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Invariant Circles for the Piecewise Linear Standard Map

Shaun Bullett

School of Mathematical Sciences, Queen Mary College, University of London, Mile End Road, London E1 4NS, United Kingdom

Abstract. We investigate invariant circles for a one-parameter family of piecewise linear twist homeomorphisms of the annulus. We show that invariant circles of all types and rotation numbers occur and we classify them into families. We compute parameter ranges in which there are no invariant circles.

1. Introduction

We investigate invariant circles for the one-parameter family $h_k (k \in \mathbb{R})$ of homeomorphisms of the annulus $S^1 \times \mathbb{R}$ defined by

$$h_k(x, y) = (x + y + kg(x), y + kg(x)), \qquad (*)$$

where $g: S^1 \to \mathbb{R}$ is the piecewise linear function $g(x) = |x - 1/2| - 1/4$, and S^1 is parametrised as $\mathbb{R}/\mathbb{Z}$.

We call h_k the *piecewise linear standard map* since it is obtained from the *standard map*

$$s_k(x, y) = \left(x + y + \frac{k}{4}\cos 2\pi x,\ y + \frac{k}{4}\cos 2\pi x\right)$$

by replacing $\cos 2\pi x$ by its crudest piecewise linear approximation.

For any continuous function g the homeomorphism h_k defined by $(*)$ satisfies the *twist condition*, that is to say, if $\tilde{h}_k$ denotes the lift of h_k to the universal cover $\mathbb{R} \times \mathbb{R}$ of the annulus and p_1 denotes the projection of $\mathbb{R} \times \mathbb{R}$ onto its first factor, then

$$p_1 \tilde{h}_k(x, y_2) > p_1 \tilde{h}_k(x, y_1) \quad \text{whenever } y_2 > y_1 .$$

Furthermore such an h_k preserves area, and if

$$\int_0^1 g(x)\,dx = 0$$

(which it does for our piecewise linear function g) then h_k has *zero flux*, that is it transports a net area of zero across $S^1 \times \{0\}$ (or equivalently across any other circle homotopic to it). The piecewise linear standard map is the simplest possible area-preserving piecewise linear twist homeomorphism of zero flux.

Area-preserving twist homeomorphisms h of the annulus arise in the study of dynamical systems [3]. A (rotational) *invariant circle* for h is a circle C embedded in $S^1 \times \mathbb{R}$ which wraps once around the S^1 factor and which satisfies $h(C)=C$. If h preserves orientation any such invariant circle is a barrier to motion under h; zero flux is an obvious necessary condition for the existence of such a circle.

Kolmogorov, Arnold, and Moser [1, 10] proved some remarkable results about the existence of invariant circles for C^∞ area-preserving twist homeomorphisms of zero flux. They showed that given any "sufficiently irrational" real number v, any such C^∞ homeomorphism h close to the simple shear $(x, y) \to (x+y, y)$ has an invariant circle of rotation number v. Rüssmann [11] and Herman [6] have extended these results to $C^{3+\varepsilon}$ in place of C^∞ and Herman [6] has shown how KAM theory fails for $C^{3-\varepsilon}$ homeomorphisms. Numerical investigation by Greene [4], Percival [8], and others has mainly centred on the standard map and questions of for what range of $k \in \mathbb{R}$ there exist invariant circles, and of what rotation numbers.

A priori there is no reason why the piecewise linear standard map h_k should have invariant circles for any k other than 0. Indeed Herman [6] has given an example of a sequence of piecewise linear twist homeomorphisms tending to the simple shear, each of which has no invariant circles. However the theory of Aubry [2], Mather [9], and Katok [7] shows that for any twist homeomorphism of zero flux there exist periodic orbits of all possible rational rotation numbers and invariant Cantor sets of all possible irrational rotation numbers. With such a simple map as the piecewise linear standard map h_k it is perhaps not surprising that there should be many values of k for which these invariant sets extend to invariant circles. We show that this is indeed so. We prove the existence of invariant circles of both types (conjugate and non-conjugate to rotations) for all rotation numbers, and we classify them into families. We also compute "windows" in the parameter range where there are no invariant circles whatever. Examples of invariant circles conjugate to rational rotations were first found by Wojtkowski [12, 13], who also carried out a detailed investigation of the mixing properties of h_k.

Our results have features in common with numerical results obtained by Hénon and Wisdom [5] for the *oval billiard*, a system which corresponds to a piecewise smooth twist homeomorphism of the annulus. For instance, with certain notable exceptions (see Sect. 6), our invariant circles contain a *cancellation orbit*, that is to say an orbit which hits both the lines where the derivative of the homeomorphism has discontinuities, and we show that cancellation orbits of any given class extend to invariant circles for a Cantor set of parameter values.

2. Summary of Results

If C is an invariant circle for a homeomorphism h of the annulus, the rotation number of $h|_C$ is defined to be

$$v = \lim_{n \to \pm\infty} (p_1 \tilde{h}^n(\tilde{x}, y) - \tilde{x})/n,$$

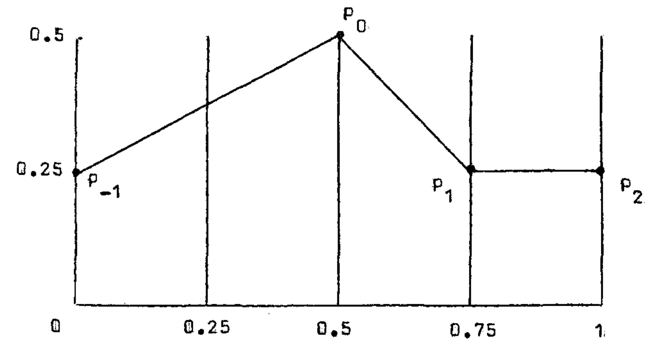

Fig. 1. $k=1$; periodic circle with $v=1/3$

where (x, y) is any point of C and $(\tilde{x}, y)$ is any lift of (x, y) to $\mathcal{R} \times \mathcal{R}$. It is a standard result that this number is independent of the choice of $(\tilde{x}, y)$ and of whether the limit is taken as $n \to \infty$ or $n \to -\infty$.

Invariant circles with rational rotation number can be of two types [7], pointwise *periodic* ($h|_C$ conjugate to a rotation of a circle), and *non-periodic* ($h|_C$ not conjugate to a rotation). The second type must still contain at least one periodic orbit (see [7]). In Sect. 3 we prove, for both types

Proposition 1. *For the piecewise linear standard map h_k any invariant circle which has rational rotation number must contain a cancellation orbit, that is an orbit meeting both lines $x=0$ and $x=1/2$, where the derivative of h_k is discontinuous. The circle is periodic if and only if the cancellation orbit is periodic.*

We can now label each *periodic* invariant circle by the number of iterations of h_k required to get from $x=0$ to $x=1/2$ and the number required to get from $x=1/2$ to $x=1$. For the circle illustrated in Fig. 1 these numbers are 1 and 2.

In Sect. 4 we investigate in detail invariant circles containing a cancellation orbit taking 1 step from $x=0$ to $x=1/2$. By a trivial calculation these are the orbits passing through $x=1/2$, $y=1/2$. We prove

Proposition 2. (i) *For each rational $0<v<1/2$ there exists a value k_v of k for which h_k has a periodic invariant circle of rotation number v, through* $(1/2, 1/2)$, *and a value k'_v of k for which it has a non-periodic circle of rotation number v, through* $(1/2, 1/2)$.

(ii) *For each irrational $0<v<1/2$ there exists a value k_v of k for which h_k has an invariant circle of rotation number v, through* $(1/2, 1/2)$.

We conjecture that Proposition 2 can be strengthened to say that the values k_v and k'_v are unique and that for the values k_v of k the restriction of h_k to the invariant circle is conjugate to a rotation. The latter is true for rational v; for irrational v it is equivalent to the cancellation orbit being dense in the circle.

The correspondence between v and k_v and k'_v is illustrated in Fig. 2. The graph is a Cantor set, with a gap at each rational v, the gap having left-hand end point $k=k'_v$ and right-hand end point $k=k_v$. For k in one of these gaps the orbit of $(1/2, 1/2)$ is no longer ordered and therefore cannot extend to an invariant circle. The k_v for irrational v correspond to accumulation points of the rational ones. Hénon and

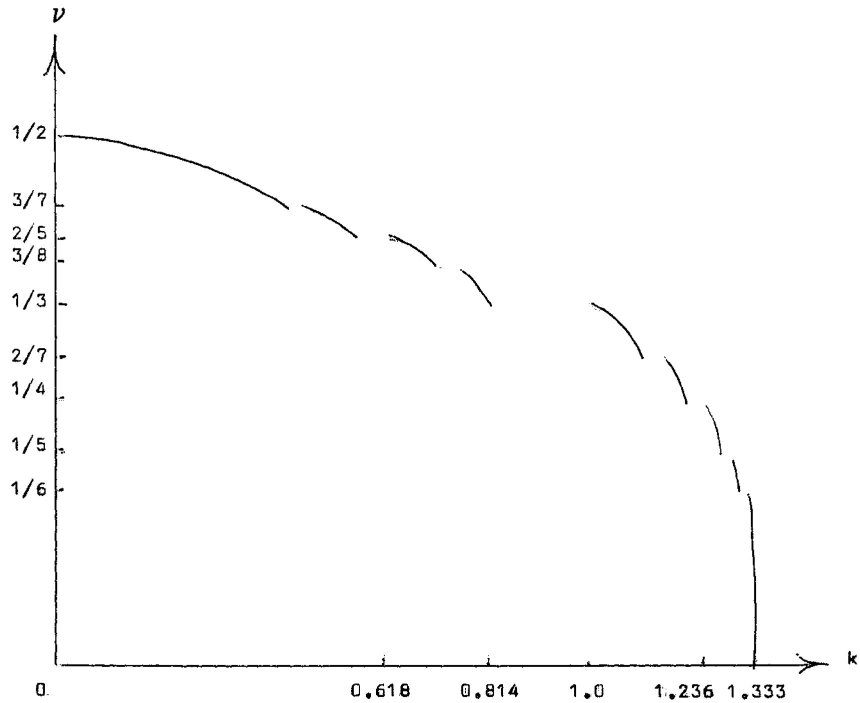

Fig. 2. Invariant circles through (1/2, 1/2); the graph is a Cantor set but only the main gaps are shown

Wisdom [5] obtain a similar Cantor set for a fixed type of cancellation orbit to extend to an invariant circle, in the oval billiard problem. In their piecewise smooth but not piecewise linear situation there is presumably an excluded interval of rotation numbers around each rational v.

