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## INVARIANT APPROXIMATIONS FOR NONCOMMUTING GENERALIZED $(\mathcal{I}, \mathcal{J})$ -NONEXPANSIVE MAPPINGS IN $q$ -NORMED SPACES

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ABSTRACT. We obtain common fixed points as invariant approximation for noncommuting generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive mappings in  $q$ -normed space which is not necessarily locally convex. Our results improve, extend and generalize various existing known results in the literature.

### 1. Introduction and Preliminaries

Let  $X$  be a linear space. A  $q$ -norm on  $X$  is a real-valued function  $\|\cdot\|_q$  on  $X$  with  $0 < q \leq 1$ , satisfying the following conditions:

- (a)  $\|x\|_q \geq 0$  and  $\|x\|_q = 0$  iff  $x = 0$ ,
- (b)  $\|\lambda x\|_q = |\lambda|^q \|x\|_q$ ,
- (c)  $\|x + y\|_q \leq \|x\|_q + \|y\|_q$ ,

for all  $x, y \in X$  and all scalars  $\lambda$ . The pair  $(X, \|\cdot\|_q)$  is called a  $q$ -normed space. It is a metric space with  $d_q(x, y) = \|x - y\|_q$  for all  $x, y \in X$ , defining a translation invariant metric  $d_q$  on  $X$ . If  $q = 1$ , we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff

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locally bounded topological linear space is given by some  $q$ -norm,  $0 < q \leq 1$ . The spaces  $l_q$  and  $L_q[0, 1]$ ,  $0 < q \leq 1$  are  $q$ -normed space. A  $q$ -normed space is not necessarily a locally convex space. Recall that, if  $X$  is a topological linear space, then its continuous dual space  $X^*$  is said to separate the points of  $X$ , if for each  $x \neq 0$  in  $X$ , there exists an  $\mathcal{I} \in X^*$  such that  $\mathcal{I}x \neq 0$ . In this case the weak topology on  $X$  is well-defined. We mention that, if  $X$  is not locally convex, then  $X^*$  need not separates the points of  $X$ . For example, if  $X = L_q[0, 1]$ ,  $0 < q < 1$ , then  $X^* = \{0\}$  ([13], page 36 and 37). However, there are some non-locally convex spaces (such as the  $q$ -normed space  $l_q$ ,  $0 < q < 1$ ) whose dual separates the points [10].

Let  $X$  be a  $q$ -normed space and let  $C$  be a nonempty subset of  $X$ . Let  $x \in X$ . An element  $y \in C$  is called a best  $C$ -approximant to  $x \in X$  if

$$\|x - y\|_q = \text{dist}_q(x, C) = \inf\{\|x - z\|_q : z \in C\}.$$

The set of best  $C$ -approximants to  $x$  is denoted by  $P_C(x_0)$  and is defined as  $P_C(x_0) = \{y \in C : \|x - y\|_q = \text{dist}_q(x, C)\}$ . Let  $\mathcal{I}, \mathcal{J} : C \rightarrow C$  be two mappings. A mapping  $\mathcal{T} : C \rightarrow C$  is called an  $(\mathcal{I}, \mathcal{J})$ -contraction if there exists  $0 \leq k < 1$  such that  $d(\mathcal{T}x, \mathcal{T}y) \leq kd(\mathcal{I}x, \mathcal{J}y)$  for any  $x, y \in C$ . If  $k = 1$ , then  $\mathcal{T}$  is called  $(\mathcal{I}, \mathcal{J})$ -nonexpansive. Also if  $\mathcal{I} = \mathcal{J}$ , we say that  $\mathcal{T}$  is called  $\mathcal{I}$ -nonexpansive. The set of fixed points of  $\mathcal{T}$  (resp.  $\mathcal{I}$ ) is denoted by  $F(\mathcal{T})$  (resp.  $F(\mathcal{I})$ ) and closure of  $C$  by  $cl(C)$ . A point  $x \in C$  is a common fixed point of  $\mathcal{I}$  and  $\mathcal{T}$  if  $x = \mathcal{I}x = \mathcal{T}x$ . The pair  $(\mathcal{I}, \mathcal{T})$  is called (1) commuting if  $\mathcal{I}\mathcal{T}x = \mathcal{T}\mathcal{I}x$  for all  $x \in C$ ; (2)  $R$ -weakly commuting if for all  $x \in C$  there exists  $R > 0$  such that  $d(\mathcal{T}\mathcal{I}x, \mathcal{I}\mathcal{T}x) \leq Rd(\mathcal{T}x, \mathcal{I}x)$ . If  $R = 1$ , then the maps are called weakly commuting. The set  $C$  is  $p$ -starshaped with  $p \in F(\mathcal{I})$  if the segment  $[p, x] = \{(1 - k)p + kx\}$  joining  $p$  to  $x$ , is contained in  $C$  for all  $x \in C$ . Suppose  $C$  is  $p$ -starshaped with  $p \in F(\mathcal{I})$  and is both  $\mathcal{T}$ - and  $\mathcal{I}$ -invariant. Then  $\mathcal{T}$  and  $\mathcal{I}$  are called  $R$ -subweakly commuting on  $C$  [18] if there exists  $R \in (0, \infty)$  such that  $\|\mathcal{T}\mathcal{I}x - \mathcal{I}\mathcal{T}x\|_q \leq R \text{dist}_q(\mathcal{I}x, [\mathcal{T}x, p])$  for all  $x \in C$ . It is well-known that commuting maps are  $R$ -subweakly commuting maps and  $R$ -subweakly commuting maps are  $R$ -weakly commuting but not conversely in general (see [17, 18]).

