Invariance principle for diffusions in degenerate and unbounded random environment.

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Alberto Chiarini
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Promotionsausschuss:

Vorsitzender: Prof. Dr. Alexander I. Bobenko
Berichter/Gutachter: Prof. Dr. Jean-Dominique Deuschel
Berichter/Gutachter: Prof. Dr. Pierre Mathieu

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Abstract

After the brilliant result of Papanicolau and Varadhan (1979) in the case of bounded stationary and ergodic environments, there has been a recent upsurge in the research of quenched homogenization of a symmetric diffusion process in random media. In particular, to identify the optimal conditions that a general stationary and ergodic environment must satisfy in order to obtain the convergence to a non-degenerate Brownian motion, is still an open problem. In this manuscript we show that, provided that the environment satisfies certain moment conditions, then both a quenched invariance principle and a quenched local central limit theorem hold for a diffusion formally generated by the divergence form operator $L^\omega = \mathrm{div}(a^\omega \nabla \cdot\,)$. Since the coefficients are not assumed to be smooth, we shall exploit Dirichlet form theory to make sense of the diffusion associated to L^ω. Both the proofs of the quenched invariance principle and of the quenched local central limit theorem rely on a priori estimates for solutions to linear partial differential equations. On one hand, with the help of the celebrated J. Moser's iteration technique, we derive a maximal inequality for solutions to degenerate elliptic PDEs which in turn gives the sublinearity of the correctors and with that the quenched invariance principle. On the other hand, relying once again on Moser's scheme, we obtain a parabolic Harnack inequality which can be used to control the oscillations of solutions to parabolic PDEs. In particular, in the diffusive limit, we are able to bound the oscillations of the transition densities of our diffusion. This successively yields the quenched local central limit theorem.

Zusammenfassung

Nach den brillanten Ergebnissen von Papanicolau und Varadhan (1979) im Fall von beschränkten, stationären und ergodischen Umgebungen gab es ein erneutes Aufflammen in der Erforschung der "quenched" Homogenisierung eines symmetrischen Diffusionsprozesses in zufälligen Medien. Insbesondere ist es immer noch ein ungelöstes Problem die optimalen Bedingungen zu bestimmen, die eine stationäre und ergodische Umgebung erfüllen muss, um die Konvergenz gegen eine nicht-degenerierte Brownsche Bewegung zu garantieren. In diesem Manuskript zeigen wir, dass, vorausgesetzt, dass die Umgebung eine bestimmte Momentenbedingung erfüllt, sowohl ein quenched Invarianzprinzip als auch ein quenched, lokaler, zentraler Grenzwertsatz für Diffusionen, die vom Divergenzform Operator $L^\omega = \mathrm{div}(a^\omega \nabla \cdot)$ generiert werden, gelten. Da die Koeffizienten nicht glatt sind, bedienen wir uns der Theorie der Dirichlet Formen, um Diffusionen, die formal mit dem Operator L^ω assoziierte sind, einen Sinn zu geben. Die Beweise für das quenched Invarianzprinzip und den quenched lokalen zentralen Grenzwertsatz verwenden a-priori-Abschätzungen von Lösung partieller Differentialgleichungen. Einerseits gelingt es uns mit Hilfe der berühmten J. Moser Iterationstechnik eine Maximalungleichung für Lösungen von elliptischen, partiellen Differentialgleichungen herzuleiten, welche wiederum die Sublinearität der Korrektoren und damit das quenched Invarianzprinzip implizieren. Andererseits, wieder dank Mosers Schema, erhalten wir eine parabolische Harnack Ungleichung, die verwendet werden kann, um die Oszillation von Lösungen parabolischer PDEs zu kontrollieren. Insbesondere, im diffusen Grenzübergang, sind wir in der Lage die Oszillation der Übergangsdichten unserer Diffusion zu beschränken. Sukzessive ergibt sich daraus ein quenched, lokaler, zentraler Grenzwertsatz.

Contents

Introduction

The theory of mathematical homogenization is a rigorous version of what in physics is known as averaging. In other words homogenization is the art of extracting homogeneous effective parameters from disordered heterogeneous media.

In many fields of science and technology one often has to solve boundary value problems in periodic media in order to determine physical quantities like temperature or electric potential. Usually the size of the period is small compared with the size of the medium and, denoting by ϵ their ratio, an analysis as ϵ goes to zero becomes very natural. What is observed is that an averaging process takes place and the high oscillating small scale structure is replaced by an effective one on the macroscopic level. This process of looking for an averaged or effective behavior through an asymptotic analysis is called homogenization.

Let us give a more precise mathematical description of the problem. We are given a conductor $\mathcal{O} \subset \mathbb{R}^d$ and heat sources determined by a function $f : \mathcal{O} \to \mathbb{R}$. Quite often in the applications the conductivity varies among the points of $\mathcal{O}$, in the simplest scenario the variation is periodic with period ϵ in each direction. One can easily think about a mixture of two conducting materials, for example, a metal matrix and a ceramic reinforcement (See Figure 1). If $a : [0,1]^d \to \mathbb{R}^{d\times d}$ is the function describing the conductivity in one of these cells, then in $\mathcal{O}$ we can describe it by $a(x/\epsilon)$, where $x \in \mathcal{O}$ is usually addressed as the *macroscopic* or fast variable, and $y := x/\epsilon$ is called the *microscopic* or slow variable. To keep things simple assume that the temperature at the boundary is kept constant and equal to zero.

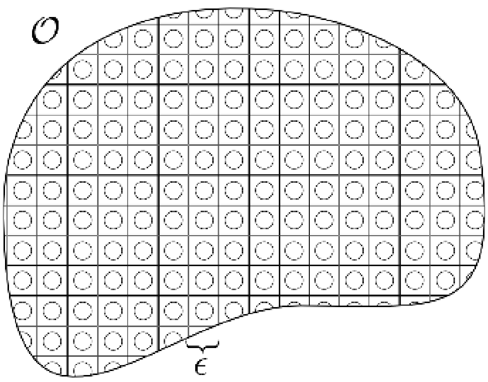

Figure 1: a conductor with small scale periodic structure.

In order to determine the temperature of the material at equilibrium $u^\epsilon(x)$ at the point $x \in \mathcal{O}$ we must solve the following boundary value problem

$$\begin{cases} \mathrm{div}(a(x/\epsilon)\nabla u^\epsilon(x)) = f(x) & x \in \mathcal{O}, \\ u^\epsilon(x) = 0 & x \in \partial\mathcal{O}. \end{cases} \tag{1}$$

The problem now is to analyze the behavior of the temperature u^ϵ as the size of the mesh goes to zero. Under suitable assumptions on a, it is found that there exists a constant matrix $\mathbf{D} := \{\mathbf{d}_{ij}\}_{i,j}$ such that if $u(x)$ is the solution of

$$\begin{cases} \mathrm{div}(\mathbf{D}\nabla u(x)) = f(x) & x \in \mathcal{O}, \\ u(x) = 0 & x \in \partial\mathcal{O}, \end{cases} \tag{2}$$

then $u^\epsilon \to u$ in $L^2(\mathcal{O})$. The tensor $\mathbf{D}$ is known as the *effective conductivity* and is implicitly given by solving some auxiliary problem. The effective conductivity captures the macroscopic properties of the conductor $\mathcal{O}$ and shows that if the oscillations of the conductivity happen on a very small scale, then there exists an homogeneous medium whose diffusive properties can approximate the heterogeneous one. It is hardly surprising that the construction of the coefficients $\mathbf{D}$ must be treated numerically, since, with the exception of trivial cases, an exact closed formula is not available.

The problem of homogenization in periodic media has been widely studied from many viewpoints with different level of generality by several authors [BLP75], [Koz85], [Lej01b], [DASC92], [ME08], [BM15] and extended also beyond the linear cases by [SRP09], [PS11] and references therein. For an exhaustive introduction on the various techniques available for the periodic homogenization problem we refer to [BLP11].

In 1979, G.C. Papanicolaou and S.R.S. Varadhan [PV81] studied the boundary value problem (1) in the case where the conductivity a is a stationary and ergodic random field. To be more precise, they assumed to have a probability space $(\Omega, \mathcal{G}, \mu)$ and a stationary and ergodic field $a : \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}$ satisfying

$$c^{-1}|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x,\omega)\xi_i\xi_j \le c|\xi|^2, \quad \forall \xi \in \mathbb{R}^d,\ \mu\text{-a.s.}$$

for some constant $c > 1$ and $x \mapsto a(x,\omega)$ smooth enough for almost all $\omega \in \Omega$. If we denote by $u^\epsilon(x,\omega)$ the solution to the Dirichlet problem with random coefficients

$$\begin{cases} \mathrm{div}(a(x/\epsilon,\omega)\nabla u^\epsilon(x,\omega)) = f(x) & x \in \mathcal{O}, \\ u^\epsilon(x,\omega) = 0 & x \in \partial\mathcal{O}, \end{cases} \tag{3}$$

then $u^\epsilon(x,\omega)$ can be interpreted as the random temperature function at the point $x \in \mathcal{O}$ in the environment $\omega \in \Omega$. In [PV81], the authors were able to prove that

$$\lim_{\epsilon\to 0} \int_{\mathcal{O}} \mathbb{E}_\mu[|u^\epsilon(x;\omega) - u(x)|^2] = 0,$$

where $\mathbb{E}_\mu$ is the expectation with respect to μ and u solves a deterministic boundary value problem of the type (2). Again, the randomly inhomogeneous conducting medium behaves like a homogeneous deterministic medium with effective conductivity $\mathbf{D}$ when ϵ is small. As in the periodic setting, an implicit formula for $\mathbf{D}$ is provided, but an exact closed formula is not given except for very simple cases. Since the seminal works [PV81], [Koz79], [Koz85] many extensions and generalizations have been explored in the framework of ergodic media; we mention [Osa83] for measurable coefficients, [Lej01a] for second order linear operators with lower order terms, and [FR09] for the study of non-linear operators.

In this manuscript we are interested in the homogenization of second order linear elliptic operators, given their beautiful and intimate connection with stochastic processes. Indeed, solutions to second order parabolic or elliptic PDEs can be interpreted as averages of functionals of the trajectory of a diffusion process. It is well known (See for example [RY99, Chapter VII]) that if the coefficients are smooth enough the operator

$$L = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{j=1}^{d} b_j(x)\partial_j$$

is the generator of the diffusion process X_t in $\mathbb{R}^d$ which solves the stochastic differential equation

$$dX_t = \sqrt{2}\sigma(X_t)dW_t + b(X_t)dt,$$

being W_t a d-dimensional Brownian motion and $\sigma\sigma^T = a$. In our work we shall look at generators in divergence form,

$$L = \frac{1}{\theta(x)}\operatorname{div}(a(x)\nabla\cdot) = \frac{1}{\theta(x)}\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \frac{1}{\theta(x)}\sum_{j=1}^{d}\Big(\sum_{i=1}^{d}\partial_j a_{ij}(x)\Big)\partial_j,$$

with $a_{ij} = a_{ji}$. As an example take $\theta(x) = e^{V(x)}$ and $a = e^{V(x)}I_d$ which corresponds, given enough regularity on V, to the diffusion

$$dX_t = \sqrt{2}dW_t + \nabla V(X_t)dt.$$

Dealing with operators in divergence form has many advantages. First there is an explicit invariant measure $\theta(x)dx$ which is absolutely continuous with respect to the Lebesgue measure. Moreover it is possible to consider coefficients which are not necessarily smooth, in this case the process formally associated to L is not in general the solution of some stochastic differential equation. If it were, the drift term would include $\sum_{i=1}^{d}\partial_j a_{ij}(x)$, which is not defined for irregular coefficients a; since Itô calculus is not available we shall exploit the stochastic calculus for processes generated by Dirichlet forms. Despite its own interest, the need to look at non-smooth coefficients is motivated by the fact that in our model they are realizations of random fields, which are naturally non-smooth objects.

Generally speaking, if X_t^ω is a diffusion associated to $L^\omega = \operatorname{div}(a(x,\omega)\nabla\cdot)$, then the process $\epsilon X^\omega_{t/\epsilon^2}$ is associated to $L^{\epsilon,\omega} = \operatorname{div}(a(x/\epsilon,\omega)\nabla\cdot)$ and the problem of homogenization of solutions $u^\epsilon(x,\omega)$ to (3) translates in identifying the asymptotic behavior of the process $\epsilon X^\omega_{t/\epsilon^2}$ as ϵ goes to zero; this is known in the literature as the diffusive limit.

Similarly to the homogenization problem (3), it can be observed that if we look at the process on long space-time scale, then the highly complex small scale structure averages in the limit, so that at the macroscopic level we will see a Brownian motion with a deterministic effective diffusivity matrix $\mathbf{D}$.

The effective diffusivity $\mathbf{D} = \{\mathbf{d}_{ij}\}_{i,j=1}^{d}$ is given precisely by the formula

$$\mathbf{d}_{ij} = 2\,\mathbb{E}_{\mu}[\langle a(0,\omega)(e_i - \nabla\chi^i(0,\omega)), e_j - \nabla\chi^j(0,\omega)\rangle], \tag{4}$$

where $\chi^k : \mathbb{R}^d \times \Omega \to \mathbb{R}$, $k = 1, \dots, d$ are functions whose gradient $\nabla\chi^k$ is a stationary and ergodic random field and e_k are the coordinate directions. The corrector field satisfies the following partial differential equation with random coefficients

$$\mathrm{div}(a(x,\omega)\nabla\chi^k(x,\omega)) = \mathrm{div}(a(x,\omega)e_k),$$

for μ-almost all $\omega \in \Omega$.

In the study of stochastic processes in random environment we have to distinguish between two different scenarios, the quenched law and the annealed law: the first is the conditional law of the process for a given realization of the environment, while the second is obtained by integrating the quenched law with respect to the environment distribution. A very important difference between the two is that while the quenched law is often Markovian (for example in the case of diffusions with generator in divergence form with random coefficients as above), this is not in general the case for the annealed law. In this manuscript we will be concerned with quenched type results.

Description of the model and main results

In this manuscript, in dealing with homogenization of second order linear operators in random media, we focus on the stochastic process point of view. More precisely, we are interested in the study of reversible diffusions associated to an infinitesimal generator L^ω in divergence form

$$L^\omega u(x) = \nabla \cdot (a^\omega(x)\nabla u(x)), \quad x \in \mathbb{R}^d, \tag{5}$$

where $a^\omega(x)$ is a symmetric d-dimensional matrix depending on a parameter ω which describes a random realization of the environment.

We model the environment as a probability space $(\Omega, \mathcal{G}, \mu)$ on which a measurable group of transformations $\{\tau_x\}_{x\in\mathbb{R}^d}$ is defined. One may think of $\tau_x\omega$ as a translation of the environment $\omega \in \Omega$ in the direction $x \in \mathbb{R}^d$. We assume that the random environment $(\Omega, \mathcal{G}, \mu, \{\tau_x\}_{x\in\mathbb{R}^d})$ is stationary and ergodic (a more precise definition is given in Section 5.1). The random field $\{a^\omega(x)\}_{x\in\mathbb{R}^d}$ will then be constructed by simply taking a random variable $a : \Omega \to \mathbb{R}^{d\times d}$ and defining $a^\omega(x) := a(\tau_x\omega)$, we will often use the notation $a(x,\omega)$ for $a^\omega(x)$ as well.

We recall from the previous section that when $x \mapsto a^\omega(x)$ is bounded and uniformly elliptic, with the same constants for μ-almost all ω, then a quenched invariance principle holds for the diffusion process X_t^ω associated with L^ω. This means that, for μ-almost all $\omega \in \Omega$, the scaled process $X_\cdot^{\epsilon,\omega} := \epsilon X_{\cdot/\epsilon^2}^\omega$ converges in distribution on $C([0,\infty),\mathbb{R}^d)$ to a Brownian motion with a non-trivial covariance structure as ϵ goes to zero; this scaling is also known in the literature as diffusive scaling. See for example the classic result of Papanicolau and Varadhan [PV81] where the coefficients are assumed to be differentiable, and [Osa83], [Lej01a] for measurable coefficients and more general operators.

Recently a lot of effort has been put into extending this result beyond the uniformly elliptic case. For example [FK97] considers a non-symmetric, but still in divergence form, diffusion with uniformly elliptic symmetric part and unbounded antisymmetric part and the recent paper [BM15] proves an invariance principle for divergence form operators of the type

$$Lu(x) = e^{V(x)} \operatorname{div}(e^{-V(x)} \nabla u(x))$$

where V is a periodic and measurable function, such that $e^V + e^{-V}$ is locally integrable. For what concerns ergodic and stationary environment a recent result has been achieved in the case of random walk in random environment in [ADS15a], [ADS15b]. In these works moments of order greater than one are needed to get an invariance principle in the diffusive limit.

The aim of this thesis is to present our new results in this direction. Namely, based on [CDa], we want to prove a quenched invariance principle for an operator L^ω of the form (5) with a random field $a^\omega(x)$ which is ergodic, stationary and possibly unbounded and degenerate. Denote by $a : \Omega \to \mathbb{R}^{d\times d}$ the $\mathcal{G}$-measurable random variable which describes the field through $a^\omega(x) = a(\tau_x\omega)$. We assume that a is symmetric and that there exist Λ and λ $\mathcal{G}$-measurable, positive and finite, satisfying the following assumptions.

Assumption a.1. For all $\omega \in \Omega$ and $\xi \in \mathbb{R}^d$

$$\lambda(\omega)|\xi|^2 \le \langle a(\omega)\xi, \xi\rangle \le \Lambda(\omega)|\xi|^2;$$

Assumption a.2. There exist $p, q \in [1,\infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\mathbb{E}_\mu[\lambda^{-q}] < \infty, \quad \mathbb{E}_\mu[\Lambda^p] < \infty;$$

Assumption a.3. As functions of x, $\lambda^{-1}(\tau_x\omega), \Lambda(\tau_x\omega) \in L^\infty_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$.

Since $a^\omega(x)$ is meant to model a random field, it is not natural to assume its differentiability in $x \in \mathbb{R}^d$. Accordingly, the operator defined in (5) does not make any sense, and the techniques coming from stochastic differential equations and Itô calculus are not very helpful neither in constructing the diffusion process, nor in performing the relevant computation.

The theory of Dirichlet forms is the right tool to approach the problem of constructing a diffusion. Instead of the operator L^ω we shall consider the bilinear form obtained by L^ω, formally integrating by parts, namely

$$\mathcal{E}^\omega(u,v) := \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}^\omega(x) \partial_i u(x) \partial_j v(x) dx \tag{6}$$

for a proper class of functions $u, v \in \mathcal{F}^\omega \subset L^2(\mathbb{R}^d, dx)$, more precisely $\mathcal{F}^\omega$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, dx)$ with respect to $\mathcal{E} + (\cdot,\cdot)_{L^2}$. It is a classical result of Fukushima [FOT94, Theorem 7.2.2] and [Röc93, Ch. II example 3b] that it is possible to associate to (6) a diffusion process $\{X^\omega, \mathbb{P}_x^\omega, x \in \mathbb{R}^d\}$ as soon as $(\lambda^\omega)^{-1}$ and Λ^ω are locally integrable. It is well known that there is a properly exceptional[1] set $\mathcal{N}^\omega \subset \mathbb{R}^d$ of X^ω such that the associated process is uniquely determined up to the ambiguity of starting points in $\mathcal{N}^\omega$, in our situation the set of exceptional points may depend on the realization of the environment. Assumption $(a.3)$ is designed to remove the ambiguity about the properly exceptional set $\mathcal{N}^\omega$. We will then prove that assumption $(a.2)$ and ergodicity of the environment are enough to grant that the process X^ω starting from any $x \in \mathbb{R}^d$ does not explode for almost all realizations of the environment.

Remark. *Moment conditions on the environment are a very natural assumption in order to achieve a quenched invariance principle for symmetric diffusions in divergence form, indeed at least the first moment of Λ and λ^{-1} is required to obtain the result. As a counterexample one can consider a periodic environment, namely the d-dimensional torus $\mathbb{T}^d$, and the following generator in divergence form*

$$Lf(x) := \frac{1}{\phi(x)}\nabla \cdot (\phi(x)\nabla f(x)),$$

where $\phi : \mathbb{T}^d \to \mathbb{R}$ is defined by $\phi(x) := 1_B(x)|x|^{-d} + 1_{B^c}(x)$, being $B \subset \mathbb{T}^d$ a ball of radius one centered in the origin. It is clear that $\phi^\alpha \in L^1(\mathbb{T}^d)$ for all $\alpha < 1$ but not for $\alpha = 1$. If we look for example at $d = 2$, then the radial part of the process associated to L above, when the radius is less than one, will be a Bessel process with parameter $\delta = 0$ which is known to have a trap in the origin [RY99, Chapter XI].

Remark. *As observed in the previous remark, if we want to prove an invariance principle, dealing with symmetric processes forces the degeneracy of the diffusion coefficient not to be too strong. Namely, the diffusion coefficient can eventually be zero only on a set of null Lebesgue measure. On the other hand, in the non-symmetric case the diffusion coefficient is allowed to vanish in open sets, as was proved in periodic environments by [ME08] and further extended and generalized in [FR09], [SRP09], [PS11]. In these works the strong degeneracy of the diffusion coefficient is compensated by the drift through the Hörmander's condition; as a result and in contrast with our setting, the coefficients need to be smooth enough.*

Once the diffusion process X^ω is constructed, the standard approach to diffusive limit theorems consists in showing the weak compactness of the rescaled process and in the identification of the limit. In the case of bounded and uniformly elliptic coefficients the compactness is readily obtained by the Aronson-Nash estimates for the heat kernel. In order to identify the limit, we use the standard technique used in [FK97], [Koz85] and [Osa83]; namely, we decompose the process X_t^ϵ into a martingale part, called the *harmonic coordinates* and a fluctuation part, called the *correctors*. The martingale part is

[1] A set $\mathcal{N} \subset \mathbb{R}^d$ is called properly exceptional if $\mathcal{N}$ is Borel, it has Lebesgue measure zero, and $\mathbb{P}_x(X_t \in \mathcal{N}$ or $X_{t-} \in \mathcal{N}$ for some $t \geq 0) = 0$ for all $x \in \mathbb{R}^d \setminus \mathcal{N}$.

supposed to capture the long time asymptotic of X_t^ϵ, and will characterize the diffusive limit.

The challenging part is to show that the correctors are uniformly small for almost all realizations of the environment. This is attained generalizing Moser's arguments [Mos61] to get a maximal inequality for positive subsolutions of uniformly elliptic, divergence form equations. In this sense the relation $1/p + 1/q < 2/d$ is designed to let the Moser's iteration scheme work. This integrability assumption firstly appeared in [EP72] in order to extend the results of De Giorgi and Nash to degenerate elliptic equations. A similar condition was also recently exploited in [Zhi13] to obtain estimates of Aronson-Nash type for solutions of degenerate parabolic equations. The author looks at generators of the form

$$\mathcal{L}u(x) = \partial_t u(x) - e^{-V(x)} \operatorname{div}(e^{V(x)} \nabla u(x)),$$

with the assumption that

$$\sup_{r \geq 1} |r|^{-d} \int_{|x| \leq r} e^{pV} + e^{-qV} dx < \infty.$$

We want to stress that condition $(a.3)$ is not needed to prove the sublinearity of the corrector, nor its existence, we used it only to have a more regular density of the semigroup associated to X^ω and avoid some technicalities due to exceptional sets in the framework of Dirichlet form theory.

Once the correctors are showed to be sublinear, the standard invariance principle for martingales gives the desired *Quenched Functional Central Limit Theorem* (QFCLT) (cf. Theorem 1.1 in [CDa]).

Theorem I (QFCLT)**.** *Assume that* $(a.1)$*,* $(a.2)$ *and* $(a.3)$ *are satisfied. Let* $\mathrm{M}^\omega := (X_t^\omega, \mathbb{P}_x^\omega)$*,* $x \in \mathbb{R}^d$*, be the minimal diffusion process associated to* $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ *on* $L^2(\mathbb{R}^d, dx)$*. Then the following hold*

(i) For μ*-almost all* $\omega \in \Omega$ *the limits*

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_0^\omega [X_t^\omega(i) X_t^\omega(j)] = \mathbf{d}_{ij} \quad i, j = 1, ..., d$$

exist and are deterministic constants.

(ii) For μ*-almost all* $\omega \in \Omega$*, the laws of the processes* $\epsilon X_{t/\epsilon^2}^\omega$*,* $\epsilon > 0$ *started at the origin over* $C([0, +\infty), \mathbb{R}^d)$ *converge weakly as* $\epsilon \to 0$ *to a Wiener measure having the covariance matrix equal to* $\mathbf{D} = [\mathbf{d}_{ij}]$*. Moreover* $\mathbf{D}$*, is a positive definite matrix.*

As a corollary of Theorem I (see Corollary 5.6.3 for the precise statement), using a time-change technique, we can establish a quenched invariance principle for diffusions formally associated to

$$L^{\theta,\omega} u(x) := \frac{1}{\theta^\omega(x)} \operatorname{div}(a^\omega(x) \nabla u(x)), \tag{7}$$

with $\theta^\omega(x)$ a locally bounded stationary and ergodic random field such that $\mathbb{E}_\mu[\theta(0)]$, $\mathbb{E}_\mu[\theta^{-1}(0)]$ are finite and with a^ω as before. In this case, the diffusion process formally associated to $L^{\theta,\omega}$ converges in distribution on $C([0,\infty),\mathbb{R}^d)$, in the diffusive scaling, to a Wiener measure having covariance matrix given by $\mathbf{D}/\,\mathbb{E}_\mu[\theta]$, where $\mathbf{D}$ was given in Theorem I.

Remark. *As a consequence of Theorem I, the continuous mapping Theorem and of the rich interplay between stochastic processes and PDEs, one could show homogenization for solutions to* (3). *Indeed, the random weak solution $u^\epsilon(x,\omega)$ to*

$$\begin{cases} \operatorname{div}(a(x/\epsilon,\omega)\nabla u^\epsilon(x,\omega)) = f(x) & x \in \mathcal{O}, \\ u^\epsilon(x,\omega) = 0 & x \in \partial\mathcal{O}, \end{cases}$$

has a probabilistic representation in terms of the process $\epsilon X^\omega_{t/\epsilon^2}$. For example, in $x=0$

$$u^\epsilon(0,\omega) = \mathbb{E}^\omega_0\left[\int_0^{\tau^\epsilon_{\mathcal{O}}} f(\epsilon X^\omega_{t/\epsilon^2})\,dt\right].$$

where $\tau^\epsilon_{\mathcal{O}}$ is the exit time of $\epsilon X^\omega_{t/\epsilon^2}$ from $\mathcal{O}$. Provided that the functional in the integral is continuous, we have

$$u^\epsilon(0,\omega) = \mathbb{E}^\omega_x\left[\int_0^{\tau^\epsilon_{\mathcal{O}}} f(\epsilon X^\omega_{t/\epsilon^2})\,dt\right] \to \mathbb{E}^W\left[\int_0^{\tau^W_{\mathcal{O}}} f(W_t)\,dt\right],$$

where $\mathbb{E}^W$ is the expectation of a Brownian motion W started at zero with covariance matrix given by $\mathbf{D}$ as in Theorem I and $\tau^W_{\mathcal{O}}$ is the exit time of W from $\mathcal{O}$.

Under the moment condition $(a.2)$ a finer result can be obtained. More precisely, we show that if a quenched invariance principle holds for the process $X^{\Lambda,\omega}$ formally associated to $L^{\Lambda,\omega}$, then under $(a.1)$ and $(a.2)$, the density of $\epsilon X^{\Lambda,\omega}_{t/\epsilon^2}$ converges uniformly on compacts to a Gaussian density for μ-almost all realizations of the environment (see Theorem 1.1 [CDb]). This type of result is known in the literature as *Quenched Local Central Limit Theorem* (QLCLT). Again, the construction of $X^{\Lambda,\omega}$ follows from Dirichlet form theory; namely $X^{\Lambda,\omega}$ is the diffusion process associated to the Dirichlet form $(\mathcal{E}^\omega,\mathcal{F}^{\Lambda,\omega})$ on $L^2(\mathbb{R}^d,\Lambda^\omega)$, where $\mathcal{E}^\omega$ is given in (6) and $\mathcal{F}^{\Lambda,\omega}$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d,\Lambda^\omega)$ with respect to $\mathcal{E}^\omega + (\cdot,\cdot)_{L^2(\mathbb{R}^d,\Lambda^\omega)}$.

To state the result we need to introduce the following assumption.

Assumption a.4. Assume that there is a positive definite symmetric d-dimensional matrix Σ such that for μ-almost all $\omega\in\Omega$ we have that for almost all $o\in\mathbb{R}^d$, all balls $B\subset\mathbb{R}^d$ and all compact intervals $I\subset(0,\infty)$

$$\lim_{\epsilon\to 0}\mathbb{P}^\omega_o(\epsilon X^{\Lambda,\omega}_{t/\epsilon^2}\in B) = \frac{1}{\sqrt{(2\pi t)^d\det\Sigma}}\int_B \exp\Big(-\frac{x\cdot\Sigma^{-1}x}{2t}\Big)\,dx,$$

uniformly in $t\in I$.

Observe that in the special case where $\lambda^\omega(\cdot)^{-1}, \Lambda^\omega(\cdot) \in L^\infty_{loc}(\mathbb{R}^d)$ μ-almost surely, assumption $(a.4)$ is satisfied for all $o \in \mathbb{R}^d$, μ-almost surely with $\Sigma = \mathbf{D}/\mathbb{E}_\mu[\Lambda]$ due to Corollary 5.6.3.

Set the notation

$$k_t^\Sigma(x) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp\Big(-\frac{x \cdot \Sigma^{-1} x}{2t}\Big)$$

for the Gaussian kernel with covariance matrix given by Σ.

Theorem II (QLCLT)**.** *Let $d \geq 2$. Assume $(a.1)$, $(a.2)$ and $(a.4)$. Let $p_t^{\Lambda,\omega}(\cdot,\cdot)$ be the density with respect to $\Lambda^\omega(x)dx$ of the semigroup $P_t^{\Lambda,\omega}$ associated to $(\mathcal{E}^\omega, \mathcal{F}^{\Lambda,\omega})$ on $L^2(\mathbb{R}^d, \Lambda^\omega)$. Let $R > 0$ and $I \subset (0,\infty)$ compact. Then for μ-almost all $\omega \in \Omega$ we have that for almost all $o \in \mathbb{R}^d$*

$$\lim_{\epsilon \to 0} \sup_{|x-o| \leq R} \sup_{t \in I} |\epsilon^{-d} p^{\Lambda,\omega}_{t/\epsilon^2}(o, x/\epsilon) - \mathbb{E}_\mu[\Lambda]^{-1} k_t^\Sigma(x)| = 0. \tag{8}$$

If we further assume that $\lambda^\omega(\cdot)^{-1}, \Lambda^\omega(\cdot) \in L^\infty_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$, then (8) *is satisfied for all $o \in \mathbb{R}^d$.*

The PDEs methods

The proofs of Theorem I and Theorem II are based on a priori estimates for solutions to PDEs. On one hand for the proof of Theorem I we need to show that the correctors are uniformly small for μ-almost all ω, and this follows from a maximal inequality for solutions to elliptic PDEs. On the other hand in order to demonstrate Theorem II we need to control the oscillations of the fundamental solution of a parabolic PDE.

Maximal inequality and sublinearity of the correctors. A key step in the proof of the QFCLT is to show that the correctors $\chi = (\chi^1, \dots, \chi^d) : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are locally sublinear, namely that

$$\limsup_{\epsilon \to 0} \sup_{|x| \leq R} \epsilon |\chi(x/\epsilon, \omega)| = 0, \quad \forall R > 0,\ \mu\text{-a.s.}$$

This helps to conclude that when we decompose the process into the sum of a martingale part and correctors the latter are converging to zero in distribution.

To obtain a priori estimates on the correctors χ we exploit the fact that they are solutions of a Poisson equation, which is formally given by

$$\nabla \cdot (a^\omega(x) \nabla \chi^k(x,\omega)) = \nabla \cdot (a^\omega(x) \nabla \pi^k(x)), \tag{9}$$

where $\pi^k(x) := x_k$ is the projection to the kth-coordinate.

The equation above has been studied extensively and generalized in many directions, also beyond the linear case. For an introduction, see for example the monographs [Eva10], [GT01] and for recent developments in the theory see [HKM06].

If the matrix a^ω is uniformly elliptic and bounded, uniformly in $\omega \in \Omega$, namely if

$$c^{-1}|\xi|^2 \leq \langle a^\omega(x)\xi, \xi\rangle \leq c|\xi|^2$$

for some $c \geq 1$, it is natural to look for weak solutions to (9) in the classical Sobolev space of square integrable functions with square integrable weak derivatives. It is a classical result due to Moser [Mos61] that an elliptic Harnack inequality holds and a result from Nash [Nas58] and De Giorgi [De 57] that solutions are Hölder continuous.

The situation changes dramatically if the coefficients are degenerate. In the most typical situation there is a positive weight $\theta : \mathbb{R}^d \to \mathbb{R}$ and a constant $c > 1$ such that

$$\theta(x)|\xi|^2 \leq \langle a^\omega(x)\xi, \xi\rangle \leq c\,\theta(x)|\xi|^2.$$

In this setting one looks for solutions to equation (9) in the weighted Sobolev space $W^{1,2}(\mathbb{R}^d, \theta)$ which is the set of weakly differentiable functions $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} |u|^2\theta dx < \infty, \text{ and } \int_{\mathbb{R}^d} |\nabla u|^2\theta dx < \infty,$$

we refer to [HKM06], [Zhi98] for more information on weighted Sobolev spaces. It was shown in [FKS82] that in order to have local regularity of solutions to (9) it is enough to have weights which are volume doubling, namely such that there exists a constant $C > 0$ for which

$$\int_{B_{2R}(x)} \theta(y)\, dy \leq C \int_{B_R(x)} \theta(y)\, dy, \quad \forall R > 0, \forall x \in \mathbb{R}^d,$$

and which satisfy weighted Sobolev and Poincaré inequalities. This weights are known in general as p-admissible (See [HKM06]), but for our discussion of the linear operator $L^\omega = \nabla \cdot (a^\omega \nabla\,)$ it is enough to look at 2-admissible weights.

Remark (Volume doubling). *In our setting it is not possible to expect the volume doubling property for small balls. The ergodic theorem ensures only that for all $x \in \mathbb{R}^d$ and μ-almost all $\omega \in \Omega$ there exist $R_0^\omega(x) > 0$ and a dimensional constant $C > 0$ such that for all $R > R_0^\omega(x)$*

$$\int_{B_{2R}(x)} \Lambda^\omega(y)\, dy \leq C \int_{B_R(x)} \Lambda^\omega(y)\, dy,$$

being $B_R(x)$ the ball of center x and radius R. We remark that the constant $R_0^\omega(x)$ cannot be taken uniformly in $x \in \mathbb{R}^d$, and $\sup_{x\in\mathbb{R}^d} R_0^\omega(x)$ may be infinite.

Examples of 2-admissible weights are the functions in the Muckenhaupt's class A_2, we refer to [FKS82], [HKM06], [Tor12] and to the original research paper [Muc72] for an exhaustive treatment on the subject. Here we briefly recall that the class A_2 is the set of all non-negative functions $\theta : \mathbb{R}^d \to [0, \infty]$ for which there exists a constant $C > 0$ such that

$$\sup_{R>0} \sup_{x\in\mathbb{R}^d} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \theta(y)\, dy \right) \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \theta^{-1}(y)\, dy \right) \leq C. \tag{10}$$

It is well known that weights in the class A_2 are volume doubling and satisfy a weighted Sobolev inequality. To be more precise, denote by $\theta(B) := \int_B \theta dx$, then there exist constants $C, \delta > 0$ such that for all $1 \le k \le d/(d-1) + \delta$

$$\left(\frac{1}{\theta(B)} \int_B |u|^{2k} \theta dx \right)^{\frac{1}{k}} \le C|B|^{\frac{2}{d}} \frac{1}{\theta(B)} \int_B |\nabla u|^2 \theta dx \quad \left(\le C|B|^{\frac{2}{d}} \frac{\mathcal{E}(u,u)}{\theta(B)} \right) \tag{11}$$

being B any ball in $\mathbb{R}^d$ and $u \in C_0^\infty(B)$.

Working with admissible weights has the advantage of being able to state Hölder continuity results for weak solutions to (9). It is still an open problem the identification of the optimal conditions that a weight has to satisfy so that the weak solutions are continuous, see the survey paper [Cav08] for details.

Many authors relied on Muckenhaupt's classes and weighted Sobolev spaces to prove homogenization results. We quote [DASC92] for the periodic case and [EPPW06] for the ergodic case. In the latter the weights are assumed to belong to a Muckenhaupt class for almost all the realizations of the environment.

In our paper, to prove the sublinearity of the corrector, we assume that the coefficient $a^\omega(x)$ satisfies

$$\lambda^\omega(x)|\xi|^2 \le \langle a^\omega(x)\xi, \xi\rangle \le \Lambda^\omega(x)|\xi|^2, \quad \mu\text{-a.s.}$$

and $\mathbb{E}_\mu[\lambda^{-q}], \mathbb{E}_\mu[\lambda^p] < \infty$ with $1/p + 1/q < 2/d$. In this case, the weights $\lambda^\omega(x) := \lambda(\tau_x\omega)$ and $\Lambda^\omega(x) := \Lambda(\tau_x\omega)$ do not belong to any of the classes mentioned above, since, as explained in the remark above, in general the measures $\lambda^\omega(x)dx$ and $\Lambda^\omega(x)dx$ are not volume doubling. The ergodicity of the environment and the fact that $\mathbb{E}_\mu[\lambda^{-1}], \mathbb{E}_\mu[\Lambda]$ are finite ensure only that

$$\sup_{x\in\mathbb{R}^d} \limsup_{R\to\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} \frac{1}{\lambda^\omega(y)}\, dy < \infty, \quad \sup_{x\in\mathbb{R}^d} \limsup_{R\to\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} \Lambda^\omega(y) dy < \infty,$$

μ-almost surely, and, contrary to (10), it is not possible to interchange the supremum and the limit staying finite. Another characterizing feature of our model is that we don't assume $\Lambda^\omega \le c\lambda^\omega$.

We cannot expect regularity for solutions to (9), however, we show that the ergodicity of the environment and the moment conditions $(a.2)$ are enough to obtain a quenched invariance principle, this is done in the same spirit of [FK97] where an unbounded but uniformly bounded away from zero non-symmetric case is considered.

Moser's method to derive a maximal inequality for solutions to (9) is based on two steps. One wants first to get a Sobolev inequality to control some L^ρ-norm in terms of the Dirichlet form and then control the Dirichlet form of any solution by a lower moment. This sets up an iteration which leads to control the supremum of the solution on a ball by a lower norm on a slightly bigger ball. In the uniformly elliptic and bounded case this is rather standard and it is possible to control the $L^{2d/(d-2)}$-norm of a solution by its L^2-norm through the classical Sobolev inequality. In the case of Muckenhaupt's weights the iteration can be set using the Sobolev inequality (11) on the weighted Sobolev space.

In our situation we are able to control locally on balls the ρ-norm of a solution by its $2p/(p-1)$-norm, with $\rho = 2qd/(q(d-2)+d)$. For Moser's iteration we need $\rho > 2p/(p-1)$ which is equivalent to the condition $1/p + 1/q < 2/d$. Indeed, by means of Hölder's inequality and the standard Sobolev inequality, for a ball B of radius $R > 0$ and center $x \in \mathbb{R}^d$, we can write

$$\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} |u|^\rho dy\right)^{\frac{2}{\rho}} \le C_{sob}\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} (\lambda^\omega)^{-q} dy\right)^{\frac{1}{q}} |B_R(x)|^{\frac{2}{d}} \frac{\mathcal{E}^\omega(u,u)}{|B_R(x)|}.$$

The Sobolev inequality above must be compared with (11). In opposition to (11), the constant in front of the inequality is strongly dependent on $x \in \mathbb{R}^d$ and $R > 0$. Therefore, the estimates we derive in Chapter 2 to control the Dirichlet form of a solution by its $2p/(p-1)$-norm, although following from very well established arguments, are a necessary step in order to clarify the dependence of the constants on

$$\frac{1}{|B_R(x)|}\int_{B_R(x)} (\lambda^\omega(y))^{-q} dy, \quad \frac{1}{|B_R(x)|}\int_{B_R(x)} (\Lambda^\omega(y))^{p} dy.$$

The maximal inequality which we obtain in Chapter 2 behaves nicely in the scaling limit, thanks to the ergodic theorem, and this will be enough to state the sublinearity of the correctors.

Remark. *It is believed that the optimal condition for a quenched invariance principle to hold is $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda] < \infty$. In periodic environment this has been proven recently in [BM15] using ideas coming from harmonic analysis and Muckenhaupt's weights. The authors consider a generator in divergence form given by $Lu = e^V \nabla \cdot (e^{-V}\nabla u)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is periodic and measurable such that $e^V + e^{-V}$ is locally integrable. Their argument relies on a time change and on the Sobolev inequality*

$$\left(\int_{\mathbb{T}^d} |u|^r w \, dx\right)^{\frac{2}{r}} \le C \int_{\mathbb{T}^d} |\nabla u|^2 e^{-V} dx$$

where $\mathbb{T}^d$ is the d-dimensional torus, $u \in C^1(\mathbb{T}^d)$ centered, $r > 2$ and w is expressed as an Hardy-Littlewood maximal function.

In this setting it is not possible to use Moser's iteration technique to prove the sublinearity of the corrector on balls, since to bound the right hand side by the $L^s(\mathbb{T}^d, w)$ norm for some $s < r$ would require further assumptions on the integrability of $e^V + e^{-V}$. In fact, they don't prove sublinearity of the correctors on balls but along the path of the process. This approach relies on a global uniform upper bound for the density of the process, which can be established due to the compactness of the periodic environment, and the fact that the process of the environment seen from the particle is just the projection of the diffusion on the torus $\mathbb{T}^d$.

Parabolic Harnack inequality and Local Central Limit Theorem. The proof of Theorem II contains as its main ingredient a parabolic Harnack inequality for solutions to the "formal" parabolic equation

$$\partial_t u(t,x) - \frac{1}{\Lambda^\omega(x)} \nabla \cdot (a^\omega(x)\nabla u(t,x)) = 0, \qquad t \in (0,\infty),\ x \in \mathbb{R}^d. \tag{12}$$

It is well known that when $x \mapsto a^\omega(x)$ and $x \mapsto \Lambda^\omega(x)$ are bounded and bounded away from zero, uniformly in $\omega \in \Omega$, then a parabolic Harnack inequality holds for solutions to (12), this is a celebrated result due to Moser [Mos64]. He showed that there is a positive constant C_{PH}, which depends only on the uniform bounds on a and Λ, such that for any positive weak solution of (12) on $(t, t+R^2) \times B_R(x)$ we have

$$\sup_{(s,z)\in Q_-} u(s,z) \le C_{PH} \inf_{(s,z)\in Q_+} u(s,z)$$

where $Q_- = (t + 1/4R^2, t + 1/2R^2) \times B_{R/2}(x)$ and $Q_+ = (t + 3/4R^2, t + R^2) \times B_{R/2}(x)$. The parabolic Harnack inequality plays a prominent role in the theory of partial differential equations, in particular to prove Hölder continuity for solutions to (12), as it was observed by Nash [Nas58] and De Giorgi [De 57], or to prove Gaussian bounds for the fundamental solution of (12) as done by Aronson [Aro67]. It is remarkable that such results do not depend neither on the regularity of a nor of Λ.