In Sect. 5 we examine some other families of invariant circles containing cancellation orbits with a specified number of steps (and circuits of S^1) from $x=0$ to $x=1/2$ or vice versa. Some of these families are illustrated in Fig. 3. By Proposition 1 every circle with rational v lies in such a family, and in two such families if it is periodic. We conjecture that every circle with irrational v occurs at an accumulation point of the (k, v) diagram of circles with rational v.

In Sect. 6 we consider the special case $k=4/3$ and we prove

Proposition 3. (i) *For each rational* $0<v\leqq 1/2$, $h_{4/3}$ *has two invariant circles of rotation number* v, *both non-periodic, but intersecting in a periodic orbit.*

(ii) *For each irrational* $0<v<1/2$, $h_{4/3}$ *has an invariant circle of rotation number* v. *This circle does not contain a cancellation orbit and* $h_{4/3}$ *is not conjugate to a rotation on it.*

(iii) *The island chains contained by intersecting pairs of rational invariant circles* (i) *together occupy full measure on the annulus.*

The orbits for $k=4/3$ are illustrated in Fig. 4. The picture is remarkable in that there is an apparent regularity for $k=4/3$. Every point on the annulus has a well-

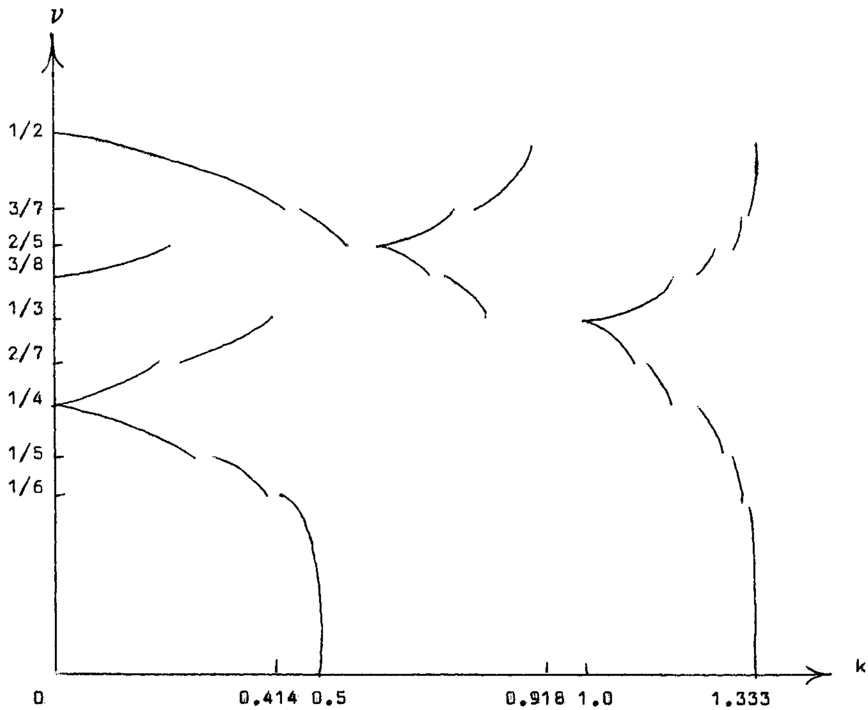

Fig. 3. The families of invariant circles discussed in Sects. 4 and 5

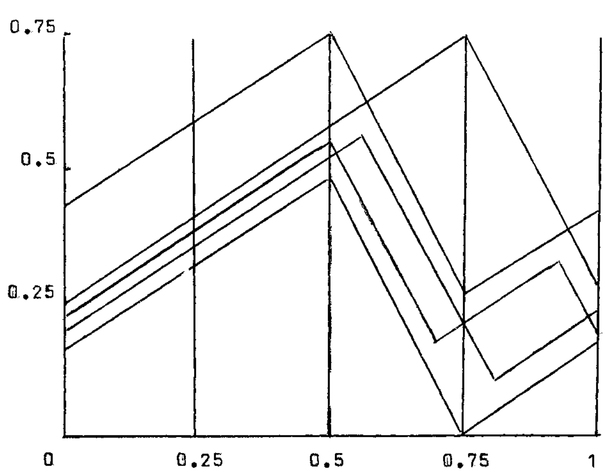

Fig. 4. $k=4/3$; some invariant circles

defined rotation number since it is either in an island or on an irrational circle. In this respect $k=4/3$ resembles $k=0$ but with the rational circles blown up from zero measure to full measure.

In Sect. 7 we code orbits by bi-infinite words, listing whether successive points of the orbit fall right or left of $x=1/2$ and we use criteria based on the corresponding tangent map to prove

Proposition 4. *There are no invariant circles for h_k when $k>4/3$ or when $0.918<k<1$.*

For $k=0.918$ (or more precisely k the root of $2k^3+4k^2-k-4=0$) and for $k=1$ there are invariant circles, so these bounds are best possible. We conjecture that there are similar "windows" arbitrarily close to $k=0$.

Finally in Sect. 8 we show how to generalise our results to the case where the two intervals on S^1 on which g is linear are allowed to differ in length, and we discuss the possibility of generalisation to maps with several linear segments.

3. Cancellation Orbits in Invariant Circles

In this section we prove Proposition 1.

We first consider the case when the invariant circle C is (pointwise) periodic of rotation number p/q and period q. Let A and B be the points where C cuts $x=0$ and $x=1/2$. Suppose, for a contradiction, that the orbits of A and B are disjoint. Any invariant circle projects one to one onto S^1 [6] and so h_k preserves the S^1 order on C. Hence between any two adjacent points on the orbit of A there is a point on the orbit of B and vice versa. Indeed if these $2q$ alternating points of the two orbits are joined by straight line segments we obtain a piecewise linear circle C' which is also periodic. Let A' and A'' be the points on the orbit of A closest to B on either side of B. The straight line $A'A''$ is bent by h_k since $A'A''$ crosses $x=1/2$. As h_k preserves area we deduce that the triangles A', B, A'' and $h_k(A')$, $h_k(B)$, $h_k(A'')$ have different areas. However h_k^{q-1} takes the second triangle to the first (without bending) and preserves area, so we have a contradiction.

We next consider the case where C is a non-periodic invariant circle of rotation number p/q. Again let A and B be the points where C cuts $x=0$ and $x=1/2$ and suppose their orbits are disjoint, for a contradiction. The possibility that A is periodic is ruled out by the same argument as above, this time applied to the area bounded by the straight line $A'A''$ and the part of C between A' and A''. If A is non-periodic the sequence $A, h_k^q(A), \dots, h_k^{nq}(A), \dots$ tends to a periodic point Q as $n\to\infty$ [7]. In a sufficiently small neighbourhood on either side of Q the map h_k^q is linear (possibly a different linear map on each side if the orbit of Q meets $x=1/2$). It follows that for large n the points $h_k^{nq}(A)$ are all on the contracting eigenvector of h_k^q at Q and indeed that C contains a straight line segment from $h_k^{nq}(A)$ to Q. Similarly if Q' denotes the limit of $h_k^{-nq}(A)$ we see that C contains a straight line segment from $h_k^{-nq}(A)$ to Q' for large n; in particular C contains the straight line segment I from $h_k^{-(m+2)q}(A)$ to $h_k^{-mq}(A)$ for some fixed large m. But $h_k^{(m+1)q}(I)$ meets $x=0$ at A, so $h_k^{nq}(I)$ is bent for all large n (by hypothesis B is not on the same orbit as A so the bending cannot be undone). However, for large n, $h_k^{nq}(I)$ is close to Q and therefore straight since it is contained in C. This gives our contradiction.

If an invariant circle is periodic, then trivially the cancellation orbit on it is periodic. For the converse, suppose that C is non-periodic but the cancellation orbit on it is periodic. Let P be any non-periodic point on C, with limit Q say under forward iteration of h_k^q. As argued above, C will contain a straight line segment I from $h_k^{nq}(P)$ to Q for n large, such that I expands under h_k^{-q}. This expansion cannot continue indefinitely so eventually some $h_k^{-m}(I)$ must meet $x=0$ or $x=1/2$ (at A or B). This contradicts the hypothesis that the orbit of A and B is periodic.