A Banach space  $X$  satisfies Opial's condition if for every sequence  $\{x_n\}$  in  $X$  weakly convergent to  $x \in X$ , the inequality  $\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$  holds for all  $y \neq x$ . Every Hilbert space and the space  $l_q$  ( $1 \leq q < \infty$ ) satisfy Opial's condition. The map  $\mathcal{T} : C \rightarrow X$  is said to be demiclosed at 0 if for every sequence  $\{x_n\}$  in  $C$  such that  $\{x_n\}$  converges weakly to  $x$  and  $\{\mathcal{T}x_n\}$  converges strongly to  $0 \in X$ , then  $0 = \mathcal{T}x$ .

Existence of fixed point have been used at many places in the field of approximation theory. One of them is to prove existence of invariant approximant with help of fixed point. In 1963, Meinardus [12] employed the

Schauder fixed-point theorem to establish the existence of an invariant approximation. Further, Brosowski [2] obtained a celebrated result and generalized the Meinardus's result. Afterwards, a number of results has been proved in the direction of Brosowski [2] (see in [5, 14, 19, 22]). In a paper, Jungck and Sessa [8] further weakened the hypothesis of Sahab, Khan and Sessa [14] by replacing the weak and strong topology for relatively nonexpansive commutative maps in the setup of normed Space. Simultaneously, Al-Thagafi [1] generalized the result of Sahab, Khan and Sessa [14] and proved some results on invariant approximations for commuting mappings. In another paper, Latif [11] extended the Theorem 6 and theorem 7 of Jungck and Sessa [8] and obtained the result in  $q$ -normed space.

Recently, Shahzad [16] initiated the study of invariant approximations for noncommutating mappings in normed space. More recently, Hussain and Jungck [6] extended Shahzad's results to generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive mappings in normed space.

The purpose of this paper is to present common fixed point theorem for noncommuting generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive mappings in  $q$ -normed space which is not necessarily locally convex. As application, various invariant approximation result are also obtained. Our results extend, improve and generalize the results of Al-Thagafi [1], Dotson [3], Habiniak [4], Hussain and Jungck [6], Jungck and Sessa [8], Khan and Khan [9], Latif [11], Sahab, Khan and Sessa [14], Sahney, Singh and Whitfield [15], Shahzad [18] and Singh [19, 20, 21].

The following common fixed point result is a consequence of Theorem 3.1 of [7] (Theorem 2.1, [6]), which will be needed in the sequel.

**Theorem 1.1.** [6, 7]. *Let  $(X, d)$  be a complete metric space, and  $\mathcal{T}, \mathcal{I}, \mathcal{J}$  be self-maps of  $X$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are continuous, the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -weakly commuting such that  $\mathcal{T}(X) \subset \mathcal{I}(X) \cap \mathcal{J}(X)$ . If there exists  $h \in (0, 1)$  such that for all  $x, y \in X$ ,*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\leq \\ &\leq h \max\{d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{T}x, \mathcal{I}x), d(\mathcal{T}y, \mathcal{J}y), \frac{1}{2}[d(\mathcal{I}x, \mathcal{T}y) + d(\mathcal{T}x, \mathcal{J}y)]\}, \end{aligned}$$

*then there is a unique point  $z$  in  $X$  such that  $\mathcal{T}z = \mathcal{I}z = \mathcal{J}z = z$ .*

## 2. Main Results

We first prove common fixed theorem for noncommutative generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive mappings in  $q$ -normed spaces.

