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INVARIANT CURVES FOR NUMERICAL METHODS*

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Abstract. The problem of finding periodic orbits of dynamical systems numerically is considered. It is shown that if a convergent, strongly stable, multi-step method is employed then under some suitable conditions, there exist invariant curves. The result also shows that the rates of convergence toward the invariant curves are roughly the same for different methods and different step sizes.

Introduction. The problem of locating the periodic orbits of dynamical systems numerically has been considered by many authors (see, e.g., [3, 4, 8]). In many instances, especially when the periodic orbits of an one-parameter family of dynamical systems are to be followed, the existence of these orbits is more important than their exact locations.

This paper arises from the author's dissertation [2]. Its purpose is to show that when a strongly stable method is employed, then under suitable conditions, there exist invariant curves. This result is stronger than the one obtained in [1] in that the method could be multi-step and the periodic orbit need not be stable. The result also shows that reducing the step size or employing more sophisticated methods do not necessarily improve the rate of convergence.

The problem. Let

$$\begin{aligned} &\dot{x} = f(x), \\ &x \in \mathbb{R}^n, \qquad f(x) \text{ is as smooth as needed,} \end{aligned} \tag{1}$$

be a dynamical system which possesses a periodic orbit Γ with characteristic multipliers $\mu_1, \ldots, \mu_n$ satisfying $\mu_1 = 1$, $|\mu_j| < 1$ for $j = 2, \ldots, l$ and $|\mu_j| > 1$ for $j = l + 1, \ldots, n$. Suppose that (1) is approximated by a convergent, k-step method of the form

$$x_{m+1} = \sum_{j=1}^{k} a_j x_{m+1-j} + h \sum_{j=1}^{k} b_j f(x_{m+1-j}) + h^2 F(h, x_m, \ldots, x_{m+1-k}) \tag{2}$$

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where

$$1-\sum_{j=1}^{k} a_j = 0 \quad \text{and} \quad \sum_{j=1}^{k} j a_j = \sum_{j=1}^{k} b_j.$$

We will also assume that the method employed is strongly stable. This means that the polynomial $g(\lambda) = \lambda^k - \sum_{j=1}^{k} a_j \lambda^{k-j}$ has k roots $\lambda_1, \ldots, \lambda_k$ satisfying $\lambda_1 = 1$, $|\lambda_j| < 1$ for $j = 2, \ldots, k$. We note that (2) includes the usual linear, multi-step methods as well as many non-linear methods such as the modified Euler's method, the Runge-Kutta method, etc. For example, the modified Euler's method applied to (1) would yield

$$x_{m+1} = x_m + hf(x_m) + h^2 F(h, x_m)$$

where $F(h, x) = (f(x + hf(x)) - f(x))/h$ for $h \neq 0$ and $F(0, x) = f'(x)f(x)$.

The moving coordinate system. The key in proving the result is to introduce the moving coordinate system. To that end, let $\Gamma = \{ p(\theta)/0 \leqslant \theta \leqslant \tau \}$ be the periodic orbit, i.e., $p(\theta)$ is a τ-periodic function satisfying $dp(\theta)/d\theta = f(p(\theta))$. The corresponding linear variational equation

$$dy(\theta)/d\theta = \partial f/\partial x(p(\theta))y(\theta) \tag{3}$$

has $p'(\theta)$ as a solution corresponding to the trivial characteristic multiplier $\mu_1 = 1$. Thus (3) has a fundamental matrix solution of the form (see, e.g., [5])

$$X(\theta) = P(\theta)e^{A\theta} \tag{4}$$

in which the first columns of $X(\theta)$ and $P(\theta)$ both equal to $p'(\theta)$, $P(\theta)$ is 2 τ-periodic and A is of the form $A = [\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}]$ where B is an $(n-1)x(n-1)$ matrix such that $e^{2B\tau}$ has eigenvalues $\mu_2^2, \ldots, \mu_n^2$. The 2 τ-periodicity is chosen here to ensure that A is a real matrix.