Finally we note that the proposition is not true for all invariant circles of irrational rotation number. Counterexamples for $k=4/3$ will be given in Sect. 6.

4. Invariant Circles Through (1/2, 1/2)

In this section we prove Proposition 2, by induction on the length of a continued fraction expansion of the rotation number ν. Throughout this section P_0 will denote (1/2, 1/2) and P_n will denote $h_k^n(P_0)$. Note that P_{-1} lies on $x=0$; it is the point $(0, 1/2-k/4)$.

Lemma 4.1. *For each integer $n \geqq 2$ there is a unique value $k_{1/n}$ of k such that the orbit of P_0 under h_k extends to a periodic invariant circle of rotation number $1/n$.*

Proof of 4.1. When $k=4/3$ the orbit of P_0 is homoclinic, of rotation number 0, as illustrated in Fig. 5. As k is reduced to 1 we pass through values of k where each P_{n-1} in turn (n decreasing) lies on $x=1$. We claim that when P_{n-1} lies on $x=1$ then $P_{n-1}=P_{-1}$. This follows from the following symmetry argument:

$$h_k = S_1 S_2 \quad \text{where} \quad S_1:(x,y)\to(y-x+1/2,\, y)$$
$$\text{and} \quad S_2:(x,y)\to(1/2-x,\, y+g(x))$$

with $S_1^2=S_2^2=$ identity, and furthermore $S_1 h_k S_1 = h_k^{-1}$ and $S_2 h_k S_2 = h_k^{-1}$ (that is S_1 and S_2 are both involutions sending forward orbits to backward ones). This is true for any $g(x)$ satisfying $g(x)=-g(1/2-x)$ and not just our piecewise linear function g.

The point P_0 lies on the S_1-symmetry line $y=2x-1/2$. Hence $P_{-1}=S_2P_0$. Given odd $n=2m+1$ we choose k (uniquely) such that P_m lies on the S_2-symmetry line $x=3/4$. Then $P_{2m}=S_2P_0=P_{-1}$. Given even $n=2m$ we choose k (again uniquely) such that P_m lies on the S_1-symmetry line $y=2x-3/2$. Then $P_{2m}=S_1P_0$ and so $P_{2m-1}=S_2P_0=P_{-1}$.

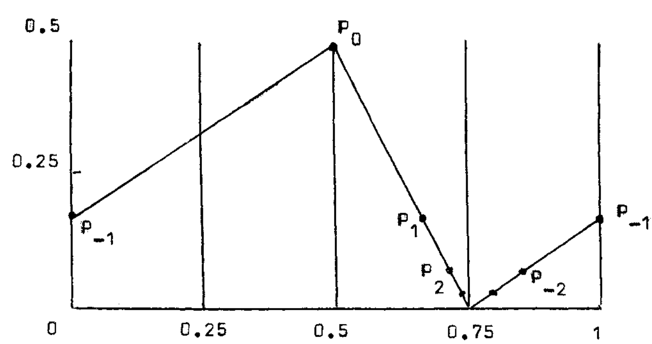

Fig. 5. $k=4/3$; the orbit of (1/2, 1/2)

Examples. It is an easy computation that while P_{n-1} has x-coordinate $\leqq 1$ we have the following values for P_n:

$$P_0 = (1/2, 1/2),$$
$$P_1 = (1 - k/4, 1/2 - k/4),$$
$$P_2 = (3/2 - k/4 - k^2/4, 1/2 - k^2/4).$$

It follows that $k_{1/3} = 1$, $k_{1/4} = \sqrt{5} - 1$, and $k_{1/5} = (\sqrt{13} - 1)/2$.

Numerical experiments suggest that there exist arbitrarily close values of k above $k_{1/n}$ for which the orbit of P_0 extends to an invariant circle, but for any k just below $k_{1/n}$ the orbit of P_0 is disordered. However if we further decrease k we reach a value $k'_{1/n}$ where P_0 again lies on an invariant circle, surprisingly of the same rotation number $1/n$ but non-periodic. In the next two lemmas we shall explain this phenomenon by showing that throughout the range $k_{1/(n-1)} \leqq k \leqq k_{1/n}$ there exists a well-behaved periodic orbit $\{Q_i\}$ of rotation number $1/n$, and that at $k'_{1/n}$ the orbit of P_0 is homoclinic to this periodic orbit. The existence of a periodic orbit of any given rational rotation number is of course guaranteed by the famous results of Poincaré and Birkhoff for twist maps; however we demand some special properties of these orbits for later use in the inductive proof of Proposition 2.

Lemma 4.2. *For each integer $n > 2$ and k in the range $k_{1/(n-1)} \leqq k \leqq k_{1/n}$ there exists a unique point Q_0 on the line $x = 1/4$ satisfying*

(I) *Q_0 is periodic of period n and rotation number $1/n$, and*

(II) *$Q_1 = h_k(Q_0), \dots, Q_{n-1} = h_k^{n-1}(Q_0)$ are all in $1/2 \leqq x \leqq 1$ and in that order (with respect to x-coordinates).*

Proof of 4.2. For $k = k_{1/n}$, Q_0 is the mid-point of the segment $P_{-1}P_0$ of periodic circle. We shall show that that as k is decreased from $k_{1/n}$ there continues to be a periodic point on the S_2-symmetry line $x = 1/4$, though its orbit will no longer extend to an invariant circle. Our construction of Q_0 will be continuous in k.

We let L_1 and L'_1 denote the S_1-symmetry lines $y = 2(x - 1/4)$ and $y = 2(x - 3/4)$ and L_2 and L'_2 denote the S_2-symmetry lines $x = 1/4$ and $x = 3/4$ (see Fig. 6).

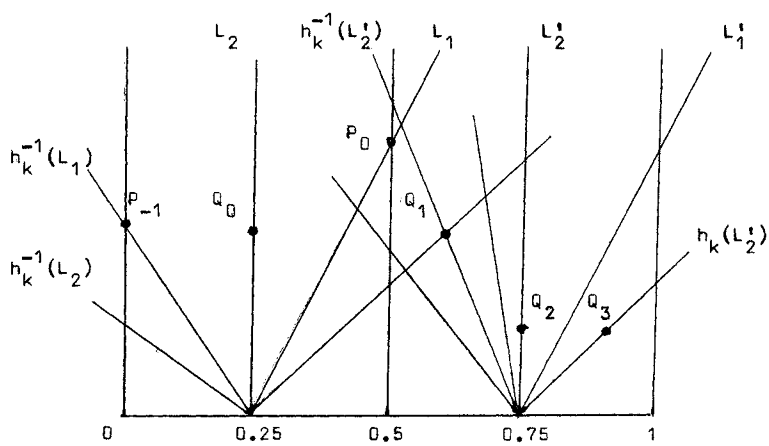

Fig. 6

Case (i): n Even. Let $n=2m$, let Q_1 be the intersection of $h_k(L_2)$ and $h_k^{1-m}(L_2')$, and let $Q_0=h_k^{-1}(Q_1)$. It is immediate from S_2-symmetry that Q_0 has period n and rotation number $1/n$. See Fig. 6 for an illustration in the case $n=4$. Since in the range of k in question P_0 lies between $h_k^{-m}(L_1')$ and $h_k^{1-m}(L_2')$, Q_1 lies to the right of $x=1/2$, and since h_k rotates L_2' clockwise about (3/4, 0) the points $\{Q_i\}$ lie in the order claimed in (II). Finally observe that Q_0 is unique since S_2-symmetry and the ordering of $Q_1, \ldots, Q_{n-1}$ require Q_1 to be the point constructed above.

Case (ii): n Odd. Let $n=2m+1$. Set Q_1 to be the point where $h_k(L_2)$ meets $h_k^{-m}(L_1')$. Then Q_{m+1} lies on L_1', so $Q_{m+1}=S_2(Q_m)$ and this implies by S_2-symmetry that Q_0 has period $n=2m+1$ and rotation number $1/n$. The remainder of the proof is analogous to that in the even case.

Lemma 4.3. *For each integer $n>2$ there is a value $k_{1/n}'$ of k such that the orbit of P_0 extends to a non-periodic invariant circle of rotation number $1/n$.*

Proof of 4.3. Let Q_0 be the point given by 4.2 for k in the range $k_{1/(n-1)} \leqq k \leqq k_{1/n}$. When $k=k_{1/n}$, Q_0 is the mid-point of $P_{n-1}P_0$ and Q_{n-1} that of $P_{n-2}P_{n-1}$, so $P_{-1}(=P_{n-1})$ is below the straight line $Q_{n-1}Q_0$. We shall show that at $k=k_{1/(n-1)}$ the point P_{-1} is above $Q_{n-1}Q_0$ and deduce there is a value $k_{1/n}'$, where P_{-1} lies on $Q_{n-1}Q_0$.

