Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex

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Abstract. In this paper, we derive an explicit group-invariant formula for the Euler-Lagrange equations associated with an invariant variational problem. The method relies on a group-invariant version of the variational bicomplex induced by a general equivariant moving frame construction, and is of independent interest.

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1. Introduction.

Most modern physical theories begin by postulating a symmetry group and then establishing field equations based on an invariant variational problem. As first recognized by Sophus Lie, [**24**], every invariant variational problem can be written in terms of the differential invariants of the symmetry group. The associated Euler-Lagrange equations inherit the symmetry group of the variational problem, and so can also be written in terms of the differential invariants. Surprisingly, to date no-one has found a general group-invariant formula that enables one to directly construct the Euler-Lagrange equations from the invariant form of the variational problem. A few specific examples, including plane curves and space curves and surfaces in Euclidean geometry, are worked out in Griffiths, [**18**], and in Anderson's notes, [**3**], on the variational bicomplex.

Example 1.1. To motivate the general formula, let us review the very simplest geometrical example. The *proper Euclidean group* consists of all orientation-preserving rigid motions. In the planar case, the group element $g = (\phi, a, b) \in \mathrm{SE}(2) \simeq \mathrm{SO}(2) \ltimes \mathbb{R}^2$ transforms a point $z = (x, u) \in \mathbb{R}^2$ to the point $w = (y, v) = g \cdot z$ with coordinates

$$y = x\cos\phi - u\sin\phi + a, \qquad v = x\cos\phi + u\sin\phi + b. \tag{1.1}$$

A complete system of differential invariants for smooth planar curves $C = \{\, z(t)\,\} \subset \mathbb{R}^2$ consists of the Euclidean curvature κ and its successive derivatives $\kappa_n = \mathcal{D}^n\kappa$, where $\mathcal{D} = D_s$ denotes invariant differentiation with respect to the standard arc length element $\omega = ds = \|\, \dot z\,\|\, dt$. Every Euclidean-invariant variational problem has the form

$$\mathcal{I}[u] = \int \widetilde{L}(\kappa, \kappa_s, \kappa_{ss}, \ldots)\, ds, \tag{1.2}$$

where the differential invariant $\widetilde{L}$ is called the *invariant Lagrangian*. Since the Euler-Lagrange equation $\mathbf{E}(L) = 0$ for the usual Lagrangian $L = \widetilde{L}\,\|\,\dot z\,\|$ is Euclidean-invariant, it can also be written in terms of the curvature invariants:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0. \tag{1.3}$$

The basic problem is to go directly from the invariant form (1.2) of the variational problem to the invariant form (1.3) of its Euler-Lagrange equation. For example, the Euler-Lagrange equation for the arc length functional $\int ds$ is $-\kappa = 0$, whose solutions are straight lines — the arc length minimizing planar curves. Similarly, the variational problem $\int \frac{1}{2}\kappa^2\, ds$ describes the Euler elastica, [**12**]. Its Euler-Lagrange equation, $\kappa_{ss} + \frac{1}{2}\kappa^3 = 0$, can be solved in terms of elliptic functions, [**26**], and was their historical origin.

Even with these particular examples in hand, the general formula connecting (1.2) to (1.3) is not at all obvious. As established in [**3, 18**], the Euler-Lagrange equation for the general Euclidean invariant variational problem (1.2) takes the form

$$(\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(\widetilde{L}) + \kappa\, \mathcal{H}(\widetilde{L}) = 0, \tag{1.4}$$

where

$$\mathcal{E}(\widetilde{L}) = \sum_n (-\mathcal{D})^n \frac{\partial L}{\partial \kappa_n}, \qquad \mathcal{H}(\widetilde{L}) = \sum_{i>j} \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \widetilde{L}}{\partial \kappa_i} - \widetilde{L},$$

are, respectively, the *invariant Eulerian*, and the *invariant Hamiltonian* of the invariant Lagrangian $\widetilde{L}$, both in direct analogy with the usual formulae for non-invariant higher order Lagrangians, cf. [**3**, **10**, **27**, **34**]. The actual Euler-Lagrange equation (1.4) is obtained by applying certain invariant differential operators to both the Eulerian and Hamiltonian.

The main purpose of this paper is to establish an analogous, general formula relating invariant variational problems to their invariant Euler-Lagrange equations for arbitrary finite-dimensional transformation groups. It turns out that, in all cases, the Euler-Lagrange equations have the invariant form

$$\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0, \tag{1.5}$$

where $\mathcal{E}(\widetilde{L})$ is the invariantized Eulerian, $\mathcal{H}(\widetilde{L})$ a suitable invariantized Hamiltonian, which in the multivariate context is, in fact, a tensor, [**34**], and $\mathcal{A}^*, \mathcal{B}^*$ certain invariant differential operators, which we name the *Eulerian* and *Hamiltonian operators*. Our methods produce an explicit computational algorithm for determining the invariant differential operators $\mathcal{A}^*, \mathcal{B}^*$, that, remarkably, can be constructed from the formulae for the infinitesimal generators of the transformation group action using only linear algebra and differentiation.

This result will be based on combining two powerful ideas in the modern, geometric approach to differential equations and the variational calculus. The first is the *variational bicomplex*, which is of fundamental importance in the study of the geometry of jet bundles, differential equations and the calculus of variations. The origins of the variational bicomplex can be traced back to work of Dedecker, [**9**]. Its modern, general form, originates with Vinogradov, [**37**, **38**, **39**], and Tulczyjew, [**36**]. The later contributions of Tsujishita, [**35**], and Anderson, [**1**, **3**], have amply demonstrated the power and efficacy of the bicomplex formalism for both local and global problems in the geometric theory of partial differential equations and the calculus of variations. The underlying construction† relies on the natural splitting of the space of differential forms on the infinite jet space into horizontal and contact components. This endows the usual deRham complex with the structure of a bicomplex, and so powerful homological algebra machinery, particularly spectral sequences, can be unleashed to compute geometric and topological quantities of interest, including conservation laws, variational structures, null Lagrangians, Euler-Lagrange equations, characteristic cohomology, characteristic classes, etc. In particular, the Euler operator or variational derivative achieves an intrinsic characterization as the corner map of the associated "edge complex".

The second ingredient in our method is Cartan's moving frame theory, [**8**, **17**, **19**, **22**], as extended and generalized in the work of the second author and Fels, [**14**, **15**]. For a general finite-dimensional transformation group G, a *moving frame* is defined as an equivariant map from an open subset of jet space to the Lie group G. Once a moving frame

† The bicomplex construction relies on a choice of local bundle structure on M. An intrinsic version can be formulated in the language of spectral sequences, based on the contact filtration of the cotangent bundle, [**39**]. However, since we shall always deal with local coordinate formulae, we have chosen to use the more down-to-earth bicomplex version here. We refer the reader to Itskov's recent thesis, [**20**, **21**], for the more abstract spectral sequence version.

is established, it provides a canonical mechanism, called *invariantization*, of associating differential functions to differential invariants. Here, we formulate a general invariantization procedure for differential forms on jet space that allows us to systematically construct the invariant counterparts of all objects of interest in the usual variational bicomplex. The resulting *invariant variational complex* provides the proper setting for studying the interplay between symmetry groups and differential equations, variational problems, conservation laws, characteristic cohomology, and so on. For non-projectable group actions, the resulting invariant complex relies on three differentials with nonstandard commutation relations — see (5.13) below — and so is no longer a bicomplex of the usual form. For lack of a better terminology, we name the structure a *quasi-tricomplex*.

The key formula relates the differentials of ordinary differential forms on the jet space to the invariant differentials of invariant forms. In general, invariantization introduces additional "correction terms" similar to the terms that distinguish covariant derivatives in Riemannian geometry from ordinary derivatives. In particular, the recurrence formulae form the basis for a complete, algorithmic classification of the syzygies and commutation formulae for differentiated invariants, as first established in [**15**]. As a byproduct of our general constgruction, the invariant version of the vertical bicomplex differential will then produce the desired formula relating invariant variational problems and invariant Euler-Lagrange equations. The final formula is not elementary; nevertheless, we establish an explicit computational algorithm based only on infinitesimal data. It is worth emphasizing that the required computations only involve linear algebra and differentiation. In particular, they do not require the explicit formulae for either the moving frame or the differential invariants, and hence can be readily automated with symbolic manipulation. (In contrast, the explicit moving frame normalizations typically require manipulating rational algebraic functions — the "Achilles heel" of all current computer algebra systems!) Our own computations have been implemented in both MATHEMATICA and MAPLE, and the results compared in order to give added assurance of their overall correctness. Since then, the first author has implemented a number of the moving frame algorithms in the general MAPLE software package VESSIOT, [**2**], developed by Ian Anderson and his students.

2. The Variational Bicomplex.

We begin with a brief review of the variational bicomplex, relying primarily on the formulation in [**1, 3, 35**]. See also [**29, 39, 40**] for basic results on jet bundles, contact forms, contact transformations, prolongation, etc.

Given a manifold M, we let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact. In particular, $\mathrm{J}^0 = M$. The infinite jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M,p)$ is defined as the inverse limit of the finite order jet bundles under the standard projections $\pi^{n+1}_n \colon \mathrm{J}^{n+1} \to \mathrm{J}^n$. The individual jet fibers $\mathrm{J}^n|_z = (\pi^n_0)^{-1}\{z\}$ are identified as generalized Grassmann manifolds, [**27**]. Differential functions, meaning functions $F\colon \mathrm{J}^n \to \mathbb{R}$ defined on an open subset of jet space, and differential forms on J^n will be routinely identified with their pull-backs to the appropriate open subset of the infinite jet space.

When we introduce local coordinates $z = (x, u)$ on M, we consider the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. The induced local coordinates on the jet bundle J^∞ are denoted by $z^{(\infty)} = (x, u^{(\infty)})$, consisting of independent variables x^i, dependent variables u^α, and their derivatives u^α_J, $\alpha = 1, \ldots, q$, $0 < \#J$, of arbitrary order. Here $J = (j_1, \ldots, j_k)$, with $1 \leq j_\nu \leq p$, is a symmetric multi-index of order $k = \#J$. Coordinates $z^{(n)} = (x, u^{(n)})$ on the jet bundle J^n are obtained by truncating at order n.

A differential form θ on $\mathrm{J}^\infty(M, p)$ is called a *contact form* if it is annihilated by all jets, so that $\theta \mid \mathrm{j}_\infty S = 0$ for every p-dimensional submanifold $S \subset M$. The subbundle of the cotangent bundle $T^*\mathrm{J}^\infty$ spanned by the contact one-forms will be called the *contact* or *vertical subbundle*, denoted by $\mathcal{C}^{(\infty)}$. In local coordinates $(x, u^{(\infty)})$, every contact one-form can be written as a linear combination of the *basic contact forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^p u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \ldots, q, \qquad 0 \leq \#J. \tag{2.1}$$

On the other hand, the coordinate one-forms dx^i span the *horizontal subbundle*, denoted by H, and thereby induce a splitting $T^*\mathrm{J}^\infty = \mathrm{H} \oplus \mathcal{C}^{(\infty)}$ of the cotangent bundle. Any one-form ϖ on J^∞ can be uniquely decomposed into horizontal and vertical (contact) components, $\varpi = \pi_H(\varpi) + \pi_V(\varpi)$, where $\pi_H : T^*\mathrm{J}^\infty \to \mathrm{H}$ and $\pi_V : T^*\mathrm{J}^\infty \to \mathcal{C}^{(\infty)}$ are the induced horizontal and vertical projections.

The splitting of $T^*\mathrm{J}^\infty$ (which is not true for finite order jet bundles) induces a bigrading of the differential forms on J^∞, known as the *variational bicomplex* owing to its importance in the calculus of variations. The differential d on J^∞ naturally splits into horizontal and vertical components, $d = d_H + d_V$, where d_H increases horizontal degree and d_V increases vertical degree. Closure, $d \circ d = 0$, implies that

$$d_H \circ d_H = 0 = d_V \circ d_V, \qquad\qquad d_H \circ d_V = -\, d_V \circ d_H.$$

In particular, the horizontal or *total differential* of a differential function $F : \mathrm{J}^\infty \to \mathbb{R}$ is the horizontal one-form

$$d_H F = \sum_{i=1}^p (D_i F)\, dx^i, \qquad \text{where} \qquad D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_J u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J} \tag{2.2}$$

denotes the usual total derivative with respect to x^i. Similarly, the vertical differential of a function $F(x, u^{(n)})$ is the contact one-form

$$d_V F = \sum_{\alpha=1}^q \sum_K \frac{\partial F}{\partial u^\alpha_K}\, \theta^\alpha_K. \tag{2.3}$$

A *total differential operator* is intrinsically defined as a vector field on J^∞ which lies in the annihilator of the contact bundle $\mathcal{C}^{(\infty)}$. The total derivatives (2.2) form a basis, and so every total differential operator has the local coordinate form

$$\mathcal{D} = \sum_{i=1}^p Q^i(x, u^{(n)})\, D_i, \tag{2.4}$$

where $Q^1, \ldots, Q^p$ are differential functions. The total differential operator (2.4) acts via Lie differentiation on the basis contact forms:

$$\mathcal{D}(\theta_J^\alpha) = \sum_{i=1}^p Q^i(x, u^{(n)})\, \theta_{J,i}^\alpha. \tag{2.5}$$

A *horizontal coframe* is a collection of p horizontal one-forms

$$\omega^i = \sum_{j=1}^p P_j^i(x, u^{(n)})\, dx^j, \qquad i = 1, \ldots, p, \tag{2.6}$$

that are defined and satisfy the linear independence criterion $\omega^1 \wedge \cdots \wedge \omega^p \neq 0$ on an open subset of J^∞. The dual total differential operators

$$\mathcal{D}_j = \sum_{i=1}^p Q_j^i(x, u^{(n)})\, D_i, \qquad j = 1, \ldots, p, \tag{2.7}$$

are defined so that

$$d_H F = \sum_{j=1}^p (\mathcal{D}_j F)\, \omega^j \tag{2.8}$$

for any differential function $F(x, u^{(n)})$, so that $\left(Q_j^i \right) = \left(P_j^i \right)^{-1}$ is the inverse coefficient matrix. This identity extends to contact one-forms ϑ (but not to horizontal one-forms!),

$$d_H \vartheta = \sum_{j=1}^p \omega^j \wedge \mathcal{D}_j \vartheta. \tag{2.9}$$

Full details of the variational bicomplex construction and a wide range of applications can be found in [**1**, **29**, **35**, **39**, **40**].

3. Moving Frames.

The second main tool in our program is the theory of moving frames as developed in [**15**]. Let us briefly review the principal constructions. The basic framework begins with an r-dimensional Lie group G acting smoothly on the manifold M. We assume, without significant loss of generality, that G acts *locally effectively on subsets*, [**30**], which means that the global isotropy subgroup $G_U^* = \{ g \in G \mid g \cdot z = z \text{ for all } z \in U \}$ of every open $U \subset M$ is a discrete subgroup of G. For analytic actions, this is equivalent to assuming G acts locally effectively on all of M, meaning that G_M^* is discrete.

Let $G^{(n)}$ denote the n^{th} prolongation of G to the jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$ induced by the action of G on p-dimensional submanifolds. The prolonged group transformations are uniquely specified by the requirement that they preserve the contact ideal and so define contact transformations on higher order jet bundles.

Remark: The methods of this paper can be easily adapted to general contact transformation groups, [**29**]. However, Bäcklund's Theorem, cf. [**29, 39**], shows that only in the case of hypersurfaces, $p = m - 1$, $q = 1$, are there (first order) contact transformations that do not arise as prolongations of point transformations. To avoid yet further complications, we have chosen to develop the machinery only in the point transformation case.

The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset where $G^{(n)}$ acts locally freely, and so has prolonged orbits of dimension $r = \dim G$. If the action of G is locally effective on all open subsets of M, then the stabilization theorem, [**30**], implies that $\mathcal{V}^n$ is nonempty for $n \gg 0$ sufficiently large, and, indeed, dense in J^n. Since the latter result is not explicitly formulated in the aforementioned reference, we provide a quick proof thereof.

Theorem 3.1. *Let G be an r-dimensional Lie group that acts locally effectively on each open subset of M. Then, for some $n \le r = \dim G$, the prolonged action is locally free on the open and dense subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

Proof: Given an open subset $U \subset M$, let $s_k = s_k(U)$ denote maximal dimension of the orbits in $J^k(U,p)$. Clearly, $s_1 \le s_2 \le \ldots \le r$, and hence the orbit dimensions stabilize at some order. We let $n = n(U)$ denote the *stabilization order*, meaning the minimal order at which $s_{n-1} < s_n = s_{n+1} = s_{n+2} = \cdots$. Theorem 5.37 in [**29**] asserts that there can exist at most one pseudo-stabilization, that is if $s_k = s_{k+1}$ and $s_m = s_{m+1}$ for $k < m$ then $n \le m$. Moreover, the stabilization theorem, [**30**], implies that the prolonged action becomes locally free on an open subset of some jet space $\mathrm{J}^m(U,p)$ of sufficiently high order m. Putting these two facts together, we conclude that $s_{n(U)} = r$ and that the stabilization order is uniformly bounded, with $n(U) \le r - s_0(U) + 1 \le r$, where the maximal orbit dimension $s_0(U) \ge 1$ since G acts locally effectively on subsets. Thus, in all cases, there is an order $n \le r$ such that the regular subset $\mathcal{V}^n \subset \mathrm{J}^n(M,p)$ is open, and, moreover, its projection $W = \pi_0^n(\mathcal{V}^n)$ is dense in M.

For each $z \in M$, the intersection $\mathcal{V}^n_z = \mathcal{V}^n \cap \mathrm{J}^n|_z$ consists of the points in the jet fiber where the prolonged infinitesimal generators are linearly independent. The prolongation formula, [**27, 29**], tells us that the prolonged infinitesimal generators depend polynomially on the derivative coordinates, and hence the fiber complement $\mathrm{J}^n|_z \setminus \mathcal{V}^n_z$ is an algebraic subset. Therefore, $\mathcal{V}^n_z$ is dense in each fiber unless it is empty. On the other hand $\mathcal{V}^n_z$ is empty if and only if $z \notin W$. Since $\mathcal{V}^n = \bigcup_{z \in W} \mathcal{V}^n_z$, we conclude that $\mathcal{V}^n$ is dense. *Q.E.D.*

Remark: A significant open problem is whether the action is, in fact, free on a (dense) open subset of J^n for large n, assuming that G acts effectively on subsets. This occurs in all known examples. However, establishing either a rigorous proof or explicit counterexample appears to be extremely difficult.