**Theorem 2.1.** *Let  $C$  be a nonempty  $p$ -starshaped subset of a  $q$ -normed space  $X$  and  $\mathcal{T}, \mathcal{I}$ , and  $\mathcal{J}$  be self-maps of  $C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ , and  $\mathcal{T}(C) \subset \mathcal{I}(C) \cap \mathcal{J}(C)$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -subweakly commuting and satisfy, for all  $x, y \in C$ ,*

$$(2.1) \quad \|\mathcal{T}x - \mathcal{T}y\|_q \leq \max\{\|\mathcal{I}x - \mathcal{J}y\|_q, \text{dist}_q(\mathcal{I}x, [\mathcal{T}x, p]), \text{dist}_q(\mathcal{J}y, [\mathcal{T}y, p]), \frac{1}{2}[\text{dist}_q(\mathcal{I}x, [\mathcal{T}y, p]) + \text{dist}_q(\mathcal{J}y, [\mathcal{T}x, p])]\},$$

then  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$  provided one of the following conditions holds:

- (i)  $C$  is complete,  $cl(\mathcal{T}(C))$  is compact and  $\mathcal{T}$  is continuous;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $C$  is weakly compact, and  $(\mathcal{I} - \mathcal{T})$  is demiclosed at 0.

*Proof.* Choose a sequence  $k_n \in (0, 1)$  with  $\{k_n\} \rightarrow 1$  as  $n \rightarrow \infty$ . Define for each  $n \geq 1$  and for all  $x \in C$ , a mapping  $\mathcal{T}_n$  by

$$\mathcal{T}_n x = (1 - k_n)p + k_n \mathcal{T}x.$$

Then, each  $\mathcal{T}_n$  is a self-mapping of  $C$  and for each  $n \geq 1$ ,  $\mathcal{T}_n(C) \subset \mathcal{I}(C) \cap \mathcal{J}(C)$  since  $\mathcal{I}$  and  $\mathcal{J}$  are affine and  $\mathcal{T}(C) \subset \mathcal{I}(C) \cap \mathcal{J}(C)$ . Now the affineness of  $\mathcal{I}$  and the  $R$ -subweak commutativity of  $\{\mathcal{T}, \mathcal{I}\}$  imply that

$$\begin{aligned} \|\mathcal{T}_n \mathcal{I}x - \mathcal{I} \mathcal{T}_n x\|_q &= (k_n)^q \|\mathcal{T} \mathcal{I}x - \mathcal{I} \mathcal{T}x\|_q \\ &\leq (k_n)^q R \text{dist}_q(\mathcal{I}x, [\mathcal{T}x, p]) \\ &\leq (k_n)^q R \|\mathcal{T}_n x - \mathcal{I}x\|_q \end{aligned}$$

for all  $x \in C$ . This implies that the pair  $\{\mathcal{T}_n, \mathcal{I}\}$  is  $(k_n)^q R$ -weakly commuting for each  $n$ . Similarly, the pair  $\{\mathcal{T}_n, \mathcal{J}\}$  is  $(k_n)^q R$ -weakly commuting for each  $n \geq 1$ . Also by (2.1),

$$\begin{aligned} \|\mathcal{T}_n x - \mathcal{T}_n y\|_q &= (k_n)^q \|\mathcal{T}x - \mathcal{T}y\|_q \\ &\leq (k_n)^q \max\{\|\mathcal{I}x - \mathcal{J}y\|_q, \text{dist}_q(\mathcal{I}x, [\mathcal{T}x, p]), \text{dist}_q(\mathcal{J}y, [\mathcal{T}y, p]), \\ &\quad \frac{1}{2}[\text{dist}_q(\mathcal{I}x, [\mathcal{T}y, p]) + \text{dist}_q(\mathcal{J}y, [\mathcal{T}x, p])]\} \\ &\leq (k_n)^q \max\{\|\mathcal{I}x - \mathcal{J}y\|_q, \|\mathcal{I}x - \mathcal{T}_n x\|_q, \|\mathcal{J}y - \mathcal{T}_n y\|_q, \\ &\quad \frac{1}{2}[\|\mathcal{I}x - \mathcal{T}_n y\|_q + \|\mathcal{J}y - \mathcal{T}_n x\|_q]\} \end{aligned}$$

for each  $x, y \in C$  and  $0 < k_n < 1$ . Thus, Theorem 1.1 guarantees that, for each  $n \geq 1$ , there exists  $x_n \in C$  such that  $x_n = \mathcal{I}x_n = \mathcal{J}x_n = \mathcal{T}_n x_n$ .