If n is even, say $n=2m$, let V_0 denote the intersection of $h_1^{-m}(L_2')$ with $x=1/2$, and let W_0 denote the intersection of $h_k^{-m-1}(L_1')$ with $x=1/2$. Thus $V_0=P_0$ when $k=k_{1/(n-1)}$, $W_0=P_0$ when $k=k_{1/n}$, and P_0 is above W_0 and below V_0 when k is between these values. V_{n-2} and W_{n-1} are on $x=1$ (being S_2-symmetric with V_0 and W_0). It follows that, for k between $k_{1/(n-1)}$ and $k_{1/n}$, P_{n-2} lies to the left of $x=1$ and P_{n-1} to the right. The same is true for n odd by a similar argument. As a consequence, in this range of k the image under h_k^n of the "bent" line $Q_{n-1}P_{-1}Q_0$ is the "bent" line $Q_{n-1}P_{n-1}Q_0$, still with a single bend. Now consider the circle made up of $Q_{n-1}P_{-1}Q_0$ and its images under the first $n-1$ iterates of h_k. The map h_k sends this circle to itself, with the exception of $Q_{n-2}P_{n-2}Q_{n-1}$, which is sent to $Q_{n-1}P_{n-1}Q_0$ rather than to $Q_{n-1}P_{-1}Q_0$. As h_k transfers a net area of zero across any circle it follows that $P_{-1}P_{n-1}$ is parallel to $Q_{n-1}Q_0$. At $k=k_{1/(n-1)}$, $P_{n-1}=P_0$ and $P_{-1}P_{n-1}$ forms part of an invariant circle of rotation number $1/(n-1)$, but as Q_0 has rotation number $1/n$, and h_k is a twist map, it follows that $P_{-1}P_{n-1}$ is now above Q_0 and hence P_{-1} is now above $Q_{n-1}Q_0$. Hence at some value of k between $k_{1/(n-1)}$ and $k_{1/n}$, P_{-1} and P_{n-1} both lie on $Q_{n-1}Q_0$; furthermore P_{n-1} then lies between Q_{n-1} and Q_0 since $Q_{n-1}P_{n-1}Q_0$ is the homeomorphic image of $Q_{n-1}P_{-1}Q_0$ under h_k^n. The circle made up of the straight line $Q_{n-1}Q_0$ and its images is then invariant under h_k; it is non-periodic but contains the periodic orbit $\{Q_i\}$ and so it has rotation number $1/n$. See Fig. 7 for an example with $\nu=1/3$; this occurs for k the root of $k^3+4k^2+k-4=0$, that is k approximately 0.814.

Remark. It is not hard to sharpen the proof of 4.3 to obtain uniqueness of $k_{1/n}'$ for each n; however our techniques for this do not easily generalise to k_ν' for ν with longer continued fraction expansions, so we omit details here.

Lemma 4.4. *For each rational ν of the form $1/(n-1/n')$, $n>2$, $n'>1$, there exists a value k_ν of k such that the orbit of P_0 extends to a periodic invariant circle of rotation number ν.*

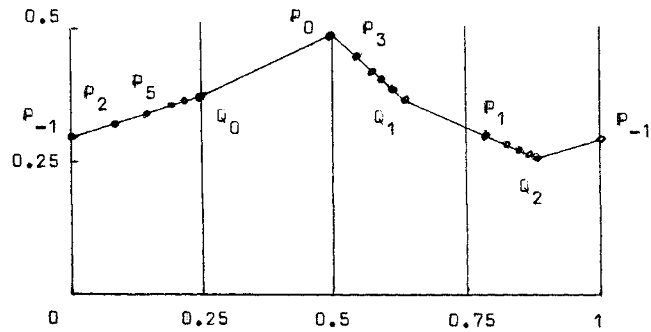

Fig. 7. $k=0.814$; non-periodic circle with $\nu=1/3$

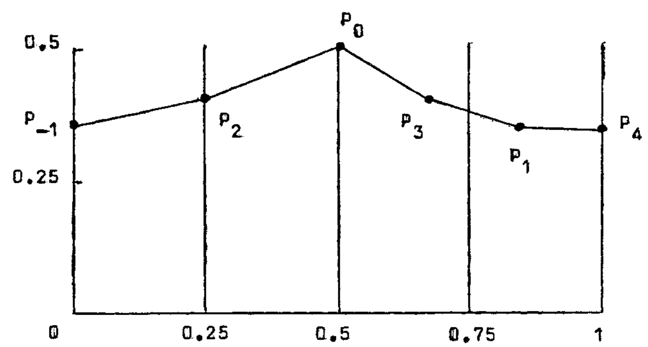

Fig. 8. $k=(\sqrt{5}-1)/2$; periodic circle with $\nu=2/5$

Proof of 4.4. We restrict attention to k in the range $k_{1/(n-1)} \leqq k \leqq k'_{1/n}$ and let Q_0 denote the point on L_2 $(x=1/4)$ of period n constructed in 4.2.

Case (i): n' Even. Let $n'=2m'$. Consider the images under $h_k^n, h_k^{2n}, \ldots, h_k^{m'n}, \ldots$ of a segment of L_2 immediately above Q_0. When $k=k'_{1/n}$ the invariance of the direction Q_0P_0 and the twist condition ensure that these images form an ordered "fan" of lines through Q_0 between L_2 and Q_0P_0. We claim that as k is decreased from $k'_{1/n}$ to $k_{1/(n-1)}$ this fan turns forward so that each line in it in turn passes through P_0. It suffices to show that for $k=k_{1/(n-1)}$ the image $h_k^n(L_2)$ of L_2 crosses $x=1/2$ below P_0. But at $k_{1/(n-1)}$ the line segment $P_{-1}P_0$ and its images under h_k form an invariant circle of period $n-1$, and thus h_k^n maps the mid-point of $P_{-1}P_0$ (on L_2) to the mid-point of P_0P_1; hence $h_k^n(L_2)$ crosses $x=1/2$ below P_0. By continuity we deduce that there is an intervening value $k_{1/(n-1/n')}$ of k, where $h_k^{m'n}(L_2)$ passes through P_0. At this value of k, $P_{-m'n}$ lies on L_2 and hence by S_2-symmetry $P_{-2m'n}=S_2P_0=P_{-1}$. Thus P_0 has period $2m'n-1=n'n-1$. Note also that for this value of k the points on the orbit of P_0 which lie between $x=1/4$ and $x=1/2$ are $P_{-m'n}, P_{-(m'-1)n}, \ldots, P_0$ in that order (with respect to x-coordinates) since h_k^{-n} acts linearly on Q_0P_0, bringing it back to the vertical L_2 after m' iterations. Thus the orbit of P_0 can be joined by straight line segments to give an invariant circle of rotation number $1/(n-1/n')$.

Figure 8 illustrates an example with $\nu=2/5=1/(3-1/2)$. This occurs for $k=(\sqrt{5}-1)/2$.

Case (ii) a: n' Odd, n Even. Let $n'=2m'+1$ and $n=2m$. The orbit of P_0 we are seeking is to have $n'-1$ points in $0<x<1/2$ and $nn'-n'-2$ points in $1/2<x<1$. As $nn'-n'-2$ is odd and the orbit is to be symmetric it will have a point on L_2' $(x=3/4)$, but as $n'-1$ is even it will have no point on L_2 $(x=1/4)$. The point Q_m also lies on L_2' (by symmetry) and our strategy is to seek a value of k such that the image under $h_k^{-n'm+1}$ of a segment of L_2' immediately above Q_m is a line from Q_1 cutting $x=1/2$ at $P_0=(1/2, 1/2)$. To show that such a value of k exists we consider a "fan" at Q_m consisting of a segment of L_2' and its images under h_k^{-n}, $h_k^{-2n}, \ldots, h_k^{-m'n}$; the image of this fan under h_k^{-m+1} is a fan at Q_1, and we examine the intersection of the latter fan with $x=1/2$. By a similar argument to that in Case (i) one can prove that as k is decreased from $k'_{1/n}$ the fan turns anticlockwise and thus that there exists a value of k at which $h_k^{-n'm+1}(L_2')$ meets $x=1/2$ at P_0. For this value of k, $P_{n'm-1}$ lies on L_2' and thus by S_2-symmetry $P_{2n'm-2}=P_{-1}$ so that P_0 has period $2n'm-1=nn'-1$. Joining the orbit by straight line segments gives the required invariant circle of rotation number $1/(n-1/n')$.

Case (ii) b: n' Odd, n Odd. Let $n'=2m'+1$ and $n=2m+1$. Our difficulty now is that no point of the sought orbit of P_0 is to lie on L_2 or L_2' and no point of the orbit of Q_0 lies on L_2'. However Q_{m+1} lies on the S_1-symmetry line L_1' $(y=2(x-3/4))$ since $Q_1=S_1(Q_0)$, and also the orbit of P_0 we seek is to have $P_{(nn'-1)/2}$ on L_1'. We find a value of k for which $P_{(nn'-1)/2}$ lies on L_1' as follows. Consider a "fan" at Q_{m+1} made up of a segment of L_1' and its images under $h_k^{-n}, \ldots, h_k^{-m'n}, \ldots$; we can move this fan to Q_1 by applying h_k^{-m} and consider the intersection of this new fan with $x=1/2$. As in the previous cases we can find a value of k such that $h_k^{-m'n}$ passes through P_0, and hence $P_{m+m'n}$ lies on L_1'. Then by symmetry $P_{2(m+m'n)}=P_0$ and so P_0 has period $2m+2m'n=nn'-1$. Joining up the points on this orbit gives an invariant circle of period $nn'-1$ and rotation number $1/(n-1/n')$.