Definition 3.2. An n^{th} order (right-equivariant) *moving frame* is a map $\rho^{(n)} \colon \mathrm{J}^n \to G$ which is (locally) G-equivariant,

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot g^{-1}, \qquad z^{(n)} \in \mathrm{J}^n, \qquad g \in G, \tag{3.1}$$

with respect to the prolonged action $G^{(n)}$ on J^n, and the right multiplication action of G on itself.

Remark: The corresponding left-equivariant moving frame is merely $\widetilde{\rho}^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1}$. Most classical geometrical moving frames are left-equivariant, but the right versions are often easier to compute and the group inversion map provides an easy mechanism for changing one to the other. To be concrete, we shall consistently use right-equivariant moving frames in this paper.

The fundamental existence theorem for moving frames follows, cf. [**15**].

Theorem 3.3. *If G acts on M, then an n^{th} order moving frame exists in a neighborhood of $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

See [**30**] for a complete characterization of *totally singular submanifolds*, meaning those whose jets are singular to all orders, $\mathrm{j}_n S \subset \mathrm{J}^n \setminus \mathcal{V}^n$, and hence admit no moving frame. Every n^{th} order moving frame automatically defines a moving frame $\rho^{(n)} \circ \pi^k_n \colon \mathrm{J}^k \to G$, $k \geq n$, on the higher order jet bundles by composition with the usual jet bundle projections $\pi^k_n \colon \mathrm{J}^k \to \mathrm{J}^n$. We will adopt a uniform notation $\rho \colon \mathrm{J}^\infty \to G$ for the induced moving frame on a suitable open subset of the infinite jet bundle.

The practical construction of a moving frame is based on Cartan's method of *normalization*, [**8**, **15**], which requires the choice of a (local) *cross-section* $\mathcal{K}^n \subset \mathcal{V}^n$ to the group orbits. For expository purposes, we shall assume $\mathcal{K}^n$ is a global cross-section, which may require shrinking the domain $\mathcal{V}^n \subset \mathrm{J}^n$ of regular jets.

Theorem 3.4. *Let G act freely, regularly on $\mathcal{V}^n \subset \mathrm{J}^n$. Let $\mathcal{K}^n \subset \mathcal{V}^n$ be a cross-section to the group orbits. Given $z^{(n)} \in \mathcal{V}^n$, let $g = \rho^{(n)}(z^{(n)})$ be the unique group element that maps $z^{(n)}$ to the cross-section: $g^{(n)} \cdot z^{(n)} = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}^n$. Then $\rho^{(n)} \colon \mathrm{J}^n \to G$ is a right moving frame for the group action.*

One typically chooses a *coordinate cross-section* $\mathcal{K}^n = \{\, z_1 = c_1, \ldots, z_r = c_r \,\}$ obtained by setting $r = \dim G$ of the components of $z^{(n)} = (x, u^{(n)})$ — either independent variables, dependent variables, or their derivatives — to equal constants. We use $w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$ to denote the explicit local coordinate formulae for the prolonged group transformations. Using the same labeling $w_1, \ldots, w_r$ for the transformed cross-section components, the moving frame in Theorem 3.4 is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{3.2}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z^{(n)}$. For simplicity, we shall always assume that we are using a coordinate cross-section to prescribe our moving frame. Non-coordinate cross-sections can also be handled, albeit with some additional complications, by a straightforward adaptation of our basic methods.

Theorem 3.5. *If $g = \rho^{(n)}(z^{(n)})$ is the moving frame solution to the normalization equations* (3.2)*, then the components of*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \tag{3.3}$$

form a complete system of differential invariants on the open subset of J^n where the moving frame is defined.

The components $I_1 = c_1, \ldots, I_r = c_r$ corresponding to the cross-section coordinates $z_1, \ldots, z_r$ are trivial, constant differential invariants, and are known as the *phantom differential invariants*, [**15**]. The remaining $\dim \mathrm{J}^n - r$ differential invariants form a complete system of functionally independent n^{th} order differential invariants for the transformation group G, and are known as the *normalized differential invariants*. Thus, any other n^{th} order differential invariant can, locally, be written uniquely as a function of the fundamental, non-phantom differential invariants of order $\leq n$.

Example 3.6. We shall use the planar action (1.1) of the Euclidean group SE(2) on plane curves $C \subset M = \mathbb{R}^2$ as a running illustrative example. The prolonged group transformations

$$\begin{aligned} y &= x\cos\phi - u\sin\phi + a, & v &= x\sin\phi + u\cos\phi + b, \\ v_y &= \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}\,, & v_{yy} &= \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}\,, \end{aligned} \tag{3.4}$$

etc., are constructed by successively applying implicit differentiation operator

$$D_y = \frac{1}{\cos\phi - u_x\sin\phi}\, D_x \tag{3.5}$$

to v. The classical Euclidean moving frame, [**19**], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{3.6}$$

Solving for the group parameters $g = (\phi, a, b)$ leads to the right-equivariant† moving frame

$$\phi = -\tan^{-1} u_x\,, \qquad a = -\frac{x + u u_x}{\sqrt{1 + u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1 + u_x^2}}\,. \tag{3.7}$$

The corresponding left moving frame is obtained by inversion, $(\widetilde{\phi}, \tilde{a}, \tilde{b}) = (\phi, a, b)^{-1} = (\tan^{-1} u_x, x, u)$. Its translation component $\tilde{a} = x, \tilde{b} = u$ is the point on the curve, while the columns of the rotation matrix with angle $\widetilde{\phi} = \tan^{-1} u_x$ consist of the unit tangent and normal vectors, and thereby recovers the classical Frenet frame, [**19**].

The fundamental normalized differential invariants for the moving frame (3.7) are

$$\begin{aligned} &y \longmapsto H = 0, && v \longmapsto I_0 = 0, && v_y \longmapsto I_1 = 0, \\ &v_{yy} \longmapsto I_2 = \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}\,, && v_{yyy} \longmapsto I_3 = \kappa_s, && v_{yyyy} \longmapsto I_4 = \kappa_{ss} + 3\kappa^3, \end{aligned} \tag{3.8}$$

and so on. In particular, H, I_0, I_1 are the phantom invariants, while $I_2 = \kappa$ is the Euclidean curvature, which forms the basic differential invariant for Euclidean plane curves. Further, $\mathcal{D} = D_s = (1 + u_x^2)^{-1/2} D_x$ is the arc length derivative, which maps differential invariants to higher order differential invariants, and can itself be constructed by applying the moving

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† Actually, this moving frame is only locally equivariant, since there remains an ambiguity of π in the prescription of the rotation angle. For simplicity, we shall ignore this technical point here, referring to [**31**] for a detailed discussion.

frame normalization (3.7) to the implicit differentiation (3.5); see below for details. Later we will see how to explicitly relate the normalized differential invariants I_n to the arc length derivatives $\kappa_m = \mathcal{D}^m \kappa$ of the curvature.

4. The Invariant Bicomplex.

We now build group invariance into the variational bicomplex through our chosen moving frame. The construction of the appropriate "invariantized" bicomplex is aided by the regularization procedure introduced in [**15**]. Given any Lie group action of G on a manifold M, consider *lifted action* $g\cdot(h,z) = (h\cdot g^{-1}, g\cdot z)$ of G on the trivial right principal bundle $\pi\colon \mathcal{B} = G \times M \to M$. No matter how complicated the original action, the lifted action is always regular and free. Moreover, a complete set of functionally independent *lifted invariants* on $\mathcal{B}$ is provided by the components of the evaluation map $w\colon \mathcal{B} \to M$, defined by $w(g,z) = g\cdot z$.

The regularization construction is immediately adapted to the prolonged action of $G^{(n)}$ on the jet bundle J^n. In the infinite jet limit, we introduce the *regularized jet bundle* $\pi\colon \mathcal{B}^\infty = G \times \mathrm{J}^\infty \to \mathrm{J}^\infty$ with lifted prolonged group action

$$g\cdot(h, z^{(\infty)}) = (h\cdot g^{-1}, g^{(\infty)}\cdot z^{(\infty)}), \qquad g \in G, \quad (h, z^{(\infty)}) \in \mathcal{B}^\infty. \tag{4.1}$$

The components of the evaluation map $w = w^{(\infty)}\colon \mathcal{B}^\infty \to \mathrm{J}^\infty$, given by

$$w(g, z^{(\infty)}) = g^{(\infty)}\cdot z^{(\infty)}, \tag{4.2}$$

provide a complete system of lifted differential invariants on $\mathcal{B}^\infty$. This endows the lifted jet space with a double fibration or groupoid, [**25, 33**], structure

$$\begin{array}{ccc} & \mathcal{B}^\infty & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle w} \\ \mathrm{J}^\infty & & \mathrm{J}^\infty. \end{array} \tag{4.3}$$

A moving frame, when prolonged to the infinite jet bundle $\rho\colon \mathrm{J}^\infty \to G$, serves to define a G-equivariant section $\sigma\colon \mathrm{J}^\infty \to \mathcal{B}^\infty$, namely $\sigma(z^{(\infty)}) = (\rho(z^{(\infty)}), z^{(\infty)})$. Now, $\pi\circ\sigma = \mathbb{1}_{\mathrm{J}^\infty}$ is the identity, whereas the components of the composition $I = w\circ\sigma\colon \mathrm{J}^\infty \to \mathrm{J}^\infty$ serve to define the fundamental normalized differential invariants (3.3), namely

$$I(z^{(\infty)}) = w(\rho(z^{(\infty)}), z^{(\infty)}) = \rho(z^{(\infty)})\cdot z^{(\infty)}. \tag{4.4}$$

If $F\colon \mathrm{J}^\infty \to \mathbb{R}$ is any differential function, we let $\widehat{F}(g, z^{(\infty)}) = w^*F = F\circ w$ denote its *lift*, which defines an invariant function on $\mathcal{B}^\infty$, and $\iota(F) = \sigma^*\widehat{F} = I^*F = F\circ I$ its *invariantization*. Geometrically, $\iota(F)$ is the unique differential invariant that agrees with F on the cross-section. In particular, $\iota(I) = I$ for any differential invariant I, and hence invariantization defines a canonical projection (depending upon the moving frame) from the space of differential functions to the space of differential invariants.

In local coordinates $z^{(\infty)} = (x, u^{(\infty)}) = (\ldots x^i \ldots u^\alpha_J \ldots)$, the normalized differential invariants (4.4) associated with the moving frame are given by invariantizing the coordinate functions, and denoted by

$$\begin{aligned} H^i(x,u^{(n)}) &= \sigma^*(y^i) = \iota(x^i), && i=1,\ldots,p, \\ I^\alpha_K(x,u^{(l)}) &= \sigma^*(v^\alpha_K) = \iota(u^\alpha_K), && \alpha=1,\ldots,q, \quad k=\#K\geq 0. \end{aligned} \tag{4.5}$$

In general, the invariantization process is given by

$$\iota\big(\, F(\,\ldots, x^i,\ldots,u^\alpha_J,\ldots\,)\,\big) = F(\,\ldots,H^i,\ldots,I^\alpha_J,\ldots\,). \tag{4.6}$$

Let $\widehat{\boldsymbol{\Omega}}^*$ denote the space of differential forms on $\mathcal{B}^\infty$, which we call *lifted differential forms*. A *coframe* or basis for $\widehat{\boldsymbol{\Omega}}^*$ consists of the pulled-back† horizontal forms $dx^1,\ldots,dx^p$, the contact one-forms θ^α_K, $\alpha=1,\ldots,q,\#K\geq 0$, and the Maurer–Cartan forms $\mu^1,\ldots,\mu^r$ on G. The Cartesian product structure on $\mathcal{B}^\infty = \mathrm{J}^\infty\times G$ induces a bigrading on $\widehat{\boldsymbol{\Omega}}^* = \bigoplus_{k,l}\widehat{\boldsymbol{\Omega}}^{k,l}$, where $\widehat{\boldsymbol{\Omega}}^{k,l}$ denotes the space of forms with k jet components — either dx^i or θ^α_K — and l Maurer–Cartan forms μ^ℓ. We accordingly decompose the differential

$$d = d_J + d_G$$

on $\mathcal{B}^\infty$ into jet and group components, so

$$d_J:\widehat{\boldsymbol{\Omega}}^{k,l}\longrightarrow \widehat{\boldsymbol{\Omega}}^{k+1,l}, \qquad\qquad d_G:\widehat{\boldsymbol{\Omega}}^{k,l}\longrightarrow \widehat{\boldsymbol{\Omega}}^{k,l+1}.$$

This decomposition induces a trivial product bicomplex structure on $\mathcal{B}^\infty$:

$$d_J^2 = 0, \qquad\qquad d_G\, d_J + d_J\, d_G = 0, \qquad\qquad d_G^2 = 0. \tag{4.7}$$

Using the bundle structure on J^∞ induced by a choice of local coordinates, we may further decompose $d_J = d_H + d_V$ into horizontal and vertical (contact) components. The latter induces the structure of the *lifted tricomplex* on $\widehat{\boldsymbol{\Omega}}^*$, with

$$d_H^2 = d_V^2 = d_G^2 = 0, \qquad d_V\, d_H + d_H\, d_V = d_G\, d_H + d_H\, d_G = d_G\, d_V + d_V\, d_G = 0. \tag{4.8}$$

Let $\widehat{\boldsymbol{\Omega}}^*_J = \bigoplus_k \widehat{\boldsymbol{\Omega}}^{k,0}$ denote the space of pure *jet forms* on $\mathcal{B}^\infty$. A jet form may depend on group parameters, but contains no Maurer–Cartan forms. Let $\pi_J:\widehat{\boldsymbol{\Omega}}^*\to\widehat{\boldsymbol{\Omega}}^*_J$ denote the *jet projection*, which annihilates all the Maurer–Cartan forms. Note that

$$\pi_J\circ d = d_J = d_J\circ\pi_J, \qquad \pi_J\circ d_G = 0, \qquad \text{but} \qquad d_G\circ\pi_J\neq 0. \tag{4.9}$$

If Ω is any differential form on J^∞, we call $\widehat{\Omega} = \pi_J w^*\Omega$ the corresponding *lifted jet form* on $\mathcal{B}^\infty$. In view of (4.2), we can identify $\widehat{\Omega}$ with the pull-back of the form Ω under the prolonged group transformations:

$$\widehat{\Omega}\big|_{(z^{(\infty)},g)} = \pi_J w^*\big(\,\Omega\big|_{z^{(\infty)}}\,\big) = g^*\big(\,\Omega\big|_{g\cdot z^{(\infty)}}\,\big), \tag{4.10}$$

† To keep the notation simple, we shall always identify a form on either J^∞ or G and its pull-back to $\mathcal{B}^\infty = G\times\mathrm{J}^\infty$ under the standard Cartesian projections.

which we formally write as $\widehat{\Omega} = g^*\Omega$. Since the group acts on $\mathcal{B}^\infty$ via the product action, both $w^*\Omega$ and its projection $\widehat{\Omega} = \pi_J w^*\Omega$ are G–invariant forms; thus

$$g^*\widehat{\Omega} = \widehat{\Omega} \qquad \text{for any} \qquad g \in G. \tag{4.11}$$

In terms of local coordinates, we introduce the *lifted horizontal forms*

$$d_J\, y^i = \pi_J(dy^i) = \pi_J w^*(dx^i), \qquad\qquad i = 1, \ldots, p, \tag{4.12}$$

and the *lifted contact forms*

$$\begin{aligned} \Theta^\alpha_K = \pi_J w^*(\theta^\alpha_K) &= \pi_J \left(dv^\alpha_K - \sum_{i=1}^p v^\alpha_{Ki}\, dy^i \right) \\ &= d_J\, v^\alpha_K - \sum_{i=1}^p v^\alpha_{Ki}\, d_J\, y^i = d_V\, v^\alpha_K - \sum_{i=1}^p v^\alpha_{Ki}\, d_V\, y^i. \end{aligned} \tag{4.13}$$

When combined with the Maurer–Cartan forms $\mu^1, \ldots, \mu^r$, the forms (4.12), (4.13) constitute a complete coframe on $\mathcal{B}^\infty$.

Warning: If the group action is non-projectable, the differential does *not* decompose properly with respect to the induced trigrading of the space of differential forms, and so this decomposition does *not* define a tricomplex. See below for additional details.

We now use the moving frame section $\sigma : \mathrm{J}^\infty \to \mathcal{B}^\infty$ to pull back the lifted jet forms to produce invariant differential forms on J^∞. The most important definition in this paper tells us how to invariantize an arbitrary differential form.

Definition 4.1. The *invariantization* of a differential form Ω on J^∞ is the invariant differential form

$$\iota(\Omega) = \sigma^*\big(\, \pi_J(w^*\, \Omega)\, \big). \tag{4.14}$$

In other words, we lift Ω to $\mathcal{B}^\infty$, then, formally, set the Maurer–Cartan forms equal to zero, and finally pull-back via our moving frame. In particular, on functions the invariantization map $\iota = (w \circ \sigma)^* = I^*$ is just pull-back by the fundamental invariants as above, (4.6). However, ι does *not* agree with pull-back on differential forms! The reason for this choice of invariantization operator is encapsulated in the following key result.