We now imbed Γ into $\mathbb{R}^{kn}$ by defining $\tilde{p}(\theta) = [p(\theta - (k-1)h)^T \cdots p(\theta)^T]^T$, here T denotes transposition. Since $p(\theta)$ is periodic, so is $\tilde{p}(\theta)$. Thus for $0 \leqslant \theta \leqslant \tau$, $\tilde{p}(\theta)$ defines a curve $\tilde{\Gamma}$ in $\mathbb{R}^{kn}$. Similarly, given a sequence $x_{m+1-k}, \ldots, x_m$ in $\mathbb{R}^n$, we define $z_m = [x_{m+1-k}^T \cdots x_m^T]^T$. The difference equation (2) then defines a mapping M from $\mathbb{R}^{kn}$ to $\mathbb{R}^{kn}$ by $M(z_m) = z_{m+1}$ where $z_{m+1} = [x_{m-k+2}^T \cdots x_{m+1}^T]^T$ and x_{m+1} is given by (2).

Due to the special form of M, any curve invariant under M in $\mathbb{R}^{kn}$ would yield a curve ψ in $\mathbb{R}^n$ which is invariant under (2) in the sense that for any point x in ψ, there exist points $x_k = x, x_{k-1}, \ldots, x_1$ on ψ such that the points generated by (2) with starting points $x_1, \ldots, x_k$ stay on ψ.

Define $\tilde{X}(\theta) = [X(\theta - (k-1)h)^T \cdots X(\theta)^T]^T$. Note that since

$$e^{-A\theta} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-B\theta} \end{bmatrix},$$

the first column of $\tilde{X}(\theta)e^{-A\theta}$ is $\tilde{p}(\theta)$. Let $Z(\theta)$ be the $(nk)x(n-1)$ matrix consisting of the remaining $n-1$ columns of $\tilde{X}(\theta)e^{-A\theta}$. Also we now assume for simplicity that the roots $\lambda_1, \ldots, \lambda_k$ of $g(\lambda) = \lambda^k - \sum_{j=1}^{k} a_j \lambda^{k-j}$ are all distinct and define

$$E(\theta) = \begin{bmatrix} \lambda_2 P(\theta) & \cdot & \lambda_k P(\theta) \\ \cdot & \cdot & \cdot \\ \lambda_2^k P(\theta) & \cdot & \lambda_k^k P(\theta) \end{bmatrix}.$$

Since $Z(\theta)$ and $E(\theta)$ are 2τ-periodic, it makes sense to build the moving coordinate system on these matrices. We have the following

LEMMA 1. There exists an open neighborhood of $\tilde{\Gamma}$ such that for every z in that neighborhood, there exist unique r in $\mathbb{R}^{n-1}$, w in $\mathbb{R}^{(k-1)n}$ and θ (unique up to $\bmod 2\tau$) such that

$$z = \tilde{p}(\theta) + Z(\theta)r + E(\theta)w. \tag{5}$$

Proof. Set $F(\theta, r, w, z) = z - (\tilde{p}(\theta) + Z(\theta)r + E(\theta)w)$. Then $F(\theta, 0, 0, \tilde{p}(\theta)) = 0$, $\partial F/\partial\theta(\theta, 0, 0, \tilde{p}(\theta)) = \tilde{p}'(\theta)$, $\partial F/\partial r(\theta, 0, 0, \tilde{p}(\theta)) = Z(\theta)$ and $\partial F/\partial w(\theta, 0, 0, \tilde{p}(\theta)) = E(\theta)$. When $h = 0$ we have

$$[\tilde{p}'(\theta), Z(\theta), E(\theta)] = \begin{bmatrix} P(\theta) & 0 & \cdot & 0 \\ 0 & P(\theta) & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & P(\theta) \end{bmatrix} \begin{bmatrix} I & \lambda_2 I & \cdot & \lambda_k I \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ I & \lambda_2^k I & \cdot & \lambda_k^k I \end{bmatrix}$$

which is invertible (here 0 and I denote the zero and identity $(n) \times (n)$ matrices). Thus continuity implies that $[\tilde{p}'(\theta), Z(\theta), E(\theta)]$ is also invertible for $h > 0$ sufficiently small. The lemma then follows from the implicit Function Theorem and the fact that $\tilde{\Gamma}$ is compact.