We shall exploit the stability of Moser's method to derive a parabolic Harnack inequality also in the case of degenerate and possibly unbounded coefficients. The technique is quite flexible and can also be applied to discrete space models for which we refer to [ADS15a] and [ADS15b].

Similarly to the derivation of the maximal inequality for the Poisson equation, with Moser's iteration technique we are able to bound the L^∞-norm of a caloric function u by its L^α-norm for some finite $\alpha > 0$, on a slightly larger ball. Since the same holds for u^{-1}, what is left to do is to link the L^α-norm of u and the L^α-norm of u^{-1}. In the uniformly elliptic case this is achieved by means of the exponential integrability of BMO functions, hence with John-Niremberg inequality. In the present work we exploit an abstract lemma due to Bombieri and Giusti [BG72] (See Lemma 3.2.1 below) for which application, besides the maximum inequality for u, we will need to establish weighted Poincaré inequalities.

Following the classic proof of Moser, with some extra care due to the different exponents, we get a local parabolic Harnack inequality for solutions to (12) in our setting. In the uniformly elliptic and bounded case the constant in front of the Harnack inequality depends only on the uniform bounds on a and Λ. In our setting we cannot expect that to be true for general weights, and the constant will strongly depend on the center and the radius of the ball, in particular we don't have any control for small balls, so that a genuine Hölder continuity result like the one of Nash is not given. Luckily, in the diffusive limit, the ergodic theorem helps to control constants and to prove Theorem II.

Remark. *Given a speed measure $\theta : \Omega \to (0, +\infty)$ one can also consider the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d, \theta^\omega)$ where $\mathcal{E}^\omega$ is given by* (6) *and $\mathcal{F}^{\theta,\omega}$ is the closure of of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \theta^\omega)$ with respect to $\mathcal{E}^\omega + (\cdot,\cdot)_{\theta^\omega}$. This corresponds to the formal generator*

$$L^\omega u(x) = \frac{1}{\theta^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(x)). \tag{13}$$

One can show along the same lines of the proof for $\theta = \Lambda$ that if

$$\mathbb{E}_\mu[\theta^r] < \infty,\ \mathbb{E}_\mu[\lambda^{-q}] < \infty,\ \mathbb{E}_\mu[\Lambda^p \theta^{1-p}] < \infty,$$

where $p, q, r \in (1, \infty]$ *are such that*

$$\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d},$$

then the parabolic Harnack inequality still works, in particular a quenched local central limit theorem can still be derived in this situation.

Observe that in the case $\theta = \Lambda$ *we find back the familiar* $1/r + 1/q < 2/d$ *while in the case* $\theta \equiv 1$, $r = \infty$ *the condition reads* $1/(p-1) + 1/q < 2/d$.

Remark. *One particular example which arises from the general form* (13) *is the one with* $\theta = e^{V^\omega(x)}$ *and* $a^\omega(x) = e^{-V^\omega(x)} I_d$, *being* I_d *the* d*-dimensional identity matrix. The generator corresponding to this choice reads*

$$L^\omega u(x) = e^{V^\omega(x)} \nabla \cdot (e^{-V^\omega(x)} \nabla u(x)).$$

Remark. *Despite the fact that condition* $1/p + 1/q < 2/d$ *given by* $(a.2)$ *seems not to be sharp for a quenched invariance principle to hold, it is morally optimal to state Theorem II. Indeed in the discrete space setting it was shown that if* $1/p + 1/q > 2/d$, *then there exists an ergodic environment for which the quenched local central limit theorem does not hold [ADS15a][See Theorem 5.4]. One could possibly construct a counterexample also in the continuous by exploiting the same ideas given in [ADS15a].*

Description of the content of the thesis. We have divided the thesis into two parts, the first being focused on Partial Differential Equations theory and the second being about diffusions in random environment. The first part is interesting on its own and will be used heavily to prove the results in the second part.

Part I. Here we deal with a priori estimates for solutions to Partial Differential Equations. We present a deterministic model, which will correspond to a diffusion where we look at a fixed random environment.

In Chapter 1 we will review some classical results in Sobolev spaces theory and extend the classical Sobolev, Nash and Poincaré inequalities for weighted spaces. The inequalities obtained there will be local, in the sense that the constants involved will strongly depend on the choice of the center and the radius of the ball of interest.

In Chapter 2 we will derive a priori estimates for weak solutions to degenerate elliptic equations. The main result of the chapter is a maximal inequality which will be used to prove the quenched invariance principle in Chapter 5. The inequality is obtained with Moser iteration technique.

In Chapter 3 we will derive a priori estimates for weak solutions to degenerate homogeneous parabolic equations. The main result of the chapter is a local parabolic Harnack inequality which will be applied in Chapter 6 to prove the quenched Local Central Limit Theorem.

Part II. This part is devoted to diffusions in degenerate and unbounded random environment.

In Chapter 4 we address the subtle question of constructing a diffusion associated to the formal generator (5). This requires the theory of Dirichlet forms which will be presented to the reader in some generality. The general theory discussed in this chapter will find a precise application to the generator (5).

In Chapter 5 we study symmetric diffusions in stationary and ergodic random environment. The chapter is devoted to the construction of the correctors and to the proof of their sublinearity. This result will lead to a proof of a quenched invariance principle, namely, the convergence in distribution of the process $\epsilon X_{\cdot/\epsilon^2}$ to a non degenerate Brownian motion as $\epsilon \to 0$.

In Chapter 6 we prove that, provided that a quenched invariance principle holds, the density of the process $\epsilon X_{\cdot/\epsilon^2}$ converges uniformly on compacts to a non-degenerate Gaussian density.

Finally, in the *Conclusions* chapter we will comment the results and present some of the natural problems and questions one could possibly address.

Part I

A priori estimates for solutions to degenerate PDEs

1

Sobolev type inequalities

In the first section we give some basic definitions and we collect, for the reader convenience, some facts from the classical theory of Sobolev spaces and some concepts from classical analysis which will be extensively used in the manuscript. Most of the attention will be put in recalling the classical Sobolev inequality and Sobolev embedding theorems. In Section 1.2 we introduce the notation and the spaces related to the symmetric form obtained formally integrating by parts the operator $L = \mathrm{div}(a(x)\nabla\cdot)$. In sections 1.3, 1.4 and 1.5 we will finally extend the classical Sobolev, Nash and Poincaré inequalities to some weighted spaces using only local integrability of the coefficients and Hölder inequality. This plan is carried out in order to get a priori estimates for solutions to degenerate elliptic and parabolic partial differential equations in the next chapters.

1.1 Basic definitions and notation

1.1 Classical analysis. Let U be an open domain in $\mathbb{R}^d$ and $\theta : U \to \mathbb{R}$ be a non-negative measurable function. If $1 \le r < \infty$, then we define the real Banach space $L^r(U, \theta)$ to be the vector space of measurable functions $u : U \to \mathbb{R}$ for which

$$\|u\|_{r,\theta} := \left(\int_U |u(x)|^r \theta(x) dx \right)^{1/r}$$

is finite. We identify two functions which coincide outside a null set for the measure $m^\theta(dx) := \theta(x)dx$ without any further comment. The space $L^\infty(U, \theta)$ is defined to be the space of measurable functions u for which

$$\|u\|_{\infty,\theta} = \min\{\lambda : m^\theta\{|u| > \lambda\} = 0\}$$

is finite. It can be shown that

$$\|u\|_{\infty,\theta} = \lim_{r\to\infty} \|u\|_{r,\theta} \tag{1.1}$$

whenever $u \in L^r(U, \theta)$ for all large enough r. When $\theta \equiv 1$, the measure m^θ coincides with the Lebesgue measure and we will simply write $L^r(U)$ for $L^r(U, \theta)$ and $\| \cdot \|_r$ for $\| \cdot \|_{r,\theta}$.

Remark 1.1.1. *For our applications θ is strictly positive almost everywhere, which implies that $\theta(x)dx$ is equivalent to the Lebesgue measure, and therefore that $L^\infty(U,\theta)$ coincides with $L^\infty(U)$.*

If $r \in [1,\infty]$, then we define $L^r_{loc}(U,\theta)$ to be the vector space of measurable functions $u : U \to \mathbb{R}$ whose restriction $u|_K$ to any compact $K \subset U$ belongs to $L^r(K,\theta)$.

1.2 Sobolev Spaces. Here we recall some of the theory on weak differentiability and Sobolev spaces. For a complete treatment on the subject we refer to [Eva10], [GT01], [SC02].

Let U be an open bounded domain in $\mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$ a multi-index. Denote by $D^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_d}$ and by $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We say that a function $u \in L^1_{loc}(U)$ has weak α^{th}-derivative if there exists a function $v \in L^1_{loc}(U)$ such that

$$\int_U u D^\alpha \eta \, dx = (-1)^{|\alpha|} \int_U v\eta \, dx, \quad \forall \eta \in C_0^\infty(U),$$

where $C_0^\infty(U)$ is the set of all infinitely differentiable functions with compact support in U. We write $D^\alpha u := v$ and note that $D^\alpha u$ is uniquely determined up to sets of measure zero, accordingly, pointwise relations involving weak derivatives will be understood to hold almost everywhere. We denote by $W^1(U)$ the set of weakly differentiable functions in U.

The next classical result allows to extend most of the properties true for classical derivatives to weak derivatives. For the proof we refer to [GT01, Theorem 7.4].

Theorem 1.1.2. *Let u, v be locally integrable on U. Then $v = D^\alpha u$ if and only if there exists a sequence of $C^\infty(U)$ functions $\{u_n\}_{n\in\mathbb{N}}$ such that $D^\alpha u_n \to v$ and $u_n \to u$ in $L^1_{loc}(U)$.*

By means of Theorem 1.1.2 we can easily prove that the product rule $D(uv) = uDv + vDu$ holds for all $u, v \in W^1(U)$ provided that uv, $vDu + uDv \in L^1_{loc}(U)$. The next proposition concerns the chain rule for weakly differentiable functions.

Proposition 1.1.3. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 function, i.e. a continuous function with piecewise continuous first derivative. Assume further $\|\Phi'\|_\infty < \infty$. If $u \in W^1(U)$ then $\Phi \circ u \in W^1(U)$. Furthermore, letting L the set of corner points of Φ, $D(\Phi \circ u) = Du(\Phi' \circ u)$ whenever $u \in L$, $D(\Phi \circ u) = 0$ otherwise.*

Let $1 \le r \le \infty$ and k a non-negative integer. We denote by

$$W^{k,r}(U) := \{u \in L^1_{loc}(U) \,:\, D^\alpha u \in L^r(U), \forall |\alpha| \le k\}.$$

The space $W^{k,r}(U)$ is clearly a linear space which becomes a Banach space once it is endowed with the norm

$$\|u\|_{W^{k,r}(U)} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^r(U)}.$$

Another Banach space is the set $W_0^{k,r}(U)$, which arises taking the closure of $C_0^\infty(U)$ in $W^{k,r}(U)$.

Remark 1.1.4. *The spaces $W^{k,r}(U)$ and $W_0^{k,r}(U)$ do not coincide when U is bounded. If $U = \mathbb{R}^d$ then they coincide.*

Local spaces $W_{loc}^{k,r}(U)$ can be defined to consist of functions which belong to $W^{k,r}(U')$ for all U' compact subset of U.

The case in which $r = 2$ is special since the norm $\|u\|_{W^{k,2}(U)}$ is associated to a scalar product, so that $W^{k,2}(U)$ and $W_0^{k,2}(U)$ are indeed Hilbert spaces. In the case $r = \infty$ Sobolev and Lipschitz spaces are related; in particular $W_{loc}^{k,\infty}(U)$ coincide with $C^{k-1,1}(U)$ and $W^{k,\infty}(U)$ with $C^{k-1,1}(\overline{U})$ for sufficiently smooth ∂U.

The next classical result states that smooth functions are dense in Sobolev spaces.

Theorem 1.1.5. *The subspace $C^\infty(U) \cap W^{k,r}(U)$ is dense in $W^{k,r}(U)$.*

This result shows that we could have characterized $W^{k,r}(U)$ as the completion of $C^\infty(U)$ with respect to the norm $\|\cdot\|_{W^{k,r}(U)}$, this is useful in many situations. If the domain U is sufficiently smooth, for example has C^1-boundary, then the same theorem holds with $C^\infty(U)$ replaced by the smaller $C^\infty(\overline{U})$.

A very handy characterization of Sobolev spaces is the so called "absolutely continuous on lines" characterization due to Nikodym [MS11, Section 1.1.3]. Let U be an open set in $\mathbb{R}^d$ and $1 \le r \le \infty$. If a function is in $W^{1,r}(U)$, then, possibly after modifying the function on a set of measure zero, the restriction to almost every line parallel to the coordinate directions in $\mathbb{R}^d$ is absolutely continuous, therefore almost everywhere differentiable; furthermore, the classical derivative along the coordinates directions are in $L^r(U)$. Conversely, if the restriction of a function u to almost every line parallel to the coordinate directions is absolutely continuous, then the pointwise gradient ∇u exists almost everywhere, and u belongs to $W^{1,r}(U)$ provided u and $|\nabla u|$ are both in $L^r(U)$. In particular, in this case the weak partial derivatives of u and the pointwise partial derivatives of u agree almost everywhere.

Classical Sobolev inequality and embedding theorems. We state here the classical Sobolev inequality as it is presented in [SC02, Theorem 1.5.2] for later reference.

Theorem 1.1.6. *Let B be an open ball of $\mathbb{R}^d$. Fix $1 \le r < d$ and let $q = rd/(d-r)$. There exists a constant $C(d,r)$ such that for all functions $u \in C_0^\infty(B)$*

$$\|u\|_{L^s(B)} \le C(d,r)|B|^{\frac{1}{d}+\frac{1}{s}-\frac{1}{r}}\|\nabla u\|_{L^r(B)} \tag{1.2}$$

for all $1 \le s \le q$. When u is a smooth function in B, $u \in C^\infty(B)$, we have instead

$$\|u-(u)_B\|_{L^s(B)} \le C(d,r)|B|^{\frac{1}{d}+\frac{1}{s}-\frac{1}{r}}\|\nabla u\|_{L^r(B)} \tag{1.3}$$

where $(u)_B := \frac{1}{|B|}\int_B u dx$ is the average of u on the ball B.

Inequality (1.2) can be generalized to any bounded domain U in $\mathbb{R}^d$. The question if this still remains true for (1.3) is more delicate and requires a subtle analysis on the regularity of the boundary of U, moreover we cannot expect that the constant does not depend on the domain.

Notice that Theorem 1.1.6 holds whenever $r < d$, what can we say about the size of smooth functions when $r > d$? We have the following theorem, for the proof we refer to [SC02][Theorem 1.4.2].

Theorem 1.1.7. *Let U be an open bounded domain in $\mathbb{R}^d$. Take $r > d$, then there exists a constant $C(d,r)$ such that for all $u \in C_0^\infty(U)$*

$$\|u\|_\infty \leq C(d,r)|U|^{\frac{1}{d}-\frac{1}{r}}\|\nabla u\|_{L^r(U)}. \tag{1.4}$$

It is clear that it is possible to extend by approximation (1.2) and (1.4) for functions in $W_0^{1,r}(U)$. These two results combined allows to state embeddings for Sobolev spaces. Theorem 1.1.6 tells us that $W_0^{1,r}(U)$ is *continuously embedded* in $L^{dr/(d-r)}(U)$ whenever $r < d$, on the other hand, Theorem 1.1.7 states that $W_0^{1,r}(U)$ is *continuously embedded* in $C^0(\overline{U})$ if $r > n$.

A further refinement is obtained through a theorem due to Morrey which shows that it is possible to improve the embedding for $r > d$ into the set of Hölder continuous functions. More precisely, there exits a constant $C(d,r)$ such that for any function $u \in W_0^{1,r}(U)$ and any ball B_R of radius $R > 0$

$$\max_{U\cap B_R} u - \min_{U\cap B_R} u \leq C(d,r)R^\alpha\|\nabla u\|_{L^r(U)},$$

where $\alpha = 1 - d/r$. Using the previous results iteratively it is possible to prove the following embedding theorems.

Theorem 1.1.8. *Let U be a bounded domain in $\mathbb{R}^d$ then,*

(i) if $kr < d$, the space $W_0^{k,r}(U)$ is continuously embedded in $L^{r^*}(U)$ being $r^* = dr/(d-kr)$, and compactly embedded in $L^q(U)$ for any $q < r^*$;

(ii) if $0 \leq m < k-d/r < m+1$, the space $W_0^{k,r}(U)$ is continuously embedded in $C^{m,\alpha}(\overline{U})$ being $\alpha = k - d/r - m$, and compactly embedded in $C^{m,\beta}(\overline{U})$ for any $\beta < \alpha$.

In general $W_0^{k,r}(U)$ cannot be replaced by $W^{k,r}(U)$ in the theorem above. However, this replacement is possible when the domain U is regular enough, e.g. if U has a Lipschitz boundary.

Theorem 1.1.9. *Let U be a $C^{0,1}$ domain in $\mathbb{R}^d$ then,*

(i) if $kr < d$, the space $W^{k,r}(U)$ is continuously embedded in $L^{r^*}(U)$ being $r^* = dr/(d-kr)$, and compactly embedded in $L^q(U)$ for any $q < r^*$;

(ii) if $0 \leq m < k-d/r < m+1$, the space $W^{k,r}(U)$ is continuously embedded in $C^{m,\alpha}(\overline{U})$ being $\alpha = k - d/r - m$, and compactly embedded in $C^{m,\beta}(\overline{U})$ for any $\beta < \alpha$.

Remark 1.1.10. *For the application we have in mind $k = 1$ and the set U is just a ball $B \subset \mathbb{R}^d$, which clearly has a smooth boundary.*

1.3 Hölder's inequality. In order to extend some of the classical inequalities valid for the flat space to spaces with weighted measure $m^\theta(x) = \theta(x)dx$, the most effective tool we exploited is Hölder's inequality. This inequality can be stated in great generality and it is an indispensable instrument for the study of L^r-spaces.

Given $r \in [1, \infty]$, we denote by $r^* = r/(r-1)$ the Hölder's conjugate, namely the only real number such that

$$\frac{1}{r} + \frac{1}{r^*} = 1.$$

The classical Hölder's inequality states that if $u, v : U \to \mathbb{R}$ are measurable functions, then

$$\|uv\|_{1,\theta} \le \|u\|_{r,\theta} \|v\|_{r^*,\theta}. \tag{1.5}$$

A slightly more sophisticated version, which is very useful at times, states that if $\nu \in (0,1)$ and $r, r_1, r_2 \in [1, \infty]$ are such that

$$\frac{1}{r} = \frac{\nu}{r_1} + \frac{1-\nu}{r_2},$$

then

$$\|uv\|_{r,\theta} \le \|u\|_{r_1,\theta}^{\nu} \|v\|_{r_2,\theta}^{1-\nu}. \tag{1.6}$$

For proofs of all these statements we refer to your favorite book in analysis, for example see [Fol99]. We will apply Hölder's inequality several times, the next lemma shows a typical application for us, since it relates the flat space $L^1_{loc}(\mathbb{R}^d)$ with the weighted space $L^r(\mathbb{R}^d, \theta)$.

Lemma 1.1.11. *Let $\theta^{-1/(r-1)} \in L^1_{loc}(\mathbb{R}^d)$. Then, $L^r(\mathbb{R}^d, \theta) \subset L^1_{loc}(\mathbb{R}^d)$ and $u_n \to u$ in $L^r(\mathbb{R}^d, \theta)$ implies $u_n \to u$ in $L^1_{loc}(\mathbb{R}^d)$.*

Proof. Take any compact $K \subset \mathbb{R}^d$, then the proof is all about Hölder's inequality

$$\int_K |u| dx = \int_K |u| \theta^{1/r} \theta^{-1/r} dx \le \left(\int_{\mathbb{R}^d} |u|^r \theta dx \right)^{1/r} \left(\int_K \theta^{-1/(r-1)} dx \right)^{1/r^*},$$

and from this it is easy to conclude. □

1.2 Symmetric forms

In this section we introduce some notation and we prove a few properties for the symmetric form which originates from the formal generator $Lu(x) = \operatorname{div}(a(x)\nabla u(x))$.

Assumption b.1. Fix $d \ge 2$. We are given a symmetric matrix $a : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $\lambda, \Lambda : \mathbb{R}^d \to [0, \infty]$ such that $\lambda^{-1}, \Lambda \in L^1_{loc}(\mathbb{R}^d)$ and for almost all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$

$$\lambda(x)|\xi|^2 \le \sum_{i,j}^{d} a_{ij}(x) \xi_j \xi_i \le \Lambda(x)|\xi|^2.$$

Let $\theta : \mathbb{R}^d \to [0, +\infty)$. We can define a symmetric bilinear form $\mathcal{E}$ on $L^2(\mathbb{R}^d, \theta)$ by

$$\mathcal{E}(u, v) := \sum_{i,j}^{d} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u(x)\partial_j v(x)\, dx, \quad u, v \in C_0^\infty(\mathbb{R}^d). \tag{1.7}$$

Observe that such a bilinear form is well defined as soon as $\Lambda \in L^1_{loc}(\mathbb{R}^d)$ and $\theta \in L^1_{loc}(\mathbb{R}^d)$. Indeed, it suffices to show that for $u \in C_0^\infty(\mathbb{R}^d)$, $\|u\|_{2,\theta}$ and $\mathcal{E}(u, u)$ are finite, and clearly

$$\int_{\mathbb{R}^d} |u|^2 \theta dx \leq \|u\|_\infty^2 \int_{\operatorname{supp} u} \theta dx, \quad \mathcal{E}(u, u) \leq \|\nabla u\|_\infty^2 \int_{\operatorname{supp} u} \Lambda dx$$

are finite being $\operatorname{supp} u$ a compact set.

We denote by $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_\theta$, where $(\cdot, \cdot)_\theta$ is the scalar product induced by $\| \cdot \|_{2,\theta}$. Then, the space $C_0^\infty(\mathbb{R}^d)$ is pre-Hilbert with respect to the scalar product $\mathcal{E}_1$. The next proposition shows that $(C_0(\mathbb{R}^d), \mathcal{E}_1)$ can be completed to become an Hilbert space.

Proposition 1.2.1. *Assume* $(b.1)$ *and that* $\theta, \theta^{-1} \in L^1_{loc}(\mathbb{R}^d)$*. Then the symmetric form* $\mathcal{E}$ *with domain* $C_0^\infty(\mathbb{R}^d)$ *is* closable*, namely if* u_n *is a sequence in* $C_0^\infty(\mathbb{R}^d)$ *such that* $\mathcal{E}(u_n - u_m, u_n - u_m) \to 0$ *as* $n, m \to \infty$ *and* $\|u_n\|_{2,\theta} \to 0$*, then* $\mathcal{E}(u_n, u_n) \to 0$ *as* $n \to \infty$*.*

Proof. As a first remark notice that $\theta^{-1}, \lambda^{-1} \in L^1_{loc}(\mathbb{R}^d)$ gives that $\lambda > 0$ and $\theta > 0$ almost everywhere. The proof is an adaptation of an argument given in [Röc93, Chapter II, example 3b] where more general conditions are given, and it is divided in two steps. We first look at the diagonal case, and then we extend the proof to our situation.

First step ($\lambda = \Lambda$). Define

$$\mathcal{E}_\lambda(u, v) := \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \lambda dx, \quad u, v \in C_0^\infty(\mathbb{R}^d).$$

Notice that $\mathcal{E}_\lambda(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$ implies that $\partial_i u_n$ is a Cauchy sequence in the complete space $L^2(\mathbb{R}^d, \lambda)$, for all $i = 1, \dots, d$, in particular there exist $v_i \in L^2(\mathbb{R}^d, \lambda)$ such that $\partial_i u_n \to v_i$ in $L^2(\mathbb{R}^d, \lambda)$. On the other hand observe that convergence in $L^2(\mathbb{R}^d, \lambda)$ or in $L^2(\mathbb{R}^d, \theta)$ implies convergence in $L^1_{loc}(\mathbb{R}^d)$ since λ^{-1} and θ^{-1} are locally integrable. Indeed, for any compact $K \subset \mathbb{R}^d$ we have by Hölder inequality

$$\int_K |u| dx \leq \left(\int_K \lambda^{-1} dx \right)^{1/2} \|u\|_{2,\lambda}$$

and similarly for θ. Next take any test function $\eta \in C_0^\infty(\mathbb{R}^d)$, by integrating by parts against η, we have

$$\int_{\mathbb{R}^d} \eta \partial_i u_n dx = - \int_{\mathbb{R}^d} \partial_i \eta u_n dx. \tag{1.8}$$

By the remark above both $u_n \to 0$ and $\partial_i u_n \to v_i$ in $L^1_{loc}(\mathbb{R}^d)$, therefore we can pass to the limit in (1.8) and get that for all $\eta \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \eta v_i dx = 0,$$

which implies that $v_i = 0$ almost everywhere and in particular that $\mathcal{E}_\lambda(u_n, u_n) \to 0$.

Second step (General case). As a consequence of $(b.1)$ we have that $\mathcal{E}_\lambda(u,u) \le \mathcal{E}(u,u)$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Therefore, the previous step implies that $\mathcal{E}_\lambda(u_n,u_n) \to 0$ as $n \to \infty$. In particular there exists a subsequence such that $\nabla u_{n_k} \to 0$ almost everywhere. By Fatou's lemma

$$\begin{aligned} 0 \le \mathcal{E}(u_n,u_n) &\le \liminf_{k\to\infty} \sum_{i,j}^{d} \int_{\mathbb{R}^d} a_{ij}\partial_i(u_n - u_{n_k})\partial_j(u_n - u_{n_k})dx \\ &= \liminf_{k\to\infty} \mathcal{E}(u_n - u_{n_k}, u_n - u_{n_k}). \end{aligned}$$

This end the proof since the quantity on the right hand side can be made arbitrarily small for n large. □

Remark 1.2.2. *More general conditions for the conclusions of Proposition 1.2.1 to hold are known in the literature, one is the Hamza condition which can be found for example in [Röc93]. In the application to diffusion in random environment the local integrability is readily granted by the moment conditions on the environment.*

The importance of Proposition 1.2.1 is that the symmetric form $\mathcal{E}$ with domain $C_0^\infty(\mathbb{R}^d)$ admits a closed extension in $L^2(\mathbb{R}^d,\theta)$, the smallest closed extension is obtained by taking as domain the limits of the $\mathcal{E}_1$-Cauchy sequences.

We denote by $\mathcal{F}^\theta$ the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d,\theta)$ with respect to $\mathcal{E}_1$, when $\theta \equiv 1$ we will omit it from the notation and simply write $\mathcal{F}$.

Lemma 1.2.3. *Let assume $(b.1)$ and that $\theta, \theta^{-1} \in L^1_{loc}(\mathbb{R}^d)$. Then $\mathcal{F}^\theta \subset W^{1,1}_{loc}(\mathbb{R}^d)$, and for all $u \in \mathcal{F}^\theta$*

$$\mathcal{E}(u,u) = \sum_{i,j}^{d} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u(x)\partial_j u(x)\, dx.$$

where the derivatives $\partial_i u$ are taken in the weak sense.

Proof. If $u \in \mathcal{F}^\theta$ then by definition there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^d)$ such that $u_n \to u$ with respect to $\mathcal{E}_1$, in particular $u_n \to u$ in $L^1_{loc}(\mathbb{R}^d)$ and there exist v_i, $i = 1,\dots,d$ such that $\partial_i u_n \to v_i$ in $L^1_{loc}(\mathbb{R}^d)$. This shows that for any smooth test function $\eta \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \partial_i \eta u dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \partial_i \eta u_n dx = -\lim_{n\to\infty} \int_{\mathbb{R}^d} \eta \partial u_n dx = -\int_{\mathbb{R}^d} \eta v_i dx,$$

therefore u is weakly differentiable with weak derivative $\partial_i u = v_i$, it follows that $\mathcal{F}^\theta \subset W^{1,1}_{loc}(\mathbb{R}^d)$. For the second part we know that $\mathcal{E}(u_n,u_n) \to \mathcal{E}(u,u)$ and we observe that along a subsequence $\nabla u_{n_k} \to \nabla u$ almost everywhere, hence by Fatou's lemma

$$\begin{aligned} &\left| \left(\int_{\mathbb{R}^d} \langle a\nabla u, \nabla u\rangle dx \right)^{1/2} - \mathcal{E}(u_n,u_n)^{1/2} \right|^2 \\ &\qquad\qquad \le \liminf_{k\to\infty} \int_{\mathbb{R}^d} \langle a(\nabla u_n - \nabla u_{n_k}), \nabla u_n - \nabla u_{n_k}\rangle dx \end{aligned}$$

and the right hand side is arbitrarily small for n large. Thus, $\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \langle a\nabla u, \nabla u\rangle dx$, which is what we wanted to prove. □

We define $\mathcal{F}^\theta_{loc}$ to be the set of functions which are locally in $\mathcal{F}^\theta$, namely, $u \in \mathcal{F}^\theta_{loc}$ if for all balls $B \subset \mathbb{R}^d$ there exist $\overline{u} \in \mathcal{F}^\theta$ such that $u \equiv \overline{u}$ on B.

In the same way we defined a quadratic form $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(\mathbb{R}^d, \theta)$, we can consider the quadratic form $\mathcal{E}$ on $C_0^\infty(B)$ where B is a ball of $\mathbb{R}^d$. The same conclusion of Proposition 1.2.1 holds and we denote by $\mathcal{F}^\theta_B$ the completion of $C_0^\infty(B)$ in $L^2(B, \theta)$ with respect to $\mathcal{E}_1$. Moreover, similarly to the lemma above one can show that $\mathcal{F}^\theta_B \subset W_0^{1,1}(B)$.

1.3 Sobolev inequalities

In the next three sections we will be interested in controlling L^r-norms of functions $u \in \mathcal{F}^\theta$ by $\mathcal{E}(u,u)$. In some sense this is a natural generalization of the theory of Sobolev spaces where the energy norm $\int_{\mathbb{R}^d} |\nabla u|^2 dx$ is replaced by $\mathcal{E}(u,u)$.

To be more precise we aim to extend Sobolev, Poincaré and Nash type inequalities for the symmetric form $\mathcal{E}$. The first and the second provide an effective tool for deriving local estimates for solutions to elliptic and parabolic degenerate partial differential equations, while the latter will be used to prove the existence of a kernel for the semigroup associated to $\mathcal{E}$ on $L^2(\mathbb{R}^d, \Lambda)$ or $L^2(\mathbb{R}^d)$.

In this particular section we will state local Sobolev inequalities for the symmetric form $\mathcal{E}$ on the flat space $L^2(\mathbb{R}^d)$ and on the weighted space $L^2(\mathbb{R}^d, \Lambda)$. We shall see that the constants appearing in the inequalities are strongly dependent on averages of λ and Λ and in particular on the ball where we focus our analysis.

Assumption b.2. We assume that there exist $p, q \in [1, \infty)$ with $1/p + 1/q < 2/d$ such that $\lambda^{-1} \in L^q_{loc}(\mathbb{R}^d)$ and $\Lambda \in L^p_{loc}(\mathbb{R}^d)$.

Assumptions $(b.1)$ and $(b.2)$ must be compared with $(a.1)$ and $(a.2)$ in the introduction. Roughly speaking, $(b.1)$ and $(b.2)$ are the "deterministic" versions of $(a.1)$ and $(a.2)$ respectively if we think that we are looking at a particular fixed environment.

Lemma 1.3.1. *Assume* $(b.1)$ *and* $(b.2)$*, then*

$$\|1_B \lambda^{-1}\|_q^{-1} \|\nabla u\|^2_{2q/(q+1)} \leq \mathcal{E}(u,u) \leq \|1_B \Lambda\|_p \|\nabla u\|^2_{2p/(p-1)}. \tag{1.9}$$

In particular $W_0^{1,2p/(p-1)}(B) \subset \mathcal{F}_B \subset W_0^{1,2q/(q+1)}(B)$ *and the embeddings are continuous.*

Proof. The proof of the second statement is a direct consequence of (1.9) and on the fact that $L^{2p/(p-1)}(B) \subset L^2(B) \subset L^{2q/(q+1)}(B)$, with the embeddings being continuous.

We are left with the proof of (1.9). By Hölder's inequality with exponents q and $q/(q+1)$ and by $(b.1)$ we can obtain the left hand side as follows

$$\begin{aligned} \|\nabla u\|^2_{2q/(q+1)} &= \Big(\int_B |\nabla u|^{\frac{2q}{q+1}} \lambda^{\frac{q}{q+1}} \lambda^{-\frac{q}{q+1}} \, dx \Big)^{\frac{q+1}{q}} \\ &\leq \|1_B \lambda^{-1}\|_q \int_B |\nabla u|^2 \lambda dx \leq \|1_B \lambda^{-1}\|_q \, \mathcal{E}(u,u), \end{aligned}$$

on the other hand with Hölder's inequality with exponents p and $p/(p-1)$ we get

$$\mathcal{E}(u,u) \le \int_B |\nabla u|^2 \Lambda dx \le \|1_B \Lambda\|_p \|\nabla u\|^2_{2p/(p-1)}.$$

□

Let $B \subset \mathbb{R}^d$ be an open bounded set. For a function $u : B \to \mathbb{R}$, $r \ge 1$ and a weight $\theta : B \to \mathbb{R}$ we denote

$$\|u\|_{r,B} := \left(\frac{1}{|B|} \int_B |u(x)|^r \, dx \right)^{\frac{1}{r}}, \quad \|u\|_{r,B,\theta} := \left(\frac{1}{|B|} \int_B |u(x)|^r \, \theta(x) dx \right)^{\frac{1}{r}}.$$

In the sequel we shall use the symbol $\lesssim$ to say that the inequality $\le$ holds up to a multiplicative positive and finite constant depending only on the dimension $d \ge 2$, or p, q as appearing in Assumption $(b.2)$.

In the next proposition it is enough to assume $\Lambda \in L^1_{loc}(\mathbb{R}^d)$ and $\lambda^{-1} \in L^q_{loc}(\mathbb{R}^d)$.

Proposition 1.3.2 (Local Sobolev inequality). *Fix an open ball $B \subset \mathbb{R}^d$. Then for all $u \in \mathcal{F}_B$*

$$\|u\|^2_{\rho,B} \lesssim C_S^B |B|^{\frac{2}{d}} \frac{\mathcal{E}(u,u)}{|B|}, \tag{1.10}$$

where $C_S^B := \|\lambda^{-1}\|_{q,B}$ and

$$\rho := \frac{2qd}{q(d-2)+d}, \tag{1.11}$$

is the Sobolev conjugate of $2q/(q+1)$.

Proof. We start proving (1.10) for $u \in C_0^\infty(B)$. Since ρ, as defined in (1.11), is the Sobolev conjugate of $2q/(q+1)$, by the classical Sobolev inequality (Theorem 1.1.6) it follows that

$$\|u\|_\rho \lesssim \|\nabla u\|_{2q/(q+1)},$$

where it is clear that we are integrating over B. By inequality (1.9) we have

$$\|\nabla u\|^2_{2q/(q+1)} \le \|1_B \lambda^{-1}\|_q \, \mathcal{E}(u,u),$$

which leads to (1.10) for $u \in C_0^\infty(B)$ after averaging over the ball B. By approximation, the inequality is easily extended to $u \in \mathcal{F}_B$. Indeed it suffices to consider a sequence $u_n \in C_0^\infty(B)$ such that $u_n \to u$ almost surely and $\mathcal{E}(u_n,u_n) \to \mathcal{E}(u,u)$, and to pass to the limit with Fatou's lemma. □

Proposition 1.3.3 (Local weighted Sobolev inequality). *Fix a ball $B \subset \mathbb{R}^d$. Then for all $u \in \mathcal{F}_B^\Lambda$*

$$\|u\|^2_{\rho/p^*,B,\Lambda} \lesssim C_S^{B,\Lambda} |B|^{\frac{2}{d}} \frac{\mathcal{E}(u,u)}{|B|}, \tag{1.12}$$

being $C_S^{B,\Lambda} := \|\lambda^{-1}\|_{q,B} \|\Lambda\|^{2p^*/\rho}_{p,B}$.

Proof. The proof follows readily from Hölder's inequality

$$\|u\|^2_{\rho/p^*,B,\Lambda} \le \|u\|^2_{\rho,B} \|\Lambda\|^{2p^*/\rho}_{p,B}$$

and the previous proposition. □

Remark 1.3.4. *From these two Sobolev inequalities, it follows that the domains $\mathcal{F}_B$ and $\mathcal{F}_B^\Lambda$ coincide for all balls $B \subset \mathbb{R}^d$. This can be deduced from* (1.10)*,* (1.12) *and the fact that due to* $(b.2)$ *we have $\rho, \rho/p^* > 2$.*

Inequalities with cutoffs. Since assumptions $(b.1)$ and $(b.2)$ only assure local integrability of λ^{-1} and Λ, we will need to work with functions that are locally in $\mathcal{F}$ or $\mathcal{F}^\Lambda$ and with cutoff functions. We recall that a function u belongs to $\mathcal{F}_{loc}^\theta$, if for all balls $B \subset \mathbb{R}^d$ there exists $u_B \in \mathcal{F}^\theta$ such that $u \equiv u_B$ almost surely on B. It follows immediately from Remark 1.3.4 that $\mathcal{F}_{loc}^\Lambda = \mathcal{F}_{loc}$ whenever $(b.2)$ is satisfied.

Let $B \subset \mathbb{R}^d$ be a ball, a cutoff on B is a function $\eta \in C_0^\infty(B)$, such that $0 \leq \eta \leq 1$. for $u, v \in \mathcal{F}_{loc}^\theta$ we define the bilinear form

$$\mathcal{E}_\eta(u,v) = \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u(x)\partial_j v(x)\eta^2(x)\,dx. \tag{1.13}$$

Lemma 1.3.5. *Let $B \subset \mathbb{R}^d$ and consider a cutoff $\eta \in C_0^\infty(B)$ as above. Then for all $u \in \mathcal{F}_{loc}$ we have that $\eta u \in \mathcal{F}_B$.*

Proof. Take $u \in \mathcal{F}_{loc}^\Lambda = \mathcal{F}_{loc}$, then there exists $\bar{u} \in \mathcal{F}^\Lambda$ such that $u = \bar{u}$ on $\overline{B}$ with $B \subset \overline{B}$. Let $\{u_n\}_\mathbb{N} \subset C_0^\infty(\mathbb{R}^d)$ be such that $u_n \to \bar{u}$ with respect to $\mathcal{E} + (\cdot,\cdot)_\Lambda$. Clearly $\eta u_n \in \mathcal{F}_B^\Lambda$ and $\eta u_n \to \eta\bar{u} = \eta u$ in $L^2(B,\Lambda)$. Moreover

$$\begin{aligned}
&\mathcal{E}(\eta u_n - \eta u_m, \eta u_n - \eta u_m) \\
&\qquad \leq 2\int_B \langle a\nabla(u_n - u_m), \nabla(u_n - u_m)\rangle dx + 2\int_B \langle a\nabla\eta, \nabla\eta\rangle |u_n - u_m|^2 dx \\
&\qquad \leq 2\mathcal{E}(u_n - u_m, u_n - u_m) + 2\|\nabla\eta\|_\infty^2 \int_B |u_n - u_m|^2 \Lambda dx.
\end{aligned}$$

Hence ηu_n is Cauchy in $L^2(B,\Lambda)$ with respect to $\mathcal{E} + (\cdot,\cdot)_\Lambda$, which implies that $\eta u \in \mathcal{F}_B^\Lambda$ (and hence $\eta u \in \mathcal{F}_B$ by Remark 1.3.4). □

The above lemma shows that we can localize functions in $\mathcal{F}_{loc}$ using cutoffs. We remark that if $1/p + 1/q > 2/d$ we cannot say a priori that given $u \in \mathcal{F}$ and a cutoff $\eta \in C_0^\infty(B)$ then $\eta u \in \mathcal{F}_B$.

Proposition 1.3.6 (Local Sobolev inequality with cutoff)**.** *Fix a ball $B \subset \mathbb{R}^d$ and a cutoff function $\eta \in C_0^\infty(B)$ as above. Then for all $u \in \mathcal{F}_{loc}$*

$$\|\eta u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2}{d}} \Big[\frac{\mathcal{E}_\eta(u,u)}{|B|} + \|\nabla\eta\|_\infty^2 \|u\|_{2,B,\Lambda}^2\Big], \tag{1.14}$$

and, for the weighted version

$$\|\eta u\|_{\rho/p^*,B,\Lambda}^2 \lesssim C_S^{B,\Lambda} |B|^{\frac{2}{d}} \Big[\frac{\mathcal{E}_\eta(u,u)}{|B|} + \|\nabla\eta\|_\infty^2 \|u\|_{2,B,\Lambda}^2\Big]. \tag{1.15}$$

Proof. We prove only (1.14), being (1.15) obtained by it through Hölder's inequality as in the proof of Proposition 1.3.3. Take $u \in \mathcal{F}_{loc}$, by Lemma 1.3.5 $\eta u \in \mathcal{F}_B$, therefore we can apply (1.10) and get

$$\|\eta u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2-d}{d}} \mathcal{E}(\eta u, \eta u).$$

To get (1.14) it suffices to compute with the product rule $\nabla(\eta u) = u\nabla\eta + \eta\nabla u$ and we easily estimate

$$\begin{aligned}\mathcal{E}(\eta u, \eta u) &= \int_{\mathbb{R}^d} \langle a\nabla(\eta u), \nabla(\eta u)\rangle dx \\ &\leq 2\int_{\mathbb{R}^d} \langle a\nabla u, \nabla u\rangle \eta^2 dx + 2\int_{\mathbb{R}^d} \langle a\nabla\eta, \nabla\eta\rangle |u|^2 dx \\ &\leq 2\mathcal{E}_\eta(u,u) + 2\|\nabla\eta\|_\infty^2 \|1_B u\|_{2,\Lambda}^2.\end{aligned}$$

Concatenating the two inequalities and writing $\|1_B u\|_{2,\Lambda}^2 = |B|\,\|u\|_{2,B,\Lambda}^2$ gives

$$\|\eta u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2-d}{d}} \Big[\mathcal{E}_\eta(u,u) + |B|\|\nabla\eta\|_\infty^2 \|u\|_{2,B,\Lambda}^2\Big].$$

Collecting $|B|$ in the right hand side ends the proof. □

1.4 Nash inequalities

In Chapter 4 we will present more in general the theory of symmetric forms. In particular we will see in Theorem 4.2.4 and Proposition 4.2.5 that to the symmetric form $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(\mathbb{R}^d, \theta)$ it is associated a strongly continuous symmetric markovian semigroup $\{P_t^\theta : t > 0\}$. The same remains true for the symmetric form $(\mathcal{E}, \mathcal{F}_B^\theta)$ on $L^2(B, \theta)$ and in this case we denote by $\{P_t^{B,\theta} : t > 0\}$ its strongly continuous symmetric markovian semigroup.