Lemma 4.2. *The invariantization map ι defines a projection, $\iota^2 = \iota$, from the space of differential forms on J^∞ onto the space of invariant differential forms on J^∞.*

Proof: The fact that $\iota(\Omega)$ is G-invariant follows from the equivariance of the moving frame section σ coupled with (4.11). To prove that invariantization defines a projection, it suffices to show that if $\Omega = g^*\,\Omega$ is any G-invariant form, then

$$\iota(\Omega) = \sigma^*(\pi_J(w^*\,\Omega)) = \Omega,$$

but this follows immediately from (4.10), which implies that[†]

$$\pi_J w^* \Omega = g^* \Omega = \Omega$$

whenever Ω is G-invariant. Applying σ^* to this identity proves the lemma. *Q.E.D.*

Example 4.3. To see the importance of the jet projection π_J for the validity of this result, consider the elementary two-parameter translation group $G \simeq \mathbb{R}^2$, acting on curves in the plane $M = \mathbb{R}^2$ via

$$y = x + a, \qquad\qquad v = u + b.$$

The group has trivial prolonged action, and so a complete set of differential invariants consists of the derivatives $u_k = D_x^k u$, $k = 1, 2, \ldots$. A (right) moving frame $\rho: M \to G$ is obtained by normalizing $y = v = 0$, so that $a = -x, b = -u$, and so the moving frame section is given by

$$\sigma(x, u, u_x, \ldots) = (-x, -u; x, u, u_x, \ldots) \in G \times \mathrm{J}^\infty.$$

The differential form dx is invariant, but

$$w^* (dx) = dx + da, \qquad \text{so} \qquad \sigma^*(w^* (dx)) = 0,$$

whereas

$$\pi_J(w^* (dx)) = dx, \qquad \text{so} \qquad \iota(dx) = \sigma^*(w^* (\pi_J(dx))) = dx.$$

Note that, were we not to include the jet projection in our definition (4.14), the invariantization map would *not* preserve invariant differential forms, and would fail to produce a complete invariant coframe.

Invariantizing the variational bicomplex on J^∞ leads to the *invariant complex*. (We refrain from using the word "invariant bicomplex" for reasons that will become apparent.) In terms of local coordinates $z^{(\infty)} = (x, u^{(\infty)})$, the *invariant horizontal one-forms* are

$$\varpi^i = \sigma^*(d_J\, y^i) = \iota(dx^i). \tag{4.15}$$

The adjective "invariant horizontal" is not meant to imply that these are purely horizontal forms. If we decompose them into horizontal and contact components

$$\varpi^i = \omega^i + \eta^i \qquad \text{where} \qquad \omega^i = \sigma^*(d_H\, y^i), \qquad \eta^i = \sigma^*(d_V\, y^i), \tag{4.16}$$

their horizontal components $\omega^i = \pi_H(\varpi^i) = \sigma^*(d_H\, y^i) \in \boldsymbol{\Omega}^{1,0}$ are the usual contact-invariant horizontal forms, [**15**]. If the group acts non-projectably, the forms ϖ^i include an additional contact "correction" $\eta^i \in \boldsymbol{\Omega}^{0,1}$ that makes them fully invariant one-forms. The fundamental *invariant contact forms* are

$$\vartheta^\alpha_J = \sigma^*(\Theta^\alpha_J) = \iota(\theta^\alpha_J). \tag{4.17}$$

These are, in all cases, genuine contact forms, and do form a basis for the full contact ideal. The invariantization map ι is an exterior algebra morphism, and so can be reconstructed from its action (4.5), (4.15), (4.17) on the fundamental coordinates and one-forms

[†] More correctly, we should write $\pi^*\Omega$ at the end of this equation since the result lives on $\mathcal{B}^\infty$, not J^∞.

Example 4.4. Consider again the planar Euclidean group SE(2) introduced in Example 3.6. To obtain the invariant differential forms, we begin with the lifted horizontal form

$$d_J\, y = (\cos\phi)\, dx - (\sin\phi)\, du = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta,$$

where $\theta = du - u_x\, dx$ is the usual contact form. Pulling back via the moving frame (3.7) leads to the invariant horizontal one-form

$$\varpi = \omega + \eta = \sqrt{1+u_x^2}\; dx + \frac{u_x}{\sqrt{1+u_x^2}}\, \theta, \tag{4.18}$$

which is a sum of the contact-invariant arc length form $\omega = ds = \sqrt{1+u_x^2}\; dx$ along with a contact correction $\eta = u_x\,(1+u_x^2)^{-1/2}\,\theta$ required to make ϖ fully Euclidean-invariant. Similarly, the invariant contact forms

$$\Theta_k = \, d_J\, v_k - v_{k+1}\, d_J\, y = \, d_V\, v_k - v_{k+1}\, d_V\, y = \pi_J(w^*\,(\theta_k))$$

are obtained by pulling back the lifted contact forms via the moving frame. In particular,

$$\begin{aligned}
\Theta &= \, d_V\, v - v_y\, d_V\, y = (\cos\phi - v_y \sin\phi)\,\theta = \frac{\theta}{\cos\phi - u_x \sin\phi}\,,\\
\Theta_y &= \, d_V\, v_y - v_{yy}\, d_V\, y = \frac{\theta_x}{(\cos\phi - u_x\sin\phi)^2} + \frac{(u_{xx}\sin\phi)\,\theta}{(\cos\phi - u_x\sin\phi)^3}\,,
\end{aligned}$$

and so on. Substituting the moving frame formula $\phi = -\tan^{-1} u_x$ produces the normalized invariant contact forms $\vartheta_k = \sigma^*(\Theta_k) = \iota(\theta_k)$, with

$$\begin{gathered}
\vartheta = \frac{\theta}{\sqrt{1+u_x^2}}\,, \qquad\qquad \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}\,,\\
\vartheta_2 = \frac{(1+u_x^2)^2\,\theta_{xx} - 3u_x u_{xx}(1+u_x^2)\,\theta_x + (3u_x^2 u_{xx}^2 - u_x(1+u_x^2)u_{xxx})\,\theta}{(1+u_x^2)^{7/2}}\,,
\end{gathered} \tag{4.19}$$

and so on.

Returning to the general picture, we note the following important fact, proved in [**15**].

Theorem 4.5. *The invariant horizontal and contact one-forms* (4.15), (4.17) *form an invariant coframe on the domain of definition* $\mathcal{V}^\infty \subset \mathrm{J}^\infty$ *of the moving frame.*

In particular, the invariant contact forms ϑ^α_K span the usual contact ideal. On the other hand, for non-projectable actions, the p-dimensional subbundle $\mathcal{H} \subset T^*\mathrm{J}^\infty$ spanned by the invariant horizontal forms is not the same as the horizontal subbundle H spanned by the dx^i. One can interpret the invariant horizontal subbundle $\mathcal{H}$ as defining an alternative, invariant connection on the infinite jet bundle, cf. [**3**].

We can uniquely decompose *any* one-form into a linear combination of the invariant horizontal one-forms $\varpi^1, \ldots, \varpi^p$ and the invariant contact forms. We will call these two components the *invariant horizontal* and *invariant vertical* components of the form. In this manner, the invariant coframe (4.15), (4.17) is used to bigrade the space of differential forms on J^∞. We let $\widetilde{\boldsymbol{\Omega}}^{r,s}$ denote the space of forms of invariant bigrade (r,s), which are linear combinations of wedge products of r invariant horizontal forms ϖ^i and s invariant contact forms ϑ^α_K, and let $\widetilde{\pi}_{r,s}\colon \boldsymbol{\Omega} \to \widetilde{\boldsymbol{\Omega}}^{r,s}$ be the associated projection. The coefficients in the linear combination are differential functions, and the form is invariant if and only if its coefficients are differential invariants.

Invariantization defines a map

$$\iota : \boldsymbol{\Omega}^{r,s} \longrightarrow \widetilde{\boldsymbol{\Omega}}^{r,s} \tag{4.20}$$

that takes an ordinary form of bigrade (r,s) and produces an invariant form of invariant bigrade (r,s). If G acts projectably, then the two bigradings are the same, $\widetilde{\boldsymbol{\Omega}}^{r,s} = \boldsymbol{\Omega}^{r,s}$, although this does *not* imply that the invariantization map (4.20) is trivial! If G acts non-projectably, the invariant bigradation is different from the standard bicomplex bigradation. However, formula (4.16) says that the horizontal and invariant horizontal forms differ only by contact forms and hence the projections,

$$\pi_{r,s}\colon \widetilde{\boldsymbol{\Omega}}^{r,s} \longrightarrow \boldsymbol{\Omega}^{r,s}, \qquad\qquad \widetilde{\pi}_{r,s}\colon \boldsymbol{\Omega}^{r,s} \longrightarrow \widetilde{\boldsymbol{\Omega}}^{r,s}, \tag{4.21}$$

are mutual inverses.

5. Recurrence Formulae.

The most important fact underlying the general construction is that the invariantization map (4.20) does not respect the exterior derivative operator. Thus, in general,

$$d\,\iota(\Omega) \neq \iota(d\,\Omega).$$

This fact is responsible for all of the complications inherent in the study of differential invariants, invariant forms, invariant variational problems, invariant differential equations, and all other quantities associated with the invariant complex. The reason is because ι is *not* a pull-back, owing to the intervening jet projection π_J. For example, if $\iota(x^i) = H^i = c$ is a phantom invariant, then $\iota(dx^i) = \varpi^i \neq d\iota(x^i) = 0$. The *recurrence formulae*, first derived in [**15**] in the particular case of differential invariants, provide the missing "correction terms", namely the difference $d\iota(\Omega) - \iota(d\,\Omega)$.

Remarkably, the required formulae can be algorithmically and explicitly constructed using only infinitesimal information! Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathfrak{g}$ be a basis for the infinitesimal generators of our transformation group. (By local effectiveness, we can unambiguously identify Lie algebra elements and their corresponding vector fields on M.) We prolong each infinitesimal generator to J^∞, and, for brevity, adopt the same notation $\mathbf{v}_\ell$ for the prolonged vector field. We use $\mathbf{v}_\ell(\Omega)$ to denote the Lie derivative of the differential form Ω on J^∞ with respect to the prolonged infinitesimal generator $\mathbf{v}_\ell$.

In local coordinates, the infinitesimal generators take the form

$$\mathbf{v}_\ell = \sum_{i=1}^{p} \xi^i_\ell(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J\geq 0} \varphi^\alpha_{J,\ell}(x,u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \ell = 1,\ldots,r. \tag{5.1}$$

The higher order coefficients $\varphi^\alpha_{J,\ell}$ are recursively constructed from the zero$^{\text{th}}$ order ones, $\xi^i_\ell, \varphi^\alpha_\ell$, by the well-known prolongation formula

$$\varphi^\alpha_{Ji,\ell} = D_i \varphi^\alpha_{J,\ell} - \sum_{j=1}^{p} u^\alpha_{Jj}\, D_i \xi^j_\ell. \tag{5.2}$$

A proof can be found in (5.31) below; see also [**27**, **29**] for the explicit, non-recursive version of this basic formula.

Let $\mu^1, \ldots, \mu^r$ denote the Maurer–Cartan forms dual to the infinitesimal generators (5.1). A key ingredient in our analysis is a generalization of a formula in [**15**; (3.8)], that exploits the duality between infinitesimal generators and Maurer–Cartan forms.

Lemma 5.1. *If $\widehat{\Omega} = \pi_J w^* \Omega$ is a lifted jet form on $\mathcal{B}^\infty$, then*

$$d_G \widehat{\Omega} = \sum_{\ell=1}^{r} \mu^\ell \wedge \pi_J w^* [\mathbf{v}_\ell(\Omega)]. \tag{5.3}$$

Remark: It is important to apply the jet projection π_J to $w^*\Omega$ first, before computing d_G. Dependence on the Maurer–Cartan forms would introduce additional terms in formula (5.3) arising from the Maurer–Cartan structure equations of G.

The proof of Lemma 5.1 straightforwardly follows from the identification (4.10) of w^* with the pull-back by a group element, along with the fact that, by duality, the coefficient of μ^ℓ in $d_G \widehat{\Omega} = d_G (g^*\Omega)$ is obtained by Lie differentiation with respect to its dual infinitesimal generator $\mathbf{v}_\ell$. Details are left to the reader.

The correction terms in the recurrence formulae arise from the moving frame pull-backs $\nu^\ell = \sigma^*\mu^\ell$ of the Maurer–Cartan forms. We invariantly decompose them as

$$\sigma^*\mu^\ell = \nu^\ell = \gamma^\ell + \varepsilon^\ell, \qquad \text{where} \qquad \gamma^\ell = \sum_{i=1}^{p} C^\ell_i\, \varpi^i \in \widetilde{\mathbf{\Omega}}^{1,0}, \quad \varepsilon^\ell = \sum_{\alpha,J} E^{\ell,J}_\alpha\, \vartheta^\alpha_J \in \widetilde{\mathbf{\Omega}}^{0,1}, \tag{5.4}$$

are, respectively, invariant horizontal and invariant contact forms, and the coefficients $C^\ell_i, E^{\ell,J}_\alpha$ are certain differential invariants. We let

$$\lambda^\ell = \alpha^\ell + \beta^\ell, \qquad \text{where} \qquad \alpha^\ell = \sum_{i=1}^{p} A^\ell_i\, dx^i \in \mathbf{\Omega}^{1,0}, \qquad \beta^\ell = \sum_{\alpha,J} B^{\ell,J}_\alpha\, \theta^\alpha_J \in \mathbf{\Omega}^{0,1}, \tag{5.5}$$

be one-forms on J^∞ whose invariantization agrees with the preceding forms:

$$\nu^\ell = \iota(\lambda^\ell), \qquad \gamma^\ell = \iota(\alpha^\ell), \qquad \varepsilon^\ell = \iota(\beta^\ell), \qquad C^\ell_i = \iota(A^\ell_i), \qquad E^{\ell,J}_\alpha = \iota(B^{\ell,J}_\alpha). \tag{5.6}$$

The one-forms $\lambda^\ell, \alpha^\ell, \beta^\ell$ are *not* uniquely determined. Since invariantization is a projection, one could, in fact, choose $\lambda^\ell = \nu^\ell$ — which, however, does *not* mean that $\alpha^\ell = \gamma^\ell$, $\beta^\ell = \varepsilon^\ell$, since these one-forms typically belong to different bigradations of J^∞. Usually, however, there are much simpler choices of α^ℓ, β^ℓ, and hence λ^ℓ, available.

We now state the key result that produces all the recurrence formulae for the invariant derivatives of functions and differential forms.

Lemma 5.2. *If Ω is any differential form on J^∞, then*

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\ell=1}^{r} \nu^\ell \wedge \iota[\mathbf{v}_\ell(\Omega)] = \iota\left(d\,\Omega + \sum_{\ell=1}^{r} \lambda^\ell \wedge \mathbf{v}_\ell(\Omega) \right). \tag{5.7}$$

Proof: This is a straightforward computation based on (4.9), (5.3)

$$\begin{aligned} d\,\iota(\Omega) &= d\,\sigma^* \pi_J w^* \Omega = \sigma^*(d\,\pi_J w^* \Omega) = \sigma^*(\,d_J\, +\, d_G\,) \pi_J w^* \Omega \\ &= \sigma^*\big(\,\pi_J\, d\, w^* \Omega + \,d_G\, \pi_J w^* \Omega\,\big) = \sigma^* \left(\pi_J w^*(d\Omega) + \sum_{\ell=1}^{r} \mu^\ell \wedge \pi_J w^*[\mathbf{v}_\ell(\Omega)] \right) \\ &= \sigma^* \pi_J w^*(d\Omega) + \sum_{\ell=1}^{r} \nu^\ell \wedge \sigma^* \pi_J w^*[\mathbf{v}_\ell(\Omega)] = \iota(d\Omega) + \sum_{\ell=1}^{r} \nu^\ell \wedge \iota[\mathbf{v}_\ell(\Omega)]. \end{aligned}$$

The definition (5.6) of λ^ℓ completes the proof. *Q.E.D.*

We now decompose (5.7) into invariant horizontal and vertical components. An important observation is that the Lie derivative operation does not — unless the vector field is projectable — preserve the bigrading of our complex. While $\mathbf{v}_\ell$ certainly maps contact forms to contact forms, we find that

$$\mathbf{v}_\ell(\,d_H\, x^i) = \mathbf{v}_\ell(dx^i) = d\xi^i_\ell = \,d_H\, \xi^i_\ell + \,d_V\, \xi^i_\ell \tag{5.8}$$

is a combination of horizontal and zero$^{\text{th}}$ order† contact forms, since $\xi^i_\ell(x,u)$ only depends on the base coordinates. Therefore,

$$\text{if } \Omega \in \mathbf{\Omega}^{r,s}, \quad \text{then} \quad \mathbf{v}_\ell(\Omega) \in \mathbf{\Omega}^{r,s} \oplus \mathbf{\Omega}^{r-1,s+1} \quad \text{while} \quad d\Omega \in \mathbf{\Omega}^{r+1,s} \oplus \mathbf{\Omega}^{r,s+1}. \tag{5.9}$$

Now, consider an invariant form $\widetilde{\Omega} = \iota(\Omega) \in \widetilde{\mathbf{\Omega}}^{r,s}$ obtained by invariantization of a differential form $\Omega \in \mathbf{\Omega}^{r,s}$. Using (5.4), (5.7), (5.9), we see that

$$d\,\widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{r+1,s} \oplus \widetilde{\mathbf{\Omega}}^{r,s+1} \oplus \widetilde{\mathbf{\Omega}}^{r-1,s+2}. \tag{5.10}$$

In fact, this decomposition holds even if $\widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{r,s}$ is not actually invariant, since we can still write it as a linear combination of differential functions multiplying wedge products of the invariant one-forms. Equation (5.10) allows us to invariantly decompose the differential into three constituents:

$$d = \,d_{\mathcal{H}}\, + \,d_{\mathcal{V}}\, + \,d_{\mathcal{W}}\,, \tag{5.11}$$

so that

† One minor complication in the generalization of these constructions to contact transformation groups is that $\xi^i_\ell(x,u^{(1)})$ can also depend on first order derivatives, and so (5.8) will contain first order contact forms.

$$
\begin{aligned}
d_{\mathcal{H}}\,\iota(\Omega) &= \iota\left(d_H\,\Omega + \sum_{\ell=1}^{r} \alpha^\ell \wedge \pi_{r,s}[\mathbf{v}_\ell(\Omega)]\right) \in \widetilde{\boldsymbol{\Omega}}^{r+1,s},\\
d_{\mathcal{V}}\,\iota(\Omega) &= \iota\left(d_V\,\Omega + \sum_{\ell=1}^{r} \left\{ \beta^\ell \wedge \pi_{r,s}[\mathbf{v}_\ell(\Omega)] + \alpha^\ell \wedge \pi_{r-1,s+1}[\mathbf{v}_\ell(\Omega)] \right\}\right) \in \widetilde{\boldsymbol{\Omega}}^{r,s+1},\\
d_{\mathcal{W}}\,\iota(\Omega) &= \iota\left(\sum_{\ell=1}^{r} \beta^\ell \wedge \pi_{r-1,s+1}[\mathbf{v}_\ell(\Omega)]\right) \in \widetilde{\boldsymbol{\Omega}}^{r-1,s+2}.
\end{aligned} \tag{5.12}
$$

The final formula requires $r \geq 1$; otherwise, $d_{\mathcal{W}}\,\iota(\Omega) = 0$ for any pure contact form (or differential function) $\Omega \in \widetilde{\boldsymbol{\Omega}}^{0,s}$. In fact, the last term in the second formula is also zero in this situation. These three fundamental identities can be used to deduce all the basic recurrence formulae!