Technically speaking, $\tilde{\Gamma}$ and the neighborhood provided by Lemma 1 depend on the step size h, but due to continuity, we may assume that (5) is valid for all $|r| < \delta$ and $|w| < \delta$ where δ is independent of h provided that h is sufficiently small. Also by assuming that the function $f(x)$ in (1) is sufficiently smooth, θ, r, w will depend as smoothly on z as needed. Let v_1, v_2 and v_3 be the appropriate functions such that $\theta = v_1(z)$, $r = v_2(z)$ and $w = v_3(z)$ for all z sufficiently close to $\tilde{\Gamma}$.

If

$$z = \tilde{p}(\theta) + Z(\theta)r + E(\theta)w \tag{6}$$

then

$$M(z) = M(\tilde{p}(\theta)) + S(\theta)[Z(\theta)r + E(\theta)w] + hg_1(h, \theta, w, r) \tag{7}$$

where $S(\theta) = \partial M/\partial z(p(\theta))$ and $g_1(h, \theta, w, r)$ has second order zero in (w, r) at $(0, 0)$. Since M arises from a convergent method, we have

$$M(\tilde{p}(\theta)) = \tilde{p}(\theta + h) + O(h^2). \tag{8}$$

Also since $X(\theta)$ is a fundamental matrix solution of (3) we have

$$S(\theta)\tilde{X}(\theta) = \tilde{X}(\theta + h) + O(h^2)$$

and hence

$$S(\theta)Z(\theta) = Z(\theta + h)e^{Bh} + O(h^2). \tag{9}$$

Finally note that

$$S(\theta)E(\theta) = E(\theta + h)\Lambda + O(h) \tag{10}$$

where

$$\Lambda = \begin{bmatrix} \lambda_2 I & 0 & \cdot & 0 \\ 0 & \lambda_3 I & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \lambda_k I \end{bmatrix}.$$

Set $\bar{z} = \tilde{p}(\theta + h) + Z(\theta + h)e^{Bh}r + E(\theta + h)\Lambda w$ then

$$M(z) - \bar{z} = hg_1(h, \theta, w, r) + hG_1(h, \theta, w) + h^2F_1(h, \theta, r).$$

The first term comes from (7), the second from (10) and the third from (8) and (9). The function $g_1(h, \theta, w, r)$ has second order zero in (w, r), $G_1(h, \theta, w)$ has first order zero in w while $F_1(h, \theta, 0) \not\equiv 0$ in general. Thus if $M(z) = \tilde{p}(\theta_1) + Z(\theta_1)r_1 + E(\theta_1)w_1$ then

$$\theta_1 = v_1(M(z)) = v_1(\bar{z}) + \partial v_1/\partial z(\bar{z})(M(z) - \bar{z}) + O(h^2).$$

The term $O(h^2)$ arises from the fact that $M(z) - \bar{z} = O(h)$. Thus

$$\theta_1 = \theta + h + hg_2(h, \theta, w, r) + hG_2(h, \theta, w, r) + h^2F_2(h, \theta, w, r) \tag{11a}$$

Similarly

$$w_1 = \Lambda w + hg_3(h, \theta, w, r) + hG_3(h, \theta, w, r) + h^2F_3(h, \theta, w, r) \tag{11b}$$

$$r_1 = e^{Bh}r + Hg_4(h, \theta, w, r) + hG_4(h, \theta, w, r) + h^2F_4(h, \theta, w, r) \tag{11c}$$

where $g_i(h, \theta, w, r)$ has second order zero in (w, r), $G_i(h, \theta, w, r)$ has first order zero in w and $F_i(h, \theta, 0, 0) \not\equiv 0$ in general.

To simplify (11), we now scale $r \to h^{1/2}r$ and $w \to hw$, i.e., we make the changes of variables $r = h^{1/2}\bar{r}$ and $w = h\bar{w}$ and call the new variables ($\bar{r}$ and $\bar{w}$) r and w again. (11) then becomes