(i) As  $cl(\mathcal{T}(C))$  is compact and  $\{\mathcal{T}x_n\}$  is a sequence in it, so  $\{\mathcal{T}x_n\}$  has a subsequence  $\{\mathcal{T}x_m\}$  converging, e.g., to  $y \in cl(\mathcal{T}(C))$ .

$$x_m = \mathcal{T}_m x_m = (1 - k_m)p + k_m \mathcal{T}x_m$$

converges to  $y$ . By the continuity of  $\mathcal{T}$ ,  $\{\mathcal{T}x_m\}$  converges to  $\mathcal{T}y$ . But  $\mathcal{T}x_m$  tends to  $y$  by the assumption. So, we have  $\mathcal{T}y = y$ . Also from the continuity of  $\mathcal{I}$ , we have

$$\mathcal{I}y = \mathcal{I}(\lim x_m) = \lim \mathcal{I}x_m = \lim x_m = y, \text{ as } m \rightarrow \infty$$

i.e.,  $\mathcal{I}y = y$ . Similarly, from the continuity of  $\mathcal{J}$ , we have  $\mathcal{J}y = y$ . Hence  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$ .

(ii) Since  $C$  is weakly compact, there is a subsequence  $\{x_m\}$  of  $\{x_n\}$  converging weakly to some  $y \in C$ . But,  $\mathcal{I}$  and  $\mathcal{J}$  being affine and continuous are weakly continuous and the weak topology is Hausdorff, so we have  $\mathcal{I}y = y = \mathcal{J}y$ . And  $M$  is bounded, so  $(\mathcal{I} - \mathcal{T})(x_m) = (1 - (k_m)^{-1})(x_m - p) \rightarrow 0$  as  $m \rightarrow \infty$ . Now the demiclosedness of  $\mathcal{I} - \mathcal{T}$  at 0 guarantees that  $(\mathcal{I} - \mathcal{T})y = 0$  and hence  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$ .  $\square$

**Theorem 2.2.** *Let  $C$  be a nonempty  $p$ -starshaped subset of a  $q$ -normed space  $X$  and  $\mathcal{T}$ ,  $\mathcal{I}$ , and  $\mathcal{J}$  be self-maps of  $C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ , and  $\mathcal{T}(C) \subset \mathcal{I}(C) \cap \mathcal{J}(C)$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -subweakly commuting and  $\mathcal{T}$  is  $(\mathcal{I}, \mathcal{J})$ -nonexpansive, then  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$ , provided one of the following conditions holds:*

- (i)  $C$  is complete, and  $cl(\mathcal{T}(C))$  is compact;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $C$  is weakly compact, and  $(\mathcal{I} - \mathcal{T})$  is demiclosed at 0;
- (iii)  $C$  is weakly compact, and  $X$  is complete with separating dual  $X^*$  satisfying Opial's condition.

*Proof.* (i) and (ii) follow from Theorem 2.1. (iii) As in (ii),  $\mathcal{I}y = y = \mathcal{J}y$  and  $\|\mathcal{I}x_m - \mathcal{T}x_m\|_q \rightarrow 0$  as  $m \rightarrow \infty$ . If  $\mathcal{I}y \neq \mathcal{T}y$ , then by Opial's condition of  $X$  and  $(\mathcal{I}, \mathcal{J})$ -nonexpansiveness of  $\mathcal{T}$  we get,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|\mathcal{I}x_m - \mathcal{J}y\|_q &= \liminf_{n \rightarrow \infty} \|\mathcal{I}x_m - \mathcal{I}y\|_q \\ &< \liminf_{n \rightarrow \infty} \|\mathcal{I}x_m - \mathcal{T}y\|_q \\ &\leq \liminf_{n \rightarrow \infty} \|\mathcal{I}x_m - \mathcal{T}x_m\|_q \\ &\quad + \liminf_{n \rightarrow \infty} \|\mathcal{T}x_m - \mathcal{T}y\|_q \\ &= \liminf_{n \rightarrow \infty} \|\mathcal{T}x_m - \mathcal{T}y\|_q \\ &\leq \liminf_{n \rightarrow \infty} \|\mathcal{I}x_m - \mathcal{J}y\|_q, \end{aligned}$$

which is a contradiction. Thus  $\mathcal{I}y = \mathcal{T}y$  and hence  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$ .  $\square$