The appearance of the extra differential $d_{\mathcal{W}}$ makes life more complicated, and prevents us from using a lot of the standard bicomplex machinery. Breaking the equation $d^2 = 0$ into its various terms in the invariant bigrading leads to the basic formulae

$$
\begin{aligned}
&d_{\mathcal{H}}^2 = 0, \qquad d_{\mathcal{H}}\,d_{\mathcal{V}} + d_{\mathcal{V}}\,d_{\mathcal{H}} = 0,\\
&d_{\mathcal{W}}^2 = 0, \qquad d_{\mathcal{V}}\,d_{\mathcal{W}} + d_{\mathcal{W}}\,d_{\mathcal{V}} = 0,
\end{aligned}
\qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}}\,d_{\mathcal{W}} + d_{\mathcal{W}}\,d_{\mathcal{H}} = 0. \tag{5.13}
$$

We will call such a structure a *quasi-tricomplex*.

Remark: If G acts projectably, then the formulae simplify. Indeed, the Lie derivative $\mathbf{v}_\ell(\Omega) \in \boldsymbol{\Omega}^{r,s}$ whenever $\Omega \in \boldsymbol{\Omega}^{r,s}$. Therefore, $d_{\mathcal{W}} = 0$, and (5.13) reduce to the usual bicomplex relations for $d_{\mathcal{H}}$, $d_{\mathcal{V}}$. Thus, for projectable group actions, the terminology "invariant variational bicomplex" is accurate.

An important observation is that, in all cases, the edge complex, [**1**, **35**], of the invariant quasi-tricomplex is a genuine complex. Closure along the horizontal edge, consisting of invariant forms of type $(r,0)$ for $0 \leq r \leq p$, is immediate from the first equation in (5.13), while closure along the vertical edge, consisting of equivalence classes $\widetilde{\boldsymbol{\Omega}}^{p,s}/\,d_{\mathcal{H}}\,\widetilde{\boldsymbol{\Omega}}^{p-1,s}$ of invariant forms of type (p,s) for $s \geq 0$, follows from the last equation and the fact that if a form $\Omega \in \widetilde{\boldsymbol{\Omega}}^{p,s}$ is of maximal horizontal degree, then $d_{\mathcal{H}}\,\Omega = 0$, and hence $d_{\mathcal{V}}^2 \equiv 0$ modulo $\operatorname{im} d_{\mathcal{H}}$. In particular, an invariant variational source form $\Omega \equiv d_{\mathcal{V}}\,\lambda \in \widetilde{\boldsymbol{\Omega}}^{p,1}/\,d_{\mathcal{H}}\,\widetilde{\boldsymbol{\Omega}}^{p-1,1}$ satisfies the invariant Helmholtz conditions $d_{\mathcal{V}}\,\Omega \equiv 0$. This edge complex is known as the *invariant variational complex*. Anderson and Pohjanpelto, [**5**], showed that, for projectable actions, the cohomology of the invariant variational complex can be identified with the Lie algebra cohomology of the transformation group; their results were extended to nonprojectable actions by Itskov, [**21**].

Let us investigate how these formulae look in local coordinates. First, if $F(x,u^{(n)}) \in \boldsymbol{\Omega}^{0,0}$ is any differential function, then

$$
d_{\mathcal{H}}\,F = \sum_{i=1}^{p} \mathcal{D}_i F \cdot \varpi^i, \qquad d_{\mathcal{V}}\,F = \sum_{\alpha,J} \mathcal{Q}_\alpha^J F \cdot \vartheta_J^\alpha, \qquad d_{\mathcal{W}}\,F = 0. \tag{5.14}
$$

Here $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are the usual invariant differential operators dual to the contact-invariant coframe, cf. (2.7), while the $\mathcal{Q}^J_\alpha$ are their vertical counterparts. Both map differential invariants to differential invariants. A complete system of higher order differential invariants can be obtained by recursively applying the $\mathcal{D}_i$ to differentiate the lower order differential invariants — specifically those of order $\leq n+1$ where n is the order of the moving frame. See [**15**] for details.

Suppose $I = \iota(F)$ is any differential invariant which is obtained by invariantizing a differential function F. The horizontal identity in (5.12) coupled with (5.5) gives

$$d_{\mathcal{H}}\, I = d_{\mathcal{H}}\, \iota(F) = \iota\left(d_H\, F + \sum_{\ell=1}^{r} \mathbf{v}_\ell(F)\, \alpha^\ell \right) = \sum_{i=1}^{p} \iota(\mathbb{D}_i F)\, \varpi^i, \tag{5.15}$$

where

$$\mathbb{D}_i = D_i + \sum_{\ell=1}^{r} A^\ell_i\, \mathbf{v}_\ell \tag{5.16}$$

is a certain group-induced modification of the total derivative. Therefore, the coefficient of ϖ^i gives the basic recurrence formulae

$$\mathcal{D}_i \iota(F) = \iota(\mathbb{D}_i F), \qquad\qquad i = 1, \ldots, p, \tag{5.17}$$

expressing the differentiated invariants in terms of the normalized invariants. We can then use equation (5.17) to determine the correction terms for the higher order derivatives of any differential invariant:

$$\mathcal{D}_K \iota(F) = \iota(\mathbb{D}_K F), \qquad \text{where} \qquad \mathcal{D}_K = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_m}, \qquad \mathbb{D}_K = \mathbb{D}_{k_1} \cdots \mathbb{D}_{k_m}. \tag{5.18}$$

Warning: Like the invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, the differential operators $\mathbb{D}_1, \ldots, \mathbb{D}_p$ do not necessarily commute.

Choosing F to be one of the coordinate functions x^i, u^α_J, we obtain the recurrence formulae for the fundamental differential invariants (4.5), namely

$$\begin{aligned} d_{\mathcal{H}}\, H^i &= \iota\left(dx^i + \sum_{\ell=1}^{r} \xi^i_\ell\, \alpha^\ell \right) = \varpi^i + \sum_{\ell=1}^{r} \Xi^i_\ell \gamma^\ell, \\ d_{\mathcal{H}}\, I^\alpha_J &= \iota\left(d_H\, u^\alpha_J + \sum_{\ell=1}^{r} \varphi^\alpha_{J,\ell}\, \alpha^\ell \right) = \sum_{i=1}^{p} I^\alpha_{Ji}\, \varpi^i + \sum_{\ell=1}^{r} \Phi^\alpha_{J,\ell}\, \gamma^\ell, \end{aligned} \tag{5.19}$$

where

$$\Xi^i_\ell = \iota(\xi^i_\ell), \qquad\qquad \Phi^\alpha_{J,\ell} = \iota(\varphi^\alpha_{J,\ell}), \tag{5.20}$$

denote the invariantizations of the prolonged infinitesimal generator coefficients (5.1), while the γ^ℓ are the invariant horizontal components of the pulled-back Maurer–Cartan forms (5.4). Therefore,

$$\mathbb{D}_i x^j = \delta^j_i + \sum_{\ell=1}^{r} A^\ell_i \xi^j_\ell, \qquad\qquad \mathbb{D}_i u^\alpha_J = u^\alpha_{Ji} + \sum_{\ell=1}^{r} A^\ell_i \varphi^\alpha_{J,\ell},$$

and so the identities (5.17) produce the known recurrence formulae, [**15**; (13.7)],

$$\mathcal{D}_i H^j = \delta_i^j + \sum_{\ell=1}^{r} C_i^\ell\, \Xi_\ell^j, \qquad\qquad \mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + \sum_{\ell=1}^{r} C_i^\ell\, \Phi_{J,\ell}^\alpha. \tag{5.21}$$

In particular, applying the identities (5.19) to the $r = \dim G$ phantom invariants, we obtain a system of r linear equations that can be used to uniquely determine the one-forms $\gamma^1, \ldots, \gamma^r$ as linear combinations of the invariant horizontal forms $\varpi^1, \ldots, \varpi^p$, and hence, (5.4), can be used to explicitly determine the differential invariant coefficients $C_i^\ell = \iota(A_i^\ell)$.

Similarly, the invariant vertical component in (5.12) yields the identity

$$d_{\mathcal{V}}\, I = d_{\mathcal{V}}\, \iota(F) = \iota\left(d_V\, F + \sum_{\ell=1}^{r} \mathbf{v}_\ell(F)\, \beta^\ell \right) = \sum_{\alpha,J} \iota(\mathbb{E}_\alpha^J F)\, \vartheta_J^\alpha, \tag{5.22}$$

where

$$\mathbb{E}_\alpha^J = \frac{\partial}{\partial u_J^\alpha} + \sum_{\ell=1}^{r} B_\alpha^{\ell,J} \mathbf{v}_\ell \tag{5.23}$$

are certain group-induced modification of the vertical differentiation operators. Therefore, the invariant vertical derivatives of a differential function are given by

$$\mathcal{Q}_\alpha^J \big[\, \iota(F) \,\big] = \iota(\mathbb{E}_\alpha^J F), \qquad\qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{5.24}$$

In particular, for the fundamental differential invariants,

$$\begin{aligned} d_{\mathcal{V}}\, H^i &= \iota\left(\sum_{\ell=1}^{r} \xi_\ell^i\, \beta^\ell \right) = \sum_{\ell=1}^{r} \Xi_\ell^i\, \varepsilon^\ell, \\ d_{\mathcal{V}}\, I_K^\alpha &= \iota\left(\theta_K^\alpha + \sum_{\ell=1}^{r} \varphi_{K,\ell}^\alpha\, \beta^\ell \right) = \vartheta_K^\alpha + \sum_{\ell=1}^{r} \Phi_{K,\ell}^\alpha\, \varepsilon^\ell. \end{aligned} \tag{5.25}$$

Furthermore,

$$\mathbb{E}_\alpha^J(x^j) = \sum_{\ell=1}^{r} B_\alpha^{\ell,J}\, \xi_\ell^j, \qquad\qquad \mathbb{E}_\alpha^J(u_K^\beta) = \delta_\alpha^\beta \delta_K^J + \sum_{\ell=1}^{r} B_\alpha^{\ell,J}\, \varphi_{J,\ell}^\alpha,$$

and so the identities (5.24) yield the explicit formulae

$$\mathcal{Q}_\alpha^J(H^j) = \sum_{\ell=1}^{r} E_\alpha^{\ell,J}\, \Xi_\ell^j, \qquad\qquad \mathcal{Q}_\alpha^J(I_K^\beta) = \delta_\alpha^\beta\, \delta_K^J + \sum_{\ell=1}^{r} E_\alpha^{\ell,J}\, \Phi_{J,\ell}^\alpha, \tag{5.26}$$

for the action of the invariant vertical differentiation operators. As above, the identities (5.25) for the r phantom invariants produce a system of r linear equations that can be used to uniquely determine the one-forms $\varepsilon^1, \ldots, \varepsilon^r$ as linear combinations of invariant contact forms, and hence, as in (5.4), can be used to explicitly determine the required differential invariant coefficients $E_\alpha^{\ell,J} = \iota(B_\alpha^{\ell,J})$.

Next we establish the recurrence formulae for the derivatives of the invariant horizontal forms. Since $\varpi^i = \iota(\,dx^i)$, equation (5.12) implies

$$\begin{aligned}
d_{\mathcal{H}}\,\varpi^i &= \iota\left(\sum_{\ell=1}^{r} \alpha^\ell \wedge d_H\,\xi^i_\ell\right) = \sum_{\ell=1}^{r} \gamma^\ell \wedge \iota(\,d_H\,\xi^i_\ell) = \sum_{\ell=1}^{r}\sum_{k=1}^{p} \iota\left(D_k \xi^i_\ell\right)\gamma^\ell \wedge \varpi^k, \\
d_{\mathcal{V}}\,\varpi^i &= \iota\left(\sum_{\ell=1}^{r} \left[\,\alpha^\ell \wedge d_V\,\xi^i_\ell + \beta^\ell \wedge d_H\,\xi^i_\ell\,\right]\right) = \sum_{\ell=1}^{r} \left[\,\gamma^\ell \wedge \iota(\,d_V\,\xi^i_\ell) + \varepsilon^\ell \wedge \iota(\,d_H\,\xi^i_\ell)\,\right] \\
&= \sum_{\ell=1}^{r}\left[\,\sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\ell}{\partial u^\alpha}\right)\gamma^\ell \wedge \vartheta^\alpha + \sum_{k=1}^{p} \iota(D_k \xi^i_\ell)\,\varepsilon^\ell \wedge \varpi^k\,\right], \qquad (5.27) \\
d_{\mathcal{W}}\,\varpi^i &= \iota\left(\sum_{\ell=1}^{r} \beta^\ell \wedge d_V\,\xi^i_\ell\right) = \sum_{\ell=1}^{r} \varepsilon^\ell \wedge \iota(\,d_V\,\xi^i_\ell) = \sum_{\ell=1}^{r}\sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\ell}{\partial u^\alpha}\right)\varepsilon^\ell \wedge \vartheta^\alpha.
\end{aligned}$$

We remark that the explicit formulae for the one-forms $\gamma^\ell, \varepsilon^\ell$ have already been determined through the use of the phantom differential invariants, cf. (5.19), (5.25). In particular, combining the first identity in (5.27) with (5.4), (5.6) produces

$$d_{\mathcal{H}}\,\varpi^i = \sum_{j<k} Y^i_{jk}\,\varpi^j \wedge \varpi^k, \qquad \text{where} \qquad Y^i_{jk} = \sum_{\ell=1}^{r} \iota\left(A^\ell_j\,D_k \xi^i_\ell - A^\ell_k\,D_j \xi^i_\ell\right). \qquad (5.28)$$

The $(2,0)$ component of this identity, namely $d_H\,\omega^i = \sum_{j<k} Y^i_{jk}\,\omega^j \wedge \omega^k$, produces the explicit commutation formulae

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = -\sum_{i=1}^{p} Y^i_{jk}\,\mathcal{D}_i = \sum_{i=1}^{p} Y^i_{kj}\,\mathcal{D}_i \qquad (5.29)$$

for the invariant differential operators, first found in [**15**; eqs. (9.11), (13.12)].

Finally, we establish the recurrence formulae for the derivatives of the invariant contact forms. Consider the Lie derivative

$$\psi^\alpha_{J,\ell} \equiv \mathbf{v}_\ell(\theta^\alpha_J) = \mathbf{v}_\ell\left(du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{Ji}\,dx^i\right) = d\varphi^\alpha_{J,\ell} - \sum_{i=1}^{p}\left[\,\varphi^\alpha_{Ji,\ell}\,dx^i + u^\alpha_{Ji}\,d\xi^i_\ell\,\right] \qquad (5.30)$$

of a basis contact form with respect to an infinitesimal generator of the group action. Since the result must be a contact form, the horizontal terms in (5.30) vanish, so

$$\sum_{i=1}^{p} \varphi^\alpha_{Ji,\ell}\,dx^i = d_H\,\varphi^\alpha_{J,\ell} - \sum_{j=1}^{p} u^\alpha_{Jj}\,d_H\,\xi^j_\ell, \qquad (5.31)$$

which, incidentally, proves the recursive prolongation formula (5.2). The remaining terms in (5.30) yield the formula

$$\psi^\alpha_{J,\ell} = d_V\,\varphi^\alpha_{J,\ell} - \sum_{i=1}^{p} u^\alpha_{Ji}\,d_V\,\xi^i_\ell. \qquad (5.32)$$

In analogy with (5.2), the contact one-forms $\psi^\alpha_{J,\ell}$ can be regarded as the "vertical prolongation coefficients" of the infinitesimal generator $\mathbf{v}_\ell$. Applying the first identity in (5.12) to the basis invariant contact form $\vartheta^\alpha_J = \iota(\theta^\alpha_J)$, we find

$$d_{\mathcal{H}}\,\vartheta^\alpha_J = \iota(\,d_H\,\theta^\alpha_J) + \sum_{\ell=1}^r \gamma^\ell \wedge \iota(\mathbf{v}_\ell(\theta^\alpha_J)) = \sum_{i=1}^p \varpi^i \wedge \vartheta^\alpha_{Ji} + \sum_{\ell=1}^r \gamma^\ell \wedge \widetilde{\psi}^\alpha_{J,\ell}, \tag{5.33}$$

where $\widetilde{\psi}^\alpha_{J,\ell} = \iota(\psi^\alpha_{J,\ell}) = \iota(\mathbf{v}_\ell(\theta^\alpha_J))$ is the invariantized vertical prolongation coefficient (5.32). On the other hand, if we apply the projection $\widetilde{\pi}_{1,1}$ to the formula (2.9), we deduce

$$d_{\mathcal{H}}\,\vartheta = \sum_{i=1}^p \varpi^i \wedge \mathcal{D}_i\,\vartheta \qquad \text{for any contact form} \qquad \vartheta \in \widetilde{\mathbf{\Omega}}^{0,1}, \tag{5.34}$$

where the invariant differential operators act by Lie differentiation on the contact form ϑ. It is worth emphasizing that formula (5.34) is only valid for *contact* one-forms. Combining (5.4), (5.33), (5.34), we deduce the recurrence formulae

$$\mathcal{D}_i \vartheta^\alpha_J = \vartheta^\alpha_{Ji} + \sum_{\ell=1}^r C^\ell_i\, \widetilde{\psi}^\alpha_{J,\ell} = \iota(\mathbb{D}_i \theta^\alpha_J) \tag{5.35}$$

for the invariant derivatives of the invariant contact forms. Again, these can be iterated to produce higher order invariant derivatives

$$\mathcal{D}_K \vartheta^\alpha_J = \iota(\mathbb{D}_K \theta^\alpha_J). \tag{5.36}$$

Finally, since $d_V\,\theta^\alpha_J = 0$, the remaining two invariant differentials act on contact forms via

$$d_{\mathcal{V}}\,\vartheta^\alpha_J = \iota\left(\sum_{\ell=1}^r \beta^\ell \wedge \mathbf{v}_\ell(\theta^\alpha_J)\right) = \sum_{\ell=1}^r \varepsilon^\ell \wedge \widetilde{\psi}^\alpha_{J,\ell}, \qquad\qquad d_{\mathcal{W}}\,\vartheta^\alpha_J = 0. \tag{5.37}$$

Let us illustrate all the preceding computations with our running example.