The local Nash inequalities follow as an easy corollary of the Sobolev inequalities (1.10) and (1.12) and provide a machinery to prove that the semigroups mentioned above admit transition kernels.

Proposition 1.4.1 (Nash inequalities). *Let $B \subset \mathbb{R}^d$ be a ball. Then for all $u \in \mathcal{F}_B$ we have*

$$\|u\|_{2,B}^{2+\frac{2}{\mu}} \lesssim C_S^B |B|^{\frac{2-d}{d}} \mathcal{E}(u,u) \|u\|_{1,B}^{\frac{2}{\mu}}, \tag{1.16}$$

where $\mu := \left(\frac{2}{d} - \frac{1}{q}\right)^{-1} > 0$, and

$$\|u\|_{2,\Lambda,B}^{2+\frac{2}{\gamma}} \lesssim C_S^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}(u,u) \|u\|_{1,\Lambda,B}^{\frac{2}{\gamma}}, \tag{1.17}$$

where $\gamma := \frac{p-1}{p}\left(\frac{2}{d} - \frac{1}{p} - \frac{1}{q}\right)^{-1} > 0$.

Proof. We prove only (1.16) being the other completely analogous. By Hölder's inequality

$$\|u\|_{2,B} \leq \|u\|_{\rho,B}^\theta \|u\|_{1,B}^{1-\theta}$$

with $\theta \in (0,1)$ and

$$\frac{1}{2} = (1-\theta) + \frac{\theta}{\rho}.$$

Now solve for θ, use (1.10) to estimate $\|u\|_{\rho,B}$ and the result is obtained. □

Note that the condition $1/p + 1/q < 2/d$ is important to have μ and γ positive, in particular $\gamma \geq d/2$, with the equality holding if $p = q = \infty$. It is well known [Dav90], [SC02, Theorem 4.4.1] that Nash inequality (1.17) for the Dirichlet form $(\mathcal{E}, \mathcal{F}_B^\Lambda)$ implies the ultracontractivity of the semigroup $P_t^{B,\Lambda}$ associated to $\mathcal{E}$ on $L^2(B, \Lambda)$. Therefore $P_t^{B,\Lambda}$ has a density $p_t^{B,\Lambda}(x, y)$ with respect to $\Lambda(x)dx$ which satisfies

$$\sup_{x,y \in B} p_t^{B,\Lambda}(x, y) \lesssim t^{-\gamma} \left[C_S^B |B|^{\frac{2}{d} - \frac{1}{\gamma}} \right]^\gamma, \tag{1.18}$$

where it is once more worthy to notice that $2/d - 1/\gamma \geq 0$, with the equality holding for the non-degenerate situation.

1.5 Poincaré inequalities

Let $B \subset \mathbb{R}^d$ be a ball. Given a measurable weight $\theta : B \to [0, \infty)$, we denote by

$$(u)_B^\theta := \int_B u\, \theta dx \Big/ \int_B \theta dx$$

the average of u with respect to the measure $\theta(x)dx$. If θ is constant we simply write $(u)_B$. Moreover, for $u \in \mathcal{F}_{loc}$ we denote by

$$\mathcal{E}_B(u, u) := \int_B \langle a \nabla u, \nabla u \rangle \, dx,$$

the restriction of $\mathcal{E}$ to B.

Proposition 1.5.1 (Poincaré inequalities). *Let $B \subset \mathbb{R}^d$ be a ball. If $u \in \mathcal{F}_{loc}$, then*

$$\|u - (u)_B\|_{2,B}^2 \lesssim C_P^B |B|^{\frac{2-d}{d}} \mathcal{E}_B(u, u), \tag{1.19}$$

being $C_P^B := \|\lambda^{-1}\|_{d/2,B}$, and

$$\|u - (u)_B^\Lambda\|_{2,B,\Lambda}^2 \lesssim C_P^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}_B(u, u), \tag{1.20}$$

being $C_P^{B,\Lambda} := \|\Lambda\|_{\bar{p},B} \|\lambda^{-1}\|_{\bar{q},B}$ with $\bar{p}, \bar{q} \in [1, \infty]$ such that $1/\bar{p} + 1/\bar{q} = 2/d$.

Proof. (1.19) follows easily from Hölder's inequality and the standard Sobolev inequality (1.3). We now prove (1.20) for $u \in C^\infty(B)$, the final result can be obtained by approximation. As first remark, notice that

$$\begin{aligned} \|u - (u)_B^\Lambda\|_{2,B,\Lambda}^2 &= \inf_{a \in \mathbb{R}} \|u - a\|_{2,B,\Lambda}^2 \\ &\leq \|\Lambda\|_{\bar{p},B} \inf_{a \in \mathbb{R}} \|u - a\|_{2\bar{p}^*,B}^2 \leq \|\Lambda\|_{\bar{p},B} \|u - (u)_B\|_{2\bar{p}^*,B}^2. \end{aligned}$$

By the classical Sobolev inequality (1.3) and by Hölder's inequality

$$\|u - (u)_B\|_{2\bar{p}^*,B}^2 \lesssim |B|^{\frac{2}{d}} \|\nabla u\|_{\beta,B}^2 \leq \|\lambda^{-1}\|_{\bar{q},B} |B|^{\frac{2-d}{d}} \mathcal{E}_B(u, u),$$

where β is such that $2\bar{p}^* d/(d + 2\bar{p}^*) = \beta = 2\bar{q}/(\bar{q} + 1)$, which is true whenever $1/\bar{p} + 1/\bar{q} = 2/d$. Concatenating the two inequalities leads to the result. $\square$

In order to get mean value inequalities for weak solutions to elliptic or parabolic PDEs and from that an Harnack inequality, we will need a Poincaré inequality with a radial cutoff. We consider a cutoff function $\eta : \mathbb{R}^d \to [0,\infty)$ which is supported in a ball $B = B(x_0, R)$ of radius $R > 0$ and center $x_0 \in \mathbb{R}^d$ and which is a radial function, namely, $\eta(x) := \Phi(|x - x_0|)$ where Φ is some non-increasing, non-negative càdlàg function non identically zero on $(R/2, R]$.

Proposition 1.5.2 (Poincaré inequalities with radial cutoff)**.** *Let $B \subset \mathbb{R}^d$ be a ball of radius $R > 0$ and center $x_0 \in \mathbb{R}^d$ and let η be a radial cutoff as above. If $u \in \mathcal{F}_{loc}$, then*

$$\|u - (u)_B^{\eta^2}\|_{2,B,\eta^2} \lesssim M^B C_P^B |B|^{\frac{2-d}{d}} \mathcal{E}_\eta(u,u) \tag{1.21}$$

where $M^B = \Phi(0)/\Phi(r/2)$, and

$$\|u - (u)_B^{\Lambda\eta^2}\|_{2,B,\Lambda\eta^2} \lesssim M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}_\eta(u,u), \tag{1.22}$$

where $M^{B,\Lambda} := M^B \|\Lambda\|_{1,B} / \|\Lambda\|_{1,B/2}$.

Proof. We give the proof only for (1.22) being (1.21) a consequence of it taking $\Lambda \equiv 1$. We want to apply [DM13, Theorem 1]. Accordingly, we define a functional $\Psi(u,s)$: $L^2(\mathbb{R}^d, \Lambda) \times (R/2, R] \to [0,\infty]$ by the position

$$\Psi(u,s) = C_P^{B_s,\Lambda} |B_s|^{\frac{2}{d}} \int_{B_s} a\nabla u \cdot \nabla u dx.$$

for $u \in \mathcal{F}^\Lambda$, and $\Psi(u,s) = \infty$ otherwise, where $B_s := B(x_0, s)$.

Such functional satisfies $\Psi(u + c, s) = \Psi(u,s)$ for all $c \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^d, \Lambda)$. Furthermore,

$$\|u - (u)_{B_s}\|_{2,\Lambda}^2 \lesssim \Psi(u,s)$$

for every $s \in (R/2, R]$ and $u \in \mathcal{F}^\Lambda$ by the Poincaré inequality (1.20). Using the fact that $C_P^{B_s,\Lambda} \lesssim C_P^{B,\Lambda}$, being $s \in (R/2, R]$, it follows from Theorem 1 in [DM13] that there exists $M > 0$ such that

$$\begin{aligned} \|u - (u)_B^{\Lambda\eta^2}\|_{2,\eta^2} &\lesssim M \int_{R/2}^R \Psi(u,s)\nu(ds) \\ &\lesssim M C_P^{B,\Lambda} |B|^{\frac{2}{d}} \int_{R/2}^R \int_B a\nabla u \cdot \nabla u 1_{B_s} dx \, \gamma(ds) \\ &= M C_P^{B,\Lambda} |B|^{\frac{2}{d}} \int_B \eta^2 a\nabla u \cdot \nabla u dx. \end{aligned}$$

Here $\gamma(ds)$ is a non-zero positive σ-finite Borel measure on $(R/2, R]$ such that

$$\eta^2(x) = \int_{R/2}^R 1_{B_s}(x)\, \nu(ds)$$

as in [DM13]. According to [DM13, Theorem 1], M is explicitly given by

$$M := \frac{\|\Lambda\|_{1,B}\Phi(0)}{\|\Lambda\|_{1,B/2}\Phi(1/2)},$$

where we recall that $\eta(x) := \Phi(|x - x_0|)$ for some non-increasing, non-negative càdlàg function Φ non identically zero on $(R/2, R]$. Of course such an inequality is local and we can extend it for $u \in \mathcal{F}_{loc}$. □

2

Elliptic second order linear PDEs

As we have briefly discussed in the introduction, our proof of the quenched invariance principle for the diffusion formally associated to $L^\omega u(x) = \operatorname{div}(a^\omega(x)\nabla u(x))$ is based on the sublinearity of the correctors. This can be proved since the corrector is constructed in such a way that for any fixed environment it is a local weak solution of an equation of the type

$$Lu(x) = \operatorname{div}(a(x)\nabla f(x)). \tag{2.1}$$

We use this information to derive a maximal inequality for the correctors, namely we aim to control the supremum on any ball of any solution by a lower moment on a slightly bigger ball.

The technique we are going to exploit goes back to Moser [Mos61] who derived an elliptic Harnack inequality for positive weak solutions to (2.1) in the case of uniformly elliptic and bounded coefficients. Moser's method to derive a maximal inequality for solutions to (2.1) is based on two steps. One wants first to get a Sobolev inequality to control some L^ρ-norm in terms of the Dirichlet form and then control the Dirichlet form of any solution by a lower moment. This sets up an iteration which leads to control the supremum of the solution on a ball by a lower norm on a slightly bigger ball. In the uniform elliptic and bounded case this is rather standard and it is possible to control the $L^{2d/(d-2)}$-norm of a solution by its L^2-norm through the classical Sobolev inequality. In our case we are able to control locally on balls the ρ-norm of a solution by its $2p/(p-1)$-norm, with $\rho = 2qd/(q(d-2)+d)$, for this we rely on the Sobolev inequalities (1.10) and (1.14). For the Moser iteration we need $\rho > 2p/(p-1)$ which is equivalent to the condition $1/p + 1/q < 2/d$.

In contrast with the classical result for uniformly elliptic operators, the constants in front of the estimates we get are not uniform either in the radius or the center of the ball in which we focus our analysis since they will depend on averages of λ^{-q} and Λ^p in such a ball. Therefore, although the proofs follow very well established arguments, they are a necessary step to understand and control the constants precisely.

2.1 Maximal inequality for Poisson equation

Let $f : \mathbb{R}^d \to \mathbb{R}$ be some function with essentially bounded weak derivatives. We say that $u \in \mathcal{F}_{loc}$ is a solution (subsolution or supersolution) of the Poisson equation, if

$$\mathcal{E}(u, \phi) = -\int_{\mathbb{R}^d} \langle a\nabla f, \nabla\phi\rangle dx \quad (\leq \text{ or } \geq) \tag{2.2}$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi \geq 0$. For a ball $B \subset \mathbb{R}^d$, we say that $u \in \mathcal{F}_{loc}$ is a solution (subsolution or supersolution) of the Poisson equation in B if (2.2) is satisfied for all test functions $\phi \in \mathcal{F}_B$, $\phi \geq 0$. Notice that the weak formulation (2.2) is obtained by (2.1) integrating by parts against the test function ϕ.

In order to get a maximal inequality for solutions to (2.2) we will use Moser's iteration scheme. To obtain the iteration step, we shall test (2.2) for $\phi = u^{2\alpha-1}\eta^2$ with $\alpha \geq 1$ and η a cutoff function in B. One technical issue that arises in working with solutions which are not smooth is that one must be careful in taking powers. Indeed, in general $u^{2\alpha-1}$ is not weakly differentiable and therefore it is not clear that $\phi \in \mathcal{F}_B$. In order to circumvent this problem, we shall consider power-like functions which have a linear growth after a certain height and then we will let the height go to infinity. The following Lemma shows that this kind of functions are suitable to be test functions for (2.2).

Lemma 2.1.1. *Let $G : (0, \infty) \to (0, \infty)$ be a Lipschitz function with Lipschitz constant $L_G > 0$. Assume also that $\lim_{t\downarrow 0} G(t) = 0$. Take $u \in \mathcal{F}$, $u \geq \epsilon$, for some $\epsilon > 0$ then $G(u) \in \mathcal{F}$.*

Proof. The result follows observing that $G(u)/L_G$ is a normal contraction of $u \in \mathcal{F}$, and by Lemma 4.1.4. □

For the proof of next proposition we follow the argument of [SC02, Lemma 2.2.1], with obvious modifications due to the degeneracy of the coefficients and the technical difficulty of dealing with an inhomogeneous equation.

Proposition 2.1.2. *Let $u \in \mathcal{F}_{loc}$ be a subsolution of* (2.2) *in B. Let $\eta \in C_0^\infty(B)$ be a cutoff function, $0 \leq \eta \leq 1$. Then for all $\alpha \geq 1$*

$$\|\eta u^+\|_{\alpha\rho,B}^{2\alpha} \lesssim \alpha^2 C_E^B |B|^{\frac{2}{d}} \left[\|\nabla\eta\|_\infty^2 \|u^+\|_{2\alpha p^*,B}^{2\alpha} + \|\nabla f\|_\infty^2 \|u^+\|_{2\alpha p^*,B}^{2\alpha-2} \right], \tag{2.3}$$

where $C_E^B := \|\lambda^{-1}\|_{q,B}\|\Lambda\|_{p,B}$.

Proof. We can assume $u \in \mathcal{F}_{2B}$ since we shall look only inside B and $u \in \mathcal{F}_{loc}$. We build here a function G to be a prototype for a power function. Let $G : (0, \infty) \to (0, \infty)$ be a piecewise C^1 function such that $G'(s)$ is bounded by a constant say $C > 0$. Assume also that G has a non-negative, non-decreasing derivative $G'(x)$ and $\lim_{s\downarrow 0} G(s) = 0$. Define $H(s) \geq 0$ by $H'(s) = \sqrt{G'(s)}$ and $\lim_{s\downarrow 0} H(s) = 0$. Observe that we have

$$G(s) \leq sG'(s), \quad H(s) \leq sH'(s).$$

Let η be a cutoff in B as in the statement of the theorem. Then, by Lemma 2.1.1 and Lemma 1.3.5, it follows that

$$\phi = \eta^2(G(u^+ + \epsilon) - G(\epsilon)) \in \mathcal{F}_B.$$

In particular, ϕ is a proper test function for the Poisson equation (2.2). In order to lighten the notation we denote $G_\epsilon(x) := G(x^+ + \epsilon) - G(\epsilon)$ and $H_\epsilon(x) := H(x^+ + \epsilon) - H(\epsilon)$ and we further observe that for all

$$G_\epsilon(x) \le x^+ G'_\epsilon(x), \quad H_\epsilon(x) \le x^+ H'_\epsilon(x).$$

Since u is a subsolution to (2.2) in B, we have by definition

$$\mathcal{E}(u, \eta^2 G_\epsilon(u)) \le -\int_{\mathbb{R}^d} \langle a\nabla f, \nabla(\eta^2 G_\epsilon(u))\rangle dx. \tag{2.4}$$

Consider first the left hand side and observe that

$$\mathcal{E}(u, \eta^2 G_\epsilon(u)) = \int_{\mathbb{R}^d} \langle a\nabla u^+, \nabla u^+\rangle G'_\epsilon(u)\eta^2 dx + 2\int_{\mathbb{R}^d} \langle a\nabla u, \nabla \eta\rangle G_\epsilon(u)\eta dx.$$

Since $\nabla H_\epsilon(u) = \nabla u^+ H'_\epsilon(u) = \nabla u^+ G'_\epsilon(u)^{1/2}$, it follows

$$\int_{\mathbb{R}^d} \langle a\nabla u^+, \nabla u^+\rangle G'_\epsilon(u)\eta^2 dx = \mathcal{E}_\eta(H_\epsilon(u), H_\epsilon(u)),$$

and moving everything else on the right hand side of (2.4), taking the absolute value, we have

$$\mathcal{E}_\eta(H_\epsilon(u), H_\epsilon(u)) \le 2\int_{\mathbb{R}^d} |\langle a\nabla u, \nabla\eta\rangle G_\epsilon(u)\eta| dx + \int_{\mathbb{R}^d} |\langle a\nabla f, \nabla(G_\epsilon(u)\eta^2)\rangle| dx. \tag{2.5}$$

The first term is estimated using $G_\epsilon(u) \le u^+ G'_\epsilon(u)$ and by Cauchy-Schwartz inequality. (We use also the fact that $u^+\nabla u = u^+\nabla u^+$).

$$\begin{aligned}\int_{\mathbb{R}^d} |\langle a\nabla u, \nabla\eta\rangle G_\epsilon(u)\eta| dx \le & \int_{\mathbb{R}^d} |\langle a\nabla u^+, \nabla\eta\rangle G'_\epsilon(u) u^+ \eta| dx \\ \le & \mathcal{E}_\eta(H_\epsilon(u), H_\epsilon(u))^{\frac{1}{2}} \|G'_\epsilon(u)(u^+)^2\|_{1,\Lambda}^{\frac{1}{2}} \|\nabla\eta\|_\infty.\end{aligned}$$

In view of the chain rule and the triangular inequality, the second term is controlled by

$$\int_{\mathbb{R}^d} |\langle a\nabla f, \nabla u^+\rangle G'_\epsilon(u)\eta^2| dx + 2\int_{\mathbb{R}^d} |\langle a\nabla f, G_\epsilon(u)\eta\nabla\eta\rangle| dx,$$

whose terms can be respectively estimated by

$$\int_{\mathbb{R}^d} |\langle a\nabla f, \nabla u^+\rangle G'_\epsilon(u)\eta^2| dx \le \|\nabla f\|_\infty \|1_B G'_\epsilon(u)\|_{1,\Lambda}^{\frac{1}{2}} \mathcal{E}_\eta(H_\epsilon(u), H_\epsilon(u))^{\frac{1}{2}}$$

and by

$$\int_{\mathbb{R}^d} |\langle a\nabla f, \nabla\eta\rangle G_\epsilon(u)\eta| dx \le \|\nabla\eta\|_\infty \|\nabla f\|_\infty \|G_\epsilon(u) 1_B\|_{1,\Lambda}.$$

Putting everything together in (2.5) we end up with the estimate

$$\begin{aligned}\mathcal{E}_\eta(H_\epsilon(u),H_\epsilon(u)) \le{}& 2\|G'_\epsilon(u)(u^+)^2\|_{1,\Lambda}^{\frac12}\|\nabla\eta\|_\infty \mathcal{E}_\eta(H_\epsilon(u),H_\epsilon(u))^{\frac12}\\ &+\|\nabla f\|_\infty \|1_B G'_\epsilon(u)\|_{1,\Lambda}^{\frac12}\mathcal{E}_\eta(H_\epsilon(u),H_\epsilon(u))^{\frac12}\\ &+2\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|G_\epsilon(u)1_B\|_{1,\Lambda},\end{aligned}$$

which after averaging over B gives, up to a universal constant,

$$\begin{aligned}\frac{\mathcal{E}_\eta(H_\epsilon(u),H_\epsilon(u))}{|B|}\lesssim{}&\|G'_\epsilon(u)(u^+)^2\|_{1,B,\Lambda}\|\nabla\eta\|_\infty^2+\|\nabla f\|_\infty^2\|G'_\epsilon(u)\|_{1,B,\Lambda}\\ &+\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|G_\epsilon(u)\|_{1,B,\Lambda}.\end{aligned}\tag{2.6}$$

At this point, it is important to observe that $H_\epsilon(u)\in\mathcal{F}$ so that we can apply the Sobolev's inequality (1.14) with cutoff function η, namely

$$\|\eta H_\epsilon(u)\|_{\rho,B}^2\lesssim C_S^B|B|^{\frac2d}\Big[\frac{\mathcal{E}_\eta(H_\epsilon(u),H_\epsilon(u))}{|B|}+\|\nabla\eta\|_\infty^2\|H_\epsilon(u)\|_{2,B,\Lambda}^2\Big].$$

Concatenating (1.14) and (2.6) yields

$$\begin{aligned}\|\eta H_\epsilon(u)\|_{\rho,B}^2\lesssim C_S^B|B|^{\frac2d}\Big[&\|\nabla\eta\|_\infty^2\|H'_\epsilon(u)^2u^2\|_{1,B,\Lambda}+\|\nabla f\|_\infty^2\|H'_\epsilon(u)^2\|_{1,B,\Lambda}\\ &+\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|G_\epsilon(u)\|_{1,B,\Lambda}+\|\nabla\eta\|_\infty^2\|H_\epsilon(u)\|_{2,B,\Lambda}^2\Big].\end{aligned}$$

Finally, it is time to fix a H,G as power-like function. Namely, for $\alpha\ge1$ we define

$$H_N(x):=\begin{cases}x^\alpha, & x\le N,\\ \alpha N^{\alpha-1}x+(1-\alpha)N^\alpha, & x>N,\end{cases}$$

which corresponds in taking

$$G_N(x)=\int_0^x H'_N(s)^2\,ds.$$

The functions $G_N(x),H_N(x)$ satisfy the properties needed by G,H, moreover $H_N(x)\uparrow$

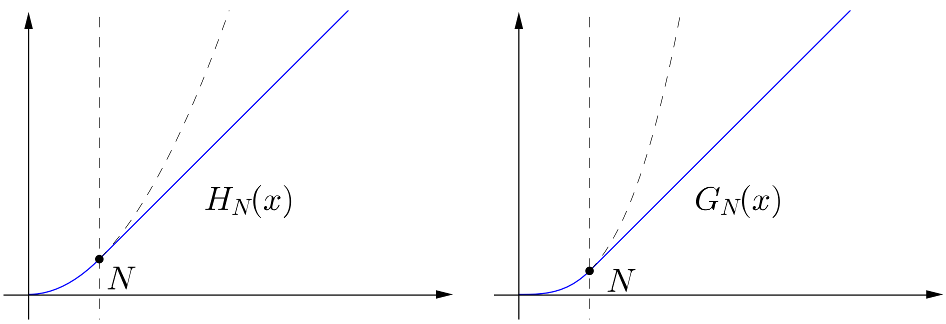

Figure 2.1: the function H_N and G_N with $\alpha=2$.

x^α and $G_N(x) \uparrow \frac{\alpha^2}{2\alpha-1}x^{2\alpha-1}$ pointwise as N goes to infinity. Therefore, letting $N \to \infty$ and using the monotone convergence theorem, we obtain

$$\|\eta(u^+ + \epsilon)^\alpha\|^2_{\rho,B} \lesssim C^B_S |B|^{\frac{2}{d}} \Big[(\alpha^2+1)\|(u^+ + \epsilon)^{2\alpha}\|_{1,B,\Lambda}\|\nabla\eta\|^2_\infty \\ + \alpha^2\|\nabla f\|^2_\infty \|(u^+ + \epsilon)^{2\alpha-2}\|_{1,B,\Lambda} + \frac{\alpha^2}{2\alpha-1}\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|(u^+ + \epsilon)^{2\alpha-1}\|_{1,B,\Lambda}\Big].$$

Taking the limit as $\epsilon \to 0$, using $\|\cdot\|_{1,B,\Lambda} \le \|\Lambda\|_{p,B}\|\cdot\|_{p^*,B}$ and $C^B_E = C^B_S\|\Lambda\|_{p,B}$, we get

$$\|\eta(u^+)^\alpha\|^2_{\rho,B} \lesssim C^B_E |B|^{\frac{2}{d}} \Big[(\alpha^2+1)\|(u^+)^{2\alpha}\|_{p^*,B}\|\nabla\eta\|^2_\infty \\ + \alpha^2\|\nabla f\|^2_\infty \|(u^+)^{2\alpha-2}\|_{p^*,B} + \frac{\alpha^2}{2\alpha-1}\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|(u^+)^{2\alpha-1}\|_{p^*,B}\Big].$$

By Jensen's inequality it holds

$$\|u^+\|_{(2\alpha-2)p^*,B} \le \|u^+\|_{2\alpha p^*,B}, \quad \|u^+\|_{(2\alpha-1)p^*,B} \le \|u^+\|_{2\alpha p^*,B}.$$

Therefore, we can rewrite

$$\|\eta u^+\|^{2\alpha}_{\alpha\rho,B} \lesssim C^B_E |B|^{\frac{2}{d}} \Big[(\alpha^2+1)\|u^+\|^{2\alpha}_{2\alpha p^*,B}\|\nabla\eta\|^2_\infty \\ + \alpha^2\|\nabla f\|^2_\infty \|u^+\|^{2\alpha-2}_{2\alpha p^*,B} + \frac{\alpha^2}{2\alpha-1}\|\nabla\eta\|_\infty\|\nabla f\|_\infty\|u^+\|^{2\alpha-1}_{B,2\alpha p^*,B}\Big].$$

Finally, using $\alpha \ge 1$ and absorbing the mixed product into the two squares we obtain exactly (3.17). $\square$

Clearly the same result also holds, with the same constant, for supersolutions with u^+ replaced by u^-. It is then clear that we can get the same type of inequality for solutions to (2.2). This is the content of the next corollary.

Corollary 2.1.3. *Let $u \in \mathcal{F}_{loc}$ be a solution of* (2.2) *in B. Let $\eta \in C^\infty_0(B)$ be a cutoff function. Then, for all $\alpha \ge 1$*

$$\|\eta u\|^{2\alpha}_{\alpha\rho,B} \lesssim \alpha^2 C^B_E |B|^{\frac{2}{d}} \Big[\|\nabla\eta\|^2_\infty\|u\|^{2\alpha}_{2\alpha p^*,B} + \|\nabla f\|^2_\infty\|u\|^{2\alpha-2}_{2\alpha p^*,B}\Big]. \tag{2.7}$$

Proof. The proof is trivial since u is both a subsolution and a supersolution of (2.2). Moreover, $u = u^+ - u^-$ and $\|u^+\|_{r,B} \vee \|u^-\|_{r,B} \le \|u\|_{r,B}$. $\square$

Theorem 2.1.4. *Let $d \ge 2$. Fix a point $x_0 \in \mathbb{R}^d$ and $R > 0$. Denote by $B(R)$ the ball of center x_0 and radius R. Suppose that u is a solution in $B(R)$ of* (2.2)*, and assume that $|\nabla f| \le C_f/R$. Then for any $p, q \in [1,\infty)$ such that $1/p + 1/q < 2/d$, there exist $\kappa := \kappa(q,p,d) \in (1,\infty)$, $\gamma := \gamma(q,p,d) \in (0,1]$ and $C_1 := C_1(q,p,d,C_f) > 0$ such that*

$$\|u\|_{\infty,B(\sigma' R)} \le C_1\left(\frac{1 \vee C^{B(R)}_E}{(\sigma-\sigma')^2}\right)^\kappa \|u\|^\gamma_{\rho,B(\sigma R)} \vee \|u\|_{\rho,B(\sigma R)}, \tag{2.8}$$

for any fixed $1/2 \le \sigma' < \sigma \le 1$.

Proof. We are going to apply inequality (2.7) iteratively. For fixed $1/2 \le \sigma' < \sigma \le 1$, and $k \in \mathbb{N}$ define

$$\sigma_k = \sigma' + 2^{-k+1}(\sigma - \sigma').$$

It is immediate that $\sigma_k - \sigma_{k+1} = 2^{-k+1}(\sigma - \sigma')$ and that $\sigma_1 = \sigma$, furthermore $\sigma_k \downarrow \sigma'$. We have already observed that $\rho > 2p^*$, where p^* is the Hölder's conjugate of p. Set $\alpha_k := (\rho/2p^*)^k$, $k \ge 1$, clearly $\alpha_k > 1$ for all $k \ge 1$. Finally consider a cutoff η_k which is constant and equal to one on $B(\sigma_{k+1}r)$ and $\eta_k = 0$ on $\partial B(\sigma_k R)$, assume that η_k has a linear decay on $B(\sigma_k R) \setminus B(\sigma_{k+1}R)$, namely choose η_k in such a way that $\|\nabla \eta_k\|_\infty \le 2^k/((\sigma - \sigma')R)$.

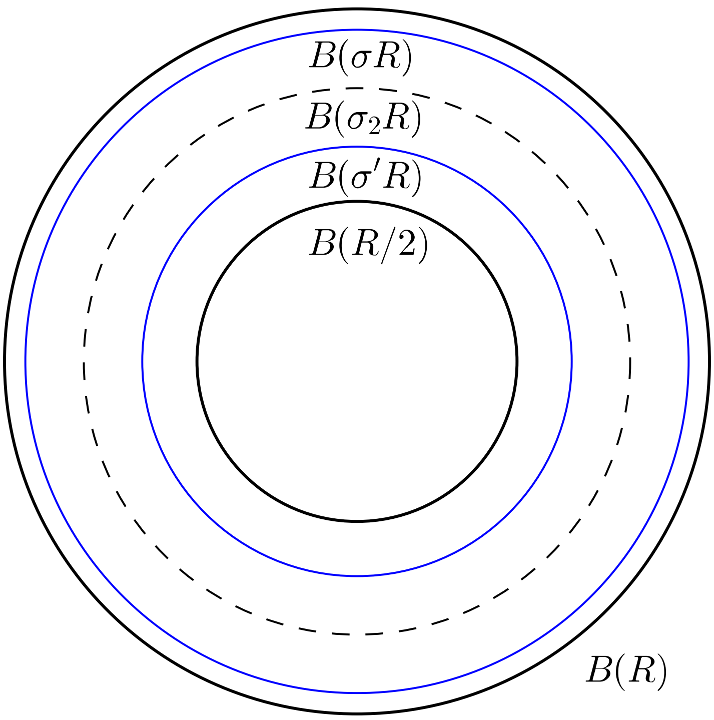

Figure 2.2: nested balls $B(\sigma_k R)$.

An application of Corollary 2.1.3 and of the relation $\alpha_k \rho = 2\alpha_{k+1}p^*$, yields for a constant $C > 0$ which depends only on the dimension, C_f and p, q, and which may change from line to line,

$$\begin{aligned} \|u\|_{2\alpha_{k+1}p^*, B(\sigma_{k+1}R)} &\le \left(C \frac{2^{2k}\alpha_k^2 |B(\sigma_k R)|^{\frac{2}{d}} C_E^{B(\sigma_k R)}}{(\sigma - \sigma')^2 R^2} \right)^{\frac{1}{2\alpha_k}} \|u\|^{\gamma_k}_{2\alpha_k p^*, B(\sigma_k R)} \\ &\le \left(C \frac{2^{2k}\alpha_k^2 C_E^{B(\sigma_k R)}}{(\sigma - \sigma')^2} \right)^{\frac{1}{2\alpha_k}} \|u\|^{\gamma_k}_{2\alpha_k p^*, B(\sigma_k R)} \end{aligned}$$

where $\gamma_k = 1$ if $\|u\|_{2\alpha_k p^*, B(\sigma_k r)} \ge 1$ and $\gamma_k = 1 - 1/\alpha_k$ otherwise. We can iterate the inequality above from $k=1$ to $k=j$ and get

$$\|u\|_{2\alpha_{j+1}p^*,B(\sigma_{j+1}R)} \leq \prod_{k=1}^{j}\left(C\frac{(\rho/p^*)^{2k}C_E^{B(\sigma R)}}{(\sigma-\sigma')^2}\right)^{\frac{1}{2\alpha_k}}\|u\|_{\rho,B(\sigma R)}^{\prod_{k=1}^{j}\gamma_k}.$$

Observe that $\kappa := \frac{1}{2}\sum 1/\alpha_k < \infty$, $\sum k/\alpha_k < \infty$ and that

$$\|u\|_{2\alpha_j p^*,B(\sigma' R)} \leq \left(\frac{|B(\sigma_k R)|}{|B(\sigma' R)|}\right)^{\frac{1}{2\alpha_j p^*}}\|u\|_{2\alpha_j p^*,B(\sigma_j R)} \leq K\|u\|_{2\alpha_j p^*,B(\sigma_j R)},$$

for some dimensional constant K and all $j\geq 1$. We also remark that $C_E^{B(\sigma R)} \lesssim 1\vee C_E^{B(R)}$. Hence, taking the limit as $j\to\infty$, gives the inequality

$$\|u\|_{B(\sigma' R),\infty} \leq C_1\left(\frac{1\vee C_E^{B(R)}}{(\sigma-\sigma')^2}\right)^{\kappa}\|u\|_{B(\sigma R),\rho}^{\prod_{k=1}^{\infty}\gamma_k},$$

for some finite $C_1>0$ which depends only on p,q,C_f and the dimension. Finally, define

$$\gamma := \prod_{k=1}^{\infty}(1-1/\alpha_k),$$

then, $0<\gamma\leq\prod_{k=1}^{\infty}\gamma_k\leq 1$ and the above inequality can be written as

$$\|u\|_{\infty,B(\sigma' R)} \leq C_1\left(\frac{1\vee C_E^{B(R)}}{(\sigma-\sigma')^2}\right)^{\kappa}\|u\|_{\rho,B(\sigma R)}^{\gamma}\vee\|u\|_{\rho,B(\sigma R)}.$$

which is (2.8). □

Since functions $u\in\mathcal{F}_{loc}$ are in $L^{\rho}_{loc}(\mathbb{R}^d)$ by the Sobolev inequality (1.10), it follows by Theorem 2.1.4 that subsolutions $u\in\mathcal{F}_{loc}$ in B of (2.2) are locally bounded in B. The previous inequality can be improved. This is what the next Corollary is about. For the proof we follow the argument of [SC02][Theorem 2.2.3].

Corollary 2.1.5. *Suppose that u satisfies the assumptions of Theorem 2.1.4. Then, for all $\alpha\in(0,\infty)$ and for any $1/2\leq\sigma'<\sigma<1$ there exist $C_2 := C_2(q,p,d,C_f)>0$, $\gamma' := \gamma'(\gamma,\alpha,\rho)$ and $\kappa' := \kappa'(\kappa,\alpha,\rho)$, such that*

$$\|u\|_{\infty,B(\sigma' R)} \leq C_2\left(\frac{1\vee C_E^{B(R)}}{(\sigma-\sigma')^2}\right)^{\kappa'}\|u\|_{\alpha,B(\sigma R)}^{\gamma'}\vee\|u\|_{\alpha,B(\sigma R)}. \tag{2.9}$$

Proof. From inequality (2.8) we get

$$\|u\|_{B(\sigma' R),\infty} \leq C_1\left(\frac{1\vee C_E^{B(R)}}{(\sigma-\sigma')^2}\right)^{\kappa}\|u\|_{\rho,B(\sigma R)}^{\gamma}\vee\|u\|_{\rho,B(\sigma R)},$$

hence, the result follows immediately for $\alpha>\rho$ by means of Jensen's inequality. For $\alpha\in(0,\rho)$ we use again an iteration argument. Consider $\sigma_k=\sigma-2^{-k}(\sigma-\sigma')$. By Hölder's inequality we get

$$\|u\|_{\rho,B(\sigma_k R)} \leq \|u\|_{\alpha,B(\sigma_k R)}^{\theta}\|u\|_{\infty,B(\sigma_k R)}^{1-\theta}$$

with $\theta = \alpha/\rho$. An application of inequality (2.8) gives

$$\|u\|_{\infty,B(\sigma_{k-1}R)} \leq 2^{2\kappa k} J \|u\|_{\alpha,B(\sigma R)}^{\gamma_k\theta} \|u\|_{\infty,B(\sigma_k R)}^{\gamma_k - \gamma_k\theta}, \tag{2.10}$$

here $\gamma_k = 1$ if $\|u\|_{\rho,B(\sigma_k R)} \geq 1$, $\gamma_k = \gamma$ otherwise and

$$J := C\left(\frac{1 \vee C_E^{B(R)}}{(\sigma - \sigma')^2}\right)^{\kappa},$$

where C is a constant depending on C_1 that can be taken greater than one.

Iterating (2.10) from $k = 1$ up to $k = i$, via similar computations as in the proof of Theorem 2.1.4, we get

$$\|u\|_{\infty,B(\sigma' R)} \lesssim (J2^{2\kappa})^{\sum_{k=1}^{i} k(1-\theta)^{k-1}} \left(\|u\|_{\alpha,B(\sigma R)}^{\gamma\theta\sum_{k=1}^{i}(\gamma-\gamma\theta)^{k-1}} \vee \|u\|_{\alpha,B(\sigma R)}^{\theta\sum_{k=1}^{i}(1-\theta)^{k-1}}\right) \|u\|_{\infty,B(\sigma R)}^{\beta_i}$$

where $\beta_i \to 0$ as $i \to \infty$. This gives the desired result taking the limit as $i \to \infty$. In particular we get $\gamma' = \gamma\theta/(1 - \gamma + \gamma\theta)$. $\square$

2.2 Comments on the elliptic Harnack inequality

In this section we make few comments on the elliptic Harnack inequality. We look at positive weak solutions u of the equation

$$\operatorname{div}(a(x)\nabla u(x)) = 0, \tag{2.11}$$

on the euclidean ball $B \subset \mathbb{R}^d$.

The classical elliptic Harnack inequality states that if the matrix $a(x)$ is symmetric, uniformly elliptic and bounded, then for all positive weak solutions to (2.11) there exists a constant C_{EH} which depends only on the ellipticity constant and the dimension such that

$$\sup_{B/2} u \leq C_{EH} \inf_{B/2} u. \tag{2.12}$$

We stress that C_{EH} does not depend on the ball B. The Harnack inequality (2.12) has the remarkable consequence to imply Hölder continuity for positive weak solutions to (2.11), that is, there exist $C_{HC} > 0$ and $\alpha > 0$, which depend only on the ellipticity constant and the dimension, such that

$$\sup_{x,y\in B/2} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C_{HC} R^{-\alpha} \|u\|_{\infty,B}$$

where R is the radius of the ball B.

Following the strategy illustrated in [SC02, Chapter 2], it is possible to prove an elliptic Harnack inequality for positive weak solutions to (2.11) also in the case in which $x \mapsto a(x)$ satisfies only $(b.1)$ and $(b.2)$. However, the Harnack inequality so obtained would be not of much use to prove Hölder regularity of solutions. Indeed it would have the form

$$\sup_{B/2} u \leq C_{EH}^{B} \inf_{B/2} u.$$

and the constant C_{EH}^{B} would depend on averages of λ^{-q} and Λ^{p} on the ball B, in particular in general it won't be uniform in the choice of B. This makes it impossible to use it to control the oscillations for small balls, unless further assumptions are given on λ and Λ.

3

Parabolic second order linear PDEs

3.1 Caloric Functions

For this section we will follow [BGK12]. Let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function such that θ^{-1}, θ are locally integrable on $\mathbb{R}^d$. Consider again the symmetric form $\mathcal{E}$ on $L^2(\mathbb{R}^d, \theta)$ with domain $\mathcal{F}^\theta$ introduced in Section 1.2,

$$\mathcal{E}(u,v) := \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u(x)\partial_j v(x)\, dx. \tag{3.1}$$

We recall that $\mathcal{F}^\theta$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \theta)$ with respect to $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_\theta$. Given an open subset G of $\mathbb{R}^d$ we will denote by $\mathcal{F}_G^\theta$ the completion of $C_0^\infty(G)$ in $L^2(G, \theta)$ with respect to $\mathcal{E}_1$.

Definition 3.1.1 (Caloric functions). *Let $I \subset \mathbb{R}$ and $G \subset \mathbb{R}^d$ an open set. We say that a function $u : I \to \mathcal{F}^\theta$ is a subcaloric (supercaloric) function in $I \times G$ if $t \to (u(t,\cdot), \phi)_\theta$ is differentiable in $t \in I$ for any $\phi \in L^2(G, \theta)$ and*

$$\frac{d}{dt}(u,\phi)_\theta + \mathcal{E}(u,\phi) \leq 0, \quad (\geq) \tag{3.2}$$

for all non-negative $\phi \in \mathcal{F}_G^\theta$. We say that a function $u : I \to \mathcal{F}^\theta$ is a caloric function in $I \times G$ if it is both sub- and supercaloric.

It is clear from the definition that if a function is subcaloric on $I \times G$ than it is subcaloric on $I' \times G'$ whenever $I' \subset I$ and $G' \subset G$.

Moreover, observe that if P_t^G is the semigroup associated to $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(G, \theta)$ and $f \in L^2(G, \theta)$, for a given open set $G \subset \mathbb{R}^d$, then the function $u(t,\cdot) = P_t^G f(\cdot)$ is a caloric function on $(0,\infty) \times G$. To complete the picture we state the following maximum principle which appeared in [GHL09]. For a real number $x \in \mathbb{R}$ we denote by $x^+ = x \vee 0$.

Lemma 3.1.2. *Fix $T \in (0,\infty]$, a set $G \subset \mathbb{R}^d$ and let $u : (0,T) \to \mathcal{F}_G^\theta$ be a subcaloric function in $(0,T) \times G$ which satisfies the boundary condition $u^+(t,\cdot) \in \mathcal{F}_G^\theta$, $\forall t \in (0,T)$ and $u^+(t,\cdot) \to 0$ in $L^2(G,\theta)$ as $t \to 0$. Then $u \leq 0$ on $(0,T) \times G$.*

As a corollary of this lemma we have the super-mean value inequality for subcaloric functions.

Corollary 3.1.3. *Fix $T \in (0, \infty]$, an open set $G \subset \mathbb{R}^d$ and $f \in L^2(G, \theta)$ non-negative. Let $u : (0, T) \to \mathcal{F}_G^\theta$ be a non-negative subcaloric function on $(0, T) \times G$ such that $u(t, \cdot) \to f$ in $L^2(G, \theta)$ as $t \to 0$. Then for any $t \in (0, T)$*

$$u(t, \cdot) \geq P_t^G f, \text{ in } G.$$

In particular for $0 < s < t < T$

$$u(t, \cdot) \geq P_{t-s}^G u(s, \cdot), \text{ in } G.$$

In order to prove maximal inequalities for sub- and supercaloric functions with parabolic partial we shall need the following technical lemma.

Lemma 3.1.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded second derivative and positive first derivative. Assume that $F'(0) = 0$. Then for any caloric (subcaloric, supercaloric) function u we have*

$$\frac{d}{dt}(F(u_t), \phi)_\theta + \mathcal{E}(u_t, F'(u_t)\phi) = 0, \quad (\leq, \geq)$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi > 0$ and $t > 0$.