If we take  $\mathcal{J} = \mathcal{I}$  in Theorem 2.1 and Theorem 2.2, then we obtain following results:

**Corollary 2.3.** *Let  $C$  be a nonempty weakly compact  $p$ -starshaped subset of a complete  $q$ -normed space  $X$  with separating dual  $X^*$  and  $\mathcal{T}$  and  $\mathcal{I}$  be self-maps of  $C$ . Suppose that  $\mathcal{I}$  is affine and continuous with  $p \in F(\mathcal{I})$  and  $\mathcal{T}(C) \subset \mathcal{I}(C)$ . If  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0, the pair  $\{\mathcal{T}, \mathcal{I}\}$  is  $R$ -subweakly commuting and satisfies*

$$(2.2) \quad \|\mathcal{T}x - \mathcal{T}y\|_q \leq \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, \text{dist}_q(\mathcal{I}x, [\mathcal{T}x, p]), \text{dist}_q(\mathcal{I}y, [\mathcal{T}y, p]), \frac{1}{2}[\text{dist}_q(\mathcal{I}x, [\mathcal{T}y, p]) + \text{dist}_q(\mathcal{I}y, [\mathcal{T}x, p])]\},$$

for all  $x, y \in C$ , then  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ .

**Corollary 2.4.** *Let  $C$  be a nonempty weakly compact  $p$ -starshaped subset of a complete  $q$ -normed space  $X$  with separating dual  $X^*$  and  $\mathcal{T}$  and  $\mathcal{I}$  be self-maps of  $C$ . Suppose that  $\mathcal{I}$  is affine and continuous with  $p \in F(\mathcal{I})$  and  $\mathcal{T}(C) \subset \mathcal{I}(C)$ . If the pair  $\{\mathcal{T}, \mathcal{I}\}$  is  $R$ -subweakly commuting and  $\mathcal{T}$  is  $\mathcal{I}$ -nonexpansive, then  $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$  provided one of the following conditions holds:*

- (a)  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0;
- (b)  $X$  satisfies Opial's condition.

As an application of Theorem 2.1, we have following results on invariant approximations:

**Theorem 2.5.** *Let  $C$  be subset of a  $q$ -normed space  $X$  and let  $\mathcal{I}, \mathcal{J}, \mathcal{T} : X \rightarrow X$  be mappings such that  $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J})$  for some  $x_0 \in X$  and  $\mathcal{T}(\partial C \cap C) \subset C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous on  $P_C(x_0)$  with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ ,  $P_C(x_0)$  is  $p$ -starshaped and  $\mathcal{I}(P_C(x_0)) = P_C(x_0) = \mathcal{J}(P_C(x_0))$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -subweakly commuting and satisfy for all  $x \in P_C(x_0) \cup \{x_0\}$ ,*

$$(2.3) \quad \|\mathcal{T}x - \mathcal{T}y\|_q \leq \begin{cases} \|\mathcal{I}x - \mathcal{J}x_0\|_q, & \text{if } y = x_0, \\ \max\{\|\mathcal{I}x - \mathcal{J}y\|_q, \text{dist}_q([\mathcal{T}x, p], \mathcal{I}x), \text{dist}_q([\mathcal{T}y, p], \mathcal{J}y), \\ \frac{1}{2}[\text{dist}_q([\mathcal{T}y, p], \mathcal{I}x) + \text{dist}_q([\mathcal{T}x, p], \mathcal{J}y)]\}, & \text{if } y \in P_C(x_0), \end{cases}$$

then  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T}) \neq \phi$ , provided one of the following conditions holds:

- (i)  $P_C(x_0)$  is complete,  $cl(\mathcal{T}(P_C(x_0)))$  is compact and  $\mathcal{T}$  is continuous;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $P_C(x_0)$  is weakly compact, and  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0.