$$\theta_1 = \theta + h + h^2H_1(h, \theta, w, r), \tag{12a}$$

$$w_1 = \Lambda w + hH_2(h, \theta, w, r), \tag{12b}$$

$$r_1 = e^{Bh}r + h^{3/2}H_3(h, \theta, w, r). \tag{12c}$$

By making the necessary transformation, we can decompose (12) as

$$\theta_1 = \theta + h + h^2H_1(h, \theta, w, y, z), \tag{13a}$$

$$w_1 = \Lambda w + hH_2(h, \theta, w, y, z), \tag{13b}$$

$$y_1 = Cy + h^{3/2}H_3(h, \theta, w, y, z), \tag{13c}$$

$$z_1 = Dz + h^{3/2}H_4(h, \theta, w, y, z), \tag{13d}$$

where $y \in \mathbb{R}^{l-1}$, $z \in \mathbb{R}^{n-1}$, C is an $(l - 1) \times (l - 1)$ matrix with eigenvalues $e^{\alpha_j h}$ for $j = 2, \ldots, l$ and D is an $(n - 1) \times (n - 1)$ matrix with eigenvalues $e^{\alpha_j h}$ for $j = l + 1, \ldots, n$. Here α_j are such that $e^{2\alpha_j} = \mu_j^2$. Note that $\|\Lambda\| \leqslant \lambda < 1$, $\|C\| < 1 - hc$ and $\|D^{-1}\| < 1 - hd$ for some constants c, $d > 0$ independent of h.

Main results. Following Lansford [7], we now introduce the spaces

$$\mathscr{B}_W = \{\text{continuous, } 2\tau\text{-periodic function } w(\theta) \text{ with values in } \mathbb{R}^{(k-1)n}$$
$$\text{such that } |w(\theta)| \leqslant 1 \text{ and } |w(\theta_1) - w(\theta_2)| \leqslant |\theta_1 - \theta_2| \text{ for all } \theta_1, \theta_2\},$$

$$\mathscr{B}_Y = \{\text{continuous, } 2\tau\text{-periodic function } y(\theta) \text{ with values in } \mathbb{R}^{l-1}$$
$$\text{such that } |y(\theta)| \leqslant 1 \text{ and } |y(\theta_1) - y(\theta_2)| \leqslant |\theta_1 - \theta_2| \text{ for all } \theta_1, \theta_2\},$$

$$\mathscr{B}_Z = \{\text{continuous, } 2\tau\text{-periodic function } z(\theta) \text{ with values in } \mathbb{R}^{n-l}$$
$$\text{such that } |z(\theta)| \leqslant 1 \text{ and } |z(\theta_1) - z(\theta_2)| \leqslant |\theta_1 - \theta_2| \text{ for all } \theta_1, \theta_2\}.$$

LEMMA 2. Given $z(\theta)$ in $\mathscr{B}_Z$, there exists a unique pair of function $(w(\theta), y(\theta))$ in $\mathscr{B}_W \times B_Y$ such that

$$w(\theta_1) = \Lambda w(\theta) + hH_2(h, \theta, w(\theta), y(\theta), z(\theta)), \tag{14a}$$
$$y(\theta_1) = Cy(\theta) + h^{3/2}H_3(h, \theta, w(\theta), y(\theta), z(\theta)), \tag{14b}$$

where

$$\theta_1 = \theta + h + h^2H_1(h, \theta, w(\theta), y(\theta), z(\theta)). \tag{14c}$$

Similarly given a pair of functions $(w(\theta), y(\theta))$ in $\mathscr{B}_W \times \mathscr{B}_y$, there exists a unique function $z(\theta)$ in $\mathscr{B}_Z$ such that

$$z(\theta_1) = Dz(\theta) + h^{3/2}H_4(h, \theta, w(\theta), y(\theta), z(\theta)) \tag{15}$$

where θ_1 is given by (14c).

Proof. Let $z(\theta)$ be a function in $\mathscr{B}_Z$. We construct a mapping P_z: $\mathscr{B}_W \times \mathscr{B}_y \to \mathscr{B}_W \times \mathscr{B}_y$ as follows: Let $u(\theta)$ and $v(\theta)$ be a pair of functions in $\mathscr{B}_W$ and $\mathscr{B}_y$ respectively. Fix θ and let $\bar{\theta}$ be such that

$$\theta = \bar{\theta} + h + h^2H_1(h, \bar{\theta}, u(\bar{\theta}), v(\bar{\theta}), z(\bar{\theta})) \pmod{2\tau} \tag{16}$$

such $\bar{\theta}$ exists since the right hand side of (16) is strictly increasing and covers an interval of length 2τ as $\bar{\theta}$ varies from 0 to 2τ. We then define $P_z(u, v) = (\tilde{u}, \tilde{v})$ where

$$\tilde{u}(\theta) = \Lambda u(\bar{\theta}) + hH_2(h, \bar{\theta}, u(\bar{\theta}), v(\bar{\theta}), z(\bar{\theta})), \tag{17a}$$
$$\tilde{v}(\theta) = Cv(\bar{\theta}) + h^{3/2}H_3(h, \bar{\theta}, u(\bar{\theta}), v(\bar{\theta}), z(\bar{\theta})). \tag{17b}$$