Proof. Let us prove the lemma when u is a caloric function, the proof for sub- and supercaloric functions is analogous. By Lemma 2.1.1 $F'(u_t) \in \mathcal{F}^\theta$ since F'' is bounded and $F'(0) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}(F(u_t), \phi)_\Lambda &= \lim_{h \downarrow 0} \frac{1}{h}(F(u_{t+h}) - F(u_t), \phi)_\theta \\ &= \lim_{h \downarrow 0} \left[\frac{1}{h}(F'(u_t)(u_{t+h} - u_t), \phi)_\Lambda + \frac{1}{h}(R(u_{t+h} - u_t), \phi)_\theta\right],\end{aligned}$$

where $|R(x)| \leq \|F''\|_\infty |x|^2$ is the remainder in the Taylor's expansion around zero. The first summand converges to $-\mathcal{E}(u_t, F'(u_t)\phi)$ since u_t solves (3.2) and $F'(u_t)\phi \in \mathcal{F}^\theta$ is non-negative. It remains to show that the second summand goes to zero. For that notice that

$$\frac{1}{h}|(R(u_{t+h} - u_t), \phi)_\theta| \leq h\|\phi\|_\infty \|F''\|_\infty \|(u_{t+h} - u_t)h^{-1}\|_{2,\theta}^2 \to 0$$

as $h \to 0$. □

3.2 An abstract lemma

Similarly to the derivation of the maximal inequality for the Poisson equation, with Moser's iteration technique we are able to bound the L^∞-norm of a caloric function u by its L^α-norm for some finite $\alpha > 0$, on a slightly larger parabolic ball. Since the same holds for u^{-1}, what is left to obtain a parabolic Harnack inequality is to link the L^α-norm of u and the L^α-norm of u^{-1}. In the uniformly elliptic case this is achieved by means of the exponential integrability of BMO functions, hence with John-Niremberg inequality. In the present work we exploit an abstract lemma due to Bombieri and Giusti (See Lemma 3.2.1 below) for which application, besides the maximal inequality for u and u^{-1}, we will need the weighted Poincaré inequalities established in Section 1.5.

Lemma 3.2.1 (Bombieri-Giusti [BG72])**.** *Consider a collection of measurable subsets U_σ, $0 < \sigma \leq 1$, of a fixed measure space $(\mathcal{X}, \mathcal{M})$ endowed with a measure γ, such that $U_{\sigma'} \subset U_\sigma$ whenever $\sigma' < \sigma$. Fix $\delta \in (0,1)$. Let κ and K_1, K_2 be positive constants and $0 < \alpha_0 \leq \infty$. Let u be a positive measurable function on $U := U_1$ which satisfies*

$$\left(\int_{U_{\sigma'}} |u|^{\alpha_0} \, d\gamma \right)^{\frac{1}{\alpha_0}} \leq \left(K_1 (\sigma - \sigma')^{-\kappa} \gamma(U)^{-1} \right)^{\frac{1}{\alpha} - \frac{1}{\alpha_0}} \left(\int_{U_\sigma} |u|^\alpha \, d\gamma \right)^{\frac{1}{\alpha}} \tag{3.3}$$

for all σ, σ' and α such that $0 < \delta \leq \sigma' < \sigma \leq 1$ and $0 < \alpha \leq \min\{1, \alpha_0/2\}$. Assume further that u satisfies

$$\gamma(\log u > \ell) \leq K_2 \, \gamma(U) \ell^{-1} \tag{3.4}$$

for all $\ell > 0$. Then

$$\left(\int_{U_\delta} |u|^{\alpha_0} \, d\gamma \right)^{\frac{1}{\alpha_0}} \leq C_{BG} \, \gamma(U)^{\frac{1}{\alpha_0}},$$

where C_{BG} depends only on δ, κ, a lower bound on α_0 and increasingly on K_1, K_2.

Proof. The proof of Bombieri-Giusti's lemma can be found in [SC02, Lemma 2.2.6]. □

The rest of the chapter will be devoted to establish the mean value inequalities (3.3) and (3.4) for u and u^{-1} with $U_\sigma = (s - \sigma R^2, s) \times B(x, \sigma R)$ for some fixed parabolic ball $(s - R^2, s) \times B(x, R) \subset \mathbb{R}^d$, we will follow the rather classical argument given in [SC02, Chapter 5]. These inequalities together with Lemma 3.2.1 will give the parabolic Harnack inequality.

3.3 Mean value inequalities for subcaloric functions

We will look at the equation (3.2) for $\theta = \Lambda$. This corresponds to the formal parabolic equation

$$\partial_t u(t,x) - \frac{1}{\Lambda(x)} \operatorname{div}(a(x) \nabla u(t,x)) = 0.$$

We recall here the assumptions $(b.1)$ and $(b.2)$ which we made on $x \mapsto a(x)$. We will assume them throughout the chapter without any further comment.

Assumption b.1. Fix $d \geq 2$. We are given a symmetric matrix $a : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $\lambda, \Lambda : \mathbb{R}^d \to [0, \infty]$ such that $\lambda^{-1}, \Lambda \in L^1_{loc}(\mathbb{R}^d)$ and for almost all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$

$$\lambda(x)|\xi|^2 \leq \sum_{i,j}^d a_{ij}(x) \xi_j \xi_i \leq \Lambda(x) |\xi|^2.$$

Assumption b.2. We assume that there exist $p, q \in [1, \infty)$ with $1/p + 1/q < 2/d$ such that $\lambda^{-1} \in L^q_{loc}(\mathbb{R}^d)$ and $\Lambda \in L^p_{loc}(\mathbb{R}^d)$.

Remark 3.3.1. *All the estimates which we will derive in this chapter can possibly be obtained also for general θ, following the same strategy, provided that*

$$\theta \in L^r_{loc}(\mathbb{R}^d), \quad \lambda^{-1} \in L^q_{loc}(\mathbb{R}^d), \quad \theta^{\frac{1}{p}-1}\Lambda \in L^p_{loc}(\mathbb{R}^d)$$

where $p, q, r \in (1, \infty]$ are such that

$$\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}.$$

To avoid the same type of technical problems that we faced in Chapter 2, we shall assume that our positive subcaloric functions u are locally bounded. It turns out that any positive subcaloric function is locally bounded; this can be proved repeating the argument below with the same type of technicalities appearing in Proposition 2.1.2.

Proposition 3.3.2. *Consider $I = (t_1, t_2) \subset \mathbb{R}$ and a ball $B \subset \mathbb{R}^d$. Let u be a locally bounded positive subcaloric function in $Q = I \times B$. Take cutoffs $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$ and $\zeta : \mathbb{R} \to [0, 1]$, $\zeta \equiv 0$ on $(-\infty, t_1]$. Set $\nu = 2 - 2p^*/\rho$. Then for all $\alpha \ge 1$*

$$\|\zeta\eta^2 u^{2\alpha}\|^\nu_{\nu, I\times B, \Lambda} \lesssim C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}} \Big[\alpha(\|\zeta'\|_\infty + \|\nabla\eta\|^2_\infty)\Big]^\nu \|u^{2\alpha}\|^\nu_{1, I\times B, \Lambda}. \tag{3.5}$$

Proof. Since $u_t > 0$ is locally bounded, the power function $|\cdot|^{2\alpha} : \mathbb{R} \to \mathbb{R}$ with $\alpha \ge 1$ satisfies the assumptions of Lemma 3.1.4. Thus, for $\eta \in C_0^\infty(B)$ as above we have

$$\frac{d}{dt}(u_t^{2\alpha}, \eta^2)_\Lambda + 2\alpha\, \mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) \le 0, \quad t \in I. \tag{3.6}$$

We can estimate

$$\begin{aligned}\mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) &= 2\int_{\mathbb{R}^d} \eta u_t^{2\alpha-1}\langle a\nabla u_t, \nabla\eta\rangle\, dx + (2\alpha-1)\int_{\mathbb{R}^d} \eta^2 u_t^{2\alpha-2}\langle a\nabla u_t, \nabla u_t\rangle\, dx \\ &\ge \frac{2\alpha-1}{\alpha^2}\,\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha) - \frac{2\|\nabla\eta\|_\infty}{\alpha}\,\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha)^{1/2}\|1_B u_t^{2\alpha}\|^{1/2}_{1,\Lambda},\end{aligned}$$

by means of Young's inequality $2ab \le (\epsilon a^2 + b^2/\epsilon)$, choosing $a = \mathcal{E}_\eta(u_t^\alpha, u_t^\alpha)^{1/2}$, $b = \|\nabla\eta\|_\infty\|1_B u_t^{2\alpha}\|^{1/2}_{1,\Lambda}$ and $\epsilon = 1/2\alpha$ and exploiting that $\alpha \ge 1$, we get

$$\mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) \ge (1/2\alpha)\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha) - 2\|\nabla\eta\|^2_\infty\|1_B u_t^{2\alpha}\|_{1,\Lambda}.$$

Plugging the estimate above in (3.6) we deduce

$$\frac{d}{dt}\|(u_t^\alpha\eta)^2\|_{1,\Lambda} + \mathcal{E}_\eta(u_t^\alpha, u_t^\alpha) \le 4\alpha\|\nabla\eta\|^2_\infty\|1_B u_t^{2\alpha}\|_{1,\Lambda}.$$

We now take a smooth cutoff in time $\zeta : \mathbb{R} \to [0, 1]$, $\zeta \equiv 0$ on $(-\infty, t_1]$, where we recall that $I = (t_1, t_2)$. We multiply the inequality above by ζ and integrate in time. This yields

$$\zeta(t)\|(u_t^\alpha\eta)^2\|_{1,\Lambda} + \int_{t_1}^t \zeta(s)\mathcal{E}_\eta(u_s^\alpha, u_s^\alpha)\, ds \le 4\alpha\Big[\|\zeta'\|_\infty + \|\nabla\eta\|^2_\infty\Big]\int_{t_1}^t \|1_B u_s^{2\alpha}\|_{1,\Lambda}\, ds,$$

after averaging in space and taking the supremum for $t \in I$ we obtain

$$\sup_{t\in I} \zeta(t)\|(\eta u_t^\alpha)^2\|_{1,B,\Lambda} + \int_I \zeta(s)\frac{\mathcal{E}_\eta(u_s^\alpha, u_s^\alpha)}{|B|}\, ds \lesssim \alpha\Big[\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2\Big] \int_I \|u_s^{2\alpha}\|_{1,B,\Lambda}\, ds. \quad (3.7)$$

We use (3.7) together with (1.15) to get (3.5). Observe that $\nu = 2-2p^*/\rho$ is greater than one, since $\rho > 2p^*$ by the condition $1/p + 1/q < 2/d$. Using Hölder's inequality (1.6) and some easy manipulation yields

$$\|(\eta u_s^\alpha)^2\|_{\nu,B,\Lambda}^\nu \le \|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2 \|(\eta u_s^\alpha)^2\|_{1,B,\Lambda}^{\nu-1}.$$

We can then integrate this inequality against $\zeta(s)^\nu$ over I and obtain

$$\frac{1}{|I|}\int_I \zeta(s)^\nu \|\eta^2 u_s^{2\alpha}\|_{\nu,B,\Lambda}^\nu\, ds \le \Big(\sup_{s\in I}\zeta(s)\|(\eta u_s^\alpha)^2\|_{1,B,\Lambda}\Big)^{\nu-1} \frac{1}{|I|}\int_I \zeta(s)\|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2\, ds. \quad (3.8)$$

In view of the Sobolev inequality (1.15) we have

$$\|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2 \lesssim C_S^{B,\Lambda}|B|^{\frac{2}{d}}\Big[\frac{\mathcal{E}_\eta(u_s^\alpha,u_s^\alpha)}{|B|} + \|\nabla\eta\|_\infty^2\|u_s^{2\alpha}\|_{1,B,\Lambda}\Big],$$

by (3.7) we can bound each of the two factors on the right hand side of (3.8). We end up with the following iterative step

$$\|\zeta\eta^2 u^{2\alpha}\|_{\nu,I\times B,\Lambda}^\nu \lesssim C_S^{B,\Lambda}\frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}}\Big[\alpha(\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2)\Big]^\nu \|u^{2\alpha}\|_{1,I\times B,\Lambda}^\nu,$$

which is what we wanted to prove. □

The main idea is to use Moser's iteration technique on a sequence of parabolic balls; the iteration step is provided by Proposition 3.3.2 once we choose proper cutoffs η, ζ and a proper parameter α. Fix a $\tau > 0$, let $x \in \mathbb{R}^d$, and $R > 0$. Consider also a parameter $\delta \in (0,1)$. Then we define the parabolic balls (see Figure 3.1)

$$\begin{aligned} Q(\tau,x,s,R) = Q &= (s-\tau R^2, s)\times B(x,R),\\ Q_\delta &= (s-\delta\tau R^2, s)\times B(x,\delta R). \end{aligned}$$

Clearly $Q_\delta \subset Q$ for all $\delta \in (0,1)$.

Theorem 3.3.3. *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Let u be a positive subcaloric function on $Q = Q(\tau,x,s,R)$. Then there exists a positive constant $C_3 := C_3(d,p,q)$ such that*

$$\sup_{Q_{\sigma'}} u(t,z) \le C_3 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}}\tau^{\frac12}\left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{2\nu-2}} \|u\|_{2,Q_\sigma,\Lambda}, \quad (3.9)$$

where $\nu = 2-2p^*/\rho$.

Proof. We want to apply (3.5) with a suitable sequence of cutoffs η_k and ζ_k, $k \in \mathbb{N}$. Set

$$\sigma_k = \sigma' + 2^{-k}(\sigma-\sigma'), \quad \delta_k = 2^{-k-1}(\sigma-\sigma')$$

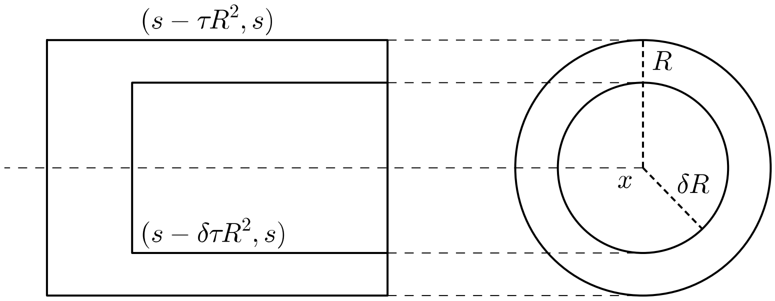

Figure 3.1: orthogonal projections of Q and Q_δ.

then $\sigma_k - \sigma_{k+1} = \delta_k$. Consider a cutoff $\eta_k : \mathbb{R}^d \to [0,1]$, such that $\operatorname{supp} \eta_k \subset B(x, \sigma_k R)$ and $\eta_k \equiv 1$ on $B(x, \sigma_{k+1} R)$, moreover assume that $\|\nabla \eta\|_\infty \le 2/(R\delta_k)$. Take also a cutoff in time $\zeta : \mathbb{R} \to [0,1]$, $\zeta_k \equiv 1$ on $I_{\sigma_{k+1}} = (s - \sigma_{k+1}\tau R^2, s)$, $\zeta_k \equiv 0$ on $(-\infty, s - \sigma_k \tau R^2)$ and $\|\zeta'\|_\infty \le 2/(R^2\tau\delta_k)$. Let $\alpha_k = \nu^k$ with $\nu = 2 - 2p^*/\rho$ as above. Then, an application of (3.5) and using the fact that $\alpha_{k+1} = \nu\alpha_k$ yields

$$\|u\|_{2\alpha_{k+1}, Q_{\sigma_{k+1}}, \Lambda} \le \left\{ c(d) C_S^{B,\Lambda} \tau^{\nu-1} \left[\frac{\alpha_k (1+\tau^{-1}) 2^{2k}}{(\sigma - \sigma')^2} \right]^\nu \right\}^{\frac{1}{2\alpha_{k+1}}} \|u\|_{2\alpha_k, Q_{\sigma_k}, \Lambda}, \tag{3.10}$$

where we used the fact that $\sigma_k / \sigma_{k+1} < 2$, and that $\sigma_k \in [1/2, 1]$. This is the starting point for Moser's iteration. Iterating inequality (3.10) from $k = 0$ up to $k = j$ we get at the price of a constant $C_3 > 0$ which depends on p, q and the dimension

$$\|u\|_{2\alpha_j, Q_{\sigma_j}, \Lambda} \le C_3 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[\frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda},$$

where we exploited the fact that $\sum_{k=0}^\infty 1/\alpha_k = \nu/(\nu-1)$ and that $\sum_{k=0}^\infty k/\alpha_k < \infty$. From the inequality above we easily get, taking C_3 larger if needed,

$$\|u\|_{2\alpha_j, Q_{\sigma'}, \Lambda} \le C_3 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[\frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda},$$

and taking the limit as $j \to \infty$ gives the result

$$\sup_{Q_{\sigma'}} u(t,z) \le C_3 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[\frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda}.$$

□

Corollary 3.3.4. *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Let u be a positive subcaloric function in $Q = Q(\tau, x, s, R)$. Then there exists a positive constant $C_4 := C_4(q, p, d)$ which depends only on the dimension and on p, q such that for all $\alpha > 0$*

$$\sup_{Q_{\sigma'}} u(t,z) \le C_4 2^{\frac{2}{\alpha^2}\frac{\nu}{\nu-1}} (C_S^{B,\Lambda})^{\frac{1}{\alpha\nu-\alpha}} \tau^{\frac{1}{\alpha}} \left[\frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\alpha\nu-\alpha}} \|u\|_{\alpha, Q_\sigma, \Lambda}. \tag{3.11}$$

Proof. To prove (3.11) one can follow the same approach in [SC02][Theorem 2.2.3] with the only difference that we will consider parabolic balls Q_σ instead of balls. Observe that for $\alpha > 2$ this is just an application of Jensen's inequality. □

We remark that (3.11) is not good for the application of Bombieri-Giusti's Lemma 3.2.1 since $2^{\frac{2}{\alpha}\frac{\nu}{\nu-1}}$ is exploding as α approaches zero. To get rid of this problem we develop in the next section the same type of inequalities for supercaloric functions.

3.4 Mean value inequalities for supercaloric functions

Theorem 3.4.1. *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Let u be a positive supercaloric function in $Q = Q(\tau, x, s, R)$. Then there exists a positive constant $C_5 := C_5(p, q, d)$ which depends only on the dimension and on p, q such that for all $\alpha \in (0, \infty)$*

$$\sup_{Q_{\sigma'}} u(t,z)^{-\alpha} \le C_5 (C_S^{B,\Lambda})^{\frac{1}{\nu-1}} \tau \left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}} \|u^{-1}\|^{\alpha}_{\alpha, Q_\sigma, \Lambda}, \tag{3.12}$$

where $\nu = 2 - 2p^*/\rho$.

Proof. We can always assume that $u > \epsilon$ by considering the supersolution $u+\epsilon$ and then sending ϵ to zero at the end of the argument. Applying Lemma 3.1.4 with the function $F(x) := -|x|^{-\beta}$ and $\beta > 0$ we obtain

$$-\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,\Lambda} + \beta\, \mathcal{E}(u_t^{-\beta-1}\eta^2, u_t) \ge 0,$$

which after some manipulation gives

$$-\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,\Lambda} - 4\frac{\beta+1}{\beta}\, \mathcal{E}_{\eta^2}(u_t^{-\beta/2}, u_t^{-\beta/2}) - 4\int_{\mathbb{R}^d} a\nabla\eta\cdot\nabla(u_t^{-\beta/2})\eta u_t^{-\beta/2} dx \ge 0.$$

By means of Young's inequality $4ab \le 3a^2 + 2b^2/3$ and using the fact that $(\beta+1)/\beta > 1$ we get after averaging

$$\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,B,\Lambda} + \frac{\mathcal{E}_{\eta^2}(u_t^{-\beta/2}, u_t^{-\beta/2})}{|B|} \lesssim \|\nabla\eta\|^2_\infty \|u_t^{-\beta}\|_{1,B,\Lambda}.$$

We now integrate against a time cutoff $\zeta : \mathbb{R} \to [0,1]$ to obtain something similar to (3.7). Hence the same approach as in Proposition 3.3.2 applies and we deduce

$$\|\zeta\eta^2 u^{-\beta}\|^{\nu}_{\nu, I\times B, \Lambda} \lesssim C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}} \left[\|\zeta'\|_\infty + \|\nabla\eta\|^2_\infty\right]^{\nu} \|u^{-\beta}\|^{\nu}_{1, I\times B, \Lambda}.$$

Moser's iteration technique with $\beta_k = \nu^k\alpha$ and $\alpha > 0$ and the same argument of Theorem 3.3.3 will finally yield

$$\sup_{Q_{\sigma'}} u(t,z)^{-\alpha} \le C_5 (C_S^{B,\Lambda})^{\frac{1}{\nu-1}} \tau \left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}} \|u^{-1}\|^{\alpha}_{\alpha, Q_\sigma, \Lambda}.$$

□

We introduce the following parabolic ball. Given $x \in \mathbb{R}^d$, $R > 0$, $\tau > 0$ and $s \in \mathbb{R}$, $\delta \in (0,1)$, we note

$$Q'_\delta = Q'_\delta(\tau, x, s, R) = (s - \tau R^2, s - (1-\delta)\tau R^2) \times B(x, \delta R).$$

Theorem 3.4.2. *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Let u be a positive supercaloric function on $Q = Q(\tau, x, s, R)$. Fix $0 < \alpha_0 < \nu$. Then there exists a positive constant $C_6 := C_6(q, p, d, \alpha_0)$ which depends only on the dimension, on p, q and on α_0 such that for all $0 < \alpha < \alpha_0 \nu^{-1}$ we have*

$$\|u\|_{\alpha_0, Q'_{\sigma'}, \Lambda} \le \left\{ C_6 \tau (1+\tau^{-1})^{\frac{\nu}{\nu-1}} \left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu-1}} \right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|u\|_{\alpha, Q'_\sigma, \Lambda} \tag{3.13}$$

where $\nu = 2 - 2p^*/\rho$.

Proof. Assume u is supercaloric on $Q = I \times B$. Applying Lemma 3.1.4 with the function $F(x) := |x|^\beta$ with $\beta \in (0,1)$ we get

$$\frac{d}{dt} \|\eta^2 u_t^\beta\|_{1,\Lambda} + \beta\, \mathcal{E}(u_t^{\beta-1} \eta^2, u_t) \ge 0$$

which after some manipulation gives

$$\frac{d}{dt} \|\eta^2 u_t^\beta\|_{1,\Lambda} + 4 \frac{\beta - 1}{\beta}\, \mathcal{E}_\eta(u_t^{\beta/2}, u_t^{\beta/2}) + 4 \int_{\mathbb{R}^d} a \nabla \eta \cdot \nabla(u_t^{\beta/2}) \eta u_t^{\beta/2} dx \ge 0.$$

Note that $(\beta - 1)$ is negative. If we take $0 < \beta < \alpha_0 \nu^{-1}$ then we have

$$\frac{1-\beta}{\beta} > 1 - \beta > 1 - \alpha_0/\nu =: \epsilon,$$

this yields after Young's inequality

$$-\frac{d}{dt} \|\eta^2 u_t^\beta\|_{1,\Lambda} + \epsilon\, \mathcal{E}_\eta(u_t^{\beta/2}, u_t^{\beta/2}) \le A \|\nabla \eta\|_\infty^2 \|1_B u_t^\beta\|_{1,\Lambda},$$

where A is a constant possibly depending on q, p, α_0 and d which will be changing throughout the proof. Here we introduce a difference with respect to the previous proofs, the time cutoff $\zeta : \mathbb{R} \to [0,1]$, $\zeta \equiv 0$ on $(t_2, \infty]$, where $I = (t_1, t_2)$, is zero at the top of the time interval and not at the bottom. This gives after integrating

$$\zeta(t) \|\eta^2 u_t^\beta\|_{1,\Lambda} + \int_t^{t_2} \zeta(s) \mathcal{E}_\eta(u_s^{\beta/2}, u_s^{\beta/2})\, ds \le A \left[\|\zeta'\|_\infty + \|\nabla \eta\|_\infty^2 \right] \int_t^{t_2} \|1_B u_s^\beta\|_{1,\Lambda}\, ds,$$

which has the same flavor of (3.7). Starting from this inequality, and repeating the argument we used for subcaloric functions, we end up with

$$\|\zeta \eta^2 u^\beta\|_{\nu, I \times B, \Lambda}^\nu \le A\, C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}} \left[\|\zeta'\|_\infty + \|\nabla \eta\|_\infty^2 \right]^\nu \|u^\beta\|_{1, I \times B, \Lambda}^\nu. \tag{3.14}$$

The idea is now to iterate inequality (3.14) with an appropriate choice of exponents, parabolic balls and cutoffs. We follow closely the argument in [SC02, Theorem 2.2.5].

Exponents: we define the exponents $\alpha_i := \alpha_0 \nu^{-i}$ and $\beta_j = \alpha_i \nu^{j-1}$ for $j = 1, \ldots, i$. Observe that $0 < \beta_j < \alpha_0 \nu^{-1}$ and thus we are in a setting where (3.14) is applicable.

Parabolic balls: for $j = 1, \ldots, i$ we fix $\sigma_j - \sigma_{j+1} = 2^{-j-1}(\sigma - \sigma')$, $\sigma_0 = \sigma$ and set

$$I_{\sigma_j} = (s - \tau R^2, s - (1 - \sigma_j)\tau R^2), \quad Q'_{\sigma_j} = I_{\sigma_j} \times B(x, \sigma_j R).$$

Cutoffs: for $j = 1, \ldots, i$ we define the cutoffs $\eta_j : \mathbb{R}^d \to [0,1]$ such that $\operatorname{supp} \eta_j \subset B(x, \sigma_j R)$, $\eta_j \equiv 1$ on $B(x, \sigma_{j+1} R)$ and $\|\nabla \eta_j\|_\infty \leq 2/(R\delta_j)$, and the cutoffs $\zeta_j : \mathbb{R} \to [0,1]$, $\zeta_j \equiv 1$ on $I_{\sigma_{j+1}}$, $\zeta_j \equiv 0$ on $(s - (1-\sigma_j)\tau R^2, \infty)$ and $\|\zeta'_j\|_\infty \leq 2/(R^2 \tau \delta_j)$.

We are ready to apply (3.14) for $j = 1, \ldots, i$ and the choices above.

$$\|u^{\alpha_i \nu^j}\|_{1, Q'_{\sigma_j}, \Lambda} \leq A\, C_S^{B,\Lambda} \tau^{\nu - 1} \left[\frac{(1 + \tau^{-1}) 2^{2j}}{(\sigma - \sigma')^2} \right]^\nu \|u^{\alpha_i \nu^{j-1}}\|^\nu_{1, Q'_{\sigma_{j-1}}, \Lambda},$$

which after an iteration from $j = 1$ to $j = i$ gives

$$\|u\|^{\alpha_0}_{\alpha_0, Q'_{\sigma_i}, \Lambda} \leq 2^{2 \sum_{j=0}^{i-1} (i-k)\nu^j} \left\{ A\, C_S^{B,\Lambda} \tau^{\nu-1} \left[\frac{1 + \tau^{-1}}{(\sigma - \sigma')^2} \right]^\nu \right\}^{\sum_{j=0}^{i-1} \nu^j} \|u^{\alpha_i}\|^{\nu^i}_{1, Q'_\sigma, \Lambda}.$$

Now observe that

$$\sum_{j=0}^{i-1} (i-j)\nu^j \leq C(\nu)(\alpha_0/\alpha_i - 1), \qquad \sum_{j=0}^{i-1} \nu^j = \frac{\nu^i - 1}{\nu - 1} = \frac{\alpha_0/\alpha_i - 1}{\nu - 1},$$

where $C(\nu) > 0$ is a constant which depends on ν but not on i. This yields the following inequality

$$\|u\|_{\alpha_0, Q'_{\sigma'}, \Lambda} \leq \left\{ A\, \tau (1 + \tau^{-1})^{\frac{\nu}{\nu-1}} \left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu-1}} \right\}^{1/\alpha_i - 1/\alpha_0} \|u\|_{\alpha_i, Q'_\sigma, \Lambda},$$

where the constant A depends only on α_0, q, p and the dimension $d \geq 2$. Replacing A by $A \vee 1$, we can assume it greater than one. Finally, we extend the inequality for $\alpha \in (0, \alpha_0 \nu^{-1})$. Let $i \geq 2$ be an integer such that $\alpha_i \leq \alpha < \alpha_{i-1}$, then from the relation $1/\alpha_i - 1/\alpha_0 \leq (1 + \nu)(1/\alpha - 1/\alpha_0)$ and by means of Jensen's inequality we deduce

$$\|u\|_{\alpha_0, Q'_{\sigma'}, \Lambda} \leq \left\{ A\, \tau (1 + \tau^{-1})^{\frac{\nu}{\nu-1}} \left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu-1}} \right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|u\|_{\alpha, Q'_\sigma, \Lambda},$$

which is what we wanted to prove. □

3.5 Mean value inequalities for the logarithm

In this section we derive mean value inequalities for $\log u$ where u is a positive supercaloric function on $Q = (s - \tau R^2, s) \times B(x, R)$, with $\tau > 0$ fixed. We denote by $m^\Lambda(dx) := \Lambda(x) dx$ and by $\gamma^\Lambda := dt \otimes m^\Lambda$.

Theorem 3.5.1. *Fix $\tau > 0$, $\kappa \in (0,1)$ and $\delta \in [1/2,1)$. For any $s \in \mathbb{R}$, $R > 0$ and any positive supercaloric function u on $Q = (s - \tau R^2, s) \times B(x,R)$, there exist a positive constant $C_7 := C_7(q,p,d,\delta)$ and a constant $k := k(u,\kappa) > 0$ such that for all $\ell > 0$*

$$\gamma^\Lambda\{(t,z) \in K^+ \mid \log u_t < -\ell - k\} \le C_7\, m^\Lambda(B)\Big[M^{B,\Lambda}|B|^{\frac{2}{d}}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1}, \tag{3.15}$$

and

$$\gamma^\Lambda\{(t,z) \in K^- \mid \log u_t > \ell - k\} \le C_7\, m^\Lambda(B)\Big[M^{B,\Lambda}|B|^{\frac{2}{d}}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1}, \tag{3.16}$$

where $K^+ = (s - \kappa\tau R^2, s) \times B(x,\delta R)$ and $K^- = (s - \tau R^2, s - \kappa\tau R^2) \times B(x,\delta R)$.

Proof. The argument of the proof is quite classical and we adopt the strategy presented in Theorem 5.4.1 of [SC02]. We can always assume $u_t \ge \epsilon$ and then send ϵ to zero in our estimates, since $u_t + \epsilon$ is still a supercaloric function. We denote as usual $B := B(x,R)$. By Lemma 3.1.4

$$\begin{aligned}\frac{d}{dt}(\eta^2, -\log u_t)_\Lambda \le \mathcal{E}(u_t^{-1}\eta^2, u_t) &= -\mathcal{E}_\eta(\log u_t, \log u_t) + 2\int \langle a\nabla\eta, \nabla u_t\rangle \eta u_t^{-1} dx \\ &\le -\mathcal{E}_\eta(\log u_t, \log u_t) + 2\mathcal{E}_\eta(\log u_t, \log u_t)^{1/2}\|\nabla\eta\|_\infty \|1_B\|_{1,\Lambda}^{1/2} \\ &\le -\frac{1}{2}\mathcal{E}_\eta(\log u_t, \log u_t) + 2m^\Lambda(B)\|\nabla\eta\|_\infty^2,\end{aligned} \tag{3.17}$$

where in the last inequality we exploited Young's inequality $2ab \le (1/2a^2 + 2b^2)$. The cutoff function η must be on the form used in (1.22), namely a radial cutoff. We take

$$\eta(z) := (1 - |x - z|/R)^+$$

where x, R are the center and the radius of the ball B as by assumption. We note

$$w_t(z) := -\log u_t(z), \quad W_t := (w_t)_B^{\Lambda\eta^2},$$

then the weighted Poincaré inequality (1.22) reads

$$\frac{|B|}{\|\eta^2\Lambda\|_1}\|w_t - W_t\|_{2,B,\Lambda\eta^2}^2 \lesssim M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\frac{\mathcal{E}(w_t,w_t)}{2\|\eta^2\Lambda\|_1},$$

rewriting (3.17) with the above estimate we get

$$\partial_t W_t + \frac{|B|}{\|\eta^2\Lambda\|_1}\Big(M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\Big)^{-1}\|w_t - W_t\|_{2,B,\Lambda\eta^2}^2 \lesssim \|\nabla\eta\|_\infty^2\frac{m^\Lambda(B)}{\|\eta^2\Lambda\|_1}.$$

By the fact that $(1-\delta)^2 m^\Lambda(B(x,\delta r)) \le \|\eta^2\Lambda\|_1 \le m^\Lambda(B)$ and $\|\nabla\eta\|_\infty^2 \lesssim |B|^{-\frac{2}{d}}$, it follows

$$\partial_t W_t + \Big(m^\Lambda(B)M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\Big)^{-1}\int_{\delta B}|w_t - W_t|^2\,\Lambda dx \le c\,M^{B,\Lambda}|B|^{-\frac{2}{d}} \tag{3.18}$$

for some constant $c > 0$ depending only on the dimension and δ. Observe that we fixed $\delta \in [1/2,1)$ to stay away from the boundary of B. What we have above resembles closely what is given in [SC02, Theorem 5.4.1], except for the dependence of the constants on B. Let us introduce the following auxiliary functions

$$\bar{w}_t := w_t - c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s'), \quad \bar{W}_t := W_t - c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s'),$$

where $s' = s - \kappa\tau R^2$ and $c > 0$ is the constant appearing in (3.18). We can now rewrite (3.18) as

$$\partial_t \bar{W}_t + \Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)^{-1} \int_{\delta B} |\bar{w}_t - \bar{W}_t|^2\, \Lambda dx \le 0. \tag{3.19}$$

Now set $k = k(u,\kappa) := \bar{W}_{s'}$ and define for $\ell > 0$ the two sets

$$D_t^+(\ell) := \{z \in B(x,\delta R) \,|\, \bar{w}(t,z) > k + \ell\},$$

$$D_t^-(\ell) := \{z \in B(x,\delta R) \,|\, \bar{w}(t,z) < k - \ell\}.$$

Since $\partial_t \bar{W}_t \le 0$ we have for $t > s'$ that $\bar{w}_t - \bar{W}_t > \ell + k - \bar{W}_t \ge \ell$ on $D_t^+(\ell)$. Using this in (3.19) we obtain

$$\partial_t \bar{W}_t + \Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)^{-1} |\ell + k - \bar{W}_t|^2\, m^\Lambda(D_t^+(\ell)) \le 0, \tag{3.20}$$

or equivalently

$$-\Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \partial_t |\ell + k - \bar{W}_t|^{-1} \ge m^\Lambda(D_t^+(\ell)). \tag{3.21}$$

Integrating from s' to s yields, for $\gamma^\Lambda = dt \otimes m^\Lambda$,

$$\gamma^\Lambda\{(t,z) \in K^+ \,|\, \bar{w}(t,z) > k + \ell\} \le m^\Lambda(B) \Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \ell^{-1}.$$

Recall that $-\log u_t = \bar{w}_t + c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s')$, therefore

$$\gamma^\Lambda\{(t,z) \in K^+ \,|\, \log u_t + c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') < -k - \ell\} \le m^\Lambda(B) \Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \ell^{-1}.$$

Finally,

$$\begin{aligned}
\gamma^\Lambda\{(t,z) \in K^+ \,|\, \log u_t < -k - \ell\}& \\
&\le \gamma^\Lambda\{(t,z) \in K^+ \,|\, \log u_t + c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') < -k - \ell/2\} \\
&+ \gamma^\Lambda\{(t,z) \in K^+ \,|\, c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') > \ell/2\} \\
&\lesssim m^\Lambda(B) \Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \ell^{-1} + m^\Lambda(B) \Big(\tau^2 M^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \ell^{-1} \\
&\lesssim m^\Lambda(B) \Big[M^{B,\Lambda} |B|^{\frac{2}{d}} (C_P^{B,\Lambda} \vee \tau^2)\Big] \ell^{-1},
\end{aligned}$$

where in the second but last step we used Markov's inequality and the fact that $\kappa < 1$. Working with $D_t^-(\ell)$ and K^- and using similar arguments proves the second inequality. $\square$

3.6 Parabolic Harnack inequality

We have all the tools to apply effectively Lemma 3.2.1 to a positive function u which is caloric in the parabolic ball $Q(\tau, s, x, R) = (s - \tau R^2, s) \times B(x, R)$. This will finally provide us the parabolic Harnack inequality. Fix $\delta \in (0, 1)$ and $\tau > 0$. For $x \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $R > 0$ denote

$$
\begin{aligned}
Q_- &= (s - (3+\delta)\tau R^2/4, s - (3-\delta)\tau R^2/4) \times B(x, \delta R), \\
Q'_- &= (s - \tau R^2, s - (3-\delta)\tau R^2/4) \times B(x, \delta R), \\
Q_+ &= (s - (1+\delta)\tau R^2/4, s) \times B(x, \delta R).
\end{aligned} \tag{3.22}
$$

Then we have the following.

Theorem 3.6.1. *Fix $\tau > 0$ and $\delta \in [1/2, 1)$. Fix $\alpha_0 \in (0, \nu)$. Let u be any positive caloric function on $Q = (s - \tau R^2, s) \times B(x, R)$. Then we have*

$$
\|u\|_{\alpha_0, Q'_-, \Lambda} \lesssim C_8 \inf_{Q_+} u(t, z), \tag{3.23}
$$

where the constant C_8 depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on $\tau, p, q, \alpha_0, d, \delta$.

Proof. For the proof we follow the argument presented in [SC02, Theorem 5.4.2]. Take $k := k(u, \kappa)$ corresponding to $\kappa = 1/2$ in Theorem 3.5.1. Set $v = e^k u$ and

$$
U = (s - \tau R^2, s - 1/2\tau R^2) \times B(x, R), \quad U_\sigma = (s - \tau R^2, s - (3-\sigma)\tau R^2/4) \times B(x, \sigma R).
$$

By Theorem 3.4.2 it follows that

$$
\|v\|_{\alpha_0, U_{\sigma'}, \Lambda} \leq \left\{ C_6 \tau (1 + \tau^{-1})^{\frac{\nu}{\nu-1}} \left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu-1}} \right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|v\|_{\alpha, U_\sigma, \Lambda}
$$

for all $1/2 \leq \sigma' < \sigma \leq 1$ and all $\alpha \in (0, \alpha_0 \nu^{-1})$, in particular notice that $\alpha_0 \nu^{-1} > \alpha_0/2$ and that $\alpha_0/2 < \nu/2 < 1$ since $\nu \in (1, 2)$. By Theorem 3.5.1 we have that

$$
\gamma^\Lambda\{(t, z) \in U \mid \log v > \ell\} \leq C_7 \, \gamma^\Lambda(U) \tau^{-1} \Big[M^{B,\Lambda}(C_P^{B,\Lambda} \vee \tau^2) \Big] \ell^{-1}.
$$

Bombieri-Giusti's Lemma 3.2.1 is applicable and we deduce

$$
\|e^\kappa u\|_{\alpha_0, Q'_-, \Lambda} \lesssim C_{BG}^{B,\Lambda}
$$

where $C_{BG}^{B,\Lambda}$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on τ, p, q, α_0, d.

On the other hand we can now fix

$$
V = (s - 1/2\tau R^2, s) \times B(x, R), \quad V_\sigma = (s - (1+\sigma)\tau R^2/4, s) \times B(x, \sigma R)
$$

and apply Theorem 3.4.1 to $v = e^{-k} u^{-1}$ where k is the same constant as above, this produces

$$
\sup_{V_{\sigma'}} v(t, z) \leq \left\{ C_5 (C_S^{B,\Lambda})^{\frac{1}{\nu-1}} \tau \left[\frac{1 + \tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu-1}} \right\}^{1/\alpha} \|v\|_{\alpha, V_\sigma, \Lambda},
$$

for all $\alpha > 0$ and $1/2 \le \sigma' < \sigma \le 1$. Since by Theorem 3.5.1 we have

$$\gamma^\Lambda\{(t,z) \in V \mid \log v > \ell\} \le C_7\, \gamma^\Lambda(V)\tau^{-1}\Big[M^{B,\Lambda}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1},$$

then Bombieri-Giusti's lemma is applicable and yields

$$\sup_{Q_+} e^{-\kappa}u^{-1} \lesssim C_{BG}^{B,\Lambda}$$

for some $C_{BG}^{B,\Lambda}$ which we can assume to be the same as before taking the maximum of the two. Putting the two inequalities together gives the result. □

Theorem 3.6.2 (Parabolic Harnack inequality)**.** *Fix $\tau > 0$ and $\delta \in [1/2, 1)$. Let u be any positive caloric function in $Q = (s - \tau R^2, s) \times B(x, R)$. Then we have*

$$\sup_{Q_-} u(t,z) \le C_{PH}^{B,\Lambda} \inf_{Q_+} u(t,z), \tag{3.24}$$

where the constant $C_{PH}^{B,\Lambda}$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on τ, p, q, d, δ.

Proof. It follows from the previous theorem for positive supercaloric functions and Corollary 3.3.4. □

In the classical situation of uniformly elliptic and bounded coefficients, or in the case of Muckenhaupt's weights, the constant $C_{PH}^{B,\Lambda}$ in front of the parabolic Harnack inequality can be taken uniformly in B. As a result one can prove that any caloric function is Hölder continuous, both in time and space.

Since we are interested in the diffusive scaling of caloric functions u, for us it will be enough to control oscillations of $u(t/\epsilon^2, x/\epsilon)$ in the limit as $\epsilon \to 0$. This translates in assuming a good behavior of the constant $C_{PH}^{B(x/\epsilon,R/\epsilon),\Lambda}$ as $\epsilon \to 0$.

As we know, $C_{PH}^{B(x/\epsilon,R/\epsilon),\Lambda}$ depends on averages of λ^{-q} and Λ^p on $B(x/\epsilon, R/\epsilon)$, therefore to get a good control over the constants and make proper use of (3.24) we will need to assume something like

$$\sup_{x\in\mathbb{R}^d} \limsup_{\epsilon\to 0} \frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} \Lambda^p(x) + \lambda^{-q}(x)\,dx < \infty.$$

This assumption will be made more precise in Chapter 6.

Part II

Central Limit Theorems

4

Elements of Dirichlet forms theory

One of the major technical issue we faced in our work is the complete absence of regularity of the conductivity matrix a. Since the entries $x \mapsto a_{ij}(x)$ are not supposed to be either continuous or weakly differentiable, the operator in divergence form

$$Lu(x) := \operatorname{div}(a(x)\nabla u(x))$$

is not well defined. In order to overcome this problem and associate a diffusion process to this "formal" generator we had to take a weak approach, namely, we considered the bilinear form $\mathcal{E}$ obtained by L formally integrating by parts $(u, -Lv)_{L^2(\mathbb{R}^d)}$

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} \langle a\nabla u, \nabla v\rangle \, dx, \quad u, v \in C_0^\infty(\mathbb{R}^d). \tag{4.1}$$

In this chapter we discuss the general theory behind this weak approach, the by now well known Dirichlet Forms theory, and we will explore the link between Dirichlet forms and stochastic processes. In particular we consider the bilinear form given in (4.1) and already introduced in Section 1.2 as our guiding example. For an exhaustive treatment on Dirichlet Forms we refer to the celebrated book of M. Fukushima, Y. Oshima and M. Takeda [FOT94].