Example 5.3. The prolonged infinitesimal generators of the planar Euclidean group SE(2) acting in the standard manner (1.1) on $M = \mathbb{R}^2$ are

$$\mathbf{v}_1 = \partial_x, \qquad\qquad \mathbf{v}_2 = \partial_u,$$
$$\mathbf{v}_3 = -\,u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + (4u_x u_{xxx} + 3u_{xx}^2)\,\partial_{u_{xxx}} + \;\cdots\;.$$

According to (5.17), to compute the invariant arc length derivative $\mathcal{D} = D_s$ of a differential invariant $I = \iota(F)$ in terms of the normalized differential invariants, we require

$$\begin{aligned} \mathcal{D}I = \mathcal{D}\iota(F) = \iota(\mathbb{D}F) &= \iota\left(\,D_x F + A^1\,\mathbf{v}_1(F) + A^2\,\mathbf{v}_2(F)) + A^3\,\mathbf{v}_3(F)\,\right) \\ &= \iota(D_x F) + C^1\,\iota(\mathbf{v}_1(F)) + C^2\,\iota(\mathbf{v}_2(F)) + C^3\,\iota(\mathbf{v}_3(F)), \end{aligned} \tag{5.38}$$

where $C^\ell = \iota(A^\ell)$ are certain differential invariants. To determine C^ℓ we apply the identity (5.38) to the three phantom invariants

$$\iota(x) = H = 0, \qquad\qquad \iota(u) = I_0 = 0, \qquad\qquad \iota(u_x) = I_1 = 0. \tag{5.39}$$

Note that

$$\begin{aligned}
&\iota(\mathbf{v}_1(x)) = \iota(1) = 1, && \iota(\mathbf{v}_2(x)) = \iota(0) = 0, && \iota(\mathbf{v}_3(x)) = \iota(-u) = 0,\\
&\iota(\mathbf{v}_1(u)) = \iota(0) = 0, && \iota(\mathbf{v}_2(u)) = \iota(1) = 1, && \iota(\mathbf{v}_3(u)) = \iota(x) = 0,\\
&\iota(\mathbf{v}_1(u_x)) = \iota(0) = 0, && \iota(\mathbf{v}_2(u_x)) = \iota(0) = 0, && \iota(\mathbf{v}_3(u_x)) = \iota(1+u_x^2) = 1,
\end{aligned}$$

which are just the invariantizations of the coefficients of the first order prolonged infinitesimal generators. Therefore, (5.38) yields the three linear equations

$$0 = \mathcal{D}H = 1 + C^1, \qquad 0 = \mathcal{D}I_0 = I_1 + C^2 = C^2, \qquad 0 = \mathcal{D}I_1 = I_2 + C^3 = \kappa + C^3,$$

and hence

$$C^1 = -1, \qquad C^2 = 0, \qquad C^3 = -\kappa = -I_2. \tag{5.40}$$

We can choose

$$A^1 = -1, \qquad A^2 = 0, \qquad A^3 = -u_{xx},$$

as representatives of the differential invariants (5.40). As a result, the differential operator (5.16) is given explicitly by

$$\mathbb{D} = D_x - \partial_x - u_{xx}\mathbf{v}_3.$$

Note that we can adopt our normalization values $x = u = u_x = 0$ in this formulae without any effect on subsequent invariantization, and so one can use the alternative expression

$$\mathbb{D} = u_{xxx}\,\partial_{u_{xx}} + (u_{xxxx} - 3u_{xx}^3)\,\partial_{u_{xxx}} + (u_{xxxxx} - 10u_{xx}^2 u_{xxx})\,\partial_{u_{xxxx}} + \cdots = \sum_{k=2}^{\infty} V_k\,\partial_{u_k},$$

to generate the recurrence formulae, where, setting $u_k = D_x^k u$,

$$\mathbb{D}u_k = V_k = u_{k+1} - \frac{1}{2}\,u_{xx}\sum_{i=2}^{k-1}\binom{k+1}{i}u_i\,u_{k-i+1}. \tag{5.41}$$

Invariantization of (5.41) gives the key recurrence formulae

$$\mathcal{D}I_k = \iota(V_k) = I_{k+1} - \frac{1}{2}\,I_2\sum_{i=2}^{k-1}\binom{k+1}{i}I_i\,I_{k-i+1},$$

of which the first few are

$$\begin{aligned}
\kappa_s &= \mathcal{D}I_2 = I_3, & \mathcal{D}I_4 &= I_5 - 10I_2^2 I_3,\\
\kappa_{ss} &= \mathcal{D}I_3 = I_4 - 3I_2^3, & \mathcal{D}I_5 &= I_6 - 15I_2^2 I_4 - 10I_2\,I_3^2.
\end{aligned} \tag{5.42}$$

These can be iteratively solved to produce the explicit formulae

$$\begin{aligned}
\kappa &= I_2, & I_2 &= \kappa,\\
\kappa_s &= I_3, & I_3 &= \kappa_s,\\
\kappa_{ss} &= I_4 - 3I_2^3, & I_4 &= \kappa_{ss} + 3\kappa^3,\\
\kappa_{sss} &= I_5 - 19I_2^2 I_3, & I_5 &= \kappa_{sss} + 19\kappa^2\kappa_s,\\
\kappa_{ssss} &= I_6 - 34I_2^2 I_4 - 48I_2 I_3^2 + 57I_2^5, & I_6 &= \kappa_{ssss} + 34\kappa^2\kappa_{ss} + 48\kappa\kappa_s^2 + 45\kappa^5,
\end{aligned} \tag{5.43}$$

relating the fundamental normalized and differentiated Euclidean differential invariants.

Similarly, the vertical recurrence formulae (5.22) can be written as

$$d_{\mathcal{V}}\, I = d_{\mathcal{V}}\, \iota(F) = \iota(\, d_V\, F) + \iota(\mathbf{v}_1(F))\, \varepsilon^1 + \iota(\mathbf{v}_2(F))\, \varepsilon^2 + \iota(\mathbf{v}_3(F))\, \varepsilon^3, \tag{5.44}$$

where the ε^ℓ are certain invariant contact forms. They can be determined by evaluating (5.44) on the phantom differential invariants (5.39):

$$0 = d_{\mathcal{V}}\, H = \varepsilon^1, \qquad 0 = d_{\mathcal{V}}\, I_0 = \vartheta + \varepsilon^2, \qquad 0 = d_{\mathcal{V}}\, I_1 = \vartheta_1 + \varepsilon^3.$$

Therefore

$$\varepsilon^1 = 0, \qquad \varepsilon^2 = -\vartheta = -\iota(\theta), \qquad \varepsilon^3 = -\vartheta_1 = -\iota(\theta_1),$$

and so the basic vertical differentiation formulae is

$$d_{\mathcal{V}}\, \iota(F) = \iota\left(\, d_V\, F - \frac{\partial F}{\partial u}\, \theta - \mathbf{v}_3(F)\, \theta_1 \right) = \iota\left(\sum_{k=1}^{\infty} \frac{\partial F}{\partial u_k}\, \theta_k - \mathbf{v}_3(F)\, \theta_1 \right). \tag{5.45}$$

In particular,

$$d_{\mathcal{V}}\, \kappa = d_{\mathcal{V}}\, I_2 = \vartheta_2, \qquad d_{\mathcal{V}}\, \kappa_s = d_{\mathcal{V}}\, I_3 = \vartheta_3 - 3\,\kappa^2\, \vartheta_1, \qquad d_{\mathcal{V}}\, I_4 = \vartheta_4 - 10\,\kappa\,\kappa_s\, \vartheta_1,$$

and, in general,

$$d_{\mathcal{V}}\, I_k = \vartheta_k - \left(\frac{1}{2} \sum_{i=2}^{k-1} \binom{k+1}{i}\, I_i\, I_{k-i+1} \right) \vartheta_1. \tag{5.46}$$

We can now apply these formulas to compute the arc length derivatives of the invariant contact forms

$$\mathcal{D}\vartheta_k = \iota(\mathbb{D}\theta_k) = \iota\big[\, (D_x - u_{xx}\mathbf{v}_3)\theta_k \,\big] = \vartheta_{k+1} - \kappa\, \iota(\psi_{k,3}) = \vartheta_{k+1} - \kappa\, \widetilde{\psi}_{k,3}.$$

The vertical prolongation coefficients (5.32) are $\psi_{k,\nu} = \mathbf{v}_\nu(\theta_k)$, and so

$$\psi_{k,1} = \psi_{k,2} = 0, \qquad \text{while} \qquad \begin{aligned} \psi_{0,3} &= \mathbf{v}_3(\theta) = u_x\theta, \\ \psi_{1,3} &= \mathbf{v}_3(\theta_x) = 2u_x\theta_x + u_{xx}\theta, \\ \psi_{2,3} &= \mathbf{v}_3(\theta_{xx}) = 3u_x\theta_{xx} + 3u_{xx}\theta_x + u_{xxx}\theta, \end{aligned}$$

and, in general,

$$\psi_{k,3} = \sum_{i=0}^{k} \binom{k+1}{i}\, u_{k+1-i}\, \theta_i, \qquad \widetilde{\psi}_{k,3} = \iota(\psi_{k,3}) = \sum_{i=0}^{k-1} \binom{k+1}{i}\, I_{k+1-i}\, \vartheta_i.$$

In particular,

$$\mathcal{D}\vartheta = \vartheta_1, \qquad \mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\, \vartheta, \qquad \mathcal{D}\vartheta_2 = \vartheta_3 - 3\,\kappa^2\, \vartheta_1 - \kappa\,\kappa_s\, \vartheta, \tag{5.47}$$

and so on. Finally, in the vertical differentiation formula (5.27), the only term that survives is

$$d_{\mathcal{V}}\, \varpi = \iota(-\,\alpha^3 \wedge \theta) = -\,\kappa\,\vartheta \wedge \varpi, \tag{5.48}$$

since, according to (5.5),

$$\alpha^3 = A^3\, dx = -\,u_{xx}\, dx, \qquad \gamma^3 = \iota(\alpha^3) = -\,\kappa\,\varpi.$$

6. Invariant Euler-Lagrange Equations.

We now apply our invariant quasi-tricomplex construction to derive the formulae for the Euler-Lagrange equation associated with an invariant variational problem. As above, we assume throughout that the variational problem is defined on the regular open subset of jet space where the moving frame is well-defined, and work exclusively thereon. One can then appeal to continuity, or, in the analytic category, analytic continuation to apply the resulting formulae to more general invariant Lagrangians.

According to Lie, [**24**, **29**], any G-invariant variational problem can be written in the form $\mathcal{I}[u] = \int \widetilde{L}\, \boldsymbol{\omega}$, where $\boldsymbol{\omega} \in \boldsymbol{\Omega}^{p,0}$ is a contact-invariant volume form and the invariant Lagrangian $\widetilde{L}$ is an arbitrary differential invariant for the group, and hence a function of the fundamental differential invariants and their invariant derivatives. The $(p,0)$ form $\lambda = \widetilde{L}\, \boldsymbol{\omega} \in \boldsymbol{\Omega}^{p,0}$ is called the *Lagrangian form* for the variational problem. The Euler-Lagrange equations admit G as a symmetry group, and so, under suitable nondegeneracy hypotheses, cf. [**29**; Theorem 6.25], can themselves be written in terms of the differential invariants. The problem is to go directly from the formula for the variational problem in terms of the fundamental differential invariants to the corresponding differential invariant formula for the Euler-Lagrange equations.

Let us recall the bicomplex construction of the Euler-Lagrange equations, referring to [**3**, **39**, **40**] for complete details. Given a Lagrangian form $\lambda \in \boldsymbol{\Omega}^{p,0}$, its differential $d\lambda = d_V \lambda \in \boldsymbol{\Omega}^{p,1}$ defines a form of type $(p,1)$. We introduce an equivalence relation on such differential forms $\Theta, \Omega \in \boldsymbol{\Omega}^{p,1}$ as follows:

$$\Theta \sim \Omega \qquad \text{if and only if} \qquad \Theta = \Omega + d_H \Upsilon \qquad \text{for some} \qquad \Upsilon \in \boldsymbol{\Omega}^{p-1,1}.$$

Let

$$\pi_* : \boldsymbol{\Omega}^{p,1} \longrightarrow \mathcal{F}^1 \equiv \boldsymbol{\Omega}^{p,1} / \sim \tag{6.1}$$

denote the induced projection onto the space of equivalence classes. The elements $\Sigma \in \mathcal{F}^1$ are known as *source forms*. A simple integration by parts argument proves that, in local coordinates, every source form has a canonical representative

$$\Sigma \simeq \sum_{\alpha=1}^{q} \Delta_\alpha\, \theta^\alpha \wedge d\mathbf{x}.$$

Thus, in local coordinates, there is a one-to-one correspondence between source forms and q-tuples of differential functions $\Delta = (\Delta_1, \ldots, \Delta_q)$. In applications, a source form is regarded as defining a system of q differential equations $\Delta_\alpha = 0$, $\alpha = 1, \ldots, q$, for the q dependent variables $u = (u^1, \ldots, u^q)$.

The composite map $\delta = \pi_* \circ d : \boldsymbol{\Omega}^{p,0} \to \mathcal{F}^1$ takes a Lagrangian form $\lambda = L[u]\, d\mathbf{x} = L\, dx^1 \wedge \cdots \wedge dx^p$ to its *variational derivative*, which is the source form

$$\delta \lambda \simeq \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\, \theta^\alpha \wedge d\mathbf{x}.$$

The components

$$\mathbf{E}_\alpha(L) = \sum_J (-D)_J \frac{\partial L}{\partial u^\alpha_J}$$

of the source form $\delta\,\lambda$ are the classical Eulerian or Euler-Lagrange expressions associated with the Lagrangian L.

It will be helpful to extend the definition of variational derivative to completely general p forms, thereby allowing Lagrangian forms with contact components. The contact components will play no role in the Euler-Lagrange equations — indeed they vanish when evaluated on jets of sections — but key invariance properties under the group action will be retained by this device. We first extend the definition of the source form projection π_* to allow arbitrary $p+1$ forms, by composition with the the projection onto $\boldsymbol{\Omega}^{p,1}$:

$$\widetilde{\pi}_* = \pi_* \circ \pi_{p,1} \,:\, \boldsymbol{\Omega}^{p+1} = \textstyle\bigoplus_{k\geq 1} \boldsymbol{\Omega}^{p+1-k,k} \longrightarrow \mathcal{F}^1. \tag{6.2}$$

Thus, $\widetilde{\pi}_*$ only uses the $(p,1)$ components of the form. Given any p-form $\widetilde{\lambda} \in \boldsymbol{\Omega}^p = \bigoplus_{k\geq 0} \boldsymbol{\Omega}^{p-k,k}$, we define

$$\delta\,\widetilde{\lambda} = \widetilde{\pi}_*(d\widetilde{\lambda}) = \widetilde{\pi}_*(\,d_V\,\widetilde{\lambda}), \tag{6.3}$$

the horizontal component $d_H\,\widetilde{\lambda}$ being annihilated by the source form projection. Therefore, the extended Euler derivative δ annihilates all contact components in $\widetilde{\lambda}$.

Lemma 6.1. *If* $\pi_{p,0}\lambda = \pi_{p,0}\widetilde{\lambda}$, *then* $\delta\,\lambda = \delta\,\widetilde{\lambda}$.

Given a Lagrangian form $\lambda \in \boldsymbol{\Omega}^{p,0}$, let $\widetilde{\lambda} = \widetilde{\pi}_{p,0}(\lambda) \in \widetilde{\boldsymbol{\Omega}}^{p,0}$ denote its fully invariant counterpart. The forms λ and $\widetilde{\lambda}$ differ only by contact forms. Explicitly, if

$$\lambda = \widetilde{L}\,\omega^1 \wedge \cdots \wedge \omega^p, \qquad \text{then} \qquad \widetilde{\lambda} = \widetilde{L}\,\varpi^1 \wedge \cdots \wedge \varpi^p,$$

where $\widetilde{L}$ is a differential invariant and $\varpi^1, \ldots, \varpi^p$ are the invariant horizontal coframe elements. Lemma 6.1 implies that $\delta\,\lambda = \delta\,\widetilde{\lambda}$ are the same Euler-Lagrange source form.

Remark: The construction of the fully invariant Lagrangian is very reminiscent of Carathéodory's construction of the Cartan form, [**7**, **34**], for a first order multivariate Lagrangian. See also [**28**] for recent developments in the higher order theory.

The computation of the Euler-Lagrange equations of an invariant Lagrangian requires an invariant version of the basic integration by parts formula. This relies on the fact that the horizontal and invariant horizontal differentials agree modulo contact forms:

Lemma 6.2. *If* $\Omega \in \boldsymbol{\Omega}^{r,s}$, *then* $d_H\,\Omega = \pi_{r+1,s}[d_{\mathcal{H}}\,\Omega]$. *Conversely, if* $\widetilde{\Omega} \in \widetilde{\boldsymbol{\Omega}}^{r,s}$, *then* $d_{\mathcal{H}}\,\widetilde{\Omega} = \widetilde{\pi}_{r+1,s}[d_H\,\widetilde{\Omega}]$.

Corollary 6.3. *Two differential forms* $\Omega, \Theta \in \boldsymbol{\Omega}^{p+1}$ *map to the same source form* $\widetilde{\pi}_*(\Omega) = \widetilde{\pi}_*(\Theta) \in \mathcal{F}^1$ *if and only if*

$$\widetilde{\pi}_{p,1}(\Omega) = \widetilde{\pi}_{p,1}(\Theta + d_{\mathcal{H}}\,\Psi) \qquad \text{for some} \qquad \Psi \in \boldsymbol{\Omega}^p. \tag{6.4}$$

As we shall see, the identity (6.4) provides the key to the required invariant integration by parts formulas.