Let N be a bound for the norms of $H_j(h, \theta, w, y, z)$ and their partial derivatives on a neighborhood of $\tilde{\Gamma}$. We have

$$|\tilde{u}(\theta)| \leqslant \lambda|u(\theta)| + hN \leqslant \lambda + hN \leqslant 1 \quad \text{if } hN \leqslant 1 - \lambda.$$

Similarly

$$|\tilde{v}(\theta)| \leqslant (1 - hc) + h^{3/2}N \leqslant 1 \text{ if } h^{1/2} \leqslant c/N.$$

Now let θ_1, θ_2 be given and let $\bar{\theta}_1$, $\bar{\theta}_2$ be such that

$$\theta_1 = \bar{\theta}_1 + h + h^2H_1(h, \bar{\theta}_1, u(\bar{\theta}_1), v(\bar{\theta}_1), z(\bar{\theta}_1)), \tag{18a}$$
$$\theta_2 = \bar{\theta}_2 + h + h^2H_1(h, \bar{\theta}_2, u(\bar{\theta}_2), v(\bar{\theta}_2), z(\bar{\theta}_2)), \tag{18b}$$

then since

$$\left|H_1\left(h,\bar{\theta}_1,u\left(\bar{\theta}_1\right),v\left(\bar{\theta}_1\right),z\left(\bar{\theta}_1\right)\right)-H_1\left(h,\bar{\theta}_2,u\left(\bar{\theta}_2\right),v\left(\bar{\theta}_2\right),z\left(\bar{\theta}_2\right)\right)\right| \leqslant (4N)\left|\bar{\theta}_1-\bar{\theta}_2\right|$$

We have

$$\left|\bar{\theta}_1-\bar{\theta}_2\right| \leqslant (1-4Nh^2)^{-1}|\theta_1-\theta_2|$$

Thus

$$\begin{aligned}|\tilde{u}(\theta_1)-\tilde{u}(\theta_2)| &\leqslant \lambda\left|u\left(\bar{\theta}_1\right)-u\left(\bar{\theta}_2\right)\right| + h\left|H_2\left(h,\bar{\theta}_1,u\left(\bar{\theta}_1\right),v\left(\bar{\theta}_1\right),z\left(\bar{\theta}_1\right)\right)\right. \\ &\qquad \left.- H_2\left(h,\bar{\theta}_2,u\left(\bar{\theta}_2\right),v\left(\bar{\theta}_2\right),z\left(\bar{\theta}_2\right)\right)\right| \\ &\leqslant (\lambda+4Nh)|\theta_1-\theta_2| \\ &\leqslant (\lambda+4Nh)(1-4Nh^2)^{-1}|\theta_1-\theta_2| \leqslant |\theta_1-\theta_2|\end{aligned}$$

if h is such that $(\lambda + 4Nh)(1 - 4Nh^2)^{-1} \leqslant 1$, which is true if h is sufficiently small since $\lambda < 1$. Thus $\tilde{u}(\theta)$ is a function in $\mathscr{B}_W$. Similarly

$$\begin{aligned}|\tilde{v}(\theta_1)-\tilde{v}(\theta_2)| &\leqslant \|C\|\left|v\left(\bar{\theta}_1\right)-\left(\bar{\theta}_2\right)\right| \\ &\quad + h^{3/2}\left|H_3\left(h,\bar{\theta}_1,u\left(\bar{\theta}_1\right),v\left(\bar{\theta}_1\right),z\left(\bar{\theta}_1\right)\right)-H_3\left(h,\bar{\theta}_2,u\left(\bar{\theta}_2\right),v\left(\bar{\theta}_2\right),z\left(\bar{\theta}_2\right)\right)\right| \\ &\leqslant (1-hc)|\theta_1-\theta_2| + 4Nh^{3/2}|\theta_1-\theta_2| \leqslant |\theta_1-\theta_2|\end{aligned}$$

if h is such that $(1 - hc + 4Nh^{3/2})(1 - 4Nh^2)^{-1} < 1$. This shows that $\tilde{v}(\theta)$ belongs to $\mathscr{B}_Y$.