4.1 Basic Definitions

Dirichlet form theory has seen a wide spread appreciation in both the analytic and probabilistic community. This success is due to the rich interplay between the theory of strongly continuous semigroups and stochastic processes. The analytic part goes back to the seminal work of A. Beurling and J. Deny [BD59], while the more probabilistic part was initiated by the fundamental work of M. Fukushima [FOT94] and M.L. Silverstein [Sil74] combining Dirichlet forms and symmetric Markov processes. The theory developed by the above mentioned authors dealt with locally compact spaces, and in this chapter we will be mostly interested in this case. More recently much of the attention has shifted to infinite dimensional spaces with weaker topological assumptions, see for example [Röc93] for an introduction.

For our purposes it is enough to work with locally compact separable spaces. Let $(\mathcal{X}, \mathcal{B}, m)$ be a σ-finite measure space, where $\mathcal{X}$ is a locally compact separable metric space and m is a positive Radon measure on $\mathcal{X}$ such that $\mathrm{supp}[m] = \mathcal{X}$. Let $L^2(\mathcal{X}, m)$ consist of the square integrable $\mathcal{B}$-measurable extended real functions on $\mathcal{X}$. It is well known that $L^2(\mathcal{X}, m)$ endowed with the inner product

$$(u, v)_m := \int_{\mathcal{X}} u(x)v(x)\, m(dx), \quad u, v \in L^2(\mathcal{X}, m)$$

is a Hilbert space.

For the application we have in mind, $\mathcal{X}$ will mostly be $\mathbb{R}^d$ or an open subset of $\mathbb{R}^d$, m a measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ of the form $m(dx) = \theta(x)\, dx$ and $\mathcal{B}$ the usual Borel σ-algebra.

Definition 4.1.1. *$\mathcal{E}$ is called a* symmetric *form on $L^2(\mathcal{X}, m)$ with domain $\mathcal{D}$ if the following conditions hold*

- *$\mathcal{E} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a dense linear subspace of $L^2(\mathcal{X}, m)$,*
- *$\mathcal{E}(u, v) = \mathcal{E}(v, u)$, $\mathcal{E}(u + w, v) = \mathcal{E}(u, v) + \mathcal{E}(w, v)$, $c\,\mathcal{E}(u, v) = \mathcal{E}(cu, v)$, $\mathcal{E}(u, u) \geq 0$, for all $u, v, w \in \mathcal{D}$ and $c \in \mathbb{R}$.*

One trivial example of symmetric form is the inner product $(\cdot, \cdot)_m$ on $L^2(\mathcal{X}, m)$ which is defined on the whole space $L^2(\mathcal{X}, m)$. An other example, and actually the most important for this thesis, is the symmetric form (cf. Section 1.2) defined by

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \langle a\nabla u, \nabla v\rangle\, dx, \quad u, v \in C_0^\infty(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d, \theta)$ with domain $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$.

Given a symmetric form $\mathcal{E}$ with domain $\mathcal{D}$ on $L^2(\mathcal{X}, m)$, the position $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_m$ defines a new symmetric form on $L^2(\mathcal{X}, m)$ for each $\alpha > 0$. Observe that $\mathcal{D}$ is a pre-Hilbert space with inner product $\mathcal{E}_\alpha$, moreover $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ define equivalent metrics on $\mathcal{D}$ for different $\alpha, \beta > 0$. If $\mathcal{D}$ is complete with respect to this metric, then $(\mathcal{E}, \mathcal{D})$ is said to be *closed*. In other words, a symmetric form $\mathcal{E}$ is said to be closed if Cauchy sequences in $\mathcal{D}$ with respect to $\mathcal{E}_1$ have limiting points in $\mathcal{D}$, that is,

- $u_n \in \mathcal{D}$, $\mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$ implies that there exists $u \in \mathcal{D}$ such that $\mathcal{E}_1(u_n - u, u_n - u) \to 0$ as $n \to \infty$.

Clearly if a symmetric form is closed, then $(\mathcal{D}, \mathcal{E}_\alpha)$ is a real Hilbert space for all $\alpha > 0$.

Given two symmetric forms $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$, with domains $\mathcal{D}_1$ and $\mathcal{D}_2$, we say that $\mathcal{E}^{(2)}$ is an *extension* of $\mathcal{E}^{(1)}$ if $\mathcal{D}_1 \subset \mathcal{D}_2$ and $\mathcal{E}^{(1)} \equiv \mathcal{E}^{(2)}$ on $\mathcal{D}_1 \times \mathcal{D}_1$. We say that a symmetric form $\mathcal{E}$ is *closable* if it has a closed extension.

Remark 4.1.2. *The bilinear form $\mathcal{E}$ on $L^2(\mathbb{R}^d, \theta)$ with domain $C_0^\infty(\mathbb{R}^d)$ introduced in Section 1.2, and given above for the reader's convenience, is closable thanks to Proposition 1.2.1. We denoted by $(\mathcal{E}, \mathcal{F}^\theta)$ its smallest closed extension on $L^2(\mathbb{R}^d, \theta)$.*

So far we didn't define what is a Dirichlet form, in order to do that we need a further concept. A symmetric form $\mathcal{E}$ on $L^2(\mathcal{X}, m)$ with domain $\mathcal{D}$ is called a *Markovian symmetric form* if for each $\epsilon > 0$ there exists a real function $\phi_\epsilon(t)$, $t \in \mathbb{R}$ such that $\phi_\epsilon(t) = t$, for all $t \in [0,1]$, $-\epsilon \leq \phi_\epsilon(t) \leq 1+\epsilon$, for all $t \in \mathbb{R}$ and $0 \leq \phi_\epsilon(t') - \phi_\epsilon(t) \leq t' - t$ whenever $t < t'$ with the property that if $u \in \mathcal{D}$, then $\phi_\epsilon(u) \in \mathcal{D}$ and $\mathcal{E}(\phi_\epsilon(u), \phi_\epsilon(u)) \leq \mathcal{E}(u,u)$.

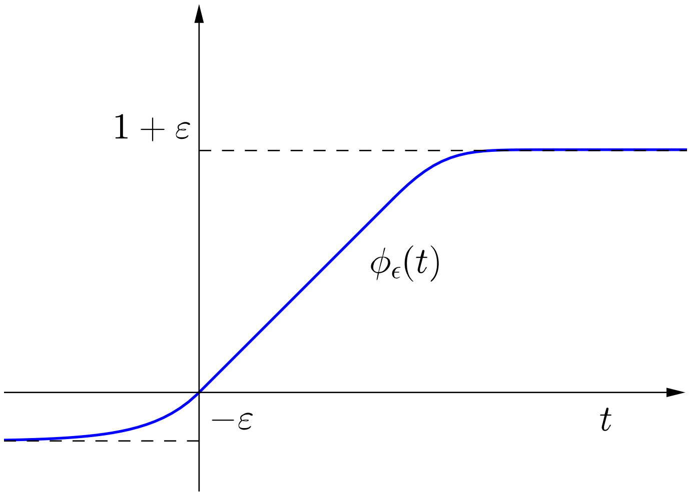

Figure 4.1: graph of $t \to \phi_\epsilon(t)$.

Proposition 4.1.3. *Assume* $(b.1)$ *and that* $\theta, \theta^{-1} \in L^1_{loc}(\mathbb{R}^d)$*, then the symmetric form given by* (4.1) *with domain* $C_0^\infty(\mathbb{R}^d)$ *on* $L^2(\mathbb{R}^d, \theta)$ *is Markovian on* $L^2(\mathbb{R}^d, \theta)$.

Proof. For each $\epsilon > 0$ we have to construct an infinitely differentiable function ϕ_ϵ which satisfies the conditions above. If this were possible, clearly for all $u \in C_0^\infty(\mathbb{R}^d)$ we would have $\phi_\epsilon(u) \in C_0^\infty(\mathbb{R}^d)$, $\nabla \phi_\epsilon(u) = \phi'_\epsilon(u) \nabla u$ and $0 \leq \phi'_\epsilon \leq 1$ so that

$$\mathcal{E}(\phi_\epsilon(u), \phi_\epsilon(u)) = \int_{\mathbb{R}^d} \langle a \nabla \phi_\epsilon(u), \nabla \phi_\epsilon(u) \rangle dx = \int_{\mathbb{R}^d} \langle a \nabla u, \nabla u \rangle (\phi'_\epsilon(u))^2 dx \leq \mathcal{E}(u,u).$$

It remains to construct ϕ_ϵ. This can be done by mollifying the non-smooth function $\psi_\epsilon(t) = ((-\epsilon) \vee t) \wedge (1+\epsilon)$ with the standard infinitely differentiable mollifier

$$\eta_\delta(t) := \begin{cases} \delta^{-1} \exp(-1/(1-(t/\delta)^2)), & |t| < \delta, \\ 0, & |t| \geq \delta, \end{cases}$$

with $\delta < \epsilon$. □

Lemma 4.1.4. *Let* $(\mathcal{E}, \mathcal{D})$ *be a closed symmetric form on* $L^2(\mathcal{X}, m)$*.* $\mathcal{E}$ *is Markovian if and only if for all* $u \in \mathcal{D}$ *and* v *normal contraction of* u *we have* $v \in \mathcal{D}$ *and* $\mathcal{E}(v,v) \leq \mathcal{E}(u,u)$*. Here* v *is called* normal contraction *of* u *if*

$$|v(x) - v(y)| \leq |u(x) - u(y)|, \quad |v(x)| \leq |u(y)|, \quad \forall x, y \in \mathcal{X}.$$

Proof. The fact that the condition is sufficient follows choosing $\phi_\epsilon(t) = (0 \vee t) \wedge 1$ and does not depend on the fact that $\mathcal{E}$ is closed. The necessity is more involved and we refer to [Röc93] and [BH91] for a proof in full generality. □

We are ready to give the most important definition of this chapter.

Definition 4.1.5. *A* Dirichlet form $(\mathcal{E}, \mathcal{D})$ *on* $L^2(\mathcal{X}, m)$ *is a closed and Markovian symmetric form on* $L^2(\mathcal{X}, m)$.

Let $C_0(\mathcal{X})$ be the set of compactly supported continuous functions on $\mathcal{X}$. A *core* of a symmetric form $\mathcal{E}$ is by definition a subset $\mathcal{C}$ of $\mathcal{D} \cap C_0(\mathcal{X})$ such that $\mathcal{C}$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}_1$ and in $C_0(\mathcal{X})$ with respect to the uniform norm.

Definition 4.1.6. *A symmetric form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$ is said to be regular if it possesses a core. A core $\mathcal{C}$ is said to be standard if $\mathcal{C}$ is a linear subspace of $C_0(\mathcal{X})$ and for any $\epsilon > 0$ there exists a function ϕ_ϵ satisfying the condition above such that $\phi_\epsilon(u) \in \mathcal{C}$ for all $u \in \mathcal{C}$. By a special standard core we mean a standard core $\mathcal{C}$ which is a dense subalgebra of $C_0(\mathcal{X})$, and such that for any compact set K and relatively compact open set G with $K \subset G$ there is $u \in \mathcal{C}$ which is non-negative, $u = 1$ on K and $u = 0$ on $\mathcal{X} \setminus G$.*

For a m-measurable function u the support of the measure $u(x)m(dx)$ will be denoted by $\mathrm{supp}[u]$. Clearly, if $u \in C(\mathcal{X})$ then $\mathrm{supp}[u]$ is just the closure of of the set of points $x \in \mathcal{X}$ where u is not zero.

Definition 4.1.7. *We say that a symmetric form $\mathcal{E}$ is* local *if for any $u, v \in \mathcal{D}$ with disjoint compact support $\mathcal{E}(u, v) = 0$. $\mathcal{E}$ is said to be* strongly local *if for any $u, v \in \mathcal{D}$ with compact support and such that v is constant on a neighborhood of* $\mathrm{supp}[u]$*, then $\mathcal{E}(u, v) = 0$.*

Remark 4.1.8. *It is clear that the Dirichlet form $\mathcal{E}$ on $L^2(\mathbb{R}^d, \theta)$ with domain $C_0(\mathbb{R}^d)$ and defined by* (4.1) *has a special standard core $\mathcal{C}$ simply given by $C_0^\infty(\mathbb{R}^d)$. In particular it is regular. Moreover, it is immediate to see that it is a strongly local symmetric form as well.*

4.2 Dirichlet forms and Markovian semigroups

In this section we want to link closed symmetric forms with the infinitesimal generators, resolvents and semigroups on L^2-spaces. The following definitions are very well known, we include them for sake of completeness.

Definition 4.2.1. *A family of symmetric bounded operators $\{T_t : t > 0\}$ on $L^2(\mathcal{X}, m)$ is called a semigroup if*

(i) (semigroup property) $T_t \circ T_s = T_{t+s}$ *for all* $s, t \in (0, \infty)$,

(ii) (contraction property) $(T_t u, T_t u)_m \leq (u, u)_m$ *for all* $u \in L^2(\mathcal{X}, m)$.

The semigroup is said strongly continuous *if in addition*

(iii) $(T_t u - u, T_t u - u)_m \to 0$ *as* $t \to 0$ *for all* $u \in L^2(\mathcal{X}, m)$.

Definition 4.2.2. *A resolvent on $L^2(\mathcal{X}, m)$ is a family of symmetric bounded operators $\{G_\alpha : \alpha > 0\}$ from $L^2(\mathcal{X}, m)$ into itself such that*

(i) (resolvent equation) $G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0$ *for all* $\alpha, \beta > 0$

(ii) (contraction property) $(\alpha G_\alpha u, \alpha G_\alpha u)_m \leq (u,u)_m$ *for all* $\alpha > 0$ *and* $u \in L^2(\mathcal{X}, m)$.

The semigroup is said strongly continuous *if in addition*

(iii) $(\alpha G_\alpha u - u, \alpha G_\alpha u - u)_m \to 0$ *as* $\alpha \to +\infty$ *for all* $u \in L^2(\mathcal{X}, m)$.

Definition 4.2.3. *A generator* L *of a strongly continuous semigroup* $\{T_t : t > 0\}$ *on* $L^2(\mathcal{X}, m)$ *is defined by*

$$Lu := \lim_{t \downarrow 0} \frac{T_t u - u}{t},$$

whenever the limit exists in $L^2(\mathbb{R}^d, m)$. *We denote by* $\mathcal{D}(L)$ *the set of* $u \in L^2(\mathcal{X}, m)$ *for which such limit exists.*

What is remarkable, and also well established, is that these three concepts are intimately linked. Indeed, given a strongly continuous semigroup $\{T_t\}_{t>0}$ we can associate to it a strongly continuous resolvent $\{G_\alpha\}_{\alpha>0}$ by the Bochner integral

$$G_\alpha u := \int_0^\infty e^{-\alpha t} T_t u \, dt$$

and vice versa given $\{G_\alpha\}_{\alpha>0}$ one can find back the semigroup by the formula

$$T_t u = \lim_{\beta \to \infty} e^{-\beta t} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n u.$$

Moreover, the generator L of $\{T_t\}_{t>0}$ can be also obtained by $Lu = \alpha u - G_\alpha^{-1} u$ with $\mathcal{D}(L) = G_\alpha(L^2(\mathcal{X}, m))$ and given a non-positive definite self-adjoint operator L on $L^2(\mathcal{X}, m)$ we can associate to it a strongly continuous resolvent with the position $G_\alpha u = (\alpha - L)^{-1} u$.

To summarize, there is a one to one correspondence among the family of non-positive definite self-adjoint operators on $L^2(\mathcal{X}, m)$, the family of strongly continuous semigroups, and the family of strongly continuous resolvents.

One of the most important result in the theory of closed symmetric forms is that they also fit in this picture.

Theorem 4.2.4. *There is a one to one correspondence between the family of closed symmetric forms on* $L^2(\mathcal{X}, m)$ *and the family of non-positive definite self-adjoint operators* L *on* $L^2(\mathcal{X}, m)$. *The correspondence is determined by*

$$\begin{cases} \mathcal{D} \subset \mathcal{D}(L) \\ \mathcal{E}(u,v) = (-Lu, v)_m, \quad u \in \mathcal{D}(L),\, v \in \mathcal{D}. \end{cases}$$

Proof. Again, for the proof we refer to [FOT94, Theorem 1.3.1]. □

More can be said if the closed symmetric form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$ is also Markovian, and therefore a Dirichlet form. A bounded linear operator S on $L^2(\mathcal{X}, m)$ is called Markovian if $0 \leq Su \leq 1$ m-almost everywhere whenever $0 \leq u \leq 1$ m-almost everywhere.

Proposition 4.2.5. *Let $(\mathcal{E}, \mathcal{D})$ be a closed symmetric form on $L^2(\mathcal{X}, m)$. Let $\{T_t : t > 0\}$ and $\{G_\alpha : \alpha > 0\}$ be the strongly continuous semigroup and the strongly continuous resolvent on $L^2(\mathcal{X}, m)$ which are associated with $\mathcal{E}$ as above. Then the followings are equivalent*

(i) $\mathcal{E}$ *is a Dirichlet form,*

(ii) T_t *is Markovian for all* $t > 0$*,*

(iii) αG_α *is Markovian for all* $\alpha > 0$*.*

Remark 4.2.6. *As a consequence of the above Proposition and Theorem 4.2.4 there is a one to one correspondence between Dirichlet Forms and Markovian semigroups.*

4.3 Closability and smallest closed extension

In this section we show that the markovianity and the local property of a symmetric form are preserved under the operation of taking the smallest closed extension. Since this is all classical theory of Dirichlet forms we state the results without proof, the interested reader can refer to [FOT94, Chapter 3] or [Röc93].

Theorem 4.3.1. *Suppose that $\mathcal{E}$ is a closable Markovian symmetric form on $L^2(\mathcal{X}, m)$. Then its smallest closed extension $\overline{\mathcal{E}}$ is again Markovian and hence a Dirichlet form.*

Theorem 4.3.2. *Assume that a closable Markovian symmetric $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$ form satisfies the following conditions (i) and (ii):*

(i) $\mathcal{D}$ *is a dense subalgebra of* $C_0(\mathcal{X})$*,*

(ii) for any compact set K and relatively compact open set $G \supset K$ there exists a non negative function u such that $u = 1$ on K and $u = 0$ on $\mathcal{X} \setminus G$.

If $\mathcal{E}$ has the local property, then so does its smallest closed extension $\overline{\mathcal{E}}$. Furthermore, $\overline{\mathcal{E}}$ is a regular Dirichlet form possessing $\mathcal{D}$ as its special standard core.

Remark 4.3.3. *It is easy to see that under the assumptions of Theorem 4.3.2 the smallest closed extension $\overline{\mathcal{E}}$ enjoys the strong local property whenever $\mathcal{E}$ does.*

As a result of the above theorems and the subsequent remark we can finally complete the information on the symmetric form $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(\mathbb{R}^d, \theta)$. Let us condense this information in the following proposition.

Proposition 4.3.4. *Let $\theta : \mathbb{R}^d \to [0, +\infty)$ be a measurable function such that $\theta, \theta^{-1} \in L^1_{loc}(\mathbb{R}^d)$ and assume $(b.1)$. Then the symmetric form*

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \langle a\nabla u, \nabla v\rangle \, dx, \quad u, v \in C_0^\infty(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d, \theta)$ is closable. Moreover, its smallest closed extension $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(\mathbb{R}^d, \theta)$ is a strong local Dirichlet form possessing $C_0^\infty(\mathbb{R}^d)$ as its special standard core.

Proof. It follows from Theorem 4.3.1, Theorem 4.3.2 and the various remarks in the previous sections of this chapter. □

4.4 Symmetric Markov processes

One of the most beautiful and important result in Dirichlet form theory is the link between stochastic processes and symmetric forms.

A Hunt process is a Markov process which possesses very useful properties such as the right continuity of sample paths, the quasi-left continuity and the strong Markov processes. One important feature of Hunt processes is that there is a one to one correspondence between regular Dirichlet forms and equivalent symmetric Hunt processes (for more on Hunt processes we refer to [FOT94, Appendix A.2]).

Theorem 4.4.1 (Theorem 7.2.2 in [FOT94]). *The following conditions are equivalent to each other for a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$:*

(i) *$\mathcal{E}$ possesses the local property;*

(ii) *there exists an m-symmetric diffusion process on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ whose Dirichlet form is the given one $\mathcal{E}$.*

We say that two m-symmetric Hunt processes are *equivalent* if they possess a common properly exceptional[1] set outside which their transition semigroups coincide.

Next theorem helps us to state a uniqueness result about Hunt processes M which share the same regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$.

Theorem 4.4.2 (Theorem 4.2.8 in [FOT94]). *Let M_1 and M_2 be two m-symmetric Hunt processes with a common regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$. Then M_1 and M_2 are equivalent.*

By the two theorems and under the assumptions of Proposition 4.3.4, it follows that there exists a θ-symmetric diffusion process associated to $(\mathcal{E}, \mathcal{F}^\theta)$, which is uniquely identified up to a properly exceptional set. In the next section we show that with further assumptions on the coefficients we can choose one particular version whose transition probability kernel is regular.

4.1 Example: minimal diffusion process. In the context of diffusions in random environment we would like to be able to fix a common starting position for almost all realizations of the environment, or alternatively to start the process from all possible positions $x \in \mathbb{R}^d$. This is not possible in general for the process associated to $(\mathcal{E}, \mathcal{F}^\theta)$ defined in Section 1.2 unless we assume stronger conditions on the local behavior of the coefficients λ and Λ. What we need is that the transition kernel $p_t^\theta(x, y)$ of the diffusion associated to $(\mathcal{E}, \mathcal{F}^\theta)$ is jointly continuous in x, y for all $t > 0$. Assumption $(b.3)$ below provides such regularity. We stress that this is not an optimal condition for local regularity and one may assume as well that λ and Λ are locally (but not globally) in some Muckenhaupt's class. More about a possible generalization to the case in which $(b.3)$ is not satisfied will be discussed in Chapter 7.

[1]A set $\mathcal{N} \subset \mathcal{X}$ is called properly exceptional for a process $(X, \mathbb{P}_x)$ if $\mathcal{N}$ is Borel, it has Lebesgue measure zero, and $\mathbb{P}_x(X_t \in \mathcal{N}$ or $X_{t-} \in \mathcal{N}$ for some $t \geq 0) = 0$ for all $x \in \mathcal{X} \setminus \mathcal{N}$.

Assumption b.3. We assume that $x \mapsto \lambda^{-1}(x)$, $x \mapsto \Lambda(x) \in L^\infty_{loc}(\mathbb{R}^d)$.

Recall that the resolvent $G^{\theta,B}_\alpha$ restricted to the ball $B \subset \mathbb{R}^d$ of a process $\mathbf{M}^\theta := (X^\theta_t, \mathbb{P}^\theta_x, \zeta^\theta)$ is defined by

$$G^{\theta,B}_\alpha u(x) := \mathbb{E}^\theta_x \left[\int_0^{\tau_B} e^{-\alpha t} u(X^\theta_t)\, dt \right], \quad u \geq 0$$

being $\tau_B = \inf\{t > 0 : X^\theta_t \notin B\}$ the exit time from B. When $\theta \equiv 1$ we will, as usual, drop it from the notation.

Theorem 4.4.3. *Assume* $(b.1)$*,* $(b.3)$*, and* $\theta, \theta^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$*. Denote* $C_\infty(B)$ *the set of continuous functions vanishing at the boundary. Then, there exists a unique standard diffusion process* $\mathbf{M}^\theta := (X^\theta_t, \mathbb{P}^\theta_x, \zeta^\theta)$*,* $x \in \mathbb{R}^d$ *whose resolvent* $G^{\theta,B}_\alpha$ *restricted to any open bounded set* B *satisfies*

$$G^{\theta,B}_\alpha u \in C_\infty(B), \quad u \in L^s(B,\theta), \quad \forall s > d$$

and $G^{\theta,B}_\alpha C_\infty(B)$ *is dense in* $C_\infty(B)$*.*

Proof. For a proof see for example [Ich78], [Kun70], [Tom82], it basically relies on the classical regularity results of Stampacchia for solutions to elliptic partial differential equations with uniformly elliptic and bounded coefficients. First one builds diffusion processes killed when exiting from bounded domains, then the processes so obtained are "glued" together to originate a diffusion process on the whole $\mathbb{R}^d$ which can start from every point. □

From now on we will consider only the process $\mathbf{M}^\theta$ constructed in Theorem 4.4.3. Fix a ball $B \subset \mathbb{R}^d$ and consider the semigroup associated to the process above killed when exiting from B, that is,

$$P^{\theta,B}_t f(x) := \mathbb{E}_x[f(X^\theta_t), t < \tau_B].$$

By Theorem 4.4.3 and Hille-Yoshida's Theorem, $P^{\theta,B}_t C_\infty(B) \subset C_\infty(B)$. Such a property turns out to be very handy to remove all the ambiguities about exceptional sets and to construct a transition kernel $p^{\theta,B}_t(x,y)$ for $P^{\theta,B}_t$ which is jointly continuous in x, y. This is a consequence of the next theorem whose proof is a minor variation of [BBCK09, Theorem 2.1].

Theorem 4.4.4. *Let* T_t *be the semigroup on* $L^2(\mathcal{X}, m)$ *such that* $T_t C_\infty(\mathcal{X}) \subset C_\infty(\mathcal{X})$*. Assume that*

$$\|T_t u\|_{\infty,m} \leq M(t)\|u\|_{1,m}, \tag{4.2}$$

for all $u \in L^1(\mathcal{X}, m)$*,* $t > 0$ *and some lower semicontinuous function* $M(t)$ *on* $(0,\infty)$*. Then there exists a positive symmetric kernel* $k_t(x,y)$ *defined on* $(0,\infty) \times \mathcal{X} \times \mathcal{X}$ *such that*

(i)

$$T_t u(x) = \int_{\mathcal{X}} u(y) k_t(x,y) m(dy),$$

for all $x \in \mathcal{X}$*,* $t > 0$*,* *(ii) for every $t, s > 0$ and $x, y \in \mathcal{X}$*

$$k_{t+s}(x,y) = \int_{\mathcal{X}} k_t(x,z) k_s(z,y) m(dz),$$

(iii) $k_t(x,y) \le M(t)$ for every $t > 0$ and $x, y \in \mathcal{X}$,

(iv) for every fixed $t > 0$, $k_t(x,y)$ is jointly continuous in $x, y \in B$.

Proof. The proof is analogous to the one presented in [BBCK09, Theorem 2.1]. One uses $T_t C_\infty(\mathcal{X}) \subset C_\infty(\mathcal{X})$ to remove all the ambiguities due to properly exceptional sets. Let $\{u_k : k \in \mathbb{N}\} \subset C_0(\mathcal{X})$ be dense both in $L^1(\mathcal{X}, m)$ and in $L^2(\mathcal{X}, m)$. The inequality (4.2) yields

$$\sup_{x \in \mathcal{X}} |T_t u_k - T_t u_h| \le M(t) \|u_k - u_h\|_{1,m},$$

since $C_0(\mathcal{X})$ is dense in $L^1(\mathcal{X}, m)$ it follows that $T_t u$ is continuous for all $u \in L^1(\mathcal{X}, m)$ and that

$$|T_t u(x)| \le M(t) \|u\|_{1,m}$$

for all $x \in \mathcal{X}$ and $t > 0$. Therefore, for all $t > 0$ and all $x \in \mathcal{X}$ there is an integrable kernel $y \to h_t(x,y)$ defined m-almost surely on $\mathcal{X}$ such that

$$T_t u(x) = \int_{\mathcal{X}} h_t(x,y) u(y)\, m(dy), \quad \forall u \in L^1(\mathcal{X}, m), \tag{4.3}$$

and $h_t(x,y) \le M(t)$, for m-almost all $y \in \mathcal{X}$.

From the semigroup property $T_t T_s = T_{t+s}$ it follows that

$$h_{t+s}(x,y) = \int_{\mathcal{X}} h_t(x,z) h_s(z,y)\, m(dy)$$

for all $x \in \mathcal{X}$, all $t, s > 0$ and for m-almost all $y \in \mathcal{X}$. By the symmetry of T_t we get that for all $t > 0$

$$h_t(x,y) = h_t(y,x),$$

for m-almost all $x, y \in \mathcal{X}$. Now we construct a kernel $k_t(x,y)$ which is defined for all $x, y \in \mathcal{X}$. For all $t > 0$ and $x, y \in \mathcal{X}$, we define for any $s < t/3$

$$k_t(x,y) := \int_{\mathcal{X}} h_s(x,z) \Big(\int_{\mathcal{X}} h_{t-2s}(z,w) h_s(y,w)\, m(dw) \Big)\, m(dz).$$

It is clear that the above definition is independent on the choice of s and that $k_t(x,y) = k_t(y,x)$ for all $x, y \in \mathcal{X}$. By the semigroup property and (4.3) we get for $\phi \ge 0$

$$\begin{aligned}
&T_t \phi(x) \\
&= \int_{\mathcal{X}} \left(\int_{\mathcal{X}} h_s(x,z) \Big(\int_{\mathcal{X}} h_{t-2s}(z,w) h_s(w,y)\, m(dw) \Big)\, m(dz) \right) \phi(y)\, m(dy) \\
&= \int_{\mathcal{X}} \left(\int_{\mathcal{X}} h_s(x,z) \Big(\int_{\mathcal{X}} h_{t-2s}(z,w) h_s(y,w)\, m(dw) \Big)\, m(dz) \right) \phi(y)\, m(dy) \\
&= \int_{\mathcal{X}} k_t(x,y) \phi(y)\, m(dy).
\end{aligned}$$

Thus $k_t(x,y)$ coincides with $h_t(x,y)$ m-almost surely on $\mathcal{X}\times\mathcal{X}$, in particular $k_t(x,y)\leq M(t)$ for every $t>0$ and $x,y\in\mathcal{X}$. One can show that (ii) and (iii) are satisfied proceeding exactly as in [BBCK09, Theorem 2.1]. □

We see that if we choose $m(dx)=\theta(x)dx$ and we assume $(b.1)$, $(b.3)$ we immediately get the existence of a transition kernel $p_t^{\theta,B}(x,y)$ for the semigroup $P_t^{\theta,B}$, jointly continuous in $x,y\in B$. Indeed assumption (4.2) is easily satisfied by $(b.3)$. In the next proposition we prove the existence of a transition kernel $p_t^{\theta}(x,y)$ for the semigroup P_t^{θ} of M^θ by a localization argument.

Proposition 4.4.5. *Assume* $(b.1)$*,* $(b.3)$ *and* $\theta,\theta^{-1}\in L^\infty_{loc}(\mathbb{R}^d)$*. Consider the semigroup* P_t^θ *associated to the minimal diffusion* M^θ*. Then, there exists a transition kernel* $p_t^\theta(x,y)$ *defined on* $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ *associated to* P_t^θ*,*

$$P_t^\theta f(x)=\int_{\mathbb{R}^d} f(y)p_t^\theta(x,y)\theta(y)\,dy,\quad \forall x\in\mathbb{R}^d,\ t>0.$$

Moreover, for all $t>0$ *and* $x,y\in\mathbb{R}^d$

$$p_t^{\theta,B_R}(x,y)\nearrow p_t^\theta(x,y)\quad R\to\infty$$

being the limit increasing in R*.*

Proof. The proof comes from the the fact that for all balls $B\subset\mathbb{R}^d$ the semigroup $\mathcal{P}_t^{B,\theta}$ satisfies (4.2), which means that P_t^θ is locally ultracontractive and from [GT12, Theorem 2.12]. □

As a further consequence of assumption $(b.3)$, more precisely from the fact that λ is locally bounded from below we can prove that M^θ is an irreducible process.

Proposition 4.4.6. *Assume* $(b.3)$ *and assume* $\theta^{-1},\theta\in L^\infty_{loc}(\mathbb{R}^d)$*. Then the process* M^θ *is irreducible.*

Proof. It follows immediately from [FOT94, Corollary 4.6.4] and the fact that the Brownian motion on $\mathbb{R}^d$ is a irreducible process. □

In the next theorem we clarify the relation between M and M^θ, namely we show that they are one the time change of the other.

Theorem 4.4.7 (Time change)**.** *Assume* $(b.3)$ *and assume* $\theta^{-1},\theta\in L^\infty_{loc}(\mathbb{R}^d)$*. Define* $\hat{\mathrm{M}}=(\hat{X}_t,\mathbb{P}_x)$ *by*

$$\hat{X}_t:=X_{\tau_t},\quad \tau_t=\inf\left\{s>0;\int_0^s\theta(X_u)\,du>t\right\}.$$

Let $f:\mathbb{R}^d\to\mathbb{R}$ *being any positive and measurable function. Then,* $\hat{P}_tf(x):=\mathbb{E}_x[f(X_{\tau_t})]=P_t^\theta f(x)$ *for almost all* $x\in\mathbb{R}^d$*,* $t>0$*.*

Proof. According to Theorem 6.2.1 of [FOT94], $\hat{P}_tf(x)=P_t^\theta f(x)$ coincide for almost all $x\in\mathbb{R}^d$ and $t>0$. □

There is a natural time change $\theta : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which makes the process $\mathbf{M}^\theta$ conservative. Namely we pick $\theta \equiv \Lambda$. The condition we give will be suitable in the setting of ergodic environment.

Proposition 4.4.8. *Assume that*

$$\limsup_{R\to\infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} \Lambda(x)\, dx < \infty.$$

Then the process $\mathbf{M}^\Lambda$ *is conservative.*

Proof. The proof is an application of Theorem 5.7.3 in [FOT94], with (in the notation of that theorem) $\rho(x) = |x|$ and $m(dx) = \Lambda(x)\, dx$. Observe that from [FOT94, Chapter 3]

$$\nu_{\langle\rho\rangle}{}^2 = 2 \sum_{i,j} \frac{x_i x_j}{|x|^2} a_{ij}(x) dx,$$

hence the density of $\nu_{\langle\rho\rangle}$ with respect to $m(dx) = \Lambda(x)dx$ can be bounded as follows

$$2 \sum_{i,j} \frac{x_i x_j}{|x|^2} \frac{a_{ij}(x)}{\Lambda(x)} \leq 2.$$

This implies in particular that $M^\rho(R+r) \leq 2$. According to [FOT94, Theorem 5.7.3] we have to prove that for all $T > 0$ and any $R > 0$

$$\liminf_{r\to\infty} \left(\int_{B(0,R+r)} \Lambda dx \right) \ell\left(\frac{r}{\sqrt{M^\rho(R+r)T}} \right) = 0, \tag{4.4}$$

where $\ell(t) := \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx$. Next observe that

$$\ell\left(\frac{r}{\sqrt{M^\rho(R+r)T}} \right) = \frac{1}{\sqrt{2\pi}} \int_{\frac{r}{\sqrt{M^\rho(R+r)T}}}^\infty e^{-x^2/2}\, dx \leq \frac{1}{\sqrt{2\pi}} \int_{\frac{r}{\sqrt{2T}}}^\infty e^{-x^2/2}\, dx \leq \frac{\sqrt{T}}{\sqrt{\pi} r} e^{-\frac{r^2}{4T}}.$$

Clearly, using the assumption, (4.4) is satisfied, and accordingly the proposition is true. $\square$

If the generator $L = \operatorname{div}(a(x)\nabla\cdot)$ were well defined and associated to a stochastic process X_t, then $u(X_t)$ would be a martingale for all functions u such that $Lu = 0$, as can be seen by a direct application of Itô's formula. Functions u such that $Lu = 0$ are called L-harmonic. In the case where not enough regularity on the coefficients is given, the weaker approach of Dirichlet forms provides an analogous characterization. A function $u \in \mathcal{D}_{loc}$ is said to be $\mathcal{E}$-harmonic on $\mathcal{X}$ if

$$\mathcal{E}(u,v) = 0, \quad \forall v \in \mathcal{C},$$

[2] $\nu_{\langle u\rangle}$ is the energy measure of u, defined to be the only positive radon measure such that

$$\int_{\mathbb{R}^d} v(x)\, d\nu_{\langle u\rangle}(dx) = 2\mathcal{E}(uv,v) - \mathcal{E}(u^2,v), \quad v \in C_0^\infty(\mathbb{R}^d).$$

being $\mathcal{C}$ a special standard core for $\mathcal{D}$. Roughly speaking, if a function u is $\mathcal{E}$-harmonic and X_t is the process associated to $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathcal{X}, m)$, then $u(X_t)$ is a martingale. For a more precise description of the link between $\mathcal{E}$-harmonic functions and martingales we refer to [FOT94, Chapter 5].

In the sequel, we will use the following theorem due to Fukushima [FNT87, Theorem 3.1].

Theorem 4.4.9. *Fix a point $x_0 \in \mathbb{R}^d$ and assume the following conditions for a conservative process $\mathbf{N} = (Z_t, \mathbb{P}_x)$ associated to $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ and for a function $u : \mathbb{R}^d \to \mathbb{R}$.*

(i) *The transition semigroup P_t of $\mathbf{N}$ satisfies $P_t 1_A(x_0) = 0$ for all $t > 0$ and $A \subset \mathbb{R}^d$ with $Cap(A)^3 = 0$.*

(ii) *$u \in \mathcal{F}_{loc}$, u is continuous and $\mathcal{E}$-harmonic.*

(iii) *Let $\nu_{\langle u \rangle}$ be the energy measure of u. We assume that $\nu_{\langle u \rangle}$ is absolutely continuous with respect to the Lebesgue measure $\nu_{\langle u \rangle} = f dx$ and that the density function f satisfies*

$$\mathbb{E}_{x_0}\left[\int_0^t f(Z_s)\, ds\right] < \infty, \quad t > 0.$$

Then $M_t = u(Z_t) - u(Z_0)$ is a $\mathbb{P}_{x_0}$-square integrable martingale with

$$\langle M \rangle_t = \int_0^t f(Z_s)\, ds, \quad t > 0, \quad \mathbb{P}_{x_0}\text{-a.s.}$$

Proof. For the proof see [FNT87, Theorem 3.1]. □

[3] [FOT94, Chapter 2]

5

Quenched Central Limit Theorem

5.1 The random environment

By a stationary and ergodic random environment, we mean a probability space $(\Omega, \mathcal{G}, \mu)$ on which is defined a group of transformations $\{\tau_x\}_{x \in \mathbb{R}^d}$ acting on Ω such that

(i) $\mu(\tau_x A) = \mu(A)$ for all $A \in \mathcal{G}$ and any $x \in \mathbb{R}^d$;

(ii) if $A \in \mathcal{G}$ and $\tau_x A = A$ for all $x \in \mathbb{R}^d$, then $\mu(A) \in \{0, 1\}$;

(iii) the function $(x, \omega) \to \tau_x \omega$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable.

The transformations $\tau_x : \Omega \to \Omega$ form a group in the sense that they satisfy $\tau_x \circ \tau_y = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$ and $\tau_0 = id_\Omega$. We will denote by $\mathbb{E}_\mu$ the expectation with respect to the probability measure μ.

Given a measurable function $\mathbf{u} : \Omega \to \mathbb{R}$, we call the stationary field associated to $\mathbf{u}$ the function $u : \mathbb{R}^d \times \Omega \to \mathbb{R}$ defined by $u(x; \omega) := \mathbf{u}(\tau_x \omega)$. As a general convention we denote by bold letters the random variables and by the corresponding normal letter the random field associated. Moreover, we will often write u^ω instead of $u(\cdot; \omega)$ to shorten the notation. The measurability in $x \in \mathbb{R}^d$ of $u(x; \omega)$, μ-almost surely, follows from (iii) above.

The following lemma, despite its simplicity, it is extremely important to relate moment conditions on the environment to integrability properties of the stationary field.

Lemma 5.1.1. *Fix $r > 0$. Let $\mathbf{u} \in L^r(\Omega, \mu)$, then the function $x \mapsto u^\omega(x) := \mathbf{u}(\tau_x \omega)$ belongs to $L^r_{loc}(\mathbb{R}^d)$, for μ-almost all $\omega \in \Omega$.*

Proof. Let $K \subset \mathbb{R}^d$ be any open bounded domain. Then, by stationarity and Fubini's theorem

$$\begin{aligned}\mathbb{E}_\mu[|\mathbf{u}|^r] &= \frac{1}{|K|} \int_K \mathbb{E}_\mu[|\mathbf{u}|^r] \, dx = \frac{1}{|K|} \int_K \mathbb{E}_\mu[|u(x; \omega)|^r] \, dx \\ &= \frac{1}{|K|} \mathbb{E}_\mu\left[\int_K |u(x; \omega)|^r \, dx\right] < \infty.\end{aligned}$$

It follows that $\int_K |u(x; \omega)|^r \, dx < \infty$ μ-almost surely. The μ-null set where this is true may depend on K, nevertheless, since we can cover $\mathbb{R}^d$ by a countable number of balls, it follows that $u^\omega \in L^r_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$. □

As a consequence of ergodicity (see (ii)) we have the following theorem, whose proof can be found in [KLO12, Theorem 11.18].

Theorem 5.1.2. *Let $\phi \in L^1(\mathbb{R}^d)$, and $\mathbf{u} \in L^1(\Omega, \mu)$, then*

$$\lim_{\epsilon \to 0} \epsilon^d \int_{\mathbb{R}^d} \phi(\epsilon x) \mathbf{u}(\tau_x \omega)\, dx = \mathbb{E}_\mu[\mathbf{u}] \int_{\mathbb{R}^d} \phi\, dx$$

in the $L^1(\Omega, \mu)$ sense. If additionally ϕ is compactly supported and bounded, then the convergence holds μ-almost surely as well.

Remark 5.1.3. *As an immediate consequence of Theorem 5.1.2 we have that for any $\mathbf{u} \in L^1(\Omega, \mu)$ and any ball $B_R \subset \mathbb{R}^d$ of radius R*

$$\lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} \mathbf{u}(\tau_x \omega)\, dx = \mathbb{E}_\mu[\mathbf{u}]$$

in $L^1(\Omega, \mu)$ and μ-almost surely. In words, Theorem 5.1.2 states that given a stationary and ergodic random field the expected value of the field in a point can be estimated by the spatial averages around that point.

The random environment $(\Omega, \mathcal{G}, \mu)$ comes naturally with a differential structure by exploiting the translations $\{\tau_x\}_{x \in \mathbb{R}^d}$. Next lemma goes in this direction.