Remark. We conjecture the k_ν given by 4.4 to be unique but we have not proved this to be the case. It would not be sufficient to prove that the "fans" discussed in the proof turn monotonically with k, since the centres of these fans also vary in position as k changes. In the rest of this section we make the following convention. Wherever we discuss an interval of values of k, such as

$$k_{1/(n-1/(n'-1))} \leqq k \leqq k_{1/(n-1/n')}$$

we shall assume the interval to be minimal, that is no intervening point could be given the same label k_ν as one of the end-points. For the values k_ν of k at the ends of such an interval, P_0 lies on a certain "spoke" of a fan at Q_1 (by the proof of 4.4). The effect of our convention is to ensure that for values of k within the interval P_0 lies between the spokes it meets at the ends of the interval.

Lemma 4.5. *For each rational of the form $1/(n-1/n')$, $n>2$, $n'>1$, and each k in the range $k_{1/(n-1/(n'-1))} \leqq k \leqq k_{1/(n-1/n')}$ there exists a unique point Q_0' on either $x=1/4$ or $x=3/4$ (depending on the parity of n and n') satisfying.*

(I) *Q_0' is periodic of period $nn'-1$ and rotation number $1/(n-1/n')$, and*

(II) *the points of the orbit of Q_0' are arranged in the same order, and lie on the same sides of $x=1/2$, as the mid-point orbit of the invariant circle for $k_{1/(n-1/n')}$.*

Proof of 4.5. For $k=k_{1/(n-1/n')}$ the orbit of P_0 either misses $x=1/4$ or $x=3/4$ since either $n'-1$ or $nn'-n'-2$ is even. We take $\{Q'_j\}$ to be the corresponding mid-point orbit; by symmetry this hits $x=1/4$ or $x=3/4$. We must show that $\{Q'_j\}$ continues to exist while k is decreased to $k_{1/(n-1/(n'-1))}$.

The range of k in question lies within $k_{1/(n-1)} \leqq k \leqq k_{1/n}$ so by 4.2 there is a periodic orbit $\{Q_i\}$ of period n and rotation number $1/n$, hitting $x=1/4$. Centred on these $\{Q_i\}$ we may take "fans" as in 4.4 and appropriate intersections of "spokes" will give the desired orbit $\{Q'_j\}$. Note that in the range $k_{1/(n-1/(n'-1))} \leqq k \leqq k_{1/(n-1/n')}$ (chosen as in the remark preceding this lemma if there is any ambiguity) P_0 lies between spokes in such a way as to ensure that the $\{Q'_j\}$ satisfy (II). We omit details, which are analogous to 4.2 except that there are various cases to consider for the various parities of n and n'.

Lemma 4.6. *For each rational v of the form $1/(n-1/n')$, $n>2$, $n'>1$, there exists a value k'_v of k such that the orbit of P_0 extends to a non-periodic invariant circle of rotation number v.*

Proof of 4.6. This follows from 4.5 in the same way that 4.3 follows from 4.2; we may just repeat the proof of 4.3 but with a pair of adjacent points from the orbit $\{Q'_j\}$ (constructed in 4.5) in place of Q_{n-1} and Q_0.

Proof of Proposition 2(i). Our strategy is to repeat the method of 4.1–4.3 and 4.4–4.6 for v with an increasing length of continued fraction expansion. Write $[n_1, n_2, \ldots, n_m]$ for the continued fraction

$$\cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{}{\ddots - \cfrac{1}{n_m}}}}.$$

We remark that using an expansion with subtraction at each stage, rather than the more conventional addition, has the advantage that truncations give approximations all from the same side.

Let v be the continued fraction above and μ be its $(m-1)^{\text{th}}$ approximant, that is $\mu=[n_1, \ldots, n_{m-1}]$. Write r_m/s_m for v as an ordinary rational and r_{m-1}/s_{m-1} for μ. We first state some useful relations.

Claim. (1) $r_m s_{m-1} - r_{m-1} s_m = 1$.

(2) $r_m + r_{m-2} = n_m r_{m-1}$ *and* $s_m + s_{m-2} = n_m s_{m-1}$.

(3) *If* n_m *is even then* $(r_{m-2}, s_{m-2}) \equiv (r_m, s_m) \bmod 2$. *If* n_m *is odd then* (r_{m-2}, s_{m-2}), (r_{m-1}, s_{m-1}) *and* (r_m, s_m) *are all different* mod 2.

Property (1) is proved by induction on m. It is true for $m=2$, and if we assume it for r_m/s_m it follows easily for $R_m/S_m = 1/(n - r_m/s_m)$ for any n. Then (2) follows by a similar induction and (3) is an elementary consequence of (1) and (2). Note that (1) shows that approximants approach v from one side.

We now attack the inductive step in the proof of Proposition 2(i). Let μ and v be as above and let $\varrho=[n_1, \ldots, n_{m-1}-1]$. Our inductive hypothesis is that at $k=k_\mu$

the orbit of P_0 can be joined by straight line segments to give a periodic invariant circle of rotation number μ, that the "mid-point" orbit $\{Q_j^{(m-1)}\}$ of these segments continues to exist as a periodic orbit as k is reduced to k_ϱ, and that at some intervening value k'_μ the orbit of P_0 extends to a non-periodic invariant circle of rotation number μ. For the inductive step we must deduce results corresponding to 4.4, 4.5, and 4.6 for v. For the first we must show that for some parameter value k_v in $[k_\varrho, k'_\mu]$ the orbit of P_0 extends to a periodic invariant circle of rotation number v. As in 4.4 there are three cases.

Case (i): $(r_m, s_m) \equiv (0, 1) \bmod 2$. For the value of k we are seeking, the orbit of P_0 is to have $r_m - 1$ points in $0 < x < 1/2$ (since P_{-1} lies on $x=0$ and P_0 lies on $x = 1/2$). By symmetry it will hit L_2; indeed the point on L_2 will be P_n, where $n = (s_m - 1)/2$. The periodic orbit $\{Q_j^{(m-1)}\}$ of rotation number $\mu = r_{m-1}/s_{m-1}$ has r_{m-1} points in $0 < x < 1/2$, and since r_{m-1} is odd (by part (1) of the Claim) this orbit also hits L_2. We next note that since $r_m s_{m-1} = 1 \bmod s_m$, the orbit $\{P_i\}$ is to have the property that each point on it is obtained from the adjacent one to the left by an application of h_k^s, where $s = s_{m-1}$. However if we construct a "fan" at $Q_0^{(m-1)}$ (on L_2) consisting of a segment of L_2 and its images under iterates of h_k^s, $s = s_{m-1}$, then the "spokes" of the fan have exactly this property. It remains only to move this fan to the point $Q_q^{(m-1)}$ of $\{Q_i^{(m-1)}\}$ nearest to $x = 1/2$ on the left, by applying h_k^q for a suitable q, and then to adjust k until the appropriate spoke passes through P_0, just as in the proof of 4.4 (Case (i)). Explicitly $q = s_{m-2}(r_{m-1} - 1)/2$ reduced mod s_{m-1}, since there are $(r_{m-1} - 1)/2$ points of the orbit $\{Q_j^{(m-1)}\}$ in $1/4 < x < 1/2$ and h_k^t, $t = s_{m-2}$ moves each to the next on the right. The "appropriate spoke" of the fan to pass through P_0 is $h_k^{q+Ns}(L_2)$, where $s = s_{m-1}$ and $Ns = (s_m + 1)/2 - q$, since we wish P_{-n}, $n = (s_m + 1)/2$, to lie on L_2. Note that $(s_m + 1)/2 - q$ (for q as above) is indeed divisible by s_{m-1} since

$$
\begin{aligned}
(s_m + 1)/2 - q &\equiv (s_m + 1 - s_{m-2} r_{m-1} + s_{m-2})/2 \quad \bmod s_{m-1} \\
&\equiv (s_m + s_{m-2} - s_{m-1} r_{m-2})/2 \quad \text{(by (1) of Claim)} \\
&\equiv (n_m + r_{m-2}) s_{m-1}/2 \quad \text{(by (2) of Claim)}
\end{aligned}
$$

and $n_m + r_{m-2}$ is even (by (3) of Claim) when $(r_m, s_m) \equiv (0, 1) \bmod 2$.

Case (ii) a: $(r_m, s_m) \equiv (1, 1) \bmod 2$. Then $s_m - r_m - 1$ is odd, so the orbit of P_0 is to hit L'_2, indeed P_n is to lie on L'_2 for $n = (s_m - 1)/2$. But by the Claim $s_{m-1} - r_{m-1}$ is odd and so $\{Q_j^{(m-1)}\}$ also hits L'_2. We may therefore proceed as in 4.4 Case (ii) a.

Case (ii) b: $(r_m, s_m) \equiv (1, 0) \bmod 2$. Then $\{p_i\}$ is to have an even number of points, and as P_0 lies on $L_1(y = 2(x - 1/4))$ the point P_n, $n = s_m/2$, is to lie on the other S_1-symmetry line L'_1. But by the Claim $\{Q_j^{(m-1)}\}$ has an odd number of points and so it too has a point on L'_1 by its construction [as in 4.2, case (ii)]. Hence we may proceed as in 4.4 Case (ii) b.

The remaining parts of the inductive step in the proof of Proposition 2(i) are the analogues of 4.5 and 4.6. The first is to show that the "mid-point orbit" $\{Q_j^{(m)}\}$ of the invariant circle for k_v continues to exist for a appropriate range of k, and the second is to show that there is a value k'_v, where P_0 is on a non-periodic circle of rotation number v. Since these proofs follow similar lines to those of 4.5 and 4.6 (indeed to 4.2 and 4.3) we omit further details.

