

$$\begin{aligned} au_0 x_0^{a-1} + pv_0 x_0^{p-1} y_0 &= \lambda_0 \overline{x_0}, \\ v_0 x_0^p &= \lambda_0 \overline{y_0}. \end{aligned}$$

These constraints, together with the assumption $u_0 x_0^a + v_0 x_0^p y_0 = 0$, imply that

$$p - a = \frac{\|x_0\|^2}{\|y_0\|^2} > 0,$$

which is impossible.

Case 3.2: $u_0 x_0^a + v_0 x_0^p y_0 \neq 0$. We have

$$\begin{aligned} \kappa + ma &= mp + n = 0, \\ \kappa + m(a-1) &\leq \delta + m, \\ mp &= \delta + n. \end{aligned}$$

Hence $\delta = -2n \geq -2m$. It follows that $n \leq m < 0$, which is in contradiction with $mp + n = 0$.

Having exhausted all cases, we have completed the proof of Lemma 3.6. □

Proof of Corollary 1.5. By Theorem 1.2, there exist $r \gg 1$ and homeomorphisms Φ and Ψ such that the following diagram commutes:

$$\begin{array}{ccc} f_0^{-1}(\mathbf{S}_r^1) & \xrightarrow{f_0} & \mathbf{S}_r^1 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ f_1^{-1}(\mathbf{S}_r^1) & \xrightarrow{f_1} & \mathbf{S}_r^1. \end{array}$$

Fix $c \in \mathbf{S}_r^1$. For each $t \in [0,1]$, let $h_t : f_0^{-1}(c) \to f_0^{-1}(ce^{2\pi it})$ be a homeomorphism induced by the fibration $f_0 : f_0^{-1}(\mathbf{S}_r^1) \to \mathbf{S}_r^1$. Then the map h_1 gives rise to the global monodromy operators $m_0(f_0)$ and $m_1(f_0)$ of f_0. Moreover, we have a commutative diagram:

$$\begin{array}{ccc} f_0^{-1}(c) & \xrightarrow{h_t} & f_0^{-1}(ce^{2\pi it}) \\ {\scriptstyle\Phi_0}\downarrow & & \downarrow{\scriptstyle\Phi_t} \\ f_1^{-1}(\Psi(c)) & \xrightarrow{\Phi_t \circ h_t \circ \Phi_0^{-1}} & f_1^{-1}(\Psi(ce^{2\pi it})), \end{array}$$

where Φ_t is the restriction of Φ on the fiber $f_0^{-1}(ce^{2\pi it})$, and thus a homeomorphism

$$f_1^{-1}(\Psi(c)) \to f_1^{-1}(\Psi(c)), \quad z \mapsto \Phi_1 \circ h_1 \circ \Phi_0^{-1}(z).$$

By definition, this map gives rise to the global monodromy operators $m_0(f_1)$ and $m_1(f_1)$ of f_1. Therefore the following diagram commutes $(q = 0, 1)$:

$$\begin{array}{ccc} H_q(f_0^{-1}(c), \mathbf{Z}) & \xrightarrow{m_q(f_0)} & H_q(f_0^{-1}(c), \mathbf{Z}) \\ {\scriptstyle \Phi_0}\downarrow & & {\scriptstyle \Phi_1}\downarrow \\ H_q(f_1^{-1}(\Psi(c)), \mathbf{Z}) & \xrightarrow{m_q(f_1)} & H_q(f_1^{-1}(\Psi(c)), \mathbf{Z}). \end{array}$$

Since $\Phi_0 \equiv \Phi_1$, this gives us what we want. □

Proof of Corollary 1.6. The proof follows from Lemma 2.6 by using the same argument in [4, Theorem 3]. We will leave to the reader to verify these facts. □

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