Lemma 5.1.4. *The group $\{\tau_x\}_{x \in \mathbb{R}^d}$ on Ω defines a group of strongly continuous unitary operators $\{T_x\}_{x \in \mathbb{R}^d}$ on $L^r(\Omega, \mu)$ for all $r \in [1, \infty)$, by the position $T_x \mathbf{u} = \mathbf{u} \circ \tau_x$.*

Proof. The proof is similar in spirit to what we proved in the previous lemma. By stationarity $\mathbb{E}_\mu[|T_x \mathbf{u}|^r] = \mathbb{E}_\mu[|\mathbf{u}|^r]$ from which it follows that $\{T_x\}_{x \in \mathbb{R}^d}$ is an unitary group of operators on $L^r(\Omega, \mu)$ for all $r \geq 1$. Let us now prove the strong continuity. Fix any ball $B \subset \mathbb{R}^d$ and for $\mathbf{u} \in L^\infty(\Omega, \mu)$, by stationarity and Fubini's theorem

$$\begin{aligned}
\mathbb{E}_\mu[|T_x \mathbf{u} - \mathbf{u}|^r] &= \frac{1}{|B|} \int_B \mathbb{E}_\mu[|T_x \mathbf{u} - \mathbf{u}|^r]\, dz \\
&= \frac{1}{|B|} \int_B \mathbb{E}_\mu[|u(x+z, \omega) - u(z, \omega)|^r]\, dz \\
&= \mathbb{E}_\mu\left[\frac{1}{|B|} \int_B |u(x+z, \omega) - u(z, \omega)|^r\, dz\right].
\end{aligned}$$

By continuity of translations in $L^r(B, dx)$ for $r \in [1, \infty)$ [Fol99, Proposition 8.5], we have that

$$\int_B |u(x+z, \omega) - u(z, \omega)|^r\, dz \to 0, \quad \mu\text{-almost surely.}$$

We can now conclude using dominated convergence, and density of bounded functions in $L^r(\Omega, \mu)$. □

By Lemma 5.1.4, $\{T_x\}_{x \in \mathbb{R}^d}$ is a group of strongly continuous contraction operators, their generators correspond to differentiation in the canonical directions e_i, $i = 1, \ldots, d$. Define for $i = 1, \ldots, d$

$$D_i \mathbf{u} := \lim_{h \to 0} \frac{T_{he_i} \mathbf{u} - \mathbf{u}}{h},$$

whenever the limit exists in $L^2(\Omega, \mu)$. Denote by $\mathcal{D}(D_i)$ the domain of D_i, that is, the set of functions $\mathbf{u} \in L^2(\Omega, \mu)$ for which such limit exists.

Lemma 5.1.5. *Let $\mathbf{u}, \mathbf{v} \in \mathcal{D}(D_i)$, then $\mathbb{E}_\mu[D_i \mathbf{u}] = 0$ and $\mathbb{E}_\mu[\mathbf{v} D_i \mathbf{u}] = -\mathbb{E}_\mu[\mathbf{u} D_i \mathbf{v}]$.*

Proof. By stationarity, and the fact that convergence in $L^2(\Omega, \mu)$ implies convergence in mean,

$$\mathbb{E}_\mu[D_i \mathbf{u}] = \lim_{h \to 0} \mathbb{E}_\mu\left[\frac{T_{he_i}\mathbf{u} - \mathbf{u}}{h}\right] = \lim_{h \to 0} \frac{\mathbb{E}_\mu[T_{he_i}\mathbf{u}] - \mathbb{E}_\mu[\mathbf{u}]}{h} = 0.$$

Again, by stationarity

$$\mathbb{E}_\mu[\mathbf{v} D_i \mathbf{u}] = \lim_{h \to 0} \mathbb{E}_\mu\left[\mathbf{v}\frac{T_{he_i}\mathbf{u} - \mathbf{u}}{h}\right] = -\lim_{h \to 0} \mathbb{E}_\mu\left[\mathbf{u}\frac{T_{-he_i}\mathbf{v} - \mathbf{v}}{-h}\right] = -\mathbb{E}_\mu[\mathbf{u} D_i \mathbf{v}].$$

□

We introduce now a set of "smooth" bounded functions in $\bigcap_i \mathcal{D}(D_i)$, they have the property that the stationary random fields associated are smooth in the classical sense.

We define

$$C_b^\infty(\Omega) := \left\{ \int_{\mathbb{R}^d} \mathbf{u}(\tau_x \omega)\phi(x)\, dx \;:\; \mathbf{u} \in L^\infty(\Omega, \mu),\ \phi \in C_0^\infty(\mathbb{R}^d) \right\}. \tag{5.1}$$

Lemma 5.1.6. *$C_b^\infty(\Omega)$ is dense in $L^r(\Omega, \mu)$ for all $r \in [1, \infty)$ and $C_b^\infty(\Omega) \subset \bigcap \mathcal{D}(D_i)$. Moreover, if $\mathbf{v} \in C_b^\infty(\Omega)$, then $v^\omega(\cdot) := \mathbf{v}(\tau_\cdot \omega) \in C_b^\infty(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$ and the following implication holds*

$$\mathbf{v}(\omega) = \int_{\mathbb{R}^d} \mathbf{u}(\tau_x \omega)\phi(x) dx \Rightarrow D_i \mathbf{v}(\omega) = -\int_{\mathbb{R}^d} \mathbf{u}(\tau_x \omega)\partial_i \phi(x) dx. \tag{5.2}$$

In particular, $\partial_i v^\omega(\cdot) = D_i \mathbf{v}(\tau_\cdot \omega)$, μ-almost surely.

Proof. It is clear that $C_b^\infty(\Omega) \subset L^\infty(\Omega, \mu)$. We start proving the density. It suffices to show that any $\mathbf{v} \in L^\infty(\Omega, \mu)$ can be approximated by functions in $C_b^\infty(\Omega)$ in $L^r(\Omega, \mu)$ for $r \geq 1$. Take any positive mollifier $\phi \in C_0^\infty(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} \phi dx = 1$ and $\epsilon > 0$. Define

$$\mathbf{v}_\epsilon(\omega) := \epsilon^{-d} \int_{\mathbb{R}^d} \phi(x/\epsilon)\mathbf{v}(\tau_x \omega)\, dx.$$

By definition $\mathbf{v}_\epsilon \in C_b^\infty(\Omega)$, furthermore

$$\mathbb{E}_\mu[|\mathbf{v}_\epsilon - \mathbf{v}|^r] \leq \mathbb{E}_\mu\left[\int_{\mathbb{R}^d} \phi(x)|\mathbf{v}(\tau_{\epsilon x}\omega) - \mathbf{v}(\omega)|^r\, dx\right] = \int_{\operatorname{supp}\phi} \phi(x)\, \mathbb{E}_\mu[|T_{\epsilon x}\mathbf{v} - \mathbf{v}|^r]\, dx$$

which goes to zero as $\epsilon \to 0$ by the continuity of translations $T_{\epsilon x}$ and by the dominated convergence theorem.

The fact that $v^\omega(\cdot) := \mathbf{v}(\tau_\cdot \omega) \in C_b^\infty(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$ is clear since $v^\omega(x) = [\mathbf{u}(\tau_\cdot \omega) * \tilde{\phi}](x)$ is the convolution with the smooth function $\tilde{\phi}(x) = -\phi(-x)$, in particular $D^\alpha v^\omega = \mathbf{u}(\tau_\cdot \omega) * D^\alpha \tilde{\phi}$, where $D^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. We now prove (5.2), for $h \to 0$, we have to compute

$$\begin{aligned}
&\mathbb{E}_\mu \left| h^{-1} \int_{\mathbb{R}^d} [\mathbf{u}(\tau_{x+he_i}\omega) - \mathbf{u}(\tau_x \omega)]\phi(x) dx + \int_{\mathbb{R}^d} \mathbf{u}(\tau_x \omega)\partial_i \phi(x) dx \right|^2 \\
&\quad = \mathbb{E}_\mu \left| h^{-1} \int_{\mathbb{R}^d} \mathbf{u}(\tau_x \omega)[\phi(x - he_i) - \phi(x) + \partial_i \phi(x) h] dx \right|^2 \\
&\quad \leq |K_\phi|\, \mathbb{E}_\mu[|\mathbf{u}|^2] \int_{\mathbb{R}^d} \left| \frac{\phi(x - he_i) - \phi(x)}{h} + \partial_i \phi(x) \right|^2 dx \to 0,
\end{aligned}$$

since $\phi \in C_0^\infty(\mathbb{R}^d)$. Here K_ϕ is a compact set large enough to contain $\operatorname{supp}\phi(\cdot - he_i)$ for all small enough, say $h \leq 1$. $\square$

5.2 Diffusions in random environment

We want to construct diffusions associated to the formal generator

$$L^\omega u(x) = \operatorname{div}(a(x,\omega)\nabla u(x)) \tag{5.3}$$

for μ-almost all $\omega \in \Omega$. To be more precise we assume that $a(x,\omega) := a(\tau_x\omega)$ where $a : \Omega \to \mathbb{R}^{d\times d}$ is symmetric and such that there exist Λ and λ $\mathcal{G}$-measurable, positive and finite, satisfying the following assumptions.

Assumption a.1. For μ-almost all $\omega \in \Omega$ and $\xi \in \mathbb{R}^d$

$$\lambda(\omega)|\xi|^2 \leq \langle a(\omega)\xi, \xi\rangle \leq \Lambda(\omega)|\xi|^2$$

Assumption a.2. There exist $p, q \in [1,\infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\mathbb{E}_\mu[\lambda^{-q}] < \infty, \quad \mathbb{E}_\mu[\Lambda^p] < \infty,$$

Assumption a.3. As functions of x, $\lambda^{-1}(\tau_x\omega), \Lambda(\tau_x\omega) \in L^\infty_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$.

Assumption $(a.1)$ and $(a.2)$ imply $(b.1)$ and $(b.2)$ for μ-almost all $\omega \in \Omega$. Indeed by translation we can rewrite

$$\lambda(x,\omega)|\xi|^2 \leq \langle a^\omega(x,\omega)\xi, \xi\rangle \leq \Lambda^\omega(x,\omega)|\xi|^2,$$

for almost all $x \in \mathbb{R}^d$, all $\xi \in \mathbb{R}^d$, μ-almost surely. Moreover, by Lemma 5.1.1, $\mathbb{E}_\mu[\lambda^{-q}] < \infty$, $\mathbb{E}_\mu[\Lambda^p] < \infty$ give $\lambda^{-1}(\cdot,\omega) \in L^q_{loc}(\mathbb{R}^d)$ and $\Lambda(\cdot,\omega) \in L^p_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$.

More is known on the behavior of λ^{-q} and Λ^p. The ergodicity of the environment and Theorem 5.1.2 entail

$$\lim_{R\to\infty} \frac{1}{|B(x,R)|}\int_{B(x,R)} \lambda^{-q}(x,\omega)\,dx = \mathbb{E}_\mu[\lambda^{-q}] \tag{5.4}$$

and

$$\lim_{R\to\infty} \frac{1}{|B(x,R)|}\int_{B(x,R)} \Lambda^{p}(x,\omega)\,dx = \mathbb{E}_\mu[\Lambda^{p}] \tag{5.5}$$

μ-almost surely and in $L^1(\Omega,\mu)$, which will be crucial in controlling constants in the estimates derived in Chapter 2 and Chapter 3.

To associate a diffusion to (5.3) we shall exploit Dirichlet form theory. Let us consider the following bilinear form

$$\mathcal{E}^\omega(u,v) := \sum_{i,j}\int_{\mathbb{R}^d} a^\omega_{ij}(x)\partial_i u(x)\partial_j v(x)dx, \quad u,v \in C_0^\infty(\mathbb{R}^d),$$

Throughout this chapter we will look at two Dirichlet forms determined by $\mathcal{E}^\omega$ above. One is the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d)$ where $\mathcal{F}^\omega$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to $\mathcal{E}_1^\omega$. The second is the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^{\Lambda,\omega})$ on $L^2(\mathbb{R}^d, \Lambda^\omega)$ where $\mathcal{F}^{\Lambda,\omega}$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \Lambda^\omega)$ with respect to $\mathcal{E}_1^\omega$.

We have already observed that $(a.1)$, $(a.2)$ and $(a.3)$ imply $(b.1)$, $(b.2)$ and $(b.3)$, for μ-almost all $\omega \in \Omega$. Therefore, by Theorem 4.4.3, we have the existence, for μ-almost all $\omega \in \Omega$, of two minimal diffusion processes, $\mathbf{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega, \zeta^\omega)$ and $\mathbf{M}^{\Lambda,\omega} = (X_t^{\Lambda,\omega}, \mathbb{P}_x^{\Lambda,\omega})$, respectively associated to $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ and $(\mathcal{E}^\omega, \mathcal{F}^{\Lambda,\omega})$. Note that $\mathbf{M}^{\Lambda,\omega}$ is conservative μ-a.s. by Proposition 4.4.8 and (5.5), the fact that also $\mathbf{M}^\omega$ is conservative will be proved at the end of next section.

Denote by P_t^ω the transition semigroup associated to $\mathbf{M}^\omega$ and by $p_t^\omega(x,y)$ its transition kernel with respect to dx. Analogously, denote by $P_t^{\Lambda,\omega}$ the transition semigroup associated to $\mathbf{M}^{\Lambda,\omega}$ and by $p_t^{\Lambda,\omega}(x,y)$ its transition kernel with respect to $\Lambda^\omega(x)dx$.

Lemma 5.2.1 (Translation Property for killed process). *Fix a ball $B \subset \mathbb{R}^d$. Then for μ-almost all $\omega \in \Omega$*

$$
\begin{aligned}
p_t^{B-z,\tau_z\omega}(x-z, y-z) &= p_t^{B,\omega}(x,y), \\
p_t^{\Lambda,B-z,\tau_z\omega}(x-z, y-z) &= p_t^{\Lambda,B,\omega}(x,y),
\end{aligned}
\tag{5.6}
$$

for all $t \geq 0$, $x, y \in B$ and $z \in \mathbb{R}^d$.

Proof. We prove property (5.6) only for the transition kernel $p_t^{B,\omega}$, being the other totally equivalent. It is known in [FOT94, Chapter 1] that the resolvent $G_\alpha^{B,\omega}$ is uniquely determined by the following equation

$$
\mathcal{E}_\alpha^\omega(G_\alpha^{B,\omega} f, v) = \int_B f(x)v(x)\,dx,
$$

for all $f \in L^2(B)$, $v \in \mathcal{F}_B^\omega$. On the other hand

$$
\begin{aligned}
\mathcal{E}_\alpha^\omega(G_\alpha^{B,\omega} f, v) &= \int_{B-z} f(x+z)v(x+z)\,dx \\
&= \mathcal{E}_\alpha^{\tau_z\omega}\left(\left[G_\alpha^{B-z,\tau_z\omega} f(\cdot+z)\right], v(\cdot+z)\right) \\
&= \mathcal{E}_\alpha^{\omega}\left(\left[G_\alpha^{B-z,\tau_z\omega} f(\cdot+z)\right](\cdot - z), v\right),
\end{aligned}
$$

for all $f \in L^2(B)$, $v \in \mathcal{F}_B^\omega$. Hence, for μ-almost all $\omega \in \Omega$

$$
\left[G_\alpha^{B-z\tau_z\omega} f(\cdot+z)\right](x-z) = G_\alpha^{B,\omega} f(x), \quad \text{a.a } x \in B, \forall z \in \mathbb{R}^d.
$$

Moving from the resolvent to the semigroup we get the relation

$$
[P_t^{B-z,\tau_z\omega} f(\cdot+z)](x-z) = P_t^{B,\omega} f(x),
$$

for all $f \in C_\infty(B)$. The equality is true for all $x \in B$ and for all $z \in \mathbb{R}^d$ by the Feller property, μ-almost surely. Finally it is easy to derive the equality for the transition kernel and get

$$
p_t^{B-z,\tau_z\omega}(x-z, y-z) = p_t^{B,\omega}(x,y), \tag{5.7}
$$

for all $z \in \mathbb{R}^d$, and almost all $x, y \in B$, μ-almost surely. Using the joint continuity of $p_t^{B,\omega}(x,y)$ in x and y (cf. (iv) Theorem 4.4.4) we get (5.7) for all $z \in \mathbb{R}^d$, $x, y \in B$, μ-almost surely. $\square$

Lemma 5.2.2 (Translation Property). *For μ-almost all $\omega \in \Omega$*

$$\begin{aligned} p_t^{\tau_z\omega}(x-z, y-z) &= p_t^{\omega}(x,y), \\ p_t^{\Lambda,\tau_z\omega}(x-z, y-z) &= p_t^{\Lambda,\omega}(x,y), \end{aligned} \tag{5.8}$$

for all $t \geq 0$ and $x, y, z \in \mathbb{R}^d$

Proof. It follows from the previous lemma, passing to the limit. Namely, take an increasing sequence of balls $B_n \uparrow \mathbb{R}^d$, then we have

$$\begin{aligned} p_t^{\tau_z\omega}(x-z, y-z) &= \lim_{n\to\infty} p_t^{B_n - z, \tau_z\omega}(x-z, y-z) \\ &= \lim_{n\to\infty} p_t^{B_n,\omega}(x,y) = p_t^{\omega}(x,y). \end{aligned}$$

□

5.3 Environment process

We shall first construct the environment process for $\mathbf{M}^{\Lambda,\omega} = (X_t^{\Lambda,\omega}, \mathbb{P}_x^{\Lambda,\omega})$, $x \in \mathbb{R}^d$, since we know that $\mathbf{M}^{\Lambda,\omega}$ is conservative μ-almost surely by Proposition 4.4.8. From this construction and the ergodic theorem we will prove that also the process $\mathbf{M}^{\omega}$ is conservative μ-almost surely.

For a fixed $\omega \in \Omega$, we define a stochastic process on Ω by

$$\psi_t^{\Lambda,\omega}(\tilde{\omega}) := \tau_{X_t^{\Lambda,\omega}(\tilde{\omega})}\omega, \quad t \geq 0$$

where $\tilde{\omega}$ is a point of the sample space of the diffusion $\mathbf{M}^{\Lambda,\omega}$. The process $\psi_t^{\Lambda,\omega}$ under the measure $\mathbb{P}_x^{\Lambda,\omega}$ is Ω-valued and it is known as the *environment process*. First, we describe the semigroup associated to $\psi_t^{\Lambda,\omega}$ under $\mathbb{P}_0^{\Lambda,\omega}$. Take any positive and bounded $\mathcal{G}$-measurable function $f : \Omega \to \mathbb{R}$ and observe that

$$\mathbf{P}_t^{\Lambda} f(\omega) := \mathbb{E}_0^{\Lambda,\omega}[f(\tau_{X_t^{\Lambda,\omega}}\omega)] = P_t^{\Lambda,\omega} f(\tau_{\cdot}\omega)(0) = \int_{\mathbb{R}^d} f(\tau_y\omega) p_t^{\Lambda,\omega}(0,y)\Lambda(\tau_y\omega)\, dy$$

Proposition 5.3.1. *$\{\mathbf{P}_t^{\Lambda}\}_{t\geq 0}$ is a symmetric strongly continuous semigroup on $L^2(\Omega, \Lambda)$, the process $t \to \psi_t^{\Lambda,\omega}$ is ergodic with respect to $\Lambda d\mu$.*

Proof. The proof of the contractivity and the symmetry of $\{\mathbf{P}_t^{\Lambda}\}_{t\geq 0}$ on $L^2(\Omega, \Lambda)$ follows from the stationarity of the environment and by (5.8), it is standard and can be found in [Osa83], [ZKO94]. We want to prove that $\mathbf{P}_t^{\Lambda}$ is strongly continuous on $L^2(\Omega, \Lambda)$. We shall show that for all $\mathbf{f} \in C_b^{\infty}(\Omega)$

$$\lim_{t\to 0} \mathbb{E}_{\mu}[|\mathbf{P}_t^{\Lambda}\mathbf{f} - \mathbf{f}|^2 \Lambda] = 0.$$

Recall that $C_b^{\infty}(\Omega) \subset L^2(\Omega, \Lambda)$ densely and that $\mathbf{f} \in C_b^{\infty}(\Omega)$ is identified by $\mathbf{u} \in L^{\infty}(\Omega)$ and $\phi \in C_0^{\infty}(\mathbb{R}^d)$ as follows

$$\mathbf{f}(\omega) = \int_{\mathbb{R}^d} \mathbf{u}(\tau_x\omega)\phi(x)\, dx.$$

We first show that
$$\lim_{t\to 0}\mathbb{E}_\mu[|\mathbf{P}_t^\Lambda \mathbf{f}-\mathbf{f}|\Lambda]=0.$$
This follows from the calculation below,

$$\begin{aligned}
&\mathbb{E}_\mu[|\mathbf{P}_t\mathbf{f}-\mathbf{f}|\Lambda]\\
&\quad=\mathbb{E}_\mu\left[\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{u}(\tau_{x+y}\omega)\phi(x)p_t^{\Lambda,\omega}(0,y)\Lambda(\tau_y\omega)dxdy-\int_{\mathbb{R}^d}\mathbf{u}(\tau_z\omega)\phi(z)dz\Big|\Lambda(\omega)\right]\\
&\quad=\mathbb{E}_\mu\left[\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{u}(\tau_z\omega)\phi(x)p_t^{\Lambda,\omega}(0,z-x)\Lambda(\tau_{z-x}\omega)dxdz-\int_{\mathbb{R}^d}\mathbf{u}(\tau_z\omega)\phi(z)dz\Big|\Lambda(\omega)\right]\\
&\quad\le\int_{\mathbb{R}^d}\mathbb{E}_\mu\left[\Big|\int_{\mathbb{R}^d}\phi(x)p_t^{\Lambda,\omega}(0,z-x)\Lambda(\tau_{z-x}\omega)dx-\phi(z)\Big||\mathbf{u}(\tau_z\omega)|\Lambda(\omega)\right]dz\\
&\quad\le\|\mathbf{u}\|_\infty\,\mathbb{E}_\mu\left[\int_{\mathbb{R}^d}|P_t^{\Lambda,\omega}\hat{\phi}(z)-\hat{\phi}(z)|\Lambda(\tau_z\omega)dz\right]\to 0,\quad\text{as } t\to 0,
\end{aligned}$$

where $\hat{\phi}(x):=\phi(-x)$, since $P_t^{\Lambda,\omega}$ is a strongly continuous semigroup on $L^2(\mathbb{R}^d,\Lambda^\omega)$ and $\hat{\phi}\in L^2(\mathbb{R}^d,\Lambda^\omega)$. This implies, by Markov's inequality, that
$$\lim_{t\to 0}\mathbb{E}_\mu[1_{|\mathbf{P}_t^\Lambda\mathbf{f}-\mathbf{f}|>\delta}\Lambda]=0,\quad\forall\delta>0,\mathbf{f}\in C_b^\infty(\Omega)$$
In particular we have
$$\begin{aligned}
\mathbb{E}_\mu[|\mathbf{P}_t^\Lambda\mathbf{f}-\mathbf{f}|^2\Lambda]&\le\delta^2\,\mathbb{E}_\mu[\Lambda]+\mathbb{E}_\mu[|\mathbf{P}_t^\Lambda\mathbf{f}-\mathbf{f}|^2 1_{|\mathbf{P}^\Lambda\mathbf{f}-\mathbf{f}|>\delta}\Lambda]\\
&\le\delta^2\,\mathbb{E}_\mu[\Lambda]+2\|\mathbf{f}\|_\infty^2\,\mathbb{E}_\mu[1_{|\mathbf{P}_t^\Lambda\mathbf{f}-\mathbf{f}|>\delta}\Lambda]
\end{aligned}$$
from which the conclusion follows. Since $C_b^\infty(\Omega)$ is dense in $L^2(\Omega,\Lambda)$ and $\mathbf{P}^\Lambda$ is a bounded operator, the proof of strong continuity is over.

The proof of the ergodicity of the process $t\to\psi_t^{\Lambda,\omega}$ with respect to $\Lambda d\mu$ can also be found in [Osa83] and it is based on the irreducibility of the process $X_t^{\Lambda,\omega}$, which was proven in Proposition 4.4.6. □

Proposition 5.3.2 (Ergodic Theorem)**.** *For all functions $\mathbf{u}\in L^p(\Omega,\Lambda)$, $p\ge 1$, set $u(x,\omega)=\mathbf{u}(\tau_x\omega)$, then*
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(X_s^{\Lambda,\omega},\omega)\,ds=\mathbb{E}_\mu[\mathbf{u}\Lambda],\quad\mathbb{P}_x^{\Lambda,\omega}\text{-a.s, a.a. } x\in\mathbb{R}^d$$
for μ-almost all $\omega\in\Omega$.

Proof. In order to have the result stated, observe that the measure $Q_0^{\tau_x\omega}$ induced by $\mathbb{P}_0^{\Lambda,\tau_x\omega}$ through $\psi_t^{\Lambda,\tau_x\omega}$ on the space of Ω-valued trajectories coincide with the measure Q_x^ω induced by $\mathbb{P}_x^{\Lambda,\omega}$ through $\psi_t^{\Lambda,\omega}$. It is then easy to show that for any ball $B\subset\mathbb{R}^d$ the two measures
$$\int_\Omega Q_0^\omega(\cdot)d\mu=\frac{1}{|B|}\int_{B\times\Omega}Q_0^{\tau_x\omega}(\cdot)dxd\mu=\frac{1}{|B|}\int_{\Omega\times B}Q_x^\omega(\cdot)d\mu dx$$
coincide; in the first equality we used the stationarity of the environment. The fact that the limiting relation hold $\int Q_0^\omega(\cdot)d\mu$-almost surely follows immediately from Proposition 5.3.1, then the result follows. □

We use Proposition 5.3.2 to control the explosion time of $\mathbf{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega, \zeta^\omega)$ in terms of the time changed process $\mathbf{M}^{\Lambda,\omega}$.

Theorem 5.3.3. *Let $\mathbf{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega, \zeta^\omega)$, $x \in \mathbb{R}^d$, be the minimal diffusion constructed in section 3.1. Then such a diffusion is conservative for all starting points.*

Proof. We consider the time change

$$\tau_t := \inf\Big\{ s > 0 : \int_0^s \frac{1}{\Lambda(X_u^{\Lambda,\omega}, \omega)}\, du > t \Big\},$$

and define the process $Y_t^\omega = X_{\tau_t}^{\Lambda,\omega}$. We know from Theorem 4.4.7 that Y_t^ω is equivalent to X_t^ω, since they possess the same Dirichlet form. It is not difficult to see that the explosion time of Y_t^ω equals $\int_0^\infty \frac{1}{\Lambda(X_u^{\Lambda,\omega},\omega)}\, du$ [FOT94, see Chapter 6]. By Proposition 5.3.2,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{\Lambda(X_s^{\Lambda,\omega}, \omega)}\, ds = \mathbb{E}_\mu[\Lambda^{-1}\Lambda] = 1, \quad \mathbb{P}_x^{\Lambda,\omega}\text{-a.s, a.a. } x \in \mathbb{R}^d$$

for μ-almost all $\omega \in \Omega$. It follows that Y_t^ω is conservative for almost all starting points $x \in \mathbb{R}^d$, μ-almost surely. Denote by $P^{Y,\omega}$ the transition semigroup of the process Y_t^ω. By Theorem 4.4.7, $P_t^\omega 1(x) = P_t^{Y,\omega} 1(x) = 1$ for almost all $x \in \mathbb{R}^d$, and since $\mathbf{M}^\omega$ is our minimal diffusion, then $P_t^\omega 1(x) = 1$ for all $x \in \mathbb{R}^d$. We can pass from almost all to all $x \in \mathbb{R}^d$ since the minimal diffusion satisfies property (4.2.9) in [FOT94], namely $P_t^\omega(x, dy)$ is absolutely continuous with respect to the Lebesgue measure for each $t > 0$ and each $x \in \mathbb{R}^d$ (see Theorem 4.5.4 in [FOT94]). □

For the rest of the chapter we will completely forget about the time changed process. Following the construction in this section it is possible to obtain an environment process for the minimal diffusion $\mathbf{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega)$, namely the process $t \to \tau_{X_t^\omega}\omega =: \psi_t^\omega$, with semigroup $\mathbf{P}_t$ given by

$$\mathbf{P}_t f(\omega) := \int_{\mathbb{R}^d} f(\tau_y \omega) p_t^\omega(0, y)\, dy$$

Proposition 5.3.4. *$\{\mathbf{P}_t\}_{t\geq 0}$ defines a symmetric strongly continuous semigroup on $L^2(\Omega, \mu)$, and $t \to \psi_t^\omega$ is ergodic with respect to μ.*

Proof. Analogous to Proposition 5.3.1. □

5.4 Construction of the corrector

4.1 The space $L^2(a)$. Fix a stationary and ergodic random medium $(\Omega, \mathcal{G}, \mu, \{\tau_x\}_{x\in\mathbb{R}^d})$ as in Section 5.1. In the construction of the corrector we rely only on assumption $(a.1)$ and $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda]$ finite being. We recall that $(a.1)$ requires that there exist $\lambda, \Lambda : \Omega \to [0, \infty]$ such that for almost all $\xi \in \mathbb{R}^d$ and μ-almost surely

$$\lambda(\omega)|\xi|^2 \leq \langle a(\omega)\xi, \xi\rangle \leq \Lambda(\omega)|\xi|^2,$$

being a a symmetric d-dimensional matrix.

Remark 5.4.1. *From $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda]$ finite and $(a.1)$, it easily follows that $0 < \lambda \leq \Lambda < \infty$ μ-almost surely.*

In order to construct the corrector, or more precisely its gradient, we introduce the following space

$$L^2(a) := \{V : \Omega \to \mathbb{R}^d : \mathbb{E}_\mu[\langle aV, V\rangle] < \infty\}$$

Such a space is clearly a pre-Hilbert space with the scalar product

$$\Theta(U, V) := \mathbb{E}_\mu[\langle aU, V\rangle].$$

Since $L^2(a)$ is isometric to $L^2(\Omega, \mu)^d$ through the map $\Psi : L^2(\Omega, \mu)^d \to L^2(a)$ given by $\Psi(V) = a^{-1/2}V$, it is also complete and accordingly an Hilbert space. Given a real non-negative random variable θ on Ω we denote by $L^r(\Omega, \theta)$ the L^r-space with respect to the measure $\theta d\mu$.

Lemma 5.4.2. *Assume $(a.1)$ and $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda] < \infty$, then we have the following continuous embeddings*

$$L^\infty(\Omega, \mu)^d \subset L^2(\Omega, \Lambda)^d \subset L^2(a) \subset L^2(\Omega, \lambda)^d \subset L^1(\Omega, \mu)^d$$

Proof. Indeed, we have by $(a.1)$

$$\mathbb{E}_\mu[|V|^2\lambda] \leq \mathbb{E}_\mu[\langle aV, V\rangle] \leq \mathbb{E}_\mu[|V|^2\Lambda]$$

and by Hölder inequality

$$\mathbb{E}_\mu[|V|]^2 \leq \mathbb{E}_\mu[\lambda^{-1}]\,\mathbb{E}_\mu[|V|^2\lambda], \quad \mathbb{E}_\mu[|V|^2\Lambda] \leq \mathbb{E}_\mu[\Lambda]\|V\|_\infty^2.$$

Notice that in the two inequalities above we need $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda] < \infty$. These inequalities together give the continuous embeddings we were looking for. □

Remark 5.4.3. *It follows immediately by this lemma that constant functions belong to $L^2(a)$. In particular the functions $\pi^k : \Omega \to \mathbb{R}^d$, $\pi^k(\omega) \equiv e_k$, with $k = 1, \dots, d$ are in $L^2(a)$, here e_k are the canonical coordinate directions in $\mathbb{R}^d$.*

Remark 5.4.4. *Observe that it is not true in general that T_x maps $L^2(\Omega, \lambda)$ into itself.*

4.2 Weyl's decomposition. For $\mathbf{v} \in C_b^\infty(\Omega)$ we can define $\nabla\mathbf{v} = (D_1\mathbf{v}, \dots, D_d\mathbf{v})$ and it is clear that $\nabla\mathbf{v} \in L^2(a)$, being $\nabla\mathbf{v} \in L^\infty(\Omega, \mu)$. We define the space of potential L^2_{pot} to be the closure of $\{\nabla\mathbf{v} | \mathbf{v} \in C_b^\infty(\Omega)\}$ in $L^2(a)$. As an immediate consequence of this definition we have the following lemma.

Lemma 5.4.5. *Let $U \in L^2_{pot}$. Then U satisfies the following properties*

(i) $\mathbb{E}_\mu[U_i] = 0$ *for all* $i = 1, \dots, d$.

(ii) *for all* $\eta \in C_0^\infty(\mathbb{R}^d)$ *and* $i, j = 1, \dots, d$

$$\int_{\mathbb{R}^d} U_i(\tau_x\omega)\partial_j\eta(x)\,dx = \int_{\mathbb{R}^d} U_j(\tau_x\omega)\partial_i\eta(x)\,dx,$$

for μ-almost all $\omega \in \Omega$.

Proof. Let start with (i). For functions of the type $\nabla \mathbf{u}$ with $\mathbf{u} \in C_b^\infty(\Omega)$, this follows immediately by Lemma 5.1.6 and Lemma 5.1.5.

For a general $U \in L^2_{pot}$, we find a sequence of functions $\mathbf{u}_n \in C_b^\infty(\Omega)$ such that $\nabla \mathbf{u}_n \to U$ in $L^2(a)$, hence in $L^1(\Omega,\mu)^d$ by Lemma 5.4.2. It follows that

$$\mathbb{E}_\mu[U] = \lim_{n\to\infty} \mathbb{E}_\mu[\nabla \mathbf{u}_n] = 0.$$

We now prove (ii). Consider again $\mathbf{u} \in C_b^\infty(\Omega)$. Then $x \mapsto u^\omega(x)$ is infinitely many times differentiable, μ-almost surely by Lemma 5.1.6. Integrating by parts we get

$$\int_{\mathbb{R}^d} D_i u^\omega(x) \partial_j \eta(x)\, dx = -\int_{\mathbb{R}^d} u^\omega(x) \partial_i \partial_j \eta(x)\, dx,$$

finally switch the partial derivatives and conclude

$$\int_{\mathbb{R}^d} D_i u^\omega(x) \partial_j \eta(x)\, dx = \int_{\mathbb{R}^d} D_j u^\omega(x) \partial_i \eta(x)\, dx.$$

For a general $U \in L^2_{pot}$ take approximations and use the fact that $\nabla \mathbf{u}_n \to U$ in $L^2(a)$ implies that along a subsequence $D_i u_n(\cdot;\omega) \to U_i(\cdot;\omega)$ in $L^1_{loc}(\mathbb{R}^d)$ μ-almost surely. $\square$

Since $L^2(a)$ is an Hilbert space and L^2_{pot} is by construction a closed subspace, we can write immediately

$$L^2(a) = L^2_{pot} \oplus (L^2_{pot})^\perp.$$

We want to decompose the bounded functions $\{\pi^k\}_{k=1}^d$, where π^k is the unit vector in the kth-direction. Since $\pi_k \in L^2(a)$, for each $k = 1,\dots,d$, there exist functions $\Xi^k \in L^2_{pot}$ and $R^k \in (L^2_{pot})^\perp$ such that $\pi^k = \Xi^k + R^k$. By definition of orthogonal projection we have

$$\mathbb{E}_\mu[\langle a\,\Xi^k, V\rangle] = \mathbb{E}_\mu[\langle a\pi^k, V\rangle], \quad \forall V \in L^2_{pot}$$

Remark 5.4.6. *By the fact that $\{\nabla \mathbf{v} : \mathbf{v} \in C_b^\infty(\Omega)\}$ is dense in L^2_{pot} and by definition of orthogonal projection, it follows in particular that*

$$\mathbb{E}_\mu[\langle a(\Xi^k - \pi_k), \Xi^k - \pi_k\rangle] = \inf_{\mathbf{v}\in C_b^\infty(\Omega)} \mathbb{E}_\mu[\langle a(\nabla \mathbf{v} - \pi_k), \nabla \mathbf{v} - \pi_k\rangle].$$

Proposition 5.4.7. *Set $\mathbf{d}_{ij} := 2\,\mathbb{E}_\mu[\langle a(\Xi^i - \pi_i), \Xi^j - \pi_j\rangle]$. Then the matrix $\{\mathbf{d}_{ij}\}_{i,j}$ is positive definite.*

Proof. Take any $\xi \in \mathbb{R}^d$, then

$$\sum_{i,j} \mathbf{d}_{ij}\xi_i\xi_j = 2\,\mathbb{E}_\mu\Big[\langle a\Big(\sum_i \xi_i \Xi^i - \xi\Big), \sum_j \xi_j \Xi^j - \xi\rangle\Big].$$

Clearly $\sum_i \xi_i \Xi^i \in L^2_{pot}$. Moreover by linearity $\sum_i \xi_i \Xi^i$ is the orthogonal projection of the constant function $\pi_\xi : \omega \to \xi$, and $\pi_\xi \in L^2(a)$. By means of the characterization in Remark 5.4.6 we can write

$$\begin{aligned}\sum_{i,j} \mathbf{d}_{ij}\xi_i\xi_j &= \inf_{\mathbf{v}\in C_b^\infty(\Omega)} 2\,\mathbb{E}_\mu[\langle a(\nabla \mathbf{v} - \xi), \nabla \mathbf{v} - \xi\rangle] \geq \sum_{i=1}^d \inf_{\mathbf{v}\in C_b^\infty(\Omega)} 2\,\mathbb{E}_\mu[\lambda |D_i \mathbf{v} - \xi_i|^2] \\ &= \sum_{i=1}^d |\xi_i|^2 \inf_{\mathbf{v}\in C_b^\infty(\Omega)} 2\,\mathbb{E}_\mu[\lambda |D_i \mathbf{v} - 1|^2], \end{aligned} \tag{5.9}$$

so that we end up with a basic one dimensional problem. Observe that by Hölder's inequality we have

$$\mathbb{E}_\mu[\lambda|D_i\mathbf{v}-1|^2] \geq \mathbb{E}[\lambda^{-1}]^{-1}\,\mathbb{E}_\mu[(D_i\mathbf{v}-1)]^2 = \mathbb{E}[\lambda^{-1}]^{-1}$$

for all $\mathbf{v} \in C_b^\infty(\Omega)$ since by Lemma 5.1.5 we have that $\mathbb{E}_\mu[D_i\mathbf{v}] = 0$.

Therefore (5.9) is bounded from below by $2\sum_{i=1}^d |\xi_i|^2\,\mathbb{E}_\mu[\lambda^{-1}]^{-1} = 2|\xi|^2\,\mathbb{E}_\mu[\lambda^{-1}]^{-1}$ and we get the bound

$$\sum_{i,j}\mathbf{d}_{ij}\xi_i\xi_j \geq 2\,\mathbb{E}_\mu[\lambda^{-1}]^{-1}|\xi|^2$$

which is what we wanted to proof. □

By means of Weyl's decomposition we were able to project π_k onto the abstract space L^2_{pot} obtaining Ξ^k. For $k = 1, ..., d$ we define the *corrector* in direction k to be the function $\chi^k : \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by

$$\chi^k(x,\omega) := \sum_{j=1}^d \int_0^1 x_j \Xi_j^k(\tau_{tx}\omega)\,dt. \tag{5.10}$$

Proposition 5.4.8. *For $k = 1, ..., d$ the function $x \mapsto \chi^k(x,\omega)$ is well defined, and it belongs to $L^1_{loc}(\mathbb{R}^d)$, moreover $\chi^k(0,\omega) = 0$ μ-almost surely and $\mathbb{E}_\mu[\chi^k(x,\omega)] = 0$ for all $x \in \mathbb{R}^d$.*

Proof. First observe that the function $t \to \Xi^k(\tau_{tx}\omega)$ is measurable for μ-almost all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$. We want to check that if $\tilde{\Xi}^k$ is another version of Ξ^k, then $\Xi^k(\tau_{tx}\omega) = \tilde{\Xi}^k(\tau_{tx}\omega)$ for almost all $t \in [0,1]$, almost all $x \in \mathbb{R}^d$, μ-almost surely. This is easily achieved noticing that

$$\begin{aligned}\mathbb{E}_\mu\left[\int_B\int_{[0,1]}|\Xi^k(\tau_{tx}\omega)-\tilde{\Xi}^k(\tau_{tx}\omega)|\,dtdx\right] &= \int_B\int_{[0,1]}\mathbb{E}_\mu[|\Xi^k(\tau_{tx}\omega)-\tilde{\Xi}^k(\tau_{tx}\omega)|]\,dtdx\\ &= \int_B\int_{[0,1]}\mathbb{E}_\mu[|\Xi^k(\omega)-\tilde{\Xi}^k(\omega)|]\,dtdx = 0,\end{aligned}$$

being B any bounded domain in $\mathbb{R}^d$. We now check that $x \mapsto \chi^k(x,\omega)$ belongs to $L^1_{loc}(\mathbb{R}^d)$, μ-almost surely; take a ball $B \subset \mathbb{R}^d$ of radius R, then

$$\mathbb{E}_\mu\left[\int_B|\chi^k|\,dx\right] \leq \sum_{j=1}^d\int_0^1\int_B|x_j|\,\mathbb{E}_\mu[|\Xi_j^k(\tau_{tx}\omega)|]\,dt \lesssim |B|R\;\mathbb{E}_\mu[|\Xi^k|] < \infty$$

which gives the result. It is obvious by definition of χ^k, Fubini theorem and Lemma 5.4.5 that $\chi^k(0,\omega) = 0$ μ-almost surely and $\mathbb{E}_\mu[\chi^k(x,\omega)] = 0$ for all $x \in \mathbb{R}^d$. □

The key result about the corrector is listed here below, and states that χ^k is weakly differentiable in k and that the gradient is identified by Ξ^k.

Proposition 5.4.9. *For $k = 1, ..., d$ the function $x \mapsto \chi^k(x,\omega)$ is weakly differentiable μ-almost surely and $\partial_i\chi^k(x,\omega) = \Xi_i^k(\tau_x\omega)$.*

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be any test function and compute

$$\int_{\mathbb{R}^d} \chi^k(x,\omega)\partial_i\eta(x)\,dx = \int_{\mathbb{R}^d} \sum_{j=1}^d \int_0^1 x_j \Xi_j^k(\tau_{tx}\omega)\,dt\,\partial_i\eta(x)\,dx.$$

By switching the order of integration and applying the change of variables $y = tx$ we get

$$\int_0^1 \sum_{j=1}^d \int_{\mathbb{R}^d} \Xi_j^k(\tau_y\omega)\frac{y_j}{t^{d+1}}\partial_i\eta\Big(\frac{y}{t}\Big)\,dydt.$$

Next observe that for $j \neq i$,

$$\frac{y_j}{t^{d+1}}\partial_i\eta\Big(\frac{y}{t}\Big) = \partial_i\Big(\frac{y_j}{t^d}\eta\Big(\frac{y}{t}\Big)\Big),$$

which together with property (ii) of Lemma 5.4.5 gives

$$\int \chi^k(x,\omega)\partial_i\eta(x)\,dx = \int \Xi_i^k(\tau_y\omega)\int_0^1 \sum_{j\neq i}\partial_j\Big(\frac{y_j}{t^d}\eta\Big(\frac{y}{t}\Big)\Big) + \frac{y_i}{t^{d+1}}\partial_i\eta\Big(\frac{y}{t}\Big)dtdy.$$

Finally, observe that for $y \neq 0$ and exploiting that η has compact support we have

$$\int_0^1 \sum_{j\neq i}\partial_j\Big(\frac{y_j}{t^d}\eta\Big(\frac{y}{t}\Big)\Big) + \frac{y_i}{t^{d+1}}\partial_i\eta\Big(\frac{y}{t}\Big)dt = -\int_0^1 \frac{d}{dt}\Big(\eta\Big(\frac{y}{t}\Big)\frac{1}{t^{d-1}}\Big)\,dt = -\eta(y).$$

This ends the proof since it follows that

$$\int_{\mathbb{R}^d} \chi^k(x,\omega)\partial_i\eta(x)\,dx = -\int_{\mathbb{R}^d} \Xi_i^k(x;\omega)\eta(x)\,dx. \tag{5.11}$$

One may think that the set of ω for which (5.11) holds, depends on η. Since $C_0^\infty(\mathbb{R}^d)$ is separable we can remove such ambiguity considering a countable dense subset $\{\eta_n\}_{n\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^d)$. □

Remark 5.4.10. *The field $\chi^k(x,\omega)$ is not stationary as can be deduced by the fact that $\chi^k(0,\omega) = 0$, nonetheless its gradient is a stationary random field.*

4.3 Harmonic coordinates and Poisson equation. Now that we have the corrector we want to construct a weak solution to the Poisson equation $L^\omega u = 0$ for μ-almost all $\omega \in \Omega$. For $k = 1,...,d$, we define the *harmonic coordinates* to be the functions $y^k : \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by $y^k(x,\omega) := x_k - \chi^k(x,\omega)$.