Proof of Proposition 2(ii). Any irrational v can be expressed uniquely as a continued fraction (of our "subtraction" form) $[n_1, n_2, \ldots, n_m, \ldots]$, where the n_m are all $\geqq 2$ and infinitely many of them are $\neq 2$. Let K_m denote the interval $[k_\varrho, k_\sigma]$, where $\varrho = [n_1, \ldots, n_m - 1]$ and $\sigma = [n_1, \ldots, n_m]$, with k_ϱ and k_σ given by Proposition 2(i) and, if there is any ambiguity, chosen so that $[k_\varrho, k_\sigma]$ satisfies the convention in the remark preceding Lemma 4.5. Let k_v denote the limit of the sequence of upper ends of these nested intervals $\{K_m\}$. Since k_v lies in K_1 the points $P_{-1}, P_0, \ldots, P_{n-2}$ $(n = n_1)$ of the orbit of P_0 lie in the correct order for an invariant circle of rotation number v. This is because for k in K_1 the point P_0 lies between appropriate spokes of fans centred at the fixed points (1/4, 0) and (3/4, 0). Next, since k_v lies in K_2 there are $n_1 n_2 - 1$ points of the orbit of P_0 in the correct order for an invariant circle of rotation number v. This is because for k in K_2 the point P_0 is between appropriate spokes of fans centred on the periodic orbit $\{Q_j\}$ of period n_1. Repeating the same argument for increasing m we deduce that for $k = k_v$ all the points of the orbit of P_0 are in the correct order for an invariant circle of rotation number v. The closure of this orbit is either an invariant circle or a Cantor set. In the latter case (which we conjecture does not occur) we can fill all the gaps with straight line segments (since h_k is linear away from $x = 0$ and $x = 1/2$). Either way we obtain an invariant circle of rotation number v.

Remarks. 1. We have not yet explained all the features of Fig. 2. For example, why is P_0 on an invariant circle for certain k arbitrarily close to $k_{1/n}$ and above it but not for k just below it? Observation suggests the following explanation. The symmetric orbit $\{Q_i\}$ of period n and rotation number $1/n$ constructed in 4.2 appears to be elliptic for $k > k_{1/n}$ and hyperbolic for $k < k_{1/n}$ (clearly it is neutral for $k = k_{1/n}$). When it is elliptic we can use the "rotating fan" argument to produce values of k just above $k_{1/n}$ for which P_0 is periodic and has rotation number $1/(n + 1/N)$ (N large). However for $k'_{1/n} < k < k_{1/n}$ the orbit $\{Q_i\}$ is hyperbolic and P_0 is below the contracting eigenvector of the nearest Q_j to its right, with the result that the orbit of P_0 is disordered. For $k < k'_{1/n}$, while $\{Q_i\}$ is still hyperbolic, P_0 is above this contracting eigenvector and its orbit is ordered, at least to a first approximation (there may disorder on a smaller scale).

2. Note that the hierarchy of symmetric periodic orbits $\{Q_j^{(m)}\}$ $m = 1, 2, \ldots$ used in the proof of Proposition 2 was constructed using symmetry arguments. This suggests such hierarchies might be proved to exist for appropriate parameter ranges for more general families of twist maps.

5. Other Families of Invariant Circles

So far we have only considered orbits passing through (1/2, 1/2), that is cancellation orbits taking one step from $x = 0$ to $x = 1/2$. In this section we look at some other families of cancellation orbits; they are illustrated in Fig. 3.

Two steps from $x = 0$ *to* $x = 1/2$.

These orbits pass through $P_0 = (1/4, 1/4)$ since then $P_{-1} = (0, 1/4 - k/4)$ lies on $x = 0$ and $P_1 = (1/2, 1/4)$ lies on $x = 1/2$.

The largest k for which there is an invariant circle containing this orbit is $k = 1/2$, when the orbit is homoclinic to the fixed point (3/4, 0) and so extends to an invariant circle of rotation number 0. As we decrease k from 1/2 we obtain invariant circles for all $0 \leqq v \leqq 1/4$, just as in Sect. 4.

Examples. (i) A periodic circle of rotation number 1/5 occurs when P_3 is on the symmetry line $y=2x-3/2$. This happens for $k=(\sqrt{13}-3)/2$.

(ii) A non-periodic circle of rotation number 1/5 occurs for k the root of $k^4-7k^2-2k+1=0$, that is $k=0.262$ approximately.

(iii) A periodic circle of rotation number 1/6 occurs when P_3 lies on $x=3/4$, that is for $k=\sqrt{2}-1$.

Two steps from $x=1/2$ *to* $x=1$. These orbits pass through (3/4, 1/4). The simplest example is that for $k=1$, $v=1/3$ already considered (Fig. 1). For this family as we reduce k the rotation number of invariant circles through $P_0=(3/4, 1/4)$ is *reduced*. The family runs from $v=1/4$ (at $k=0$) to $v=1/2$ (at $k=4/3$).

Example. We compute $k'_{1/3}$ for this family. From the proof of 4.2 there is a period 3 orbit $\{Q_0, Q_1, Q_2\}$ with $Q_0=(1/4, d)$ where $d=(k+2)/(2k+6)$; Q_1 has the same y-coordinate. $P_2=(1/2, 1/4+k/4)$, and so it is an easy calculation that P_2 lies on Q_0Q_1 when $k=\sqrt{2}-1$. This gives us the non-periodic circle of rotation number 1/3 in this family. Note that we now have two invariant circles for $k=\sqrt{2}-1$, one periodic of rotation number 1/6, and one non-periodic of rotation number 1/3.

Four steps and 3/2 *circuits of* S^1 *from* $x=1/2$ *to* $x=1$. Recall the circle through (1/2, 1/2) for $k=(\sqrt{5}-1)/2$, periodic with $v=2/5$ (Fig. 8). We considered P_0 as having an orbit taking one step from $x=0$ to $x=1/2$ but we could equally well regard it as taking 4 steps and 3/2 circuits of S^1 to get from $x=1/2$ to $x=1$. We now consider orbits of the latter type. By an elementary computation these all pass through $P_0=(1/2, e)$, where

$$e=\frac{k^2+3k+3}{4(k+2)}.$$

Examples. For $k=0$ we obtain a periodic circle of rotation number 3/8. At the other extreme in this family we have a non-periodic invariant circle with $v=1/2$ as illustrated in Fig. 9. We can compute the value of k for which this occurs, by finding when P_2P_0 cuts the symmetry line $y=2(x-1/2)$ at a point Q_0 of period 2. The slope of P_2P_0 is

$$\frac{k(k+1)}{k+2}$$

and P_2P_0 therefore cuts the symmetry line at $Q_0=(1/2+b, 1/2+2b)$, where

$$b=\frac{k^2+k-1}{4(4+k-k^2)}.$$

For $(1/2+b, 1/2+2b)$ to have period 2 requires

$$b=\frac{k}{4(k+4)},$$

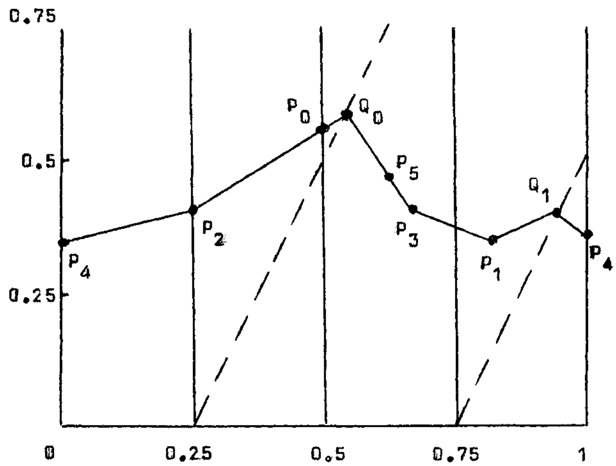

Fig. 9. $k=0.918$; non-periodic circle with $v=1/2$

and thus the circle occurs for k the solution of $2k^3+4k^2-k-4=0$, that is for k approximately 0.918.

Other Families. For any m and $n \geqq 2m$ we can consider the family taking n steps and $m+1/2$ circuits of the annulus to go from $x=0$ to $x=1/2$, or equally the family taking n steps and $m+1/2$ circuits to go from $x=1/2$ to $x=1$. Families of the first kind have a downward slope on their (k, v) diagram and those of the second kind have an upward slope.

Remarks. 1. By Proposition 1 every periodic circle occurs at the intersection of a family of the first kind and a family of the second kind. At the corresponding point on the (k, v) diagram there is a bifurcation to the right and a gap in both branches to the left.

2. The individual families all start at $k=0$ but may end at $k=4/3$ or elsewhere. For example the family taking 4 steps and 3/2 circuits from $x=1/2$ to $x=1$ ends at $k=0.918$, as we have just seen.

3. An individual rational v will occur in several ways for periodic orbits. For example $v=2/5$ occurs for $k=(\sqrt{5}-1)/2$ (one step from $x=0$ to $x=1/2$) and also for $k=(\sqrt{13}-1)/2$ (two steps from $x=1/2$ to $x=1$).

4. The (k, v) diagram can be continued to the range outside $0 \leqq v \leqq 1/2$ simply by reflection about $v=0$ and reflection about $v=1/2$.

5. We conjecture that invariant circles with irrational v occur precisely at the accumulation points of the full (k, v) diagram of circles of rational v. We have already seen in Sect. 4 that accumulation points on individual families give circles with irrational v. In Sect. 6 we shall see examples of accumulation points not lying on a single family of rational circles.