*Proof.* We follow Shahzad [18]. Let  $x \in P_C(x_0)$ . Then,  $x \in P_C(x_0)$  and hence  $\|x - x_0\|_q = \text{dist}_q(x_0, C)$ . Note that for any  $k \in (0, 1)$ ,

$$\|kx_0 + (1 - k)x - x_0\|_q = (1 - k)^q \|x - x_0\|_q < \text{dist}_q(x_0, C).$$

It follows that the line segment  $\{kx_0 + (1 - k)x : 0 < k < 1\}$  and the set  $C$  are disjoint. Thus  $x$  is not in the interior of  $C$  and so  $x \in \partial C \cap C$ . Since  $\mathcal{T}(\partial C \cap C) \subset C$ ,  $\mathcal{T}x$  must be in  $C$ . Also since  $\mathcal{I}x \in P_C(x_0)$ ,  $x_0 = \mathcal{T}x_0 = \mathcal{J}x_0$  and from (2.3), we have

$$\|\mathcal{T}x - x_0\|_q = \|\mathcal{T}x - \mathcal{T}x_0\|_q \leq \|\mathcal{I}x - \mathcal{J}x_0\|_q = \|\mathcal{I}x - x_0\|_q = \text{dist}_q(x_0, C).$$

Thus,  $\mathcal{T}x \in P_C(x_0)$ . Consequently,  $\mathcal{T}(P_C(x_0)) \subset P_C(x_0) = \mathcal{I}(P_C(x_0)) = \mathcal{J}(P_C(x_0))$ . Now Theorem 2.1 guarantees that  $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \neq \phi$ . This completes the proof.  $\square$

**Corollary 2.6.** *Let  $C$  be subset of a  $q$ -normed space  $X$  and let  $\mathcal{I}, \mathcal{J}, \mathcal{T} : X \rightarrow X$  be mappings such that  $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J})$  for some  $x_0 \in X$  and  $\mathcal{T}(\partial C \cap C) \subset C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous on  $P_C(x_0)$  with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ ,  $P_C(x_0)$  is  $p$ -starshaped and  $\mathcal{I}(P_C(x_0)) = P_C(x_0) = \mathcal{J}(P_C(x_0))$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -subweakly commuting and  $\mathcal{T}$  is  $(\mathcal{I}, \mathcal{J})$ -nonexpansive on  $P_C(x_0) \cup \{x_0\}$ , then  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T}) \neq \phi$ , provided one of the following conditions holds:*

- (i)  $P_C(x_0)$  is complete,  $\text{cl}(\mathcal{T}(P_C(x_0)))$  is compact and  $\mathcal{T}$  is continuous;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $P_C(x_0)$  is weakly compact,  $X$  is complete and  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0;
- (iii)  $P_C(x_0)$  is weakly compact and  $X$  is complete with separating dual  $X^*$  satisfying Opial's condition.

Following Al-Thagafi [1], we define  $D = P_C(x_0) \cap \mathcal{D}_C^{\mathcal{I}, \mathcal{J}}(x_0)$ , where  $\mathcal{D}_C^{\mathcal{I}}(x_0) = \{x \in C : \mathcal{I}x \in P_C(x_0)\}$  and  $\mathcal{D}_C^{\mathcal{I}, \mathcal{J}}(x_0) = \mathcal{D}_C^{\mathcal{I}}(x_0) \cap \mathcal{D}_C^{\mathcal{J}}(x_0)$ .

**Theorem 2.7.** *Let  $C$  be subset of a  $q$ -normed space  $X$  and  $\mathcal{I}, \mathcal{J}, \mathcal{T} : X \rightarrow X$  be mappings such that  $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J})$  for some  $x_0 \in X$  and  $\mathcal{T}(\partial C \cap C) \subset C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous on  $D$  with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ ,  $D$  is  $p$ -starshaped and  $\mathcal{I}(D) = D = \mathcal{J}(D)$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are commuting and  $\mathcal{T}$  is  $(\mathcal{I}, \mathcal{J})$ -nonexpansive on  $D \cup \{x_0\}$ , then  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T}) \neq \phi$ , provided one of the following conditions holds:*

- (i)  $D$  is complete and  $\text{cl}(\mathcal{T}(D))$  is compact;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $D$  is weakly compact, and  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0;
- (iii)  $D$  is weakly compact and  $X$  is complete with separating dual  $X^*$  satisfying Opial's condition.