Given the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d)$, we say that a function $u \in \mathcal{F}^\omega_{loc}$ is $\mathcal{E}^\omega$-harmonic if $\mathcal{E}^\omega(u,\phi) = 0$, $\forall\phi \in C_0^\infty(\mathbb{R}^d)$. To show that $y^k(x,\omega)$ are $\mathcal{E}^\omega$-harmonic we first have to show that $y^k(\cdot,\omega) \in \mathcal{F}^\omega_{loc}$, μ-almost surely. Our proof of this fact requires Sobolev's embedding theorems and $(a.2)$, namely that for $1/p + 1/q < 2/d$ $\mathbb{E}_\mu[\lambda^{-q}]$, $\mathbb{E}_\mu[\Lambda^p] < \infty$.

Let us start with a simple consequence of $(a.2)$.

Lemma 5.4.11. *Assume* $(a.1)$ *and* $(a.2)$*, then we have the following continuous embeddings*

$$L^{2p/(p-1)}(\Omega,\mu)^d \subset L^2(\Omega,\Lambda)^d \subset L^2(a) \subset L^2(\Omega,\lambda)^d \subset L^{2q/(q+1)}(\Omega,\mu)^d.$$

Proof. Indeed, we have by $(a.1)$

$$\mathbb{E}_\mu[|V|^2\lambda] \leq \mathbb{E}_\mu[\langle aV, V\rangle] \leq \mathbb{E}_\mu[|V|^2\Lambda]$$

and by Hölder inequality

$$\mathbb{E}_\mu[|V|^2\Lambda] \leq \mathbb{E}_\mu[\Lambda^p]^{1/p}\,\mathbb{E}[|V|^{2p^*}]^{1/p^*}, \quad \mathbb{E}[|V|^{2q/(q+1)}]^{(q+1)/q} \leq \mathbb{E}_\mu[\lambda^{-q}]^{1/q}\,\mathbb{E}_\mu[|V|^2\lambda]\,.$$

Notice that in the two inequalities above we exploited $\mathbb{E}_\mu[\lambda^{-q}]$, $\mathbb{E}_\mu[\Lambda^p] < \infty$. Clearly, these inequalities give the continuous embeddings we were looking for. $\square$

Proposition 5.4.12. *Assume* $(a.1)$ *and* $(a.2)$*, then the corrector* $\chi^k(\cdot,\omega) \in \mathcal{F}^\omega_{loc}$ *for* μ*-almost all* $\omega \in \Omega$*. In particular,* $y^k(\cdot,\omega) \in \mathcal{F}^\omega_{loc}$ *for* μ*-almost all* $\omega \in \Omega$*.*

Proof. By construction, there exists a sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset C_b^\infty(\Omega)$ such that $\nabla\mathbf{u}_n \to \Xi^k$ in $L^2(a)$. This implies that for any ball $B \subset \mathbb{R}^d$ along a subsequence if necessary

$$\int_B \langle a^\omega(x)(\nabla\mathbf{u}_n(\tau_x\omega) - \nabla\chi^k(x,\omega)), \nabla\mathbf{u}_n(\tau_x\omega) - \nabla\chi^k(x,\omega)\rangle\, dx \to 0, \tag{5.12}$$

μ-almost surely. Observe that $g_n(x,\omega) = \mathbf{u}_n(\tau_x\omega) - \mathbf{u}_n(\omega)$ belongs to $C^\infty(\mathbb{R}^d)$ and satisfies

$$g_n(x,\omega) = \sum_{i=1}^d \int_0^1 x_j\partial_j\mathbf{u}_n(\tau_{tx}\omega)\, dt.$$

We first prove that that $g_n \to \chi^k$ in $W^{1,2q/(q+1)}(B)$ for any ball $B \subset \mathbb{R}^d$. The convergence of $\nabla g_n \to \nabla\chi^k$ in $L^{2q/(q+1)}(B)$ follows directly by (5.12) above and assumption $(a.2)$. By Lemma 5.4.11 one sees easily that $g_n \to \chi^k$ in $L^{2q/(q+1)}(B)$. We claim that $\eta g_n \to \eta\chi^k$ in $L^2(\mathbb{R}^d)$ with respect to $\mathcal{E}^\omega_1$, for any function $\eta \in C_0^\infty(\mathbb{R}^d)$ and μ-almost surely, which by definition proves $\chi^k(\cdot,\omega) \in \mathcal{F}^\omega_{loc}$. Indeed

$$\int_{\mathbb{R}^d} \langle a\nabla(\eta g_n) - \nabla(\eta\chi^k), \nabla(\eta g_n) - \nabla(\eta\chi^k)\rangle\, dx \leq$$

$$2\int_B \langle a\nabla g_n - \nabla\chi^k, \nabla g_n - \nabla\chi^k\rangle\, dx + 2\|\nabla\eta\|_\infty^2 \int_B \Lambda|g_n - \chi^k|^2\, dx \to 0$$

where the last integral goes to zero by $g_n \to \chi^k$ in $W^{1,2q/(q+1)}(B)$, and by means of the Sobolev's embedding theorem $W^{1,2q/(q+1)}(B) \hookrightarrow L^{2p^*}(B)$. $\square$

Theorem 5.4.13. *For* $k = 1, .., d$*, the harmonic coordinates* $x \mapsto y^k(x,\omega)$ *are* $\mathcal{E}^\omega$*-harmonic* μ*-almost surely with respect to the Dirichlet form* $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ *on* $L^2(\mathbb{R}^d)$*.*

Proof. We have to prove that μ-almost surely, for all $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\mathcal{E}^\omega(y^k,\phi) = \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x,\omega)\partial_i y^k(x,\omega)\partial_j\phi(x)\, dx = 0.$$

By construction of the corrector, the stationarity of the environment and the fact that $T_x C_b^\infty(\Omega) = C_b^\infty(\Omega)$, we have that

$$\sum_{i,j} \mathbb{E}_\mu[a_{ij}(x,\omega)\partial_i y^k(x,\omega) D_j \mathbf{u}(\omega)] = 0, \quad \forall x \in \mathbb{R}^d, \forall \mathbf{u} \in C_b^\infty(\Omega).$$

Now fix $\phi \in C_0^\infty(\mathbb{R}^d)$ and integrate against the equality above. We deduce that for all $\mathbf{u} \in C_b^\infty(\Omega)$

$$\begin{aligned} 0 &= \sum_{i,j} \int_{\mathbb{R}^d} \phi(x)\, \mathbb{E}_\mu[a_{ij}(x,\omega)\partial_i y^k(x,\omega) D_j \mathbf{u}(\omega)]\, dx \\ &= \sum_{i,j} \mathbb{E}_\mu\Big[a_{ij}(0,\omega)\partial_i y^k(0,\omega) \int_{\mathbb{R}^d} D_j \mathbf{u}(\tau_{-x}\omega)\phi(x)\, dx\Big] \\ &= \mathbb{E}_\mu\Big[\mathbf{u}(\omega) \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x,\omega)\partial_i y^k(x,\omega)\partial_j \phi(x)\, dx\Big]. \end{aligned}$$

We know from Lemma 5.1.6 that $C_b^\infty(\Omega) \subset L^r(\Omega,\mu)$ for all $r \geq 1$ densely. Using the same strategy of that proof, we denote by ψ a standard compactly supported mollifier and for $A \in \mathcal{G}$ we note

$$\mathbf{u}_n(\omega) := n^d \int_{\mathbb{R}^d} 1_A(\tau_x\omega)\psi(nx)\, dx$$

then $\mathbf{u}_n \to 1_A$ in $L^1(\Omega,\mu)$ as $n \to \infty$, in particular, along a subsequence if necessary, $\mathbf{u}_n \to 1_A$ for μ almost all $\omega \in \Omega$ and by construction $|\mathbf{u}_n| \leq 1$. Furthermore,

$$\sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x,\omega)\partial_i y^k(x,\omega)\partial_j \phi(x)\, dx$$

is in $L^1(\Omega,\mu)$, being $\nabla y^k(x,\omega) \in L^2_{pot}$, therefore, by the dominated convergence theorem we can prove that for all $A \in \mathcal{G}$

$$\mathbb{E}_\mu\left[1_A(\omega) \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x,\omega)\partial_i y^k(x,\omega)\partial_j \phi(x)\, dx\right] = 0.$$

We can conclude that

$$\sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x,\omega)\partial_i y^k(x,\omega)\partial_j \phi(x)\, dx = 0, \quad \mu\text{-a.s.} \tag{5.13}$$

which ends the proof. To be precise, one should observe that $C_0^\infty(\mathbb{R}^d)$ is separable, which ensures that (5.13) is satisfied for all $\phi \in C_0^\infty(\mathbb{R}^d)$, μ-almost surely. □

Remark 5.4.14. *Observe that we didn't use the neither $(a.2)$ nor $(a.3)$ in the construction of the gradients of the correctors and of the harmonic coordinates, nor in showing that the Poisson equation is satisfied. The only important assumption was that $\mathbb{E}[\lambda^{-1}]$ and $\mathbb{E}[\Lambda]$ are finite. However we needed $(a.2)$ to grant that the harmonic coordinates belong to $\mathcal{F}^\omega_{loc}$. We claim that assumption $(a.2)$ can be dropped if we look at a time changed process, namely if we ask that the harmonic coordinates are $\mathcal{E}^\omega$-harmonic with respect to $(\mathcal{E}^\omega, \mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d, \theta^\omega)$ for an appropriate choice of θ^ω.*

So far, in this section, we didn't exploit the fact that the random environment $(\Omega, \mathcal{G}, \mu)$ is ergodic. Such a property can be used to obtain an upper bound on integrals of the harmonic coordinates in the diffusive limit, as we show in the remark below.

Remark 5.4.15. *Define $y_\epsilon^k(x,\omega) := \epsilon y^k(x/\epsilon,\omega)$ and let $B_R \subset \mathbb{R}^d$ be any ball of radius $R > 0$, then an application of the ergodic theorem (Theorem 5.1.2) yields*

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{B_R} \langle a(x/\epsilon,\omega)\nabla y_\epsilon^k(x,\omega), \nabla y_\epsilon^k(x,\omega)\rangle \, dx \\ = \lim_{\epsilon\to 0} \epsilon^{-d} \int_{B_{R/\epsilon}} \langle a(x,\omega)\nabla y^k(x,\omega), \nabla y^k(x,\omega)\rangle \, dx \\ = |B_R| \, \mathbb{E}_\mu[\langle a(\pi_k - \Xi^k), \pi_k - \Xi^k\rangle] < \infty, \end{aligned} \tag{5.14}$$

which in view of $(a.2)$ implies that

$$\limsup_{\epsilon\to 0} \|\nabla y_\epsilon^k\|^2_{2q/(q+1),B_R} \le \mathbb{E}_\mu[\langle a(\pi_k - \Xi^k), \pi_k - \Xi^k\rangle] < \infty, \quad \mu\text{-a.s.}$$

and by Sobolev embedding theorem (Theorem 1.1.9) implies that

$$\limsup_{\epsilon\to 0} \|y_\epsilon^k\|_{\rho,B_R} < \infty, \quad \mu\text{-a.s.}, \tag{5.15}$$

being $\rho = 2qd/(q(d-2)+d)$ the Sobolev conjugate of $2q/(q+1)$.

5.5 Harmonic coordinates and martingales

If $L^\omega u = \mathrm{div}(a^\omega \nabla u)$ were well defined and associated to X_t^ω, then $L^\omega y(x,\omega) = 0$ would imply that $y(X_t^\omega,\omega)$ is a martingale by Itô's formula. In our case we lack the regularity to use the theory coming from stochastic differential equations and we must rely on Dirichlet forms technique. We have proved in Theorem 5.4.13 that $y^k(x,\omega)$ is $\mathcal{E}^\omega$-harmonic, which in a weaker sense, is analogous to say that y^k is L^ω-harmonic.

We want to apply Theorem 4.4.9 to the function

$$u(x,\omega) = \sum_k \lambda_k y^k(x,\omega),$$

being $\mathcal{E}^\omega$-harmonic for μ-almost all $\omega \in \Omega$, and to the minimal diffusion process $\mathrm{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$. We fix the starting point to be $x_0 = 0$. Some attention is required to check that every assumption of Theorem 4.4.9 is satisfied for μ-almost all $\omega \in \Omega$.

By construction, since $\mathrm{M}^\omega = (X_t^\omega, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$ is the minimal diffusion for almost all $\omega \in \Omega$, it follows that $P_t 1_A(0) = \int_A p_t^\omega(0,y)\,dy = 0$ whenever the Lebesgue measure of A is zero. Therefore since every set with zero capacity has Lebesgue measure zero [FOT94, See page 68] assumption (i) holds.

Assumption (ii) is satisfied μ-almost surely in view of Proposition 5.4.12, Theorem 5.4.13 and $(a.3)$, which ensures the continuity of $x \mapsto y^k(x,\omega)$ for μ-almost all $\omega \in \Omega$

by classical results in elliptic partial differential equations with locally uniformly elliptic and bounded coefficients [GT01].

Let us prove (iii). According to [FOT94, Theorem 3.2.2] and using the fact that y^k are weakly differentiable, the density $\rho_u(x,\omega)$ of $\nu_{\langle u\rangle}$ with respect to the Lebesgue measure is given by

$$\begin{aligned}\rho_u(x,\omega) &= 2\sum_{i,j}\partial_i u(x,\omega)\partial_j u(x,\omega)\, a_{ij}(x,\omega)\\ &= 2\sum_{k,h}\lambda_k\lambda_h\Big(\sum_{i,j}\partial_i y^k(x,\omega)\partial_j y^h(x,\omega)\, a_{ij}(x,\omega)\Big),\end{aligned}$$

which we can rewrite as $\rho_u(x,\omega) = 2\langle q(x,\omega)\lambda,\lambda\rangle$, with

$$q^{hk}(\omega) := \sum_{i,j}\partial_i y^k(0,\omega)\partial_j y^h(0,\omega)\, a_{ij}(\omega) = \sum_{i,j}(\Xi_i^k(\omega)-\delta_{ik})(\Xi_j^h(\omega)-\delta_{jh})\, a_{ij}(\omega).$$

Next, using the fact that μ is the invariant measure for the environment process, we compute

$$\int_\Omega \mathbb{E}_0^\omega\Big[\int_0^t \rho_u(X_s^\omega,\omega)\,ds\Big]\,d\mu = 2\int_\Omega \mathbb{E}_0^\omega\Big[\int_0^t \langle q(\psi_s^\omega)\lambda,\lambda\rangle\,ds\Big]\,d\mu = 2t\int_\Omega\langle q(\omega)\lambda,\lambda\rangle\,d\mu$$

which is finite by construction, since $\Xi^k\in L^2(a)$ for all $k=1,...,d$. In particular (iii) is satisfied. It follows the following theorem:

Theorem 5.5.1. *Assume* $(a.1)$*,* $(a.2)$ *and* $(a.3)$*. Then for μ-almost all $\omega\in\Omega$, the process $t\to y(X_t^\omega,\omega)$ is a $\mathbb{P}_0^\omega$-square integrable martingale with quadratic covariation given by*

$$\langle y^k(X_t^\omega,\omega), y^h(X_t^\omega,\omega)\rangle_t = 2\int_0^t\sum_{i,j}a_{ij}(X_s^\omega,\omega)(\partial_i\chi^k(X_s^\omega,\omega)-\delta_{ik})(\partial_j\chi^h(X_s^\omega,\omega)-\delta_{jh})\,ds$$

Proof. Above. □

5.6 Proof of the quenched invariance principle

In Section 5.4 we constructed the functions $\chi, y:\mathbb{R}^d\times\Omega\to\mathbb{R}^d$ in a way that we can decompose the process X^ω as

$$X_t^\omega = y(X_t^\omega,\omega)+\chi(X_t^\omega,\omega),$$

in particular, we proved in Theorem 5.5.1 that $y(X_t^\omega,\omega)$ is a $\mathbb{P}_0^\omega$-square integrable martingale martingale. In order to get a quenched invariance principle for the process $\epsilon X^\omega_{t/\epsilon^2}$ we will need to prove that $\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)$ is converging to zero in law and that the quadratic variation of the martingale $\epsilon y(X^\omega_{t/\epsilon^2},\omega)$ is converging to a constant μ-almost surely as $\epsilon\to 0$.

As first result on the decay of the correctors we have the following lemma.

Lemma 5.6.1. *For all $R > 0$ and for μ-almost all $\omega \in \Omega$*

$$\lim_{\epsilon\to 0} \|\epsilon y^k(x/\epsilon,\omega) - x_k\|_{2p^*,B_R} = \lim_{\epsilon\to 0} \|\epsilon\chi^k(x/\epsilon;\omega)\|_{2p^*,B_R} = 0.$$

Proof. Set for convenience of notation

$$y_\epsilon(x,\omega) := \epsilon y(x/\epsilon,\omega), \quad \chi_\epsilon(x,\omega) := \epsilon\chi(x/\epsilon,\omega).$$

To prove the lemma it is enough to show that for any $\eta \in C_0^\infty(B_R)$ we have

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} y_\epsilon^k(x,\omega)\eta(x)\,dx = \int_{\mathbb{R}^d} x_k\eta(x)\,dx.$$

Indeed the above property implies the weak convergence $y_\epsilon^k \rightharpoonup x_k$ in $L^2(B_R)$. This gives the strong convergence in $L^{2p^*}(B_R)$, because $W^{1,2q/(q+1)}(B_R)$ is compactly embedded in $L^{2p^*}(B_R)$ (Theorem 1.1.9 and $2p^* < \rho$) and the family $\{y_\epsilon\}_{\epsilon>0}$ is bounded in $W^{1,2q/(q+1)}(B_R)$ by (5.14).

Since $\partial_j y^k(x;\omega) = \delta_{jk} - \Xi_j^k(\tau_x\omega)$,and $\mathbb{E}_\mu[\Xi_j^k] = 0$, the ergodic theorem implies that for each $\delta > 0$ arbitrary, μ-almost surely, there exists $\epsilon(\omega) > 0$ such that for all $\epsilon, s > 0$ with $s > \epsilon/\epsilon(\omega)$

$$\left|\sum_j \int_{B_R} \partial_j y_\epsilon^k(sx,\omega)x_j\eta(x)\,dx - \int_{\mathbb{R}^d} x_k\eta(x)\,dx\right| \le \delta. \tag{5.16}$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^d} y_\epsilon^k(x,\omega)\eta(x)\,dx &= \sum_j \int_{B_R}\int_0^1 \partial_j y_\epsilon^k(tx,\omega)x_j\eta(x)\,dt\,dx \\ &= \sum_j \int_0^1\int_{B_R} \partial_j y_\epsilon^k(tx,\omega)x_j\eta(x)\,dx\,dt. \end{aligned}\tag{5.17}$$

We split the right hand side of (5.17) in the sum

$$\sum_j \int_0^{\epsilon/\epsilon(\omega)}\int_{B_R} \partial_j y_\epsilon^k(tx,\omega)x_j\eta(x)\,dx\,dt + \sum_j \int_{\epsilon/\epsilon(\omega)}^1\int_{B_R} \partial_j y_\epsilon^k(tx,\omega)x_j\eta(x)\,dx\,dt,$$

now we estimate each of the two terms. We can rewrite the second term as

$$(1-\epsilon/\epsilon(\omega))\int_{B_R} x_j\eta(x)\,dx + \int_{\epsilon/\epsilon(\omega)}^1 r_{\epsilon/t}\,dt,$$

where the second integral is bounded by δ, in view of (5.16). For what concerns the first part, we can easily compute

$$\sum_j \int_0^{\epsilon/\epsilon(\omega)}\int_{B_R} \partial_j y_\epsilon^k(tx)x_j\eta(x)\,dx = \epsilon/\epsilon(\omega)\int_{B_R} \epsilon(\omega)y^k(x/\epsilon(\omega))\eta(x)\,dx.$$

Hence the first part is bounded by $c\cdot(\epsilon/\epsilon(\omega))$ for some constant $c > 0$. Finally, this yields

$$\limsup_{\epsilon\to 0}\left|\int_{\mathbb{R}^d} y_\epsilon^k(x,\omega)\eta(x)\,dx - \int_{\mathbb{R}^d} x_k\eta(x)\,dx\right| \le \delta$$

with δ arbitrarily chosen. □

Proposition 5.6.2 (Sublinearity of the correctors.). *For all $R > 0$,*

$$\lim_{\epsilon\to 0}\sup_{|x|\le R}\epsilon|\chi(x/\epsilon,\omega)| = 0, \quad \mu\text{-a.s.} \tag{5.18}$$

Proof. Observe that $\chi^k_\epsilon(x,\omega) := \epsilon\chi^k(x/\epsilon,\omega)$ is a solution on $B = B(0,R)$ for all $\epsilon > 0$ and $R > 0$ of

$$\sum_{i,j}\int_B a^\omega_{ij}(x/\epsilon)\partial_i\chi^k_\epsilon(x,\omega)\partial_j\phi(x)\,dx = \sum_{i,j}\int_B a^\omega_{ij}(x/\epsilon)\partial_i f_k(x)\partial_j\phi(x)\,dx,$$

where $f_k(x) = x_k$ and $\phi \in C^\infty_0(B)$. Clearly $|\nabla f_k(x)| \le 1$ for all $x \in \mathbb{R}^d$ and uniformly in $\epsilon > 0$. By Lemma 5.6.1, we get that

$$\lim_{\epsilon\to 0}\|\chi^k_\epsilon(x;\omega)\|_{2p^*,B_R} = 0.$$

Therefore, we can obtain 5.18 applying (2.9) with $\alpha = 2p^*$,

$$\|\chi^k_\epsilon\|_{\infty,B(R)} \le C_2\Big[1 \vee C_E^{B(2R/\epsilon)}\Big]^{\kappa'}\|\chi^k_\epsilon\|^{\gamma'}_{2p^*,B(2R)} \vee \|\chi^k_\epsilon\|_{2p^*,B(2R)}$$

which goes to zero as $\epsilon \to 0$ by Lemma 5.6.1. Notice that we can bound

$$C_E^{B(2R/\epsilon)} = \|\lambda^{-1}\|_{q,B(2R/\epsilon)}\|\Lambda\|_{p,B(2R/\epsilon),p},$$

by a constant, by means of $(a.2)$ and the ergodic theorem. □

We can now turn to the proof of Theorem I, namely of the quenched invariance principle for the diffusions $\epsilon X^\omega_{t/\epsilon^2}$.

Proof Theorem I. With the help of Proposition 5.6.2 the proof of this theorem is very similar to [FK97, Theorem 1], the main difference being the quadratic variation of the martingale part.

We only sketch (i) being totally analogous of [FK97, Theorem 1, part (i)]. We prove that there exist deterministic constants $\mathbf{d}_{hk}$ such that for μ-almost all $\omega \in \Omega$

$$\lim_{t\to\infty}\frac{\mathbb{E}^\omega_0[X^\omega_h(t)X^\omega_k(t)]}{t} = \mathbf{d}_{hk}.$$

The process X^ω_t can be decomposed as the sum of $y(X^\omega_t,\omega)$ and $\chi(X^\omega_t,\omega)$. We proved in Theorem 5.5.1 that $M^\omega_t = y(X^\omega_t,\omega)$ is a $\mathbb{P}^\omega_0$-square integrable continuous martingale whose quadratic variation is given by

$$\langle M^\omega_h, M^\omega_k\rangle_t = \int_0^t 2\sum_{i,j}a_{ij}(X^\omega_s,\omega)(\partial_i\chi^k(X^\omega_s,\omega)-\delta_{ik})(\partial_j\chi^h(X^\omega_s,\omega)-\delta_{jh})\,ds.$$

In particular, it follows that

$$\begin{aligned}\frac{\mathbb{E}^\omega_0[M^\omega_h(t)M^\omega_k(t)]}{t} &= \frac{2}{t}\int_0^t\mathbb{E}^\omega_0\Big[\langle a(X^\omega_s,\omega)(\nabla\chi^k(X^\omega_s,\omega)-e_k),\nabla\chi^h(X^\omega_s,\omega)-e_h\rangle\Big]\,ds\\ &= \frac{2}{t}\int_0^t\mathbf{P}_s[\langle a(\Xi^k-e_k),\Xi^h-e_h\rangle](\omega)\,ds,\end{aligned}$$

where we recall that $\mathbf{P}_t$ is the semigroup of the environmental process $t \to \psi_t^\omega = \tau_{X_t^\omega}\omega$. Since the environmental process is ergodic and stationary with respect to μ as proved in Proposition 5.3.4, it follows by an application of Birkoff's ergodic theorem that

$$\lim_{t\to\infty} \frac{\mathbb{E}_0^\omega[M_h^\omega(t)M_k^\omega(t)]}{t} = 2\,\mathbb{E}_\mu[\langle a(\Xi^k - e_k), \Xi^h - e_h\rangle] =: \mathbf{d}_{hk} \tag{5.19}$$

μ-almost surely. At this point one proceeds exactly as in [FK97] with the help of Proposition 5.6.2, to transfer statement (5.19) from the martingale $M^\omega(t)$ to the process $X^\omega(t)$.

We prove part (ii) of the Theorem I. Again, we make use of the decomposition

$$\epsilon X^\omega_{t/\epsilon^2} = \epsilon y(X^\omega_{t/\epsilon^2}, \omega) + \epsilon\chi(X^\omega_{t/\epsilon^2}, \omega).$$

and the fact that $M^{\epsilon,\omega} = \epsilon y(X^\omega_{t/\epsilon^2}, \omega)$ is a $\mathbb{P}_0^\omega$-square integrable continuous martingale μ-almost surely. Its quadratic variation is given by

$$\langle M_h^{\epsilon,\omega}, M_k^{\epsilon,\omega}\rangle_t = \epsilon\int_0^{t/\epsilon^2} 2\sum_{i,j} a_{ij}(X_s^\omega,\omega)(\partial_i\chi^k(X_s^\omega,\omega) - \delta_{ik})(\partial_j\chi^h(X_s^\omega,\omega) - \delta_{jh})\; ds.$$

An application of the ergodic theorem for the environmental process shows that

$$\lim_{\epsilon\to 0}\langle M_h^{\epsilon,\omega}, M_k^{\epsilon,\omega}\rangle_t = \mathbf{d}_{hk}t,$$

$\mathbb{P}_0^\omega$-almost surely, but also in the L^1-sense for almost all $\omega \in \Omega$. We can now apply the central limit for martingales [Hel82, Theorem 5.4] to conclude that the martingale $M^{\epsilon,\omega}$ converges in distribution under $\mathbb{P}_0^\omega$ to a Wiener measure with covariances given by $\mathbf{D} = \{\mathbf{d}_{hk}\}_{h,k=1}^d$ for μ-almost all $\omega \in \Omega$. The matrix is non degenerate by Proposition 5.4.7.

It remains to show that the correctors $\epsilon\chi(X^\omega_{t/\epsilon^2}, \omega)$ converge to zero in probability. For that the sublinearity of the corrector will play a major role.

Let $T > 0$ be a fixed time horizon. We claim that for all $\delta > 0$

$$\lim_{\epsilon\to 0}\mathbb{P}_0^\omega\Big(\sup_{0\le t\le T}|\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| > \delta\Big) = 0, \quad \mu\text{-a.s.} \tag{5.20}$$

Denote by $\tau_R^{\epsilon,\omega}$ the exit time of $\epsilon X^\omega_{t/\epsilon^2}$ from the ball B of radius $R > 1$ centered at the origin. Observe that

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\mathbb{P}_0^\omega\Big(\sup_{0\le t\le T}|\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| > \delta\Big)\\
&\qquad \le \limsup_{\epsilon\to 0}\mathbb{P}_0^\omega\Big(\sup_{0\le t\le \tau_R^{\epsilon,\omega}}|\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| > \delta\Big) + \limsup_{\epsilon\to 0}\mathbb{P}_0^\omega\Big(\sup_{0\le t\le T}|\epsilon X^\omega_{t/\epsilon^2}| > R\Big).
\end{aligned}$$

Let us show that both contributions converge to zero as $\epsilon \to 0$.

First addendum: By Proposition 5.6.2

$$\lim_{\epsilon\to 0}\sup_{0\le t\le \tau_R^{\epsilon,\omega}}|\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| = 0.$$

and therefore μ-almost surely

$$\limsup_{\epsilon\to 0} \mathbb{P}_0^\omega\Big(\sup_{0\le t\le \tau_R^{\epsilon,\omega}} |\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| > \delta\Big) = 0.$$

Second addendum: we use again Proposition 5.6.2 to say that there exists $\bar\epsilon(\omega) > 0$, which may depend on ω such that for all $\epsilon < \bar\epsilon(\omega)$ we have $\sup_{0\le t\le \tau_R^{\epsilon,\omega}} |\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| < 1$. For such ϵ we have μ-almost surely

$$\begin{aligned}\mathbb{P}_0^\omega\Big(\sup_{0\le t\le T} |\epsilon X^\omega_{t/\epsilon^2}| \ge R\Big) &= \mathbb{P}_0^\omega\Big(\tau_R^{\epsilon,\omega} \le T\Big)\\ &= \mathbb{P}_0^\omega\Big(\tau_R^{\epsilon,\omega} \le T, \sup_{0\le t\le \tau_R^{\epsilon,\omega}} |\epsilon y(X^\omega_{t/\epsilon^2},\omega)| > R-1\Big)\\ &\le \mathbb{P}_0^\omega\Big(\sup_{0\le t\le T} |\epsilon y(X^\omega_{t/\epsilon^2},\omega)| > R-1\Big).\end{aligned}$$

Since $\epsilon y(X^\omega_{\cdot/\epsilon^2},\omega)$ converges in distribution under $\mathbb{P}_0^\omega$ to a non-degenerate Brownian motion with deterministic covariance matrix given by $\mathbf{D}$ we have that [CZ95, Proposition 1.16] there exist positive constants c_1, c_2 independent on ω such that

$$\limsup_{\epsilon\to 0} \mathbb{P}_0^\omega\Big(\sup_{0\le t\le T} |\epsilon y(X^\omega_{t/\epsilon^2},\omega)| > R-1\Big) \le c_1 e^{-c_2 R},$$

from which it follows

$$\limsup_{\epsilon\to 0} \mathbb{P}_0^\omega\Big(\sup_{0\le t\le T} |\epsilon X^\omega_{t/\epsilon^2}| > R\Big) \le c_1 e^{-c_2 R}.$$

Therefore

$$\limsup_{\epsilon\to 0} \mathbb{P}_0^\omega\Big(\sup_{0\le t\le T} |\epsilon\chi(X^\omega_{t/\epsilon^2},\omega)| > \delta\Big) \le c_1 e^{-c_2 R}$$

and since $R > 1$ was arbitrary, the claim (5.20) follows, namely the corrector converges to zero in probability under $\mathbb{P}_0^\omega$, μ-almost surely.

The claim (5.20) combined with the fact that $\epsilon y(X_{\cdot/\epsilon^2},\omega)$ satisfies an invariance principle μ-almost surely implies also that the family $\epsilon X^\omega_{\cdot/\epsilon^2}$ under $\mathbb{P}_0^\omega$ satisfies an invariance principle μ-almost surely with the same limiting law. □

As a simple consequence of the quenched invariance principle above and a time-change technique we can easily derive a quenched invariance principle for a larger family of processes.

Corollary 5.6.3. *Let $\theta : \Omega \to \mathbb{R}$ be a $\mathcal{G}$-measurable function and assume that $\theta(\tau_\cdot\omega)$, $\theta(\tau_\cdot\omega)^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$ and that $\mathbb{E}_\mu[\theta], \mathbb{E}_\mu[\theta^{-1}] < \infty$. Let $\mathbf{M}^{\theta,\omega} := (X^{\theta,\omega}_t, \mathbb{P}^{\theta,\omega}_x)$, $x \in \mathbb{R}^d$ the minimal diffusion process associated to $(\mathcal{E}^\omega, \mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d, \theta dx)$. Then, for μ-almost all $\omega \in \Omega$, the laws of the processes $\epsilon X^{\theta,\omega}_{t/\epsilon^2}$ over $C([0,\infty),\mathbb{R}^d)$ converge weakly as $\epsilon \to 0$ to a Wiener measure with covariance matrix given by $\mathbf{D}/\,\mathbb{E}_\mu[\theta]$, where $\mathbf{D}$ was given in Theorem I.*

Proof. Let us define the time change

$$\hat{X}_t^\omega := X_{\tau_t^\omega}^\omega, \quad \tau_t^\omega = \inf\left\{ s > 0;\, A_s^\omega := \int_0^s \theta(X_u^\omega, \omega)\, du > t \right\}$$

To get asymptotic for $\epsilon^2 A_{t/\epsilon^2}$ it is easy by means of the ergodic theorem for the environmental process. We can prove as in [BM15, Lemma 15] that

$$\lim_{\epsilon \to 0} \sup_{s \in [0,t]} |\epsilon^2 A_{s/\epsilon^2}^\omega - s\, \mathbb{E}_\mu[\theta]| = 0, \quad \mathbb{P}_x^\omega\text{-a.s, a.a. } x \in \mathbb{R}^d, \tag{5.21}$$

for μ-almost all $\omega \in \Omega$. Observe that $\epsilon \hat{X}_{A^\omega(t/\epsilon^2)}^\omega = \epsilon X_{t/\epsilon^2}^\omega$, then the convergence in distribution for $\epsilon \hat{X}_{t/\epsilon^2}^\omega$ under $\mathbb{P}_x^\omega$, for almost all $x \in \mathbb{R}^d$, for μ-almost all $\omega \in \Omega$ follows from Theorem I and (5.21). On the other hand the processes $\hat{X}_t^\omega$ and $X_t^{\theta,\omega}$ are equivalent, since they possess the same Dirichlet form, see Theorem 6.2.1 in [FOT94]. Hence the same convergence holds for $\epsilon X_{t/\epsilon^2}^{\theta,\omega}$. □

6

Quenched local Central Limit Theorem

6.1 A deterministic version of ergodicity

In this first section we put aside the random environment and we discuss a set of assumptions on the weights $x \mapsto \lambda(x)$ and $x \mapsto \Lambda(x)$ which grant a local central limit theorem for the fundamental solution of equation (3.2) introduced in Chapter 3.

Recall that equation (3.2) is the weak formulation of the ill posed

$$\partial_t u(t,x) - \frac{1}{\Lambda(x)} \operatorname{div}(a(x)\nabla u(t,x)) = 0.$$

We assume that $(b.1)$ and $(b.2)$ are satisfied. As we know from Chapter 3, positive caloric functions satisfy a local parabolic Harnack inequality (cf. Theorem 3.6.2)

$$\sup_{Q_-} u(t,z) \le C_{PH}^{B,\Lambda} \inf_{Q_+} u(t,z),$$

where the constant $C_{PH}^{B,\Lambda}$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$. We recall here the explicit form of these constants (cf. Chapter 2):

- $C_S^{B,\Lambda} := \|\lambda^{-1}\|_{q,B} \|\Lambda\|_{p,B}^{2p^*/\rho}$,
- $C_P^{B,\Lambda} := \|\lambda^{-1}\|_{\bar{q},B} \|\Lambda\|_{\bar{p},B}$ where $\bar{p} \le p$ and $\bar{q} \le q$ are such that $1/\bar{p} + 1/\bar{q} = 2/d$,
- $M^{B,\Lambda} := \|\Lambda\|_{1,B} / \|\Lambda\|_{1,B/2}$.

Observe that the constants depend on averages over B of λ^{-1} and Λ. Before embarking in the proof of a local central limit theorem, we show that if for large balls the averages of λ^{-1} and Λ are bounded, then we can derive on-diagonal upper bounds and control the oscillations of the transition kernel $p_t^\Lambda(x,y)$ of the semigroup P_t^Λ associated to $(\mathcal{E}, \mathcal{F}^\Lambda)$ on $L^2(\mathbb{R}^d, \Lambda)$. Next assumption goes exactly in this direction.

Assumption c.1. For the same p, q as in assumption $(b.2)$

$$\limsup_{R\to\infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} \Lambda^p + \lambda^{-q} \, dz < \infty.$$

Lemma 6.1.1. *Under* $(c.1)$*, for all* $x \in \mathbb{R}^d$

$$\limsup_{R\to\infty} \frac{1}{|B(x,R)|} \int_{B(x,R)} \Lambda^p + \lambda^{-q}\, dz < \infty,$$

and the limit does not depend on x.

Proof. Observe that for $R > |x|$, $B(0, R-|x|) \subset B(x,R) \subset B(0, R+|x|)$, therefore the following inequality holds

$$\begin{aligned}\left(\frac{R-|x|}{R}\right)^d \frac{1}{|B(0,R-|x|)|} \int_{B(0,R-|x|)} \Lambda^p + \lambda^{-q}\, dz \\ &\leq \frac{1}{|B(x,R)|}\int_{B(x,R)} \Lambda^p + \lambda^{-q}\, dz \\ &\leq \left(\frac{R+|x|}{R}\right)^d \frac{1}{|B(0,R+|x|)|} \int_{B(0,R+|x|)} \Lambda^p + \lambda^{-q}\, dz,\end{aligned}$$

and taking the limit $R \to +\infty$ both sides gives the result. □

Lemma 6.1.2. *Assume* $(c.1)$*. Then, there exist finite positive constants* $C_S^{*,\Lambda}$*,* $C_P^{*,\Lambda}$ *and* $M^{*,\Lambda}$*, independent of* $x \in \mathbb{R}^d$ *such that*

$$\limsup_{R\to\infty} C_S^{B(x,R),\Lambda} = C_S^{*,\Lambda}, \quad \limsup_{R\to\infty} C_P^{B(x,R),\Lambda} = C_P^{*,\Lambda},$$

$$\sup_{x\in\mathbb{R}^d} \limsup_{R\to\infty} M^{B(x,R),\Lambda} = M^{*,\Lambda}.$$

In particular, for all $\delta > 0$ *and* $x \in \mathbb{R}^d$ *there exists* $s(x,\delta) \geq 1$ *such that for all* $R > s(x,\delta)$

$$C_S^{B(x,R),\Lambda} < C_S^{*,\Lambda}(1+\delta), \quad C_P^{B(x,R),\Lambda} < C_P^{*,\Lambda}(1+\delta), \quad M^{B(x,R),\Lambda} < M^{*,\Lambda}(1+\delta).$$

Proof. The existence of a finite limit for $C_S^{B(x,R),\Lambda}$ and $C_P^{B(x,R),\Lambda}$ as $R \to \infty$ is a direct consequence of $(c.1)$. In the case of $M^{B(x,R),\Lambda}$ one must be slightly more careful since $\|\Lambda\|_{1,B/2}$ appears in the denominator. It suffices to observe that

$$\limsup_{R\to\infty} M^{B(x,R),\Lambda} \leq \limsup_{R\to\infty} \frac{\|\Lambda\|_{1,B(x,R)}}{\|\Lambda\|_{1,B(x,R/2)}} \leq \limsup_{R\to\infty} \|\Lambda\|_{1,B(x,R)} \|\Lambda^{-1}\|_{1,B(x,R/2)} < \infty.$$

The independence of the limits from $x \in \mathbb{R}^d$ can be obtained as in Lemma 6.1.1. The second statement is an immediate consequence of the first part. □

Through $(c.1)$ and Lemma 6.1.2 we are provided with some control over the constants above for large balls, therefore we can apply Theorem 3.3.3 to obtain a global on-diagonal heat kernel upper bound, as it is done in the next proposition.

Proposition 6.1.3. *Let* $f \in L^2(\mathbb{R}^d, \Lambda)$*, and assume that* $(b.1)$*,* $(b.2)$ *and* $(c.1)$ *are satisfied, then there exists a constant* $C_9 := C_9(q,p,d,C_S^{*,\Lambda}) > 0$ *such that for all* $x \in \mathbb{R}^d$ *and* $t > 0$ *the following inequality holds*

$$P_t f(x) \leq C_9 t^{-\gamma}(s(0,1) + |x| + \sqrt{t})^{\gamma - d/2} \int_{\mathbb{R}^d} (s(0,1) + |y| + \sqrt{t})^{\gamma-d/2} |f(y)|\Lambda(y)dy.$$

where γ *was defined in 1.17 and* $s(x,\delta)$ *was defined in Lemma 6.1.2.*

Proof. We want to apply Theorem 3.3.3. Fix $\tau \in (0,2]$, $x = 0$, $R > 0$, $s = \tau R^2$, $\sigma = 1$ and $\sigma' = 1/2$. It follows that

$$Q_1 = (0, \tau R^2) \times B(0,R), \quad Q_{1/2} = \tau R^2(1/2,1) \times B(0,R/2).$$

We choose $R := s(0,1) + 2|z| + \sqrt{t}$ where $s(0,1)$ was defined in Lemma 6.1.2. In this way $C_S^{B(0,R),\Lambda} \leq 2C_S^{*,\Lambda}$ and we can read inequality (3.9) for $u(s,z) := P_s^\Lambda f(z)$ as follows

$$\sup_{Q_{1/2}} P_s^\Lambda f(z) \leq c(C_S^{*,\Lambda})^{\gamma/2} \frac{\tau^{-\gamma/2}}{R^{d/2}} \|f\|_{2,\Lambda},$$

with $c = c(p,q,d)$ changing throughout the proof. By definition of R we can find $\tau \in (0,2]$ such that $3/4\tau R^2 = t$ and in particular such that $(t,z) \in Q_{1/2}$. This gives

$$P_t^\Lambda f(z) \leq ct^{-\gamma/2}(s(0,1) + |z| + \sqrt{t})^{\gamma - d/2} \|f\|_{2,\Lambda},$$

for all $z \in \mathbb{R}^d$ and $t > 0$, where now $c = c(p,q,d,C_S^{*,\Lambda})$ depends on $C_S^{*,\Lambda}$ as well. Set $b_t(z) = (s(0,1) + |z| + \sqrt{t})^{\gamma - d/2}$. It follows that

$$\|b_t^{-1} P_t^\Lambda f\|_\infty \leq ct^{-\gamma/2} \|f\|_{2,\Lambda},$$

from which $\|b_t^{-1} P_t^\Lambda\|_{2 \to \infty} \leq ct^{-\gamma/2}$. By duality we get $\|P_t^\Lambda b_t^{-1}\|_{1 \to 2} \leq ct^{-\gamma/2}$. Hence

$$\|P_t^\Lambda f\|_{2,\Lambda} \leq ct^{-\gamma/2} \|b_t f\|_{1,\Lambda}.$$

Now it is left to use the semigroup property and classical techniques [Dav90, Chapter 2] to finally get the bound. □

It is now standard to get global on-diagonal estimates for the kernel $p_t^\Lambda(x,y)$ of the semigroup P_t^Λ associated to $(\mathcal{E}, \mathcal{F}^\Lambda)$ on $L^2(\mathbb{R}^d, \Lambda)$. Namely we obtain that for almost all $x, y \in \mathbb{R}^d$ and for all $t > 0$

$$p_t^\Lambda(x,y) \leq C_9 t^{-\gamma}(s(0,1) + |x| + \sqrt{t})^{\gamma - d/2}(s(0,1) + |y| + \sqrt{t})^{\gamma - d/2}. \tag{6.1}$$

Remark 6.1.4. *By Lemma 6.1.2 it follows that there exists $C_{PH}^{*,\Lambda} < \infty$ and $R_{PH}(x) \geq 1$ such that for all $R \geq R_{PH}(x)$ we have $C_{PH}^{B(x,R),\Lambda} \leq C_{PH}^{*,\Lambda}$.*

Theorem 6.1.5 (Hölder continuity). *Let $x \in \mathbb{R}^d$, and $R_{PH}(x) \geq 1$ as above. Let $R > R_{PH}(x)$ and $\sqrt{t} \geq R$. Define $t_0 := t + 1$ and $R_0 := \sqrt{t_0}$. If u is a positive caloric function on $(0,t_0) \times B(x,R_0)$ then for all $z, y \in B(x,R)$ we have*

$$u(t,z) - u(t,y) \leq c\left(\frac{R}{\sqrt{t}}\right)^\theta \sup_{[3t_0/4, t_0] \times B(x, \sqrt{t_0}/2)} u, \tag{6.2}$$

where θ, c are constants which depend only on $C_{PH}^{*,\Lambda}$.