6. The Special Case $k=4/3$

In this section we prove Proposition 3.

In $0<x<1/2$, $h_{4/3}$ is a linear map with fixed point $(1/4, 0)$. Let L denote the matrix of this map (with respect to origin the fixed point) and let R denote the corresponding matrix for $h_{4/3}$ in $1/2<x<1$ (with origin the other fixed point $(3/4, 0)$). Formally L and R are the derivative of $h_{4/3}$ on the two halves of the annulus.

$$L=\begin{pmatrix} -1/3 & 1 \\ -4/3 & 1 \end{pmatrix}, \quad R=\begin{pmatrix} 7/3 & 1 \\ 4/3 & 1 \end{pmatrix}.$$

The eigenvectors of R are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (eigenvalue 3) and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (eigenvalues 1/3). The key property of $h_{4/3}$ is that L applied to $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ gives $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. This is the reason for the existence of the homoclinic circle illustrated in Fig. 10a; each straight line segment I_n is mapped by $h_{4/3}$ to I_{n+1}.

Given any rational number $v=p/q$ we can take the same collection of line segments $\{I_n\}$ and rearrange them in the order we would get on a non-periodic

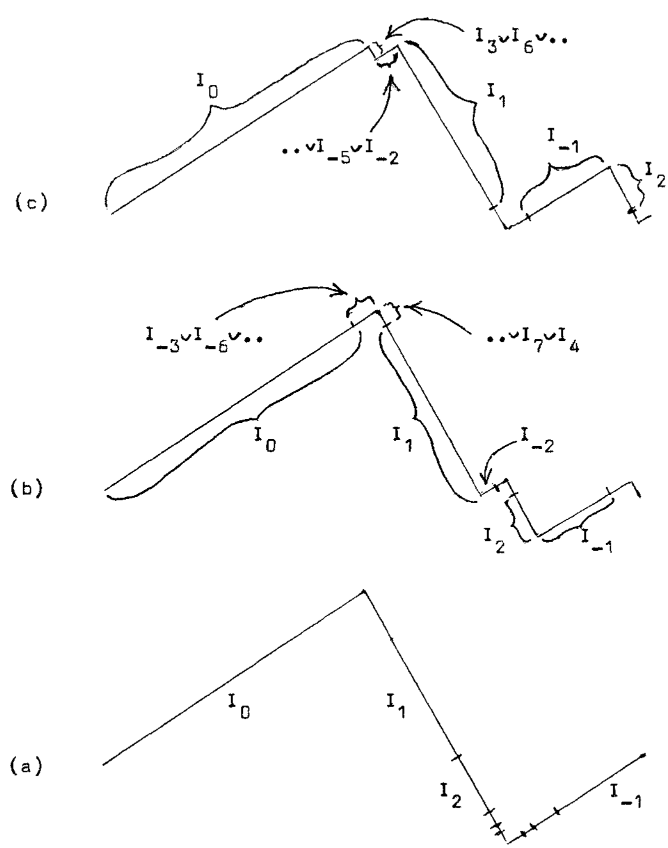

Fig. 10a–c

invariant circle of rotation number v containing a single periodic orbit. In fact for each rational there are two such orders, one corresponding to a "drift forward" and the other to a "drift backward" between adjacent points of the periodic orbit. The orders for $v=1/3$ are illustrated in Fig. 10b and c; they differ in that in one I_3 comes just after I_0 and in the other it comes just before. In making these rearrangements we have simply translated each I_n of Fig. 10a to a parallel new position, without changing its length, and so the rearranged segments still make up a closed circle on the annulus. In the new order we keep I_0 with its ends on $x=0$ and $x=1/2$ and we adjust the height above the x-axis as follows. Let $\{I_j\}_{j\in J}$ be the segments between I_0 and I_1 in the new circle. If $j\in J$ then so does $1-j$ since the cyclic order on the circle is reversed by the involution $I_j \to I_{1-j}$. As I_j and I_{1-j} have the same length in the vertical direction, but I_{1-j} goes up whereas I_j goes down, we deduce that I_0 and I_1 are at the same height above the x-axis. We may now alter the height of the whole circle above the axis until $h_{4/3}(I_0)=I_1$ (in its new position). We claim that the circle is now invariant. It clearly suffices to prove that $h_{4/3}(I_n)=I_{n+1}$ for all the I_n in their new positions. Consider the segment I_q (attached to I_0 on one side or the other, as $v=p/q$); $h_{4/3}(I_q)$ is an interval parallel to I_{q+1} and equal in length to it (being the vector RI_q), but it is also attached to I_1 (by continuity, since $h_{4/3}(I_0)=I_1$) and hence it is I_{q+1}. Repeating the argument with I_q in place of I_0 we deduce that $h_{4/3}(I_{2q})=I_{2q+1}$ and so on. We can continue around the circle and deduce for all n that $h_{4/3}(I_n)=I_{n+1}$. The only difficulty is at accumulation points of intervals I_n, that is at periodic points on the circle, and we deal with these by the argument given below for irrational rotation numbers.

An irrational rotation number v determines uniquely the order in which the segments I_n must be arranged. The order is that of a single orbit for a rigid rotation of a circle through an angle $2\pi v$. Once again we may arrange the height of our candidate invariant circle above the x-axis so that $h_{4/3}(I_0)=I_1$, but this time we cannot argue directly by continuity around the circle since no I_n is directly attached to I_0 (if it were then it is easily seen that the circle would have rational rotation number, the rational having n as denominator). Instead each end of I_0 is an accumulation point of smaller intervals I_n. However, as before let $\{I_j\}_{j\in J}$ be the segments between I_0 and I_1. Each $h_{4/3}(I_j)$ is a segment of the same vertical and horizontal lengths as I_{j+1}. Thus by continuity $h_{4/5}(I_1)$ has horizontal and vertical distances from $h_{4/3}(I_0)=I_1$ the sums of those for $\{I_{j+1}\}_{j\in J}$; but so does I_2 by our construction of the circle. Hence $h_{4/3}(I_1)=I_2$. We can continue around the circle and deduce $h_{4/3}(I_n)=I_{n+1}$ for all n. Note that in the irrational case the ends of I_0 are on separate orbits (else I_0 would be attached to an I_n) so these circles give our first examples not containing cancellation orbits. Note also that the orbit of an end-point of I_0 is not dense in the circle (it is a Cantor set) so $h_{4/3}$ is not conjugate to a rotation on the circle.

It remains to prove (iii) of Proposition 3, that the rational island chains occupy full measure for $k=4/3$. The two invariant circles for $v=p/q$ differ in that one has I_q just before I_0 and the other has I_q just after it. Thus one of the parallelogram islands trapped between these circles has sides $I_0 \cup I_{-q} \cup I_{-2q} \cup \ldots$ and $I_q \cup I_{2q} \cup I_{3q} \cup \ldots$. The vertical width of this island where it crosses $x=0$ is

$$\frac{4}{3(3^q-1)},$$

and an elementary calculation shows that the area of the island is

$$\frac{2.3^{q-1}}{(3^q-1)^2}.$$

Since $h_{4/3}$ is area-preserving, the total area of the island chain is

$$\frac{2q3^{q-1}}{(3^q-1)^2}.$$

To find the total area of all the island chains we sum this over all p/q between 0 and 1/2. Such sums are easier to compute over all p/q between 0 and 1, so we do this first. Let $\phi(q)$ denote Euler's function, namely the number of integers m with $1 \leqq m < q$ and m coprime to q. Then

$$\begin{aligned}\frac{2}{3}\sum_{0<p/q<1}\frac{q3^q}{(3^q-1)^2} &= \frac{2}{3}\sum_{1<q<\infty}\frac{q\phi(q)3^q}{(3^q-1)^2} = \frac{2}{3}\sum_{1\leqq m}\sum_{1<q<\infty}\frac{q\phi(q)m}{3^{mq}} \\ &= \frac{2}{3}\sum_{1<n<\infty}\frac{n}{3^n}\left(\sum_{\substack{q|n \\ 1<q\leqq n}}\phi(q)\right) = \frac{2}{3}\sum_{1<n<\infty}\frac{n(n-1)}{3^n} \\ &= \frac{2}{27}\sum_{1<n<\infty}\frac{n(n-1)}{3^{n-2}} = \frac{4}{27(1-1/3)^3} = \frac{1}{2}.\end{aligned}$$

Since the area of the chain corresponding to $p/q = 1/2$ is 3/16, we deduce that the total area of island chains for $0 < p/q < 1/2$ is

$$\frac{1}{2}\left(\frac{1}{2} - \frac{3}{16}\right) = \frac{5}{32}.$$

However it is an easy exercise to check that the area between the circle for $v = 0$ and the lower circle for $v = 1/2$ is also 5/32, completing the proof of Proposition 3(iii).

The motion for $k = 4/3$ is not completely regular. There are integrable zones around the centres of islands (indeed the map is a linear one there) but near the edges of islands there are hierarchies of smaller island chains generated by the overlapping of $x = 0$ and $x = 1/2$ by the large islands. However for $k = 4/3$ we do have the remarkable situation that every orbit has a well-defined rotation number and that those with rational rotation number occupy full measure.

I am indebted to Dr. M. Shirvani for showing me how to perform summations of the type in this section.