*Proof.* Let  $x \in D$ , then proceeding as in the proof of Theorem 2.2, we obtain  $\mathcal{T}x \in P_C(x_0)$ . Moreover, since  $\mathcal{T}$  commutes with  $\mathcal{I}$  on  $D$  and  $\mathcal{T}$  is  $(\mathcal{I}, \mathcal{J})$ -nonexpansive,

$$\|\mathcal{I}\mathcal{T}x - x_0\|_q = \|\mathcal{T}\mathcal{I}x - \mathcal{T}x_0\|_q \leq \|\mathcal{I}^2x - \mathcal{J}x_0\|_q = \|\mathcal{I}^2x - x_0\|_q = \text{dist}_q(x_0, C),$$

and similarly,  $\|\mathcal{J}\mathcal{T}x - x_0\|_q \leq \|\mathcal{J}^2x - x_0\|_q = \text{dist}_q(x_0, C)$ , since  $\mathcal{T}$  commutes with  $\mathcal{J}$ . Thus  $\mathcal{I}\mathcal{T}x$  and  $\mathcal{J}\mathcal{T}x$  are in  $P_C(x_0)$  and so  $\mathcal{T}x \in D_C^{\mathcal{I}, \mathcal{J}}(x_0)$ . Hence  $\mathcal{T}x \in D$ . Consequently,  $\mathcal{T}(D) \subset D = \mathcal{I}(D) = \mathcal{J}(D)$ . Now Theorem 2.1 guarantees that  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T})$ .  $\square$

Let  $D_C^{R, \mathcal{I}, \mathcal{J}}(x_0) = P_C(x_0) \cap G_C^{R, \mathcal{I}}(x_0) \cap G_C^{R, \mathcal{J}}(x_0)$ , where  $G_C^{R, \mathcal{I}}(x_0) = \{x \in C : \|\mathcal{I}x - x_0\|_q \leq (2R + 1)\text{dist}_q(x_0, C)\}$ .

**Theorem 2.8.** *Let  $C$  be subset of a  $q$ -normed space  $X$  and  $\mathcal{I}, \mathcal{J}, \mathcal{T} : X \rightarrow X$  be mappings such that  $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I}) \cap F(\mathcal{J})$  for some  $x_0 \in X$  and  $\mathcal{T}(\partial C \cap C) \subset C$ . Suppose that  $\mathcal{I}$  and  $\mathcal{J}$  are affine and continuous on  $D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)$  with  $p \in F(\mathcal{I}) \cap F(\mathcal{J})$ ,  $D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)$  is  $p$ -starshaped and  $\mathcal{I}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)) = D_C^{R, \mathcal{I}, \mathcal{J}}(x_0) = \mathcal{J}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0))$ . If the pairs  $\{\mathcal{T}, \mathcal{I}\}$  and  $\{\mathcal{T}, \mathcal{J}\}$  are  $R$ -subweakly commuting and satisfy for all  $x \in D_C^{R, \mathcal{I}, \mathcal{J}}(x_0) \cup \{x_0\}$ ,*

$$\|\mathcal{T}x - \mathcal{T}y\|_q \leq \begin{cases} \|\mathcal{I}x - \mathcal{J}x_0\|_q, & \text{if } y = x_0, \\ \max\{\|\mathcal{I}x - \mathcal{J}y\|_q, \text{dist}_q([\mathcal{I}x, p], \mathcal{I}x), \text{dist}_q([\mathcal{T}y, p], \mathcal{J}y), \\ \frac{1}{2}[\text{dist}_q([\mathcal{T}y, p], \mathcal{I}x) + \text{dist}_q([\mathcal{I}x, p], \mathcal{J}y)]\}, & \\ \text{if } y \in D_C^{R, \mathcal{I}, \mathcal{J}}(x_0), \end{cases}$$

then  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T}) \neq \emptyset$ , provided one of the following conditions holds:

- (i)  $D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)$  is complete,  $\text{cl}(\mathcal{T}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)))$  is compact and  $\mathcal{T}$  is continuous;
- (ii)  $X$  is complete with separating dual  $X^*$ ,  $D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)$  is weakly compact, and  $\mathcal{I} - \mathcal{T}$  is demiclosed at 0.