7. Variational Problems for Plane Curves.

Before tackling the completely general situation, let us begin with the simplest possible situation: variational problems for plane curves. Therefore, $M = \mathbb{R}^2$, with $p = q = 1$, i.e., both x and u are scalar variables. As before, we denote derivatives of u by $u_k = D_x^k u$.

An r-dimensional transformation group G acting on the plane is called *ordinary* if its $(r-2)^{\text{nd}}$ prolongation acts transitively on an open subset $\mathcal{V}^{r-2} \subset \mathrm{J}^{r-2}$. Most group are ordinary, [**29**]. Moreover, at the end of the following section, we will learn how to handle the exceptions. By transitivity, we can choose a cross-section to be a point in $\mathcal{V}^{r-2}$, and therefore assume — also for simplicity — that we adopt a "standard normalization"

$$y = c, \qquad u_k = c_k, \qquad k = 0, \ldots, r-2, \tag{7.1}$$

where $c, c_0, \ldots, c_{r-2}$ are the normalization constants, for constructing the moving frame. Let $\kappa = \iota(u_{r-1})$ denote the resulting fundamental differential invariant — the group-invariant curvature. The basic invariant horizontal one-form is

$$\varpi = \iota(dx) = \omega + \eta, \qquad \text{where} \qquad \omega = P(x, u^{(r-2)})\, dx \tag{7.2}$$

is the fundamental contact-invariant one-form — the group-invariant arc length — and $\eta \in \mathbf{\Omega}^{0,1}$ is a contact correction. The dual invariant differentiation operator, or *arc length derivative*, is given by $\mathcal{D} = (1/P) D_x$.

Every contact-invariant Lagrangian has the form

$$\lambda = \widetilde{L}(\kappa^{(n)})\, \omega = L(x, u^{(n)})\, dx, \qquad \text{where} \qquad L = \widetilde{L}\, P, \tag{7.3}$$

cf. (7.2). Here we use $\kappa^{(n)}$ to denote the arc length derivatives $\kappa, \kappa_s, \kappa_{ss}$ of the curvature invariant up to order n. The Euler-Lagrange equation is also G-invariant, and so, under suitable nondegeneracy conditions, is equivalent to an equation

$$\widetilde{F}(\kappa^{(m)}) = 0 \tag{7.4}$$

involving the curvature and its derivatives. The goal is to find a way of computing the function $\widetilde{F}$ directly from the invariant Lagrangian $\widetilde{L}$.

Applying the usual *Euler operator*

$$\mathbf{E} = \sum_{i=0}^{\infty} (-D_x)^i \frac{\partial}{\partial u_i} \tag{7.5}$$

to a (non-invariant) Lagrangian $L(x, u^{(n)})$ produces the Euler-Lagrange equation $\mathbf{E}(L) = 0$. The invariantized *Euler operator* is obtained from (7.5) by replacing total x derivatives by arc length derivatives and the derivatives of u by the corresponding arc length derivatives of κ, so

$$\mathcal{E} = \sum_{i=0}^{\infty} (-\mathcal{D})^i \frac{\partial}{\partial \kappa_i}\,. \tag{7.6}$$

The invariant Euler-Lagrange expression (7.4) is *not* obtained by applying $\mathcal{E}$ to the Lagrangian $\widetilde{L}(\kappa, \kappa_1, \kappa_2, \ldots)$ — see Example 1.1 — although this is one of the ingredients in the final formula.

In order to compute the associated source form in an invariant manner, we need to establish an invariant integration by parts formula. If $\alpha, \beta \in \mathbf{\Omega}^2$ are two-forms then we use the notation $\alpha \equiv \beta$ to indicate that they have the same source form $\widetilde{\pi}_*(\alpha) = \widetilde{\pi}_*(\beta)$, where $\widetilde{\pi}_* \colon \mathbf{\Omega}^2 \to \mathcal{F}^1$ is the source form projection (6.2). According to (6.4), this is equivalent to the invariant condition

$$\widetilde{\pi}_{1,1}(\alpha) = \widetilde{\pi}_{1,1}(\beta + d_{\mathcal{H}}\, \sigma) \qquad \text{for some} \qquad \sigma \in \mathbf{\Omega}^1.$$

If F is any differential function and σ a differential one-form, then

$$d_{\mathcal{H}}\,(F\,\sigma) = d_{\mathcal{H}}\,F \wedge \sigma + F\, d_{\mathcal{H}}\, \sigma, \qquad \text{and so} \qquad -F\, d_{\mathcal{H}}\, \sigma \equiv d_{\mathcal{H}}\, F \wedge \sigma. \tag{7.7}$$

In particular, if we choose $\sigma = d_{\mathcal{V}}\, H$ for some differential function H, then, by (5.13),

$$d_{\mathcal{H}}\, \sigma = d_{\mathcal{H}}\, d_{\mathcal{V}}\, H = -\, d_{\mathcal{V}}\, d_{\mathcal{H}}\, H = -\, d_{\mathcal{V}}\,(\mathcal{D}H\, \varpi).$$

Therefore, (7.7) takes the form

$$F\, d_{\mathcal{V}}\,(\mathcal{D}H) \wedge \varpi \equiv -\,\mathcal{D}F\, d_{\mathcal{V}}\, H \wedge \varpi - F\, \mathcal{D}H\, d_{\mathcal{V}}\, \varpi. \tag{7.8}$$

The identity (7.8) is our basic invariant integration by parts formula.

We begin by replacing the contact-invariant Lagrangian form (7.3) by its fully invariant counterpart

$$\widetilde{\lambda} = \widetilde{L}(\kappa^{(n)})\, \varpi = \widetilde{L}(\kappa^{(n)})\,(\omega + \eta) = \widetilde{\pi}_{1,0}(\lambda) \in \widetilde{\mathbf{\Omega}}^{1,0}. \tag{7.9}$$

Since λ and $\widetilde{\lambda}$ produce the same Euler-Lagrange source form, we can work directly with fully invariant version when computing the Euler-Lagrange equations. In accordance with (6.3), we need to compute

$$d\widetilde{\lambda} = d_{\mathcal{V}}\, \widetilde{\lambda} = d_{\mathcal{V}}\,(\widetilde{L}\, \varpi) = d_{\mathcal{V}}\, \widetilde{L} \wedge \varpi + \widetilde{L}\, d_{\mathcal{V}}\, \varpi = \sum_i \frac{\partial \widetilde{L}}{\partial \kappa_i}\, d_{\mathcal{V}}\, \kappa_i \wedge \varpi + \widetilde{L}\, d_{\mathcal{V}}\, \varpi. \tag{7.10}$$

We apply our integration by parts formula (7.8) repeatedly to the first term. The first iteration uses $F = \partial \widetilde{L}/\partial \kappa_i$ and $H = \kappa_{i-1}$ so that $\mathcal{D}H = \kappa_i$. Therefore,

$$\frac{\partial \widetilde{L}}{\partial \kappa_i}\, d_{\mathcal{V}}\, \kappa_i \wedge \varpi \equiv -\mathcal{D}\left(\frac{\partial \widetilde{L}}{\partial \kappa_i}\right) d_{\mathcal{V}}\, \kappa_{i-1} \wedge \varpi - \frac{\partial \widetilde{L}}{\partial \kappa_i}\, \kappa_i\, d_{\mathcal{V}}\, \varpi.$$

Continuing to integrate the first term by parts, we eventually arrive at the formula

$$d_{\mathcal{V}}\, \widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L})\, d_{\mathcal{V}}\, \kappa \wedge \varpi - \mathcal{H}(\widetilde{L})\, d_{\mathcal{V}}\, \varpi. \tag{7.11}$$

In the first term, $\mathcal{E}(\widetilde{L})$ denotes the invariantized Euler-Lagrange derivative (7.6) of the invariant Lagrangian $\widetilde{L}$, while the second term involves

$$\mathcal{H}(\widetilde{L}) = \sum_{i > j \geq 0} \kappa_{i-j}(-\mathcal{D})^j\, \frac{\partial \widetilde{L}}{\partial \kappa_i} - \widetilde{L} \tag{7.12}$$

which will be called the *invariantized Hamiltonian* of $\widetilde{L}$. This expression forms the invariant counterpart of the usual Hamiltonian

$$\mathbf{H}(L) = \sum_{i>j\geq 0} u_{i-j}(-D_x)^j \frac{\partial L}{\partial u_i} - L \tag{7.13}$$

associated with a (non-invariant) higher order Lagrangian $L(x, u^{(n)})$, cf. [**3**, **10**].

The final step is to use our recurrence formulae to determine explicit formulae for the two remaining $(1,1)$ forms in (7.11). First, since $\kappa = I_{r-1} = \iota(u_{r-1})$, we can use the vertical differentiation formula in (5.25), which reads

$$d_{\mathcal{V}}\, \kappa = \vartheta_{r-1} + \sum_{\ell=1}^{r} \Phi_{r-1,\ell}\, \varepsilon^\ell, \quad \text{where} \quad \varepsilon^\ell = \sum_j E_j^\ell\, \vartheta_j = \sum_j E_j^\ell\, \mathcal{F}_j(\vartheta) \equiv \mathcal{G}^\ell(\vartheta), \tag{7.14}$$

is given in (5.4). The invariant differential operators $\mathcal{F}_j$ express the normalized invariant contact forms $\vartheta_j = \iota(\theta_j)$ as arc length derivatives of the basic invariant zero$^{\text{th}}$ order contact form $\vartheta = \vartheta_0$, and are explicitly determined by iterating the recurrence formulae (5.35) for the invariant contact forms. Therefore, we find

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta), \qquad \text{where} \qquad \mathcal{A} = \mathcal{F}_{r-1} + \sum_{\ell=1}^{r} \Phi_{r-1,\ell}\, \mathcal{G}^\ell \tag{7.15}$$

is a certain invariant differential operator, which will be named the *Eulerian operator* for our transformation group.

On the other hand, according to (5.27),

$$d_{\mathcal{V}}\, \varpi = \sum_{\ell=1}^{r} \left[\iota\left(\frac{\partial \xi_\ell}{\partial u}\right) \gamma^\ell \wedge \vartheta + \iota\,(D_x \xi_\ell)\; \varepsilon^\ell \wedge \varpi \right]. \tag{7.16}$$

Equation (5.4) implies that $\gamma^\ell = C^\ell\, \varpi$, where $C^1, \ldots, C^r$ are certain differential invariants. Therefore, using (7.14),

$$d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi, \qquad \text{where} \qquad \mathcal{B} = \sum_{\ell=1}^{r} \left[\iota(D_x \xi_\ell)\, \mathcal{G}^\ell - \iota\left(\frac{\partial \xi_\ell}{\partial u}\right) C^\ell \right] \tag{7.17}$$

is another invariant differential operator, which we name the *Hamiltonian operator* for the transformation group. Substituting (7.15), (7.17) into (7.11), we find

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\equiv\; \mathcal{E}(\widetilde{L})\, \mathcal{A}(\vartheta) \wedge \varpi - \mathcal{H}(\widetilde{L})\, \mathcal{B}(\vartheta) \wedge \varpi. \tag{7.18}$$

The final stage in the procedure is to integrate both terms by parts in order to move the invariant differential operators $\mathcal{A}, \mathcal{B}$ onto the invariant Eulerian and Hamiltonian in (7.18). For this purpose, formula (5.34) tells us that

$$d_{\mathcal{H}}\, \psi = \varpi \wedge \mathcal{D}\, \psi \tag{7.19}$$

for any contact one-form ψ. Applying this to (7.7), with $\sigma = \psi$ we find

$$F\,\mathcal{D}\psi \wedge \varpi \equiv -\,\mathcal{D}F\,\psi \wedge \varpi \tag{7.20}$$

for any contact one-form ψ and any differential function F. Repeating this process, we see that if

$$\mathcal{P} = \sum_i P_i\,\mathcal{D}^i$$

is an invariant differential operator, where the coefficients P_i are differential invariants, then

$$F\,\mathcal{P}(\vartheta) \wedge \varpi \equiv \mathcal{P}^*(F)\,\vartheta \wedge \varpi, \qquad \text{where} \qquad \mathcal{P}^* = \sum_i (-\mathcal{D})^i \cdot P_i \tag{7.21}$$

is the *formal invariant adjoint* of $\mathcal{P}$ — in direct analogy with the usual formal adjoint of a total differential operator, cf. [**27**].

Applying this result to (7.18), we finally arrive at the desired identity

$$d\widetilde{\lambda} \equiv \big[\,\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L})\,\big]\,\vartheta \wedge \varpi = \delta\,\widetilde{\lambda}, \tag{7.22}$$

where $\mathcal{A}^*, \mathcal{B}^*$ are the formal adjoints of the Eulerian and Hamiltonian differential operators (7.15), (7.17). We conclude that the Euler-Lagrange equation of our invariant Lagrangian is equivalent to the G-invariant differential equation

$$\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L}) = 0. \tag{7.23}$$

Indeed, if we write

$$\pi_{1,1}(\vartheta \wedge \varpi) = W\,\theta \wedge dx, \tag{7.24}$$

where $W(x, u^{(n)})$ is a certain relative differential invariant, cf. [**13**], then (7.22) implies

$$\mathbf{E}(L) = \mathbf{E}(\widetilde{L}\,P) = W \cdot \big[\,\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L})\,\big]. \tag{7.25}$$

Therefore, canceling the extraneous factor W produces the differential invariant form (7.23) the invariant Euler-Lagrange equation in the planar case.

Remark: Since ϑ, ϖ form part of the invariant coframe, they are linearly independent, and hence $W(x, u^{(n)}) \neq 0$, on the domain of definition of the moving frame. Thus, only singular extremals can cause $W(x, u^{(n)}) = 0$ to vanish. When restricted to regular extremals, the Euler-Lagrange equation and its invariant counterpart are completely equivalent differential equations, and so, on nonsingular extremals, the Euler-Lagrange equation can be written in terms of the fundamental differential invariants. The investigation of singular extremals and their role in specific examples would be of interest. In particular, are there examples of invariant variational problems all of whose extremals are singular, and hence cannot be expressed in terms of the differential invariants? For instance, the differential equation $u_{xx} = 0$ admits the projective group SL(3) as a symmetry group, but cannot be written in terms of projective differential invariants, [**29**]. However, the associated variational problem $\int \frac{1}{2} u_x^2\,dx$ is not projectively invariant, and so does not provide such an example.

Example 7.1. For the Euclidean group SE(2), equation (7.15) is given in (5.46), namely $d_{\mathcal{V}}\,\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\vartheta$. Therefore the Eulerian operator is $\mathcal{A} = \mathcal{D}^2 + \kappa^2 = \mathcal{A}^*$, which happens to be "invariantly self-adjoint". On the other hand, formula (7.17) is given in (5.48), so $d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$. Therefore, the Hamiltonian operator $\mathcal{B} = -\,\kappa = \mathcal{B}^*$ is a multiplication operator, and is also invariantly self-adjoint. According to (4.19), (4.18), $\vartheta \wedge \varpi = \theta \wedge dx$, and so $W = 1$ in (7.24). Therefore, the invariant Euler-Lagrange formula (7.25) reduces to the known formula (1.4) for Euclidean plane curves.

Example 7.2. A more substantial example is provided by the geometry of equi-affine planar curves, [**19**]. The equi-affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acts on $M = \mathbb{R}^2$ as area-preserving affine transformations

$$g \cdot (x,u) = (\alpha x + \beta u + a, \gamma x + \delta u + b), \qquad \alpha\delta - \beta\gamma = 1. \tag{7.26}$$

The coordinate cross-section $x = u = u_x = 0, u_{xx} = 1, u_{xxx} = 0$, leads to the classical equi-affine moving frame, cf. [**14**]. The fundamental differential invariant is the equi-affine curvature

$$\kappa = \iota(u_{xxxx}) = \frac{u_{xx}u_{xxxx} - \frac{5}{3}u_{xxx}^2}{u_{xx}^{8/3}}\,. \tag{7.27}$$

The corresponding invariant horizontal form is

$$\varpi = \iota(dx) = \omega + \eta, \qquad \text{where} \qquad \omega = ds = u_{xx}^{1/3}\,dx, \qquad \eta = \frac{u_{xxx}}{3u_{xx}^{5/3}}\,\theta,$$

are, respectively, the standard contact-invariant arc length element and the contact correction required to make ϖ fully equi-affine invariant. The dual invariant differential operator $\mathcal{D} = u_{xx}^{-1/3}\,D_x$ is the equi-affine arc length derivative. All higher order differential invariants are obtained as arc length derivatives of the curvature. We emphasize that the explicit formulae for κ and $\mathcal{D}$ are *not* required to perform the ensuing computations.