7. Parameter Ranges where there are no Invariant Circles

We are concerned here with Proposition 4. We code each orbit by a bi-infinite word $\ldots l^{m_1}r^{n_1}\ldots l^{m_p}r^{n_p}\ldots$, where the m_i and n_i are positive integers, listing whether successive iterates land in the left-hand half or right-hand half of $0 \leqq x \leqq 1$. Several orbits may have the same word. We ignore difficulties concerning orbits which hit either $x = 0$ or $x = 1/2$; in practice we can avoid these.

Let L denote the derivative of h_k in $0<x<1/2$ and R that in $1/2<x<1$ (as before). Then

$$L=\begin{pmatrix}1-k & 1\\ -k & 1\end{pmatrix}, \quad R=\begin{pmatrix}1+k & 1\\ k & 1\end{pmatrix}.$$

Corresponding to the orbit $\dots l^{m_1}r^{n_1}\dots l^{m_p}r^{n_p}\dots$ the tangent map has matrix $\dots R^{n_p}L^{m_p}\dots R^{n_1}L^{m_1}\dots$, since we have adopted the usual conventions of listing an orbit from left to right and writing matrices as acting on the left, that is on column vectors.

Lemma 7.1. *For $k>4/3$ there are no invariant circles.*

Proof of 7.1. This result is well-known [6] but we include a proof here as the method motivates our other proofs.

Let C be an invariant circle and let S be any point on it in $0<x<1/2$. Let T be a point on C just to the right of S. Let v be the vector ST. Since any invariant circle C projects $(1-1)$ onto the x-axis the iterates of v must all have a positive x-component. We shall show this to be impossible for $k>4/3$.

Note that the matrix L is elliptic (it turns all vectors clockwise) and that R is hyperbolic with contracting eigenvector of slope $-k/2-\sqrt{k+k^2/4}$ and expanding eigenvector of slope $-k/2+\sqrt{k+k^2/4}$. If the direction of Lv is below the contracting eigenvector for R then any sequence of R's and L's applied to Lv will eventually turn the x-component negative since each matrix will twist it further clockwise. Similarly if v is above the expanding eigenvector for R any inverse sequence of R's and L's will eventually turn the x-component negative. Thus for an invariant circle to exist L must not turn the expanding eigenvector of R below the contracting one. It is elementary to check that this condition corresponds to $k\leqq 4/3$.

Remark. The same argument shows that no invariant Cantor set can have a point in $0<x<1/2$ for $k>4/3$.

Lemma 7.2. *For $k>1/2$ any invariant circle crosses $x=1/2$ at or above $y=1/2$.*

Proof of 7.2. For $k>1/2$, L^2 turns the expanding eigenvector of R below the contracting one (an easy calculation). Hence no invariant circle can contain an orbit with l^2 in its word. Thus any invariant circle crosses $x=1/2$ at or above $y=1/2$.

Lemma 7.3. *For $0.918<k<1$ there is no invariant circle.*

Proof of 7.3. For $k>0.918$ (to be precise the root of $2k^3+4k^2-k-4=0$) the matrix LRL turns the expanding eigenvector of R below the contracting one. Hence no invariant circle can contain an orbit with lrl in its word. But by 7.2 any invariant circle crosses $x=1/2$ above $y=1/2$ and an easy check shows that if $k<1$ then a point just to the left of this crossing has lrl as its first three iterates.

Remark. The limit 0.918 is achieved by a circle with rotation number 1/2 (Fig. 9) and the limit 1 is achieved by a circle of rotation number 1/3 (Fig. 1).

We conjecture that there are "windows" in the range of k with no circles (like that of 7.3) arbitrarily close to $k=0$. These are likely to be short and therefore difficult to detect numerically. Indeed there seem to be conspiracies to block windows; for example the window below $k=\sqrt{2}-1$ in the "2 steps from $x=0$ to $x=1/2$" family is blocked by the "2 steps from $x=1/2$ to $x=1$" family.

8. Generalisation to Other Piecewise Linear Maps

We first consider the case where $g(x)$ is still made up of two linear segments, but this time of unequal length. Explicitly

$$g(x)=\begin{cases} a-x & 0\leqq x\leqq 2a \\ \dfrac{2a(x-1/2-a)}{(1-2a)} & 2a\leqq x\leqq 1 \end{cases}$$

with

$$h_k(x,y)=(x+y+kg(x),\, y+kg(x)).$$

Note that $a=1/4$ is the piecewise linear standard map already considered.

This time the discontinuity lines are $x=0$ and $x=2a$. An orbit taking one step from the first line to the second passes through $P_0=(2a,2a)$. The image of P_0 is $P_1=((4-k)a,(2-k)a)$. This orbit is homoclinic to the fixed point (and thus can be joined up to form an invariant circle of rotation number 0) if and only if

$$\frac{ka}{(2-k)a}=\frac{2a}{(1/2-a)} \quad \text{that is} \quad k=\frac{4a}{1/2+a}.$$

The arguments of Sect. 6 show that for this value of k we obtain invariant circles of all v and those of Sect. 7 show that for k greater than this value we have *no* invariant circles. For each fixed a one can make an analysis of all the invariant circles, just as we did in Sects. 4 and 5, and obtain a similar overall picture. We can also find a sequence of functions with both a and k tending to zero all with no invariant circles; such a sequence, suitably smoothed, is used by Herman for his C^1-topology counterexample in [6].

Finally we consider what happens when $g(x)$ is the piecewise linear function illustrated in Fig. 11 with 4 points where the derivative is discontinuous. For this

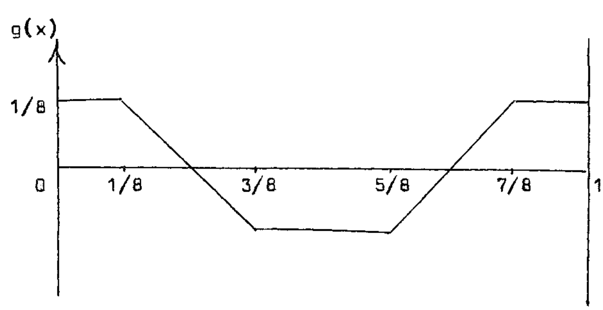

Fig. 11. Piecewise linear g with four segments

family we still find many invariant circles of irrational v but those of rational v are rare. They still occasionally exist; for example for $k=1$, $P_0=(1/2, 1/2)$ is on a periodic circle of rotation number 3/7. However they are rare because by the method of Sect. 3 they would have to contain a cancellation orbit hitting all four discontinuity lines or else a pair of cancellation orbits of the same v. To recover periodic circles for all rational v we must allow the discontinuity lines to move and consider a two parameter family of twist homeomorphisms $h_{k,a}$ with discontinuity lines at $x=a, 1/2-a$, $1/2+a$ and $1-a$. Then irrational circles should occur precisely at the closure points of the (k, a, v) diagram of rational circles. In principle one should be able to repeat the analysis with increasing numbers of linear segments in g, but the details would be complicated.

Concluding Remark. The methods of Sects. 3–5 also apply to cancellation orbits for piecewise smooth maps such as that corresponding to the oval billiard of Hénon and Wisdom [5]. The method provides orbits rather than circles for rational v, but the accumulation points of individual families should give invariant circles of irrational v just as in Sect. 4.

References

1. Arnold, V.I.: Proof of A.N. Kolmogorov's theorem on the preservation of quasi-periodic motions under small perturbations of the Hamiltonian. Usp. Mat. Nauk USSR **18**, 13–40 (1963) (Russ. Math. Surv. **18**, 9–36)
2. Aubry, S., Le Daeron, P.Y.: The discrete Frankel-Kontorova model and its extensions. Physica **8**D, 381–422 (1983)
3. Birkhoff, G.D.: Surface transformations and their dynamical applications. Acta Math. **43**, 1–119 (1922)
4. Greene, J.M.: A method for determining a stochastic transition. J. Math. Phys. **20**, 1183–1201 (1979)
5. Hénon, M., Wisdom, J.: The Benettin-Strelcyn oval billiard revisited. Physica **8**D, 157–169 (1983)
6. Herman, M.R.: Sur les courbes invariantes par les diffeomorphismes de l'anneau. Asterisque Vol. 103–104 (1983)
7. Katok, A.: Periodic and quasi-periodic orbits for twist maps. Lecture Notes in Physics, Vol. 179, pp. 47–65. Berlin, Heidelberg, New York: Springer 1983
8. MacKay, R.S., Meiss, J.D., Percival, I.C.: Transport in Hamiltonian systems. Physica **13**D, 55–81 (1984)
9. Mather, J.N.: Existence of quasi-periodic orbits for twist homeomorphisms of the annulus. Topology **21**, 457–468 (1982)
10. Moser, J.: On invariant curves of area-preserving mappings of an annulus. Nachr. Akad. Wiss. Gött. II. Math. Phys. Kl, 1–20 (1962)
11. Rüssmann, H.: On the existence of invariant curves of twist mappings of an annulus. Lecture Notes in Mathematics, Vol. 1007, pp. 677–718. Berlin, Heidelberg, New York: Springer 1983
12. Wojtkowski, M.: A model problem with the coexistence of stochastic and integrable behaviour. Commun. Math. Phys. **80**, 453–464 (1981)
13. Wojtkowski, M.: On the ergodic properties of piecewise linear perturbations of the twist map. Ergodic Theory Dyn. Syst. **2**, 525–542 (1982)

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