*Proof.* We follow Shahzad [18]. Let  $x \in D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)$ . Then, as in the proof of Theorem 2.1,  $\mathcal{T}x \in P_C(x_0)$ . From the  $R$ -subweak commutativity of the pair  $\{\mathcal{T}, \mathcal{I}\}$  and (2.4), it follows that

$$\begin{aligned} \|\mathcal{I}\mathcal{T}x - x_0\|_q &= \|\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x + \mathcal{T}\mathcal{I}x - \mathcal{T}x_0\|_q \leq R\|\mathcal{T}x - \mathcal{I}x\|_q \\ &\quad + \|\mathcal{I}^2x - \mathcal{J}x_0\|_q \\ &= R\|\mathcal{T}x - x_0 + x_0 - \mathcal{I}x\|_q + \|\mathcal{I}^2x - x_0\|_q \\ &\leq R(\|\mathcal{T}x - \mathcal{T}x_0\|_q + \|\mathcal{I}x - x_0\|_q) + \|\mathcal{I}^2x - x_0\|_q \\ &\leq R(\|\mathcal{I}x - x_0\|_q + \|\mathcal{I}x - x_0\|_q) + \|\mathcal{I}^2x - x_0\|_q \\ &\leq (2R + 1)\text{dist}_q(x_0, C). \end{aligned}$$



Similarly,  $\|\mathcal{J}\mathcal{T}x - x_0\|_q \leq (2R + 1)dist_q(x_0, C)$ . Thus  $\mathcal{T}x \in G_C^{R, \mathcal{I}}(x_0) \cap G_C^{R, \mathcal{J}}(x_0)$ . Consequently,  $\mathcal{T}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)) \subset D_C^{R, \mathcal{I}, \mathcal{J}}(x_0) = \mathcal{I}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0)) = \mathcal{J}(D_C^{R, \mathcal{I}, \mathcal{J}}(x_0))$ . Now by Theorem 2.1 we obtain,  $P_C(x_0) \cap F(\mathcal{I}) \cap F(\mathcal{J}) \cap F(\mathcal{T}) \neq \phi$ .  $\square$

*Remark 2.9.* Theorem 2.1 - Theorem 2.8 extend the Theorem 2.2 - Theorem 2.12 of Hussain and Jungck [6] in  $q$ -normed space.

*Remark 2.10.* Theorem 2.1 extend and generalize Theorem 2.2 of Shahzad [17, 18] using generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive maps in  $q$ -normed space.

*Remark 2.11.* Theorem 2.1- Theorem 2.5 extend and generalize Theorem 2, and Theorem 2.5 - Theorem 2.8 extend and generalize Theorem 4 of Khan and Khan [9].

*Remark 2.12.* Theorem 2.1(ii), Theorem 2.2((ii) and (iii)), Corollary 2.3, Corollary 2.4((a) and (b)) extend and generalize Theorem 2.1, and Theorem 2.5(ii), Corollary 2.6((ii) and (iii)), Theorem 2.7((ii) and (iii)) and Theorem 2.8(ii) extend and generalize Theorem 2.4 of Latif [11].

*Remark 2.13.* Theorem 2.1- Theorem 2.5 extend and generalize Theorem 3 and Theorem 6, and Theorem 2.5 - Theorem 2.8 extend and generalize Theorem 7 of Jungck and Sessa [8] in  $q$ -normed space.

*Remark 2.14.* Theorem 2.5(ii), Corollary 2.6((ii) and (iii)), Theorem 2.7((ii) and (iii)) and Theorem 2.8(ii) extend and generalize Theorem 3.6(i) of Sahney, Singh and Whitfield [15] in  $q$ -normed space which is not necessarily locally convex space.

*Remark 2.15.* Theorem 2.1- Theorem 2.5 extend and generalize Theorem 2.2 of Al-Thagafi [1] in the sense that the non-commuting generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive or generalized  $\mathcal{I}$ -nonexpansive maps defined in  $q$ -normed space is used in place of relatively nonexpansive commuting maps.

*Remark 2.16.* Theorem 2.5 - Theorem 2.8 extend and generalize Theorem 3.2 of Al-Thagafi [1], Theorem 3 of Sahab, Khan and Sessa [14] and theorem of Singh [19, 20, 21] in the sense that noncommuting generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive or generalized  $\mathcal{I}$ -nonexpansive maps defined in a domain which is not necessarily locally convex space is used in place of linear  $\mathcal{I}$ -nonexpansive maps.

*Remark 2.17.* Theorems 2.8 extend and generalize Theorems 2.5 of Shahzad [18] for generalized  $(\mathcal{I}, \mathcal{J})$ -nonexpansive mappings defined in a domain which is not necessarily locally convex space.

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