We now show that P_z is a contraction map. To that end, let (u_1, v_1) and (u_2, v_2) be two pairs of functions in $\mathscr{B}_W \times \mathscr{B}_Y$. Fix θ and let θ_1 and θ_2 be such that

$$\theta = \theta_1 + h + h^2 H_1(h,\theta_1,u_1(\theta_1),v_1(\theta_1),z(\theta_1)), \tag{18a}$$

$$\theta = \theta_2 + h + h^2 H_1(h,\theta_2,u_2(\theta_2),v_2(\theta_2),z(\theta_2)). \tag{18b}$$

Subtracting we obtain

$$|\theta_1-\theta_2| \leqslant h^2 N\left(4|\theta_1-\theta_2| + \|u_1-u_2\| + \|v_1-v_2\|\right);$$

here $\|\cdot\|$ denotes the sup norm, i.e., $\|u\| = \max_{0\leqslant\theta\leqslant 2\tau}|u(\theta)|$. Thus $|\theta_1 - \theta_2| \leqslant h^2K(\|u_1 - u_2\| + \|v_1 - v_2\|)$ where $K = N(1 - 4Nh^2)^{-1}$. From

$$\tilde{u}_1(\theta) = \Lambda u_1(\theta_1) + hH_2(h,\theta_1,u_1(\theta_1),v_1(\theta_1),z(\theta_1)), \tag{19a}$$

$$\tilde{u}_2(\theta) = \Lambda u_2(\theta_2) + hH_2(h,\theta_2,u_2(\theta_2),v_2(\theta_2),z(\theta_2)), \tag{19b}$$

we obtain

$$\begin{aligned}|\tilde{u}_1(\theta)-\tilde{u}_2(\theta)| &\leqslant \lambda\left(\|u_1-u_2\| + |\theta_1-\theta_2|\right) \\ &\qquad + hN\left(4|\theta_1-\theta_2| + \|u_1-u_2\| + \|v_1-v_2\|\right).\end{aligned}$$

Thus

$$\|\tilde{u}_1-\tilde{u}_2\| \leqslant \lambda_1\|u_1-u_2\| + hK_1\|v_1-v_2\| \tag{20}$$

where $\lambda_1 = \lambda + O(h)$ and $K_1 > 0$. Similarly we also have

$$\|\tilde{v}_1 - \tilde{v}_2\| \leqslant h^2 K_2 \|u_1 - u_2\| + (1 - hc_1)\|v_1 - v_2\|$$

where $c_1 = c + O(h^{1/2})$ and $K_2 > 0$. For h sufficiently small, the matrix

$$\begin{bmatrix} \lambda_1 & hK_1 \\ h^2 K_2 & 1 - hc_1 \end{bmatrix}$$

has spectrum inside the unit circle. Thus P_z is a contraction map. The fixed point of P_z is a pair of functions $(w(\theta), y(\theta))$ in $\mathscr{B}_W \times \mathscr{B}_Y$ which satisfy (14).

To prove the second part of Lemma 2, we rewrite (15) as

$$z(\theta) = D^{-1}z(\theta_1) - h^{3/2}D^{-1}H_4(h, \theta, w(\theta), y(\theta), z(\theta))$$

and then proceed as in the proof of the first part. The details will be omitted for simplicity.

If Γ is stable, i.e., the characteristic multipliers $\mu_2, \ldots, \mu_n$ of Γ satisfy $|\mu_j| < 1$ for $j = 2, \ldots, n$, Lemma 2 provides an invariant curve for (13).