Applying our computational algorithm, but suppressing the details, we first find that

$$d_{\mathcal{V}}\,\kappa = \vartheta_4 - \tfrac{5}{3}\,\kappa\,\vartheta_2,$$

where $\vartheta_j = \iota(\theta_j)$ are the normalized invariant contact forms. On the other hand, the recursion formulas imply that

$$\begin{aligned}
&\vartheta_1 = \mathcal{D}\vartheta, \qquad\qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \tfrac{1}{3}\,\kappa\,\vartheta = \big(\mathcal{D}^2 + \tfrac{1}{3}\,\kappa\big)\,\vartheta,\\
&\vartheta_3 = \mathcal{D}\vartheta_2 + \kappa\,\vartheta_1 = \big(\mathcal{D}^3 + \tfrac{4}{3}\kappa\,\mathcal{D} + \tfrac{1}{3}\,\kappa_s\big)\,\vartheta,\\
&\vartheta_4 = \mathcal{D}\vartheta_3 + 2\,\kappa\,\vartheta_2 + \tfrac{1}{3}\,\kappa^2\,\vartheta = \big(\mathcal{D}^4 + \tfrac{10}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \kappa^2\big)\,\vartheta.
\end{aligned}$$

Thus we obtain the equi-affine Eulerian operator as

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta), \qquad \text{where} \qquad \mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2.$$

On the other hand,

$$d_{\mathcal{V}}\varpi = -\tfrac{1}{3}\kappa\,\vartheta + \tfrac{1}{3}\,\vartheta_2 = \mathcal{B}(\vartheta), \qquad \text{where} \qquad \mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa$$

is the Hamiltonian operator. Remarkably, both the Eulerian and Hamiltonian operators are invariantly self-adjoint: $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{B} = \mathcal{B}^*$. Therefore, the Euler-Lagrange equation for an equi-affine invariant Lagrangian $\widetilde{L}(\kappa, \kappa_s, \ldots)\, ds$ takes the invariant form

$$\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L}) = \left(\mathcal{D}^4 + \tfrac{5}{3}\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\kappa_s\mathcal{D} + \tfrac{1}{3}\kappa_{ss} + \tfrac{4}{9}\kappa^2\right)\mathcal{E}(\widetilde{L}) - \left(\tfrac{1}{3}\mathcal{D}^2 - \tfrac{2}{9}\kappa\right)\mathcal{H}(\widetilde{L}) = 0.$$

The equi-affine arc-length functional $\int ds$ with $\widetilde{L} = 1$ has $\mathcal{E}(\widetilde{L}) = 0$, $\mathcal{H}(\widetilde{L}) = -1$, and hence the Euler-Lagrange equation is

$$\mathcal{A}^*(0) - \mathcal{B}^*(-1) = -\tfrac{2}{9}\,\kappa = 0.$$

We conclude that the minimal equi-affine curves are those with zero equi-affine curvature — the conic sections. (The reader might be tempted to wrongly speculate that this is always true: for any planar transformation group G, the minimal G-invariant curves have zero G-invariant curvature. The projective group provides one explicit counterexample.) As another example, the variational problem $\int \kappa\, ds$ has Euler-Lagrange equation

$$\mathcal{A}^*(1) - \mathcal{B}^*(-\kappa) = \tfrac{2}{3}\kappa_{ss} + \tfrac{2}{9}\kappa^2 = 0,$$

the solution to which, [**23**], gives κ as an elliptic function of s. Unlike Euclidean geometry, then, $\lambda = \kappa\, ds$ is not a null Lagrangian, and its value cannot be determined by boundary conditions on the curve.

8. Variational Problems for Curves in Higher Dimensional Manifolds.

Let us next generalize our constructions to the case of curves in higher dimensional manifolds. Thus, we continue to have only $p = 1$ independent variable, but are allowing $q \geq 1$ dependent variables, so $\dim M = 1 + q$. In general, the moving frame construction provides us with a certain number, say m, generating differential invariants $I^1, \ldots, I^m$, such that all higher order differential invariants are obtained by invariant differentiation, $I^\alpha_{,k} = \mathcal{D}^k I^\alpha$, with respect to the contact-invariant one-form ω, which can be viewed as the G-invariant arc length element. The comma in the subscript is to remind us that $I^\alpha_{,k}$ is *not* the same as the normalized differential invariant $I^\alpha_k = \iota(u^\alpha_k)$. We use the notation $I^{(n)}$ to denote the collection of all differentiated invariants $I^\alpha_{,k}$ up to some prescribed order $k \leq n$. It is known, [**29**], that in most situations (the technical hypothesis is that the group acts transitively on M and its prolonged actions do not pseudo-stabilize) $m = q = \dim M - 1$, so there are the same number of generating differential invariants as dependent variables. However, the precise number of generating differential invariants turns out not to be important for our computation.

A general invariant Lagrangian defines a contact-invariant horizontal one-form $\lambda = \widetilde{L}(I^{(n)})\,\omega \in \mathbf{\Omega}^{1,0}$. Let $\varpi = \omega + \eta = \iota(dx)$ be the fully invariant one-form obtained by the moving frame normalization, so that the modified Lagrangian form $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\,\varpi \in \widetilde{\mathbf{\Omega}}^{1,0}$ is fully G-invariant. As before, our goal is to construct the Euler-Lagrange equations directly from the invariant form of the Lagrangian. The same invariant integration by parts method (7.8) applies to the present situation, leading to the initial identity

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \sum_{\alpha=1}^{m} \mathcal{E}_\alpha(\widetilde{L})\, d_{\mathcal{V}}\, I^\alpha \wedge \varpi - \mathcal{H}(\widetilde{L})\, d_{\mathcal{V}}\,\varpi, \tag{8.1}$$

where

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,n}} \,, \qquad\qquad \alpha = 1, \ldots, m, \tag{8.2}$$

is the *invariantized Eulerians* of $\widetilde{L}$, while

$$\mathcal{H}(\widetilde{L}) = \sum_{\alpha=1}^{m} \sum_{i>j} I^\alpha_{,i-j} (-\mathcal{D})^j \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}} \, - \widetilde{L} \tag{8.3}$$

is the *invariantized Hamiltonian*.

In the second stage of the computation, we apply the infinitesimal moving frame calculus to determine the formulae

$$d_{\mathcal{V}} \, I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}^\alpha_\beta(\vartheta^\beta), \qquad d_{\mathcal{V}} \, \varpi = \sum_{\beta=1}^{q} \mathcal{B}_\beta(\vartheta^\beta) \wedge \varpi, \qquad \alpha = 1, \ldots, m, \tag{8.4}$$

for the invariant vertical differentials of the fundamental differential invariants and the invariant horizontal one-form. The *Eulerian operator* $\mathcal{A} = \left(\mathcal{A}^\alpha_\beta \right)$ is an $m \times q$ matrix of invariant differential operators, while the *Hamiltonian operator* $\mathcal{B} = \left(\mathcal{B}_\beta \right)$ is a $1 \times q$ vector of invariant differential operators. Substituting (8.4) into (8.1) and then integrating by parts based on (7.21) leads to the key formula $d\widetilde{\lambda} \equiv \delta\, \widetilde{\lambda}$, where

$$\delta\, \widetilde{\lambda} = \left(\sum_{\alpha=1}^{m} \sum_{\beta=1}^{q} (\mathcal{A}^\alpha_\beta)^* \mathcal{E}_\alpha(\widetilde{L}) - \sum_{\beta=1}^{q} (\mathcal{B}_\beta)^* \mathcal{H}(\widetilde{L}) \right) \vartheta^\beta \wedge \varpi = \left[\, \mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) \,\right] \boldsymbol{\vartheta} \wedge \varpi. \tag{8.5}$$

We conclude that the Euler-Lagrange equations are equivalent to the invariant system of differential equations

$$\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0. \tag{8.6}$$

More explicitly,

$$\mathbf{E}(\lambda) = W \cdot \left[\, \mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) \,\right], \tag{8.7}$$

where the multiplicative matrix-valued relative invariant $W = \left(W^\alpha_\beta \right)$ is obtained from writing

$$\vartheta^\alpha \wedge \varpi \equiv \sum_{\beta=1}^{q} W^\alpha_\beta \, \theta^\beta \wedge dx \tag{8.8}$$

in terms of the non-invariant coframe.

Example 8.1. Consider the usual action $w = R\, z + a$, $z = (x, u, v) \in \mathbb{R}^3$, of the proper Euclidean group $(R, a) \in \mathrm{SE}(3) \simeq \mathrm{SO}(3) \ltimes \mathbb{R}^3$ on space curves $C \subset \mathbb{R}^3$. In order to keep the computation simple, we assume that the curve is parametrized by $(x, u(x), v(x))$. The final formulae, however, do not rely on this special parametrization. As is well known, [**19**], the coordinate cross-section

$$x = u = v = u_x = v_x = v_{xx} = 0 \tag{8.9}$$

produces the classical moving frame. The translation component of its left-equivariant counterpart $(\widetilde{R}, \tilde{a}) = \rho^{(n)}(z^{(n)})^{-1}$ is the point on the curve, $\tilde{a} = z$, while the columns of the rotation matrix $\widetilde{R}$ are the unit tangent, normal, and binormal vectors, cf. [**19**]. (As in the planar case, we are ignoring sign ambiguities; see [**31**] for the complete story.) The fundamental differential invariants consist of the usual Euclidean curvature and torsion invariants

$$\kappa = \iota(u_{xx}), \qquad\qquad \tau = \iota\left(\frac{v_{xxx}}{u_{xx}}\right).$$

The slight complication, in that classical differential geometry chooses a ratio of normalized differential invariants for the second fundamental invariant, can easily be handled during the subsequent computation. The other third order derivative coordinate produces the first differentiated invariant $\iota(u_{xxx}) = \mathcal{D}\kappa = \kappa_s$. Here $\mathcal{D} = D_s$ denotes the derivative with respect to the usual arc length form ds, which is horizontal component of the invariant horizontal one-form $\varpi = \iota(dx)$.

Any Euclidean-invariant variational problem

$$\mathcal{I}[u] = \int \widetilde{L}(\kappa^{(n)}, \tau^{(n)})\, ds$$

can be rewritten in terms of the curvature, torsion and their arc-length derivatives. To determine the Euler-Lagrange equations, we must construct the Eulerian and Hamiltonian operators. Application of our infinitesimal moving frame calculus leads to the required formulae. First,

$$\begin{aligned} d_{\mathcal{V}}\,\kappa &= \iota(\theta^u_{xx}) = D_s^2\vartheta^u + (\kappa^2 - \tau^2)\vartheta^u - 2\tau D_s\vartheta^v - \tau_s\vartheta^v, \\ d_{\mathcal{V}}\,\tau &= \iota\left(\frac{u_{xx}\theta^v_{xxx} - u_{xxx}\theta^v_{xx} - v_{xxx}\theta^u_{xx}}{u_{xx}^2}\right) = \frac{2\tau}{\kappa}D_s^2\vartheta^u + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2}D_s\vartheta^u + \\ &\quad + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2}\,\vartheta^u + \frac{1}{\kappa}D_s^3\vartheta^v - \frac{\kappa_s}{\kappa^2}\,D_s^2\vartheta^v + \frac{\kappa^2 - \tau^2}{\kappa}\,D_s\vartheta^v + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2}\,\vartheta^v, \end{aligned}$$

where $\vartheta^\alpha = \iota(\theta^\alpha)$. Consequently, the Eulerian operator and its adjoint are

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa}D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2}D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa}D_s^3 - \frac{\kappa_s}{\kappa^2}D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa}D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix},$$

$$\mathcal{A}^* = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & \frac{2\tau}{\kappa}D_s^2 + \frac{\kappa\tau_s - 2\kappa_s\tau}{\kappa^2}D_s + 2\kappa\tau \\ 2\tau D_s + \tau_s & -\frac{1}{\kappa}D_s^3 + \frac{2\kappa_s}{\kappa^2}D_s^2 + \frac{\kappa\kappa_{ss} - 2\kappa_s^2 + \tau^2\kappa^2 - \kappa^4}{\kappa^3}D_s - \kappa_s \end{pmatrix}.$$

Second, the simpler formula

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta^u \wedge \varpi$$

produces the Hamiltonian operator

$$\mathcal{B} = \big(-\kappa, 0\big) \qquad \text{so that} \qquad \mathcal{B}^* = \begin{pmatrix} -\kappa \\ 0 \end{pmatrix}.$$

The resulting invariant Euler-Lagrange formula

$$\delta\,\widetilde{\lambda} = \mathcal{A}^* \begin{pmatrix} \mathcal{E}_\ell(\widetilde{L}) \\ \mathcal{E}_\tau(\widetilde{L}) \end{pmatrix} - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0$$

agrees with that derived in Anderson, [**3**]. It is worth emphasizing that these computations do *not* require the explicit formulae for the moving frame and the curvature and torsion invariants, but only rely on the normalization equations (8.9) and the prolonged infinitesimal generators. Extending these constructions to other examples, e.g., space curves in affine geometry or projective geometry, is straightforward.

An important observation is that the computation leading to (8.6) does not require that the differentiated invariants be functionally independent, and so there can be syzygies (functional relations) among them without affecting the final formula. Moreover, there is no restriction on the underlying invariant horizontal one-form $\varpi = \omega + \eta$ provided we use the associated invariant differential operator $\mathcal{D}$. The most natural choice, of course, is to let $\omega = ds$ be the arc length element. However, as first pointed out to us by V. Itskov, [**21**], choosing $\varpi = dK$ where K is a differential invariant leads to a significant simplification in the final result. Indeed, since $d\varpi = d^2 K = 0$, the Hamiltonian operator $\mathcal{B} = 0$ vanishes, and so *there is no Hamiltonian contribution* to (8.6)!

In this case, the dual differential operator D_K to $\varpi = dK$ is differentiation with respect to K, i.e.,

$$D_K F = \frac{dF}{dK} = \frac{D_x F}{D_x K}\,.$$

We supplement the basic differential invariant K with a suitable generating set $I^1, \ldots, I^m$ of differential invariants such that a complete system of higher order differential invariants is given by the derivatives of the I^α with respect to K, namely $I^\alpha_{,n} = D_K^n I^\alpha$. We can regard the differential invariant K as an "independent variable" and the additional differential invariants as "dependent variables", and hence adopt the notation $(K, I^{(n)})$ to denote the collection of all differentiated invariants $K, I^\alpha_{,j}$ for $j \leq n$.

In applying this idea, it is important to keep in mind that the use of

$$dK = d_{\mathcal{H}}\, K + d_{\mathcal{V}}\, K = \mathcal{D}K \cdot \varpi + d_{\mathcal{V}}\, K, \tag{8.10}$$

instead of $\varpi = \iota(\,dx)$ as our basic invariant horizontal form introduces a different invariant bigrading of the variational bicomplex. We denote the consequent invariant horizontal and vertical differentials as $\widetilde{d}_{\mathcal{H}}\,, \widetilde{d}_{\mathcal{V}}\,$, respectively. In particular, if I is a differential invariant, then, in view of (8.10),

$$\widetilde{d}_{\mathcal{H}}\, I = \frac{dI}{dK}\; dK, \qquad\qquad \widetilde{d}_{\mathcal{V}}\, I = d_{\mathcal{V}}\, I - \frac{dI}{dK}\; d_{\mathcal{V}}\, K. \tag{8.11}$$

Theorem 8.2. *The Euler-Lagrange equations for the invariant Lagrangian* $\widetilde{\lambda} = \widehat{L}(K, I^{(n)})\, dK$ *have the invariant form*

$$\widetilde{\mathcal{A}}^*\, \mathcal{E}\big(\,\widehat{L}\,\big) = 0, \tag{8.12}$$

where the invariant Eulerian $\mathcal{E}\big(\widehat{L}\big)$ *has components*

$$\mathcal{E}_\alpha\big(\widehat{L}\big) = \sum_{j=0}^{n} \left(-\frac{d}{dK}\right)^j \frac{\partial \widehat{L}}{\partial I^\alpha_{,j}}\,, \qquad\qquad \alpha = 1,\ldots,m, \tag{8.13}$$

while the Eulerian operator $\widetilde{\mathcal{A}}$ *is defined by*

$$\widetilde{d}_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^{q} \widetilde{\mathcal{A}}^\alpha_\beta(\vartheta^\beta), \qquad\qquad \alpha = 1,\ldots,m. \tag{8.14}$$

Note that if we use the moving frame formulae

$$d_{\mathcal{V}}\, K = \mathcal{K}(\vartheta), \qquad\qquad d_{\mathcal{V}}\, I = \mathcal{L}(\vartheta),$$

where $\mathcal{K}, \mathcal{L}$ are certain invariant differential operators, then (8.11) implies that

$$\widetilde{\mathcal{A}} = \mathcal{L} - \frac{dI}{dK}\ \mathcal{K}, \tag{8.15}$$

in which we replace the original invariant differential operator $\mathcal{D}$ by $(\mathcal{D}K)D_K$. The Euler-Lagrange equations can be obtained from their invariant counterparts by applying the Eulerian operator $\widetilde{\mathcal{A}}$ and multiplying by the appropriate relative invariant factor, which, since

$$\vartheta^\alpha \wedge dK \equiv \mathcal{D}K \cdot \vartheta^\alpha \wedge \varpi,$$

is obtained by multiplying the previous relative invariant (8.8) by $\mathcal{D}K$.

Example 8.3. As an example, we revisit the planar Euclidean case discussed in Example 7.1. Let us take $K = \kappa$ as our "independent differential invariant" and $I = \kappa_s$ as our "dependent differential invariant". The higher order differential invariants are now obtained by successive differentiation of I with respect to κ, which we denote by

$$I_{,n} = \frac{d^n I}{d\kappa^n}\,, \qquad\qquad n = 0, 1, 2, \ldots.$$

These higher order differential invariants are related to the arc length derivatives of κ through the chain rule formula

$$\frac{d}{ds} = \frac{d\kappa}{ds}\,\frac{d}{d\kappa} = I\,\frac{d}{d\kappa}\ .$$

For example, $I_{,1} = I_\kappa = \kappa_{ss}/\kappa$. Applying (5.46), (5.47), we have

$$d_{\mathcal{V}}\, \kappa = (\mathcal{D}^2 + \kappa^2)\vartheta, \qquad\qquad d_{\mathcal{V}}\, \kappa_s = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\vartheta,$$

and so the Eulerian operator (8.15) takes the form

$$\widetilde{\mathcal{A}} = \mathcal{D}^3 + \kappa^2\mathcal{D}^2 + 3\,\kappa\,\kappa_s - \frac{\kappa_{ss}}{\kappa_s}(\mathcal{D}^2 + \kappa^2) = I^3 D^3_\kappa + 2I^2 I_\kappa D^2_\kappa + (I^2 I_{\kappa\kappa} + \kappa^2 I) D_\kappa + 3\,\kappa\, I - \kappa^2 I_\kappa.$$

We write the invariant Lagrangian in the alternative form

$$\widetilde{\lambda} = \widehat{L}(\kappa, I, I_{\kappa}, I_{\kappa\kappa}, \ \ldots\ )\, d\kappa = \widehat{L}(\kappa, I, I_{\kappa}, I_{\kappa\kappa}, \ \ldots\ )\kappa_s\, ds. \tag{8.16}$$

Dropping an overall minus sign, according to Theorem 8.2, the Euler-Lagrange equation has the invariant form

$$\begin{aligned} 0 = -\,\widetilde{\mathcal{A}}^*\, \mathcal{E}\big(\,\widehat{L}\,\big) = \big[\, & I^3 D_\kappa^3 + 7\, I^2 I_\kappa D_\kappa^2 + (6\, I^2 I_{\kappa\kappa} + 10\, I\, I_\kappa^2 + \kappa^2 I) D_\kappa \\ & + 2\, I^3 I_{\kappa\kappa\kappa} + 8\, I\, I_\kappa I_{\kappa\kappa} + 2 I_\kappa^3 + 2\, \kappa^2 I_\kappa - \kappa\, I \,\big]\, \mathcal{E}\big(\,\widehat{L}\,\big), \end{aligned} \tag{8.17}$$

where the invariant Euler expression is

$$\mathcal{E}\big(\,\widehat{L}\,\big) = \sum_{n \geq 0} \left(-\,\frac{d}{d\kappa} \right)^n \frac{\partial \widehat{L}}{\partial I_{,n}} \ .$$

This result was also derived by Itskov, [**21**], using an exterior differential systems approach. Formula (8.17) provides an interesting and attractive alternative to the known version (1.4) of the Euler-Lagrange equation.