Proof. Set $R_k := 2^{-k} R_0$ and let

$$Q_k := (t_0 - R_k^2, t_0) \times B(x, R_k),$$

Q_k^- and Q_k^+ be accordingly defined as in (3.22) with $\delta = 1/2$ and $\tau = 1$,

$$Q_k^- := (t_0 - 7/8R_k^2, t_0 - 5/8R_k^2) \times B(x, 1/2R_k), \quad Q_k^+ := (t_0 - 1/4R_k^2, t_0) \times B(x, 1/2R_k).$$

Notice that $Q_{k+1} \subset Q_k$ and actually $Q_{k+1} = Q_k^+$. We set

$$v_k = \frac{u - \inf_{Q_k} u}{\sup_{Q_k} u - \inf_{Q_k} u}.$$

Clearly v_k is a caloric on Q_k, in particular $0 \le v_k \le 1$ and

$$\operatorname{osc}(v_k, Q_k) := \sup_{Q_k} v_k - \inf_{Q_k} v_k = 1.$$

This implies that replacing v_k by $1 - v_k$ if necessary $\sup_{Q_k^-} v_k \ge 1/2$. Now for all k such that $R_k \ge R_{PH}(x)$ we can apply the parabolic Harnack inequality with common constant $C_{PH}^{*,\Lambda}$ and get

$$\frac{1}{2} \le \sup_{Q_k^-} v_k \le C_{PH}^{*,\Lambda} \inf_{Q_k^+} v_k.$$

Since by construction $Q_k^+ = Q_{k+1}$, we deduce that

$$\begin{aligned}
\operatorname{osc}(u, Q_{k+1}) &= \frac{\sup_{Q_{k+1}} u - \inf_{Q_{k+1}} u}{osc(u, Q_k)} \operatorname{osc}(u, Q_k) \\
&= \left(\frac{\sup_{Q_{k+1}} u - \inf_{Q_k} u}{\operatorname{osc}(u, Q_k)} - \inf_{Q_{k+1}} v_k \right) \operatorname{osc}(u, Q_k),
\end{aligned}$$

which yields $\operatorname{osc}(u, Q_{k+1}) \le (1 - \delta) \operatorname{osc}(u, Q_k)$ with $\delta^{-1} = 2C_{PH}^{*,\Lambda}$. We can now iterate the inequality up to k_0 such that $R_{k_0} \ge R > R_{k_0+1}$ and get

$$\operatorname{osc}(u, Q_{k_0}) \le (1 - \delta)^{k_0 - 1} \operatorname{osc}(u, Q_0^+).$$

Finally since $B(x, R) \subset B(x, R_{k_0})$, $t \in (t_0 - R_{k_0}^2, t_0)$ and $-k_0 \le \log_2(R/\sqrt{t}) + 1$ the claim is proved. $\square$

Starting from (6.2) and knowing that $p_t^\Lambda(z, \cdot)$ is positive and caloric on the whole $\mathbb{R}^d$ for almost all $z \in \mathbb{R}^d$ we get the following corollary.

Corollary 6.1.6. *Let $x \in \mathbb{R}^d$, and $R_{PH}(x) \ge 1$ as above. Let $R > R_{PH}(x)$ and $\sqrt{t} \ge R$. Then we have that for almost all $o \in \mathbb{R}^d$*

$$\sup_{z,y \in B(x,R)} |p_t^\Lambda(o, z) - p_t^\Lambda(o, y)| \le c \left(\frac{R}{\sqrt{t}} \right)^\theta t^{-d/2}, \tag{6.3}$$

where θ, c are positive constants which depend only on $C_{PH}^{*,\Lambda}$.

Proof. We have just to bound the right hand side of (6.2). Define $t_0 = t + 1$ as in the previous theorem. By Harnack inequality applied to the caloric function $p_t^\Lambda(o, \cdot)$ we have

$$\begin{aligned}
\sup_{[3t_0/4, t_0] \times B(x, \sqrt{t_0}/2)} p_s^\Lambda(o, u) &\le C_{PH}^{*,\Lambda} \inf_{[3/2t_0, 7/4t_0] \times B(x, \sqrt{t_0}/2)} p_s(o, u) \\
&\le C_{PH}^{*,\Lambda} \Big[|B(x, \sqrt{t_0}/2)| \|\Lambda\|_{1, B(x, \sqrt{t_0}/2)} \Big]^{-1} \int_{B(x, \sqrt{t_0}/2)} p_t^\Lambda(o, u) \Lambda(u) \, du,
\end{aligned}$$

where $\bar{t} \in [3/2t_0, 7/4t_0]$.

Clearly $\int_{B(x,\sqrt{t_0}/2)} p_{\bar{t}}^{\Lambda}(o,u)\Lambda(u)\,du \le 1$. For $\sqrt{t_0} > R > R_{PH}(x)$, we can bound $\|\Lambda\|_{1,B(x,\sqrt{t_0}/2)}$ by a constant which does not depend on x or t_0, hence we finally get the desired bound. $\square$

We want to stress that Corollary (6.1.6) is not a true Hölder continuity result, since we cannot bound the variations for arbitrarily small balls, and indeed it is not even possible to prove continuity of the density with this technique.

1.1 Diffusive scaling and LCLT. We are interested in controlling the oscillations of caloric functions in the scaling $(t,x) \to (t/\epsilon^2, x/\epsilon)$ as ϵ goes to zero. This means that given a positive caloric function $u : [0,\infty) \times \mathbb{R}^d \to [0,+\infty)$ we are interested in the behavior of $u(t/\epsilon^2, x/\epsilon)$. Clearly $u(t/\epsilon^2, x/\epsilon)$ is still caloric with respect to $\partial_t - \Lambda(x/\epsilon)^{-1}\operatorname{div}(a(x/\epsilon)\nabla\cdot)$. It follows that we have to control the constants $C_S^{B,\Lambda}$, $C_P^{B,\Lambda}$, $M^{B,\Lambda}$ on moving balls $B = B(x/\epsilon, R/\epsilon)$ in the limit $\epsilon \to 0$. With this in mind we introduce the assumption below.

Assumption c.2. For the same p, q as appearing in $(b.2)$

$$\sup_{x\in\mathbb{R}^d} \limsup_{\epsilon\to 0} \frac{1}{|B(x/\epsilon, 1/\epsilon)|} \int_{B(x/\epsilon,1/\epsilon)} \Lambda^p + \lambda^{-q}\,dz < \infty.$$

It is clear that assumption $(c.2)$ implies $(c.1)$; Indeed the latter can be obtained by the former choosing $x = 0$.

Lemma 6.1.7. *Let $F : \mathbb{R}^d \to [0,+\infty)$ and let $\delta, R_0 > 0$. Assume that*

$$\sup_{x\in\mathbb{R}^d} \limsup_{\epsilon\to 0} \frac{1}{|B(x/\epsilon, 1/\epsilon)|} \int_{B(x/\epsilon,1/\epsilon)} F\,dz =: K < \infty. \tag{6.4}$$

Then, there exists a constant $\epsilon_1(x, R_0, \delta) > 0$ such that for all $x \in \mathbb{R}^d$ and all $\epsilon < \epsilon_1(x, R_0, \delta)$

$$\sup_{R\ge R_0} \frac{1}{|B(x/\epsilon, R/\epsilon)|} \int_{B(x/\epsilon,R/\epsilon)} F\,dz < K(1+\delta).$$

Proof. Fix $x \in \mathbb{R}^d$. First we observe that it is enough to prove the statement for $R_0 = 1$ and $x \neq 0$, being the case $R_0 \neq 1$ completely analogous and the case $x = 0$ immediate. Let $0 < \delta_0 < |x|$. We split the supremum into two parts

$$\begin{aligned}
&\sup_{R\ge 1} \frac{1}{|B(x/\epsilon, R/\epsilon)|} \int_{B(x/\epsilon,R/\epsilon)} F\,dz \\
&\quad = \sup_{1\le R\le |x|/\delta_0} \frac{1}{|B(x/\epsilon, R/\epsilon)|} \int_{B(x/\epsilon,R/\epsilon)} F\,dz \vee \sup_{R\ge |x|/\delta_0} \frac{1}{|B(x/\epsilon, R/\epsilon)|} \int_{B(x/\epsilon,R/\epsilon)} F\,dz.
\end{aligned}$$

We first deal with the second part.

$$\sup_{R\geq |x|/\delta_0} \frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz \\ \leq \sup_{R\geq |x|/\delta_0}\left(1+\frac{|x|}{R}\right)^d \frac{1}{|B(0,(R+|x|)/\epsilon)|}\int_{B(0,(R+|x|)/\epsilon)} F\,dz.$$

Recall that by (6.4), for all $R>0$ there exists $\epsilon_2(x,R,\delta_0)>0$ such that for all $\epsilon<\epsilon_2(x,R,\delta_0)$

$$\frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz < K(1+\delta_0). \tag{6.5}$$

In particular, recalling that $R\geq |x|/\delta_0>1$, for all $\epsilon<\epsilon_2(0,1,\delta_0)$

$$\sup_{R\geq |x|/\delta_0} \frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz \leq K(1+\delta_0)^{d+1}$$

The first part is a bit more delicate. For all $\epsilon>0$ define $\Psi_\epsilon:[1,\infty)\to[0,\infty)$ by

$$\Psi_\epsilon(R) := \frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz.$$

Let $1\leq R_- < R_+ \leq \bar{R} := |x|/\delta_0$ and $R\in[R_-,R_+]$, then we have

$$\Psi_\epsilon(R)-\Psi_\epsilon(R_+) \leq d\bar{R}^{2d-1}\cdot \frac{1}{|B(x/\epsilon, \bar{R}/\epsilon)|}\int_{B(x/\epsilon,\bar{R}/\epsilon)} F dz\cdot (R_+-R_-)$$

and for all $x\in\mathbb{R}^d$ we can find $\epsilon_2(x,\bar{R},\delta_0)>0$ such that for all $\epsilon<\epsilon_2(x,\bar{R},\delta_0)$

$$\Psi_\epsilon(R)-\Psi_\epsilon(R_+) \leq dK(1+\delta_0)\bar{R}^{2d-1}(R_+-R_-)$$

Now take a partition $1=r_0,...,r_m=:\bar{R}$ of in such a way that

$$|r_i-r_{i-1}|\leq \delta_0/(d\bar{R}^{2d-1}(1+\delta_0))$$

for all $i=1,...,m$. Define $\epsilon_3(x,\delta_0):=\epsilon_2(x,\bar{R},\delta_0)\wedge\min_{i=1,\ldots,m}\epsilon_2(x,r_i,\delta_0)>0$. Then for all $x\in\mathbb{R}^d$ and all $\epsilon\leq\epsilon_3(x,\delta_0)$, we have that for all $R\in[1,\bar{R}]$

$$\frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz = \Psi_\epsilon(R) \\ = \Psi_\epsilon(r_{i(R)}) + (\Psi_\epsilon(R)-\Psi_\epsilon(r_{i(R)})) \leq K(1+\delta_0)+K\delta_0 = K(1+2\delta_0), \tag{6.6}$$

where $i(R)$ is such that $0\leq r_{i(R)}-R\leq \delta_0/(d\bar{R}^{2d-1}(1+\delta_0))$. Putting together (6.5) and (6.6), and defining $\epsilon_1(x,\delta_0):=\epsilon_2(0,1,\delta_0)\wedge\epsilon_3(x,\delta_0)$, we can deduce that for all $x\in\mathbb{R}^d$ and all $\epsilon<\epsilon_1(x,\delta_0)$

$$\sup_{R\geq 1} \frac{1}{|B(x/\epsilon, R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)} F\,dz \leq K(1+2\delta_0)\wedge K(1+\delta_0)^{d+1}.$$

Finally, the statement follows by an appropriate choice of δ_0 small enough and taking into account the dependence on $R_0>0$ for the case $R_0\neq 1$. □

Remark 6.1.8. *Assumption* $(c.2)$ *and Lemma 6.1.7 allow to control* $C_S^{B,\Lambda}$, $C_P^{B,\Lambda}$, $M^{B,\Lambda}$ *uniformly on balls* $B = B(x/\epsilon, R/\epsilon)$ *when* ϵ *is small enough . In particular, since* $C_{PH}^{B,\Lambda}$ *depends increasingly on* $C_S^{B,\Lambda}$, $C_P^{B,\Lambda}$, $M^{B,\Lambda}$, *for all* $x \in \mathbb{R}^d$ *and* $R_0 > 0$ *we can find* $\epsilon_{PH}(x, R_0) > 0$ *and a finite constant* C_{PH}^{Λ} *independent of* x, R_0 *such that for all* $\epsilon < \epsilon_{PH}(x, R_0)$ *and* $r \geq R_0$

$$C_{PH}^{B(x/\epsilon, r/\epsilon),\Lambda} \leq C_{PH}^{\Lambda}.$$

Lemma 6.1.9. *Let* $R > 0$ *and* $t > 0$ *such that* $\sqrt{t}/2 \geq R$. *Then, for all* $\epsilon < \epsilon_{PH}(x, R)$ *we have that for all* $x \in \mathbb{R}^d$

$$\sup_{z,y \in B(x,R)} \epsilon^{-d} |p_{t/\epsilon^2}^{\Lambda}(o, z/\epsilon) - p_{t/\epsilon^2}^{\Lambda}(o, y/\epsilon)| \leq c \left(\frac{R}{\sqrt{t}} \right)^{\theta} t^{-d/2}, \tag{6.7}$$

where θ, c *are positive constants which depend only on* C_{PH}^{Λ}.

Proof. The proof is very close to the one of Theorem 6.1.5. Take $t \in I$ and set $t_0 := t+1$, $R_0 := \sqrt{t_0}$. Observe that if $\epsilon < \epsilon_{PH}(x, R)$, then for all $r \geq R$

$$C_{PH}^{B(x/\epsilon, r/\epsilon),\Lambda} \leq C_{PH}^{\Lambda}.$$

Next we set $R_k := 2^{-k} R_0$,

$$Q_k := \epsilon^{-2}(t_0 - R_k^2, t_0) \times B(x/\epsilon, R_k/\epsilon),$$

and Q_k^-, Q_k^+ accordingly. Since for all $k \in \mathbb{N}$ such that $R_k \geq R$, and all $\epsilon < \epsilon_{PH}(x, R)$ we have a common constant C_{PH}^{Λ} for the Harnack inequality on the sets Q_k, we can proceed as in Theorem 6.1.5 and get for $z, y \in B(x, R)$,

$$p_{t/\epsilon^2}^{\Lambda}(o, z/\epsilon) - p_{t/\epsilon^2}^{\Lambda}(o, y/\epsilon) \leq c \left(\frac{R}{\sqrt{t}} \right)^{\theta} \sup_{[3t_0/4, t_0] \times B(x, \sqrt{t_0}/2)} p_{\cdot/\epsilon^2}^{\Lambda}(o, \cdot/\epsilon),$$

where θ, c are constants which depends only on C_{PH}^{Λ}. At this point one proceeds as in Corollary 6.1.6 to get the final result. $\square$

From the Hölder continuity estimates (6.7) we get the key tool to prove a local central limit theorem. The statement of next proposition must be compared with [CH08, Assumption 2].

Proposition 6.1.10. *For almost all* $o \in \mathbb{R}^d$ *and all* $R > 0$

$$\lim_{R_0 \to 0} \limsup_{\epsilon \to 0} \sup_{\substack{x,y \in B(o,R) \\ |x-y| < R_0}} \sup_{t \in I} \epsilon^{-d} |p_{t/\epsilon^2}^{\Lambda}(o, x/\epsilon) - p_{t/\epsilon^2}^{\Lambda}(o, y/\epsilon)| = 0.$$

Proof. Let us denote by $t_1 := \inf I$. Fix $\delta > 0$ and set

$$R_0 := \frac{\sqrt{t_1}}{2} \wedge \left(\frac{t_1^{d/2} \delta}{2c} \right)^{1/\theta} \sqrt{t_1},$$

being θ and c as appearing in (6.7). Since $B(o,R)$ is compact we can cover it by a finite set of balls $\{B(x,R_0/2)\}_{x\in\mathcal{X}}$ of radius $R_0/2$ and centers $x\in\mathcal{X}\subset B(o,R)$. Take $\bar\epsilon:=\min_{x\in\mathcal{X}}\epsilon_{PH}(x,R_0)$. An application of (6.7) gives for all $\epsilon<\bar\epsilon$

$$\sup_{x\in\mathcal{X}}\sup_{|x-y|<R_0}\sup_{t\in I}\epsilon^{-d}|p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)-p^{\Lambda}_{t/\epsilon^2}(o,y/\epsilon)|\le c\Big(\frac{R_0}{\sqrt{t_1}}\Big)^{\theta}t_1^{-d/2}\le\frac{\delta}{2}.$$

We can use this bound to conclude. Namely, fix any $z\in B(o,R)$, and take $x\in\mathcal{X}$ such that $|z-x|<R_0/2$, then

$$\begin{aligned}\sup_{|z-y|\le R_0/2}\sup_{t\in I}&\epsilon^{-d}|p^{\Lambda}_{t/\epsilon^2}(o,z/\epsilon)-p^{\Lambda}_{t/\epsilon^2}(o,y/\epsilon)|\\&\le\sup_{t\in I}\epsilon^{-d}|p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)-p^{\Lambda}_{t/\epsilon^2}(o,z/\epsilon)|\\&+\sup_{|y-x|\le R_0}\sup_{t\in I}\epsilon^{-d}|p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)-p^{\Lambda}_{t/\epsilon^2}(o,y/\epsilon)|\le\delta\end{aligned}$$

and this ends the proof since we showed that the bound is uniform in $z\in B(o,R)$. $\square$

We finally give the main application of the computations we have developed in the preceding sections. The approach we exploit is the one in [CH08, Theorem 1] (the same approach is presented also in [BH09]).

We denote by $k_t^{\Sigma}(x)$, $x\in\mathbb{R}^d$ the Gaussian kernel with covariance matrix $\Sigma\in\mathbb{R}^{d\times d}$, namely

$$k_t^{\Sigma}(x):=\frac{1}{\sqrt{(2\pi t)^d\det\Sigma}}\exp\Big(-\frac{x\cdot\Sigma^{-1}x}{2t}\Big).$$

We need here two further assumptions.

Assumption c.3. For almost all $x\in\mathbb{R}^d$ and all $R>0$

$$\lim_{\epsilon\to0}\frac{1}{|B(x/\epsilon,R/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)}\Lambda\,dx=:a_{\Lambda}<\infty.$$

Assumption c.4. There exists a positive define symmetric matrix Σ such that for almost all $o\in\mathbb{R}^d$, for any compact interval $I\subset(0,\infty)$, almost all $x\in\mathbb{R}^d$ and $r>0$

$$\lim_{\epsilon\to0}\frac{1}{\epsilon^d}\int_{B(x,r)}p^{\Lambda}_{t/\epsilon^2}(o,y/\epsilon)\,\Lambda(y/\epsilon)dy\to\int_{B(x,r)}k_t^{\Sigma}(y)\,dy,$$

uniformly in $t\in I$.

Theorem 6.1.11. *Fix a compact interval $I\subset(0,\infty)$ and $R>0$. Assume $(b.1)$, $(b.2)$ and $(c.2)$, $(c.3)$, $(c.4)$, then for almost all $o\in\mathbb{R}^d$*

$$\lim_{\epsilon\to0}\sup_{x\in B(o,R)}\sup_{t\in I}|\epsilon^{-d}p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)-a_{\Lambda}^{-1}k_t^{\Sigma}(x)|=0.$$

Proof. The proof presented here is a slight variation of the one in [CH08] due to the fact that we work on $\mathbb{R}^d$ rather than on graphs. For $x \in B(o,R)$ and $R_0 > 0$ denote

$$J(t,\epsilon) := \frac{1}{\epsilon^d} \int_{B(x,R_0)} p^{\Lambda}_{t/\epsilon^2}(o, y/\epsilon)\, \Lambda(y/\epsilon) dy - \int_{B(x,R_0)} k_t(y)\, dy,$$

where $k_t := k^{\Sigma}_t$ from assumption $(c.4)$ is the Gaussian kernel with covariance matrix Σ. Now, we can split $J(t,\epsilon) = J_1(t,\epsilon) + J_2(t,\epsilon) + J_3(t,\epsilon) + J_4(t,\epsilon)$ where

$$\begin{aligned}
J_1(t,\epsilon) &:= \int_{\frac{1}{\epsilon}B(x,R_0)} [p^{\Lambda}_{t/\epsilon^2}(o,y) - p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)]\Lambda(y)dy,\\
J_2(t,\epsilon) &:= \int_{\frac{1}{\epsilon}B(x,R_0)} \Lambda(y)dy\, \Big[p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon) - \epsilon^d a_{\Lambda}^{-1} k_t(x)\Big],\\
J_3(t,\epsilon) &:= k_t(x)\Big[\epsilon^d a_{\Lambda}^{-1} \int_{\frac{1}{\epsilon}B(x,R_0)} \Lambda(y)dy - |B(x,R_0)|\Big],\\
J_4(t,\epsilon) &:= \int_{B(x,R_0)} (k_t(x) - k_t(y))dy.
\end{aligned}$$

Fix $\delta > 0$. By the continuity of k_t we can choose $R_0 \in (0,1)$ small enough such that

$$\sup_{\substack{x,y \in B(0,R+1)\\ |x-y| \le R_0}} \sup_{t \in I} |k_t(y) - k_t(x)| \le \delta, \tag{6.8}$$

from which we can easily obtain the bound $\sup_{t\in I} |J_4(t,\epsilon)| \le \delta |B(x,R_0)|$. Taking R_0 smaller if needed, thanks to Proposition 6.1.10, we can find $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$

$$\sup_{\substack{x,y \in B(o,R+1)\\ |x-y| \le R_0}} \sup_{t \in I} \frac{1}{\epsilon^d} |p^{\Lambda}_{t/\epsilon^2}(o,y/\epsilon) - p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon)| \le \delta, \tag{6.9}$$

which immediately implies that $\sup_{t\in I} |J_1(t,\epsilon)| \le \delta |B(x,R_0)|$. Furthermore, by assumption $(c.3)$ taking $\bar{\epsilon}$ smaller if needed we get $\sup_{t\in I} |J_3(t,\epsilon)| \le \delta |B(x,R_0)|$ for all $\epsilon \le \bar{\epsilon}$. Finally, assumption $(c.4)$ readily gives $\sup_{t\in I} |J(t,\epsilon)| \le \delta |B(x,R_0)|$ for ϵ small enough.

These estimates can be then used to control $|J_2(t,\epsilon)|$ for $\epsilon \le \bar{\epsilon}$ uniformly in $t \in I$. Namely, one gets

$$\sup_{t\in I} |\epsilon^{-d} p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon) - a_{\Lambda}^{-1} k_t(x)| \le 4\delta \bigg(\frac{\epsilon^d}{|B(x,R_0)|} \int_{\frac{1}{\epsilon}B(x,R_0)} \Lambda(y)dy \bigg)^{-1}$$

and we can take $\bar{\epsilon}$ even smaller to have by means of assumption $(c.3)$

$$\bigg(\frac{\epsilon^d}{|B(x,R_0)|} \int_{\frac{1}{\epsilon}B(x,R_0)} \Lambda(y)dy \bigg)^{-1} \le (\delta + a_{\Lambda}).$$

This implies that for almost all $x \in R^d$

$$\lim_{\epsilon\to 0} \sup_{t\in I} |\epsilon^{-d} p^{\Lambda}_{t/\epsilon^2}(o,x/\epsilon) - a_{\Lambda}^{-1} k_t(x)| = 0.$$

Consider now $R > 0$ and $\delta > 0$, and let $R_0 \in (0, 1)$ be chosen as before. Since $B(o, R)$ is compact there exists a finite covering $\{B(z, R_0)\}_{z\in\mathcal{X}}$ of $B(o, R)$ with $\mathcal{X} \subset B(o, R)$. Since $\mathcal{X}$ is finite, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \leq \bar{\epsilon}$

$$\sup_{z\in\mathcal{X}} \sup_{t\in I} |\epsilon^{-d} p^{\Lambda}_{t/\epsilon^2}(o, z/\epsilon) - a_{\Lambda}^{-1} k_t(z)| \leq \delta$$

Next if $x \in B(o, r)$ then $x \in B(z, r_0)$ for some $z \in \mathcal{X}$ and we can write

$$\begin{aligned}\sup_{t\in I} |\epsilon^{-d} p^{\Lambda}_{t/\epsilon^2}(o, x/\epsilon) - a_{\Lambda}^{-1} k_t(x)| \leq& \sup_{t\in I} \epsilon^{-d} |p^{\Lambda}_{t/\epsilon^2}(o, x/\epsilon) - p^{\Lambda}_{t/\epsilon^2}(o, z/\epsilon)| \\ &+ \sup_{t\in I} |\epsilon^{-d} p^{\Lambda}_{t/\epsilon^2}(o, z/\epsilon) - a_{\Lambda}^{-1} k_t(z)| \\ &+ a_{\Lambda}^{-1} \sup_{t\in I} |k_t(x) - k_t(z)|.\end{aligned}$$

Since $x, z \in B(o, R+1)$ and $|x-z| \leq R_0$, inequality (6.8) implies that the last addendum is bounded by δ, the second term is also bounded uniformly by δ since $z \in \mathcal{X}$. We can finally bound the first term uniformly by δ by means of (6.9). This ends the proof. $\square$

6.2 Application to diffusions in random environment

In this section we finally apply Theorem 6.1.11 to obtain Theorem II.

Proof of Theorem II. By Theorem 6.1.11, it is enough to show that assumptions $(b.1)$, $(b.2)$ and $(c.2)$, $(c.3)$, $(c.4)$ are satisfied for μ-almost all realizations of the environment.

By construction $(a.1)$ implies $(b.1)$ for μ-almost all $\omega \in \Omega$. Assumption $(a.2)$ together with Lemma 5.1.1 gives easily $(b.2)$, μ-almost surely. The ergodic theorem (cf. Theorem 5.1.2) is giving $(c.2)$, $(c.3)$; in particular the constant a_Λ appearing in $(c.3)$ is given by $\mathbb{E}_\mu[\Lambda]$. Finally $(c.4)$ for μ-almost all $\omega \in \Omega$ follows directly from $(a.4)$.

The second part of the statement follows readily since, if we assume that $\lambda^\omega(\cdot)^{-1}$ and $\Lambda^\omega(\cdot)$ are locally bounded for μ-almost all $\omega \in \Omega$, Theorem 6.1.11 holds for all $o \in \mathbb{R}^d$, μ-almost surely. Indeed, the density $p_t^{\Lambda,\omega}(x, y)$ is a continuous function of x and y by classical results in PDE theory [GT01]. $\square$

7

Conclusions

7.1 Summary

In this manuscript we proved a *quenched functional central limit theorem* and a *quenched local central limit theorem* for symmetric diffusions in a degenerate, stationary, ergodic random environment. We showed it provided that the moment condition $(a.2)$, namely that $\mathbb{E}_\mu[\Lambda^p]$ and $\mathbb{E}_\mu[\lambda^{-q}]$ with $1/p + 1/q < 2/d$, is satisfied.

Despite the fact that our condition seems not to be optimal for a quenched invariance principle to hold, this is the case for a quenched central limit theorem (cf. [ADS15a, Theorem 5.4] for a counterexample in the discrete setting). Both the proofs of the quenched invariance principle and of the quenched central limit theorem relied on a priori estimates for solutions to linear partial differential equations. On one hand, with the help of the celebrated J. Moser's iteration technique, we derived a maximal inequality for solutions to degenerate elliptic PDEs which in turn gives the sublinearity of the correctors and with that the quenched invariance principle. On the other hand, relying once again on Moser's scheme, we obtained a parabolic Harnack inequality which could be used to control the oscillations of solutions to parabolic PDEs. In particular, in the diffusive limit, we were able to bound the oscillations of the transition densities of our diffusion. This successively yielded the quenched local central limit theorem.

1.1 Main technical limitation. It is likely that in Theorem I assumption $(a.3)$, namely the fact that the realizations of the coefficients $x \mapsto \lambda^{-1}(\tau_x\omega)$ and $x \mapsto \Lambda(\tau_x\omega)$ are locally bounded, can be relaxed. Indeed, the construction of the harmonic coordinates and the proof of the maximal inequality for the correctors don't rely on it. Moreover, in the framework of Dirichlet form theory it is still possible, as we showed in Chapter 4, to construct a diffusion process associated to $L^\omega := \mathrm{div}(a^\omega(x)\nabla\cdot)$, for almost all realizations of the environment, as soon as $(\lambda^\omega)^{-1}$ and Λ^ω are locally integrable. It is well known that there is a properly exceptional set $\mathcal{N}^\omega \subset \mathbb{R}^d$ such that the associated process is uniquely determined up to the ambiguity of starting points in $\mathcal{N}^\omega$, in our situation the set of exceptional points may depend on the realization of the environment, and, for example, the measurability of $\cup_{\omega\in\Omega}\mathcal{N}^\omega \times \{\omega\} \subset \mathbb{R}^d \times \Omega$, and the proof of property (5.6) are very delicate issues, which require a very careful and technical analysis. We assumed $(a.3)$ to overcome this uncomfortable situation and remove any ambiguity due to the presence of exceptional sets in the Dirichlet form theory. One could possibly approach the non-locally bounded case in a rigorous framework by considering the abstract symmetric form

$$\mathfrak{E}(u,v) := \mathbb{E}_\mu\left[\int_{\mathbb{R}^d} \langle a(\tau_x\omega)\nabla u(x,\omega), \nabla v(x,\omega)\rangle\, dx\right],$$

with $u, v \in \mathfrak{D} := C_0^\infty(\mathbb{R}^d) \otimes L^\infty(\Omega)$. Provided that $(\mathfrak{E}, \mathfrak{D})$ is closable in $L^2(\mathbb{R}^d \times \Omega, dx \otimes \mu)$ and quasi-regular (see [Röc93]), there would exist a $dx \otimes \mu$-symmetric Hunt process associated to it, uniquely determined up to a properly exceptional set in $\mathbb{R}^d \times \Omega$. Morally, the process associated to the smallest closed extension of $(\mathfrak{E}, \mathfrak{D})$ on $L^2(\mathbb{R}^d \times \Omega, dx \otimes \mu)$ would be (X_t^ω, ω), where X_t^ω is the diffusion associated to L^ω and the second component is constant in time.

An other problem that arises dropping $(a.3)$ is that it is not a priori clear that the process associated to $L^\omega = \operatorname{div}(a^\omega(x)\nabla\cdot\,)$ is irreducible. In the case of locally bounded coefficients this is absolutely trivial, since we can compare locally the process with the Brownian motion (cf. Proposition 4.4.6). We can possibly overcome this issue proving the quenched invariance principle for the process associated to $L^{\Lambda,\omega} = 1/\Lambda^\omega(x)\operatorname{div}(a^\omega(x)\nabla\cdot\,)$. In this case, non-explosion would follow from Proposition 4.4.8 and irreducibility from the Harnack inequality (3.24). Given an invariance principle for the process associated to $L^{\Lambda,\omega}$, we can always go back to the process on the flat space through a time-change (cf. Corollary 5.6.3).

For what concerns Theorem II we stated it in a way that it would follow directly once $(a.1)$, $(a.2)$ and a quenched invariance principle are satisfied.

7.2 Open problems

2.1 Beyond the p,q-condition. For symmetric diffusions in general stationary and ergodic random environments moment conditions are a natural assumption to make in order to prove a quenched invariance principle. We have seen in Section 5.4 that $\mathbb{E}_\mu[\lambda^{-1}]$, $\mathbb{E}_\mu[\Lambda] < \infty$ is sufficient to construct the harmonic coordinates, the correctors and to prove that the limiting covariance matrix is non-degenerate (cf. Proposition 5.4.7). Additionally, in the framework of Dirichlet forms theory, we have seen in Chapter 1 and Chapter 4 that first moments are enough to make sense of a diffusion process associated to $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d)$. It is common belief that first moments are a necessary and sufficient condition for a quenched invariance principle to hold in a general stationary and ergodic random environment.

A brilliant approach to possibly obtain this optimal condition seems to come from [BM15], where the authors deal with the special case of periodic environment. The very nice observation is that it is enough to prove the quenched invariance principle for a time-change of the original process, and that it is not necessary to show sublinearity of the correctors on balls and it is sufficient to show the sublinearity of the correctors along the trajectories of the process. We also observed in Corollary 5.6.3 that the quenched invariance principle is very stable under time-change, thus, following in spirit the idea

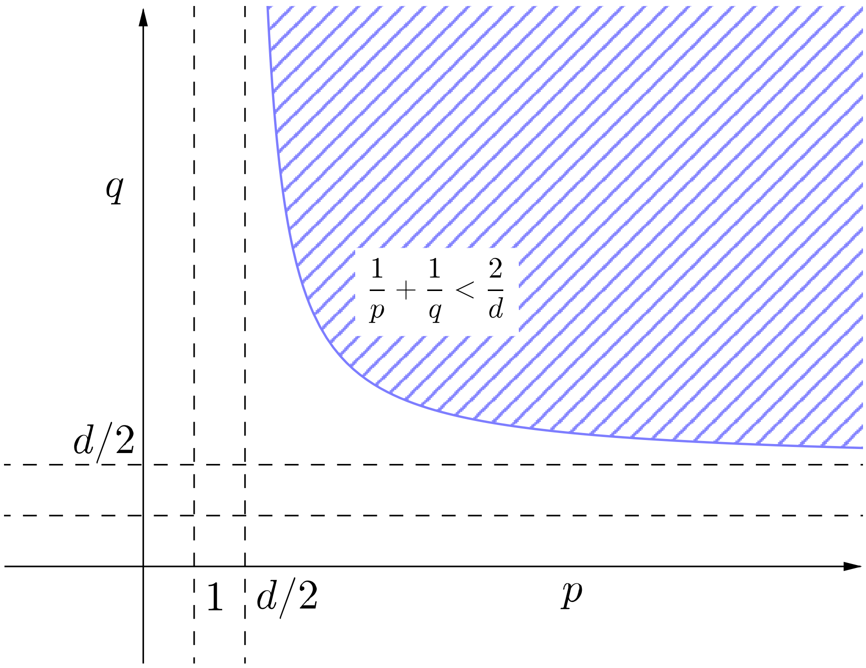

Figure 7.1: in the shaded area the p, q-condition is satisfied.

of [BM15], one could look at

$$\theta(\omega) := \frac{1}{M\Lambda^{-1}(\omega)},$$

where M is the maximal operator

$$M\Lambda^{-1}(\omega) := \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} \Lambda^{-1}(\tau_x\omega)\, dx.$$

At this point one should try to prove a quenched invariance principle for the diffusion formally associated to

$$L^{\theta,\omega}u(x) := \frac{1}{\theta^\omega(x)} \operatorname{div}(a^\omega(x)\nabla u(x)),$$

being $\theta^\omega(x) := \theta(\tau_x\omega)$ as usual. The nice thing about the weight θ^ω is that it is bounded and belongs to some Muckenhaupt's class. As it was shown in [BM15] with the help of harmonic analysis for Muckenhaupt's weights, also in this case one can obtain a weighted Sobolev inequality of the type (1.12). It remains to understand how to take advantage of it. In [BM15] it was used to get a uniform bound on the transition kernel of the process, this seems not to work as nicely in the stationary and ergodic case due to the fact that the environment process cannot be identified with the process itself as in the periodic setting. Still the Sobolev inequality for the time-changed process seems to be the right starting point to finally prove the quenched invariance principle for environments with only the first moments.

2.2 Non-symmetric diffusions. Throughout the whole manuscript the random field $\{a^\omega(x)\}_{x\in\mathbb{R}^d}$ was assumed to be a symmetric matrix. This is a very convenient assumption which allows us to work with symmetric Dirichlet forms rather than with non-symmetric Dirichlet forms.

In [FK97] a non-symmetric divergence form operator is considered and a quenched invariance principle is proved. More precisely the authors look at

$$L^\omega u(x) := \mathrm{div}((I + H(x,\omega))\nabla u(x)),$$

being $H : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ a stationary ergodic random field with smooth realizations, most important H is assumed to be antisymmetric, with zero mean and unbounded. Observe that the symmetric part is just the identity matrix, and therefore the coefficients are bounded from below. The authors prove that the correctors are sublinear employing Moser's scheme. To be able to go through the iteration they require

$$\mathbb{E}_\mu[|H|^p] < \infty, \quad p > d.$$

A more general model to consider would then be

$$L^\omega u(x) := \mathrm{div}(A(x,\omega))\nabla u(x)),$$

where A can be decomposed as $A = a + H$, being a symmetric and H antisymmetric. It is clear that the same estimates as in [FK97] would go through in this situation if we assume a to be bounded from below and from above, that $\mathbb{E}_\mu[|H|^p] < \infty$, for $p > d$ and that $\mathbb{E}_\mu[H] = 0$. The situation is less transparent if we allow the symmetric part a to be possibly degenerate and unbounded. The questions which naturally arise are how to construct the corrector field and which moment conditions are required to obtain its sublinearity? We believe that putting together the construction of the corrector in Chapter 5 and the cutoff argument for the antisymmetric part provided in [FK97] the construction of the correctors could be carried out. For what concerns the sublinearity of the correctors we claim that to let Moser's iteration working we need a to satisfy the moment conditions as in $(a.2)$, that is, $\mathbb{E}_\mu[\lambda^{-q}]$ and $\mathbb{E}_\mu[\Lambda^p]$ finite with $1/p + 1/q < 2/d$, and in addition we need $\mathbb{E}[|\sqrt{a^{-1}}H|^{2p}]$ to be finite. Observe that these conditions coincide with the one in [FK97] for the special case $a = I$.

Provided that some regularity of the coefficients is given, in the non-symmetric case, the diffusion coefficient is actually allowed to vanish in open sets, as it was proven in the periodic environment by [ME08] and further extended and generalized in [FR09], [SRP09], [PS11]. In these works the strong degeneracy of the diffusion coefficient is compensated by the drift through the Hörmander's condition; as a result the coefficients need to be smooth enough.

2.3 Time dependent environments. A natural model to consider is the one where the diffusion coefficients depend on time. In this case the random environment $(\Omega, \mathcal{G}, \mu)$ is provided with the ergodic and stationary group of shifts $\{\tau_{t,x} : (t,x) \in \mathbb{R}^{d+1}\}$. It is well known that if the coefficients are bounded from below and above, then a quenched invariance principle holds for the diffusion associated to $L^\omega = \partial_t + \mathrm{div}(a(t,x,\omega)\nabla\cdot)$;

this is essentially [LOY98]. A result in the case of degenerate coefficients is provided in [Rho07], here it is assumed that the coefficients a can be bounded from above and below by time-independent coefficients. One may ask whether a quenched invariance principle still holds in the more general case where the matrix $a(t,x,\omega) := a(\tau_{t,x}\omega)$ satisfies only $(a.2)$. One problem which arises immediately is the construction of the correctors. We may overcame this point using the techniques adopted in [FK99]. After that could possibly show that the correctors are sublinear employing similar computations as in Chapter 3, in this case for inhomogeneous parabolic PDEs rather than homogeneous. A major technical issue is to make sense of the stochastic process associated to the operator L^ω in the case that no regularity of the coefficients is assumed. Unfortunately, processes with parabolic generators are out of the framework of symmetric Dirichlet forms and therefore one must rely on the more sophisticated theory of time-dependent Dirichlet forms [Osh04].

2.4 Non-divergence form operators. To prove a quenched invariance principle for diffusions in random environment, whose generator is not in divergence form is a quite challenging problem. The major issue is that in general an invariant measure, absolutely continuous with respect to the probability measure describing the "statistics" of the environment, is not known. In 1982, Papanicolau and Varadhan [PV82] proved a quenched central limit theorem for diffusions with a symmetric generator and without drift term. More precisely, they considered

$$L^\omega u(x) := \sum_{i,j} a_{ij}(x,\omega)\partial_i\partial_j u(x), \tag{7.1}$$

where $a : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ is a stationary and ergodic random field. Furthermore, they assumed that $a_{ij} = a_{ji}$ and that there exists a constant $c > 1$ such that

$$c^{-1}|\xi|^2 \leq \langle a(x,\omega)\xi,\xi\rangle \leq c|\xi|^2, \quad \forall \xi, x \in \mathbb{R}^d,$$

for almost all realizations of the environment. In this case the corresponding diffusion is a martingale and therefore the question of proving the invariance principle can be reduced to the problem of showing that the quadratic covariation of those martingales converges. This can be achieved employing the ergodic theorem for the process of the environment seen from the particle, provided that we know that the environment process has a stationary ergodic distribution $gd\mu$ which is absolutely continuous with respect to the environment measure μ. Indeed, we would have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau_{X_t^\omega}\omega)\,dt = \mathbb{E}_\mu[ag], \quad \mathbb{P}_0^\omega\text{-a.s}, \quad \mu\text{-a.s.}$$

The main ingredient to show the existence of an ergodic invariant measure for the environment process, which is absolutely continuous with respect to the environment measure, is given by the Alexandroff-Bakelman-Pucci inequality (ABP) [Puc66]. Loosely speaking, it allows to bound the L^∞-norm of the solution u to $L^\omega u = g$ by the L^d-norm of g. Since one has the L^d-norm as an upper bound, there is room enough to plug in the weight Λ as soon as $\mathbb{E}_\mu[\Lambda^p] < \infty$ with $p > d$. This may provide a quenched invariance principle also in the case of unbounded coefficients.

In the discrete setting this problem has been treated in [Law82] for the bounded case, in [GZ12] for the degenerate and unbounded case and in [DGR] for the time-dependent case. However, at the author's best knowledge the degenerate case is still open for diffusions in random environment.

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Alberto Chiarini
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