In the more general case when not all μ_j satisfy $|\mu_j| < 1$, we actually need a stronger result than Lemma 2. More precisely, let $z_1(\theta)$ and $z_2(\theta)$ be two functions in $\mathscr{B}_z$. Let $(w_1(\theta), y_1(\theta))$ and $(w_2(\theta), y_2(\theta))$ be the corresponding functions as provided by Lemma 2. Fix θ and let θ_1 and θ_2 be such that

$$\theta = \theta_1 + h + h^2 H_1(h, \theta_1, w_1(\theta_1), y_1(\theta_1), z_1(\theta_1)), \tag{21a}$$

$$\theta = \theta_2 + h + h^2 H_1(h, \theta_2, w_2(\theta_2), y_2(\theta_2), z_2(\theta_2)), \tag{21b}$$

Solving for $|\theta_1 - \theta_2|$, we obtain

$$|\theta_1 - \theta_2| \leqslant h^2 N\big(4|\theta_1 - \theta_2| + \|w_1 - w_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\big),$$

hence from

$$w_1(\theta) = \Lambda w_1(\theta_1) + hH_2(h, \theta_1, w_1(\theta_1), y_1(\theta_1), z_1(\theta_1)), \tag{22a}$$

$$w_2(\theta) = \Lambda w_2(\theta_2) + hH_2(h, \theta_2, w_2(\theta_2), y_2(\theta_2), z_2(\theta_2)), \tag{22b}$$

we obtain

$$\|w_1 - w_2\| \leqslant \lambda_1 \|w_1 - w_2\| + hK_1\|y_1 - y_2\| + hK_2\|z_1 - z_2\|$$

for some $\lambda_1 < 1$ and K_1, $K_2 > 0$. Hence

$$\|w_1 - w_2\| \leqslant h(1 - \lambda_1)^{-1}\big(K_1\|y_1 - y_2\| + K_2\|z_1 - z_2\|\big). \tag{23}$$

Similarly we also have

$$\|y_1 - y_2\| \leqslant (1 - hc_1)\|y_1 - y_2\| + h^{3/2}\big(K_3\|w_1 - w_2\| + K_4\|z_1 - z_2\|\big)$$

or

$$\|y_1 - y_2\| \leqslant \big(h^{1/2}/c_1\big)\big(K_3\|w_1 - w_2\| + K_4\|z_1 - z_2\|\big). \tag{24}$$

(23) and (24) implies there exists a constant $K > 0$ such that

$$\|w_1 - w_2\| \leqslant hK\|z_1 - z_2\|, \tag{25a}$$

$$\|y_1 - y_2\| \leqslant h^{1/2}K\|z_1 - z_2\|. \tag{25b}$$

Conversely, suppose two pairs (w_1, y_1) and (w_2, y_2) in $\mathscr{B}_W \times \mathscr{B}_Y$ are given, then Lemma 2 provides two functions z_1, z_2 in $\mathscr{B}_W$ which satisfy

$$\|z_1 - z_2\| \leqslant (1 - hd_1)\|z_1 - z_2\| + h^{3/2}\big(K_5\|w_1 - w_2\| + K_6\|y_1 - y_2\|\big)$$

hence

$$\|z_1 - z_2\| \leqslant h^{1/2}K\big(\|w_1 - w_2\| + \|y_1 - y_2\|\big) \tag{26}$$

where $K > 0$ is a constant which is chosen to be the same as the constant given in (25) for simplicity.

We now construct a sequence of functions $(w^{(k)}, y^{(k)}, z^{(k)})$ in $\mathscr{B}_W \times \mathscr{B}_Y \times \mathscr{B}_Z$ as follows:

First set $z^{(0)}(\theta) \equiv 0$. Applying the first part of Lemma 2 to $z^{(0)}(\theta)$, we obtain a pair of functions $w^{(0)}(\theta)$ and $y^{(0)}(\theta)$ in $\mathscr{B}_W$ and $\mathscr{B}_Y$ respectively. Applying the second part of Lemma 2 to $w^{(0)}(\theta)$ and $y^{(0)}(\theta)$, we obtain a function $z^{(1)}(\theta)$ in $\mathscr{B}_Z$. The above process is repeated to obtain a sequence $(w^{(k)}(\theta), y^{(k)}(\theta), z^{(k)}(\theta))$ in $\mathscr{B}_W \times \mathscr{B}_Y \times \mathscr{B}_Z$.