9. Invariant Multivariate Lagrangians.

Let us finally tackle the general case of invariant variational problems corresponding to higher dimensional submanifolds. We now allow several independent variables, $p \geq 1$, and several dependent variables $q \geq 1$. Let

$$\varpi^i = \iota(dx^i) = \omega^i + \eta^i, \qquad \omega^i \in \mathbf{\Omega}^{1,0}, \qquad \eta^i \in \mathbf{\Omega}^{0,1}, \qquad i = 1, \ldots, p,$$

be the invariant horizontal coframe obtained by normalization using the moving frame. Let $\boldsymbol{\varpi} = \varpi^1 \wedge \cdots \wedge \varpi^p$ the corresponding fully invariant volume form, whose horizontal component $\boldsymbol{\omega} = \pi_{p,0}(\boldsymbol{\varpi}) = \omega^1 \wedge \cdots \wedge \omega^p$ is the basic contact-invariant volume form. Define the $(p-1)$–forms

$$\boldsymbol{\varpi}_{(j)} = \mathcal{D}_j \,\lrcorner\, \boldsymbol{\varpi} = (-1)^{j-1}\, \varpi^1 \wedge \cdots \wedge \varpi^{j-1} \wedge \varpi^{j+1} \wedge \cdots \wedge \varpi^p \in \widetilde{\mathbf{\Omega}}^{p-1,0}, \qquad j = 1, \ldots, p. \tag{9.1}$$

Note that

$$\varpi^i \wedge \boldsymbol{\varpi}_{(j)} = \delta^i_j\, \boldsymbol{\varpi}, \tag{9.2}$$

where δ^i_j is the usual Kronecker delta symbol.

We need to determine the invariant horizontal derivatives of the forms (9.1). Since $d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(j)} \in \widetilde{\mathbf{\Omega}}^{p,0}$, it must be a multiple of the invariant volume form, and we write

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(j)} = Z_j\, \boldsymbol{\varpi}, \qquad\qquad j = 1, \ldots, p, \tag{9.3}$$

where $Z_1, \ldots, Z_p$ are certain differential invariants, which we will call the *twist invariants*. Note that these quantities, which will cause additional complications in the subsequent formulae, do not appear in the one-variable (curve) case because $\boldsymbol{\varpi}_{(1)} = 1$ has trivial differential! The explicit formulae for the twist invariants can be found using (5.28), which implies that

$$Z_j \;=\; -\sum_{i=1}^{p} Y^i_{ij} \;=\; \sum_{i=1}^{p} Y^i_{ji} \tag{9.4}$$

is the trace of the invariant commutator tensor field Y^i_{jk}.

Every form of invariant type $(p-1,0)$ can be written as a linear combination

$$\Omega = \sum_{j=1}^{p} Q^j\, \boldsymbol{\varpi}_{(j)} \in \widetilde{\boldsymbol{\Omega}}^{p-1,0}$$

of the forms (9.1), and so can be identified with a vector $Q = (Q^1, \ldots, Q^p)$, which is a vector of differential invariants if and only if Ω is an invariant form. Applying (9.2), (9.3), we find that the invariant horizontal differential is

$$d_{\mathcal{H}}\,\Omega = \left[\; \sum_{j=1}^{p} (\mathcal{D}_j + Z_j) Q^j \;\right] \cdot \boldsymbol{\varpi} \;\in\; \widetilde{\boldsymbol{\Omega}}^{p,0}. \tag{9.5}$$

The resulting formula should be identified as the *invariant divergence* of the vector field Q, the additional Z_j factors providing a "twist" to the invariant derivatives $\mathcal{D}_j$. Equation (9.5) shows that an invariant Lagrangian which can be written as an invariant divergence, $\widetilde{L} = \sum_j (\mathcal{D}_j + Z_j) Q^j$ defines a null Lagrangian form $\lambda = \widetilde{L}\,\boldsymbol{\varpi}$, meaning that it has identically zero Eulerian: $\delta\lambda \equiv 0$.

Viewing the divergence as the dual of the invariant gradient, as defined in (5.14), we are led to the following important definition.

Definition 9.1. The *twisted invariant adjoint* of the invariant differential operator $\mathcal{D}_j$ is defined as

$$\mathcal{D}_j^{\dagger} = -(\mathcal{D}_j + Z_j), \tag{9.6}$$

where Z_j is the twist invariant given in (9.3).

More generally, if

$$\mathcal{P} = \sum_K P_K\, \mathcal{D}_K = \sum_K P_K\, \mathcal{D}_{k_1} \mathcal{D}_{k_2} \cdots \mathcal{D}_{k_m} \tag{9.7}$$

is any scalar invariant differential operator, we define its *twisted invariant adjoint* to be

$$\begin{aligned} \mathcal{P}^{\dagger} = \sum_K \mathcal{D}^{\dagger}_{\widetilde{K}} \cdot P_K &= \sum_K \mathcal{D}^{\dagger}_{k_m} \mathcal{D}^{\dagger}_{k_{m-1}} \cdots \mathcal{D}^{\dagger}_{k_1} \cdot P_K \\ &= \sum_K (-1)^m (\mathcal{D}_{k_m} + Z_{k_m})(\mathcal{D}_{k_{m-1}} + Z_{k_{m-1}}) \cdots (\mathcal{D}_{k_1} + Z_{k_1}) \cdot P_K. \end{aligned} \tag{9.8}$$

Note the reversal in order of the twisted adjoint operators, indicated by $\widetilde{K} = (k_m, \ldots, k_1)$. The order is important because the invariant differential operators do not necessarily commute, cf. (5.29). Finally, if $\mathcal{A} = \left(\mathcal{A}^{\alpha}_{\beta}\right)$ is any matrix of invariant differential operators, we define its twisted invariant adjoint to be the matrix $\mathcal{A}^{\dagger} = \left((\mathcal{A}^{\beta}_{\alpha})^{\dagger}\right)$, which is the transpose of the matrix of twisted invariant adjoint operators.

Lemma 9.2. *If F is any differential function, $\psi \in \widetilde{\mathbf{\Omega}}^{0,1}$ any invariant contact one-form, and $\mathcal{P}$ any invariant differential operator, then*

$$F\,\mathcal{P}(\psi) \wedge \boldsymbol{\varpi} \equiv \mathcal{P}^{\dagger}(F)\,\psi \wedge \boldsymbol{\varpi} \tag{9.9}$$

project via $\widetilde{\pi}_*$ to the same source form in $\mathcal{F}^1$.

Proof: Formulae (5.14), (9.3) imply that, for any one-form σ,

$$\begin{aligned} d_{\mathcal{H}}\,(F\,\sigma \wedge \boldsymbol{\varpi}_{(j)}) &= d_{\mathcal{H}}\,F \wedge \sigma \wedge \boldsymbol{\varpi}_{(j)} + F\,d_{\mathcal{H}}\,\sigma \wedge \boldsymbol{\varpi}_{(j)} - F\,\sigma \wedge\, d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(j)} \\ &= -(\mathcal{D}_j + Z_j)F\,\sigma \wedge \boldsymbol{\varpi} + F\,d_{\mathcal{H}}\,\sigma \wedge \boldsymbol{\varpi}_{(j)}. \end{aligned} \tag{9.10}$$

Using (9.6), we can write (9.10) as

$$-\,F\,d_{\mathcal{H}}\,\sigma \wedge \boldsymbol{\varpi}_{(j)} \equiv (\mathcal{D}_j^{\dagger} F)\,\sigma \wedge \boldsymbol{\varpi}. \tag{9.11}$$

If we let $\sigma = \psi$ be a contact-one form, and apply (5.34), we deduce that the basic integration by parts formula

$$F(\mathcal{D}_j\psi) \wedge \boldsymbol{\varpi} \equiv -\,(\mathcal{D}_j + Z_j)F\,\psi \wedge \boldsymbol{\varpi} = (\mathcal{D}_j^{\dagger} F)\,\psi \wedge \boldsymbol{\varpi} \tag{9.12}$$

holds for any contact one-form ψ. The lemma now follows by iteration. *Q.E.D.*

On the other hand, if we choose $\sigma = d_{\mathcal{V}}\,H$ where H is a differential function in (9.11), then

$$d_{\mathcal{H}}\,\sigma = -\,d_{\mathcal{V}}\,d_{\mathcal{H}}\,H = -\sum_{i=1}^{p} d_{\mathcal{V}}\,(\mathcal{D}_i H\,\varpi^i).$$

This results in the alternative multivariate integration by parts formulae

$$F\,d(\mathcal{D}_j H) \wedge \boldsymbol{\varpi} = (\mathcal{D}_j^{\dagger} F)\,d_{\mathcal{V}}\,H \wedge \boldsymbol{\varpi} - \sum_{i=1}^{p} F\,(\mathcal{D}_i H)\,d_{\mathcal{V}}\,\varpi^i \wedge \boldsymbol{\varpi}_{(j)}. \tag{9.13}$$

that assumes the role of its univariate counterpart (7.8).

Now, let $I^1, \ldots, I^m$ denote a fundamental set of differential invariants, which means that the differentiated invariants

$$I^{\alpha}_{,K} = \mathcal{D}_{\widetilde{K}} I^{\alpha} = \mathcal{D}_{k_m}\mathcal{D}_{k_{m-1}}\cdots\mathcal{D}_{k_1} I^{\alpha}, \qquad \text{where} \qquad K = (k_1, \ldots, k_m), \tag{9.14}$$

contain a complete system of higher order differential invariants. The I^{α} might be normalized differential invariants arising from a moving frame, but, as we observed at the end of the preceding section, this is not necessary for the initial computation. The comma indicates invariant differentiation, and serves to distinguish $I^{\alpha}_{,K}$ from the normalized differential invariant $I^{\alpha}_K = \iota(u^{\alpha}_K)$. Since the invariant differential operators do not commute, the formula for $I^{\alpha}_{,K}$ depends on the order of the multi-index K. Furthermore, the differentiated invariants are typically not functionally independent. The complete classification of their syzygies follows from the infinitesimal moving frame calculus, [**15**].

Consider an invariant variational problem $\mathcal{I}[u] = \int \widetilde{L}(I^{(n)})\,\boldsymbol{\omega}$, where the invariant Lagrangian $\widetilde{L}$ is a function of the differential invariants (9.14). We form the fully invariant

Lagrangian form $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\,\boldsymbol{\varpi}$, obtained by replacing the contact-invariant volume form $\boldsymbol{\omega}$ by its invariant counterpart $\boldsymbol{\varpi} = \widetilde{\pi}_{p,0}(\boldsymbol{\omega})$. The fact that we are allowed to invariantly differentiate I^α in any order — not to mention the possible occurrence of additional syzygies among the differentiated invariants — imply that there can exist many redundancies in our formula for the Lagrangian. Remarkably, these play no significant role in the ensuing computation.

Definition 9.3. The *invariant Eulerian* of an invariant Lagrangian $\widetilde{L}(I^{(n)})$ with respect to the differential invariant I^α is

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_K \mathcal{D}_K^\dagger \frac{\partial \widetilde{L}}{\partial I^\alpha_{,K}},$$

where (9.8) (but where the multi-index K has the reverse order) is used to compute the twisted invariant adjoints of differential operators.

As we saw in the scalar case, besides the invariant version of the Euler expressions, we also require an invariant version of the Hamiltonian associated with the variational problem. In the multi-dimensional Hamiltonian framework, [**34**], the Hamiltonian is no longer a scalar differential form, but rather a $p \times p$ matrix of differential forms.

Definition 9.4. The *Hamiltonian tensor* $\mathbf{H} = (H^i_j)$ associated with a Lagrangian $\lambda = L(x, u^{(n)})\, dx^1 \wedge \cdots \wedge dx^p$ has components

$$H^i_j(L) = -\,L\,\delta^i_j + \sum_{\alpha=1}^{q} \sum_{J,K} u^\alpha_{J,j} (-D)_K \frac{\partial L}{\partial u^\alpha_{J,i,K}}, \tag{9.15}$$

where the sum is over all pairs of multi-indices $J = (j_1, \ldots, j_r)$, $K = (k_1, \ldots, k_s)$, either of which may be empty, so $r \geq 0, s \geq 0$. The final subscript denotes to the concatenated multi-index $(J, i, K) = (j_1, \ldots, j_r, i, k_1, \ldots, k_s)$.

Remark: For an x-independent Lagrangian, the Hamiltonian tensor provides the conservation laws of linear momentum corresponding to the translation independence of the Lagrangian, resulting in the Noether divergence identity, [**27**],

$$\sum_{i=1}^{p} D_i(H^i_j) = \sum_{\alpha=1}^{q} u^\alpha_j\, \mathbf{E}_\alpha(\widetilde{L}). \tag{9.16}$$

Definition 9.5. Given a differential invariant $\widetilde{L}(I^{(n)})$, we define its *invariant Hamiltonian tensor* to have components

$$\mathcal{H}^i_j(\widetilde{L}) = -\,\widetilde{L}\,\delta^i_j + \sum_{\alpha=1}^{m} \sum_{J,K} I^\alpha_{,J,j}\, \mathcal{D}_K^\dagger \frac{\partial \widetilde{L}}{\partial I^\alpha_{,J,i,K}}. \tag{9.17}$$

Note particularly that, due to the noncommutativity of the invariant differential operators, the order of the multi-indices in (9.17) remains important!

Remark: An amazing fact is that the final Euler-Lagrange expression will not depend upon the choice of differential invariants, or their syzygies. On the other hand, according to Itskov, [**21**], the individual invariant Eulerian and Hamiltonian *do* depend on how the syzygies among differential invariants are implemented.

Proposition 9.6. *Given an invariant Lagrangian form* $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\,\boldsymbol{\varpi}$, *then*

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \sum_{\alpha=1}^{m} \mathcal{E}_{\alpha}(\widetilde{L})\, d_{\mathcal{V}}\, I^{\alpha} \wedge \boldsymbol{\varpi} - \sum_{i,j=1}^{p} \mathcal{H}^{i}_{j}(\widetilde{L})\, d_{\mathcal{V}}\, \varpi^{j} \wedge \boldsymbol{\varpi}_{(i)}. \tag{9.18}$$

Proof: We begin by computing

$$d_{\mathcal{V}}\,\widetilde{\lambda} = \sum_{\alpha,K} \frac{\partial \widetilde{L}}{\partial I^{\alpha}_{,K}}\; d_{\mathcal{V}}\, I^{\alpha}_{,K} \wedge \boldsymbol{\varpi} + \widetilde{L}\; d_{\mathcal{V}}\,\boldsymbol{\varpi}.$$

The second term will be rewritten in the form

$$\widetilde{L}\; d_{\mathcal{V}}\,\boldsymbol{\varpi} = \sum_{i,j=1}^{p} \delta^{i}_{j}\, \widetilde{L}\, d_{\mathcal{V}}\, \varpi^{j} \wedge \boldsymbol{\varpi}_{(i)}. \tag{9.19}$$

As for the first term, we invoke our integration by parts formula (9.13) to move the invariant differentiations onto the partial derivatives of $\widetilde{L}$. For the first step, we write $I^{\alpha}_{,K} = \mathcal{D}_{\widetilde{K}} I^{\alpha} = \mathcal{D}_{k_m} I^{\alpha}_{,J}$ where $J = (k_1, \ldots, k_{m-1})$, and so

$$\begin{aligned} \frac{\partial \widetilde{L}}{\partial I^{\alpha}_{,K}}\; d_{\mathcal{V}}\, I^{\alpha}_{,K} \wedge \boldsymbol{\varpi} &= \frac{\partial \widetilde{L}}{\partial I^{\alpha}_{,K}}\; d_{\mathcal{V}}\,(\mathcal{D}_{k_m} I^{\alpha}_{,J}) \wedge \boldsymbol{\varpi} \\ &\equiv \mathcal{D}^{\dagger}_{k_m} \frac{\partial \widetilde{L}}{\partial I^{\alpha}_{,K}}\; d_{\mathcal{V}}\,(I^{\alpha}_{,J}) \wedge \boldsymbol{\varpi} - \sum_{i=1}^{p} \frac{\partial \widetilde{L}}{\partial I^{\alpha}_{,K}}\, I^{\alpha}_{,J,i}\, d_{\mathcal{V}}\, \varpi^{i} \wedge \boldsymbol{\varpi}_{(k_m)}. \end{aligned}$$

The second term contributes to the invariant Hamiltonian tensor (9.17), while we continue to integrate the first term by parts. In the end, when the invariant differentiations are all applied to the partial derivative of $\widetilde{L}$, we produce the final formula (9.18). *Q.E.D.*

The second phase of the computation requires, in analogy with (7.15), (7.17), the formulae for the $(p,1)$ forms appearing on the right hand side of (9.18),

$$d_{\mathcal{V}}\, I^{\alpha} = \sum_{\beta=1}^{q} \mathcal{A}^{\alpha}_{\beta}(\vartheta^{\beta}), \qquad\qquad d_{\mathcal{V}}\, \varpi^{j} = \sum_{i=1}^{p} \sum_{\beta=1}^{q} \mathcal{B}^{j}_{i,\beta}(\vartheta^{\beta}) \wedge \varpi^{i}, \tag{9.20}$$

which follow directly from the moving frame recurrence formulae. We let