Equations (25) and (26) imply that $w^{(k)}(\theta)$, $y^{(k)}(\theta)$ and $z^{(k)}(\theta)$ converge to some functions $w(\theta)$, $y(\theta)$ and $z(\theta)$ respectively. These functions satisfy (12). This means that there are 2τ-periodic functions $r(\theta)$ and $w(\theta)$ which satisfy (11). This, in turn, implies the following

THEOREM 1. Under suitable conditions, the difference equation (2) possesses an invariant curve ψ in the sense that for any point $x = x_k$ on ψ, there exist $(k - 1)$ points $x_{k-1}, \ldots, x_1$ on ψ such that the pointed generated by (2) with starting points $x_1, \ldots, x_k$ stay on ψ.

Remarks. (1) From (12), it follows that the invariant curve satisfies $r \leqslant NH^{1/2}$ and $w \leqslant Nh$ for some $N > 0$ in the scaled moving coordinate system. This implies that ψ is in some $0(h)$-neighborhood of Γ and hence will converge to Γ as $h \to 0$.

(2) The assumption that the method employed in (2) is explicit is only for convenience. Since the existence of the invariant curve is based on the Contraction Mapping Principle, the proof of Theorem 1 still goes through even if an implicit method is used. Similarly, if the polynomial $g(\lambda) = \lambda^k - \sum_{j=1}^{k} a_j \lambda^{k-j}$ has some repeated roots, we only need to change $E(\theta)$ in Lemma 1. This can be done by letting $\eta_i = [v_{i1} \cdots v_{ik}]^\Gamma$, $i = 1, \ldots, k$ be the generalized eigenvectors of the companion matrix of $g(\lambda)$. By rearranging the subscripts if necessary, we may assume that η_1 is the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. We then define

$$E(\theta) = \begin{bmatrix} v_{21}P(\theta) & \cdot & \cdot & v_{k1}P(\theta) \\ \vdots & \vdots & \vdots & \vdots \\ v_{2k}P(\theta) & \cdot & \cdot & v_{kk}P(\theta) \end{bmatrix}.$$

The matrix Λ in (10) is no longer diagonal but still satisfies $\|\Lambda\| \leqslant \lambda \leqslant 1$. The proofs of Lemma 2 and Theorem 1 still go through as before.

(3) If the characteristic multipliers of Γ satisfy $|\mu_j| < 1$ for $j = 2,\dots,k$ then for sufficiently small h, ψ is attracting in the sense that if the starting points are compatible (see [6]) and suffiiently close to ψ, then the points generated by (2) will spiral toward ψ. It is then natural to discuss about the rate of convergence. As can be seen from the proof of Lemma 2, this rate is determined by the matrix

$$\begin{bmatrix} \lambda_1 & hK_1 \\ h^2K_2 & 1 - hc_1 \end{bmatrix}$$

whose spectral radius is

$$1 - hc_1 + O(h^2) = 1 - hc + O(h^{3/2}).$$

This means that different methods yield roughly the same rate of convergence.

Similarly, let $L = [\tau/h]$ = greatest integer $\leqslant \tau/h$. The mapping M^L is governed by the matrix

$$\begin{bmatrix} \lambda_1 & hK_1 \\ h^2K_2 & 1 - hc_1 \end{bmatrix}^L$$

whose spectral radius is $e^{-c} + O(h)$. This implies that the rate of convergence is also roughly the same if the step size is varied.

References

[1] M. Braun and J. Hershenov, *Periodic solutions of finite difference equations*, Quart. Appl. Math. **35**, 139–147 (1977)

[2] H. T. Doan, *Invariant curves for numerical methods and the Hopf bifurcation*, Ph.D. dissertation, Michigan State University, 1982

[3] K. Georg, *Numerical integration of the equation*, Numerical Solutions of Nonlinear Equations, Springer Lecture Notes, vol. 878, 1981

[4] K. P. Hadeler, *Computation of periodic orbits and bifurcation diagrams*, Numer. Math. **34**, 439–455 (1980)

[5] J. K. Hale, *Ordinary differential equations*, Wiley-Interscience, New York, 1969

[6] P. Henrici, *Discrete variable methods in ordinary differential equations*, John Wiley and Sons, New York, 1962

[7] O. E. Lansford, *Bifurcation of periodic solutions into invariant tori: the work of Ruelle and Takens*, Nonlinear Problems in the Physical Sciences and Biology, Springer Lecture Notes, vol. 322, 1972

[8] I. B. Schwartz, *Existence of unstable periodic orbits*, SIAM J. Numer. Anal. **20**, 107–120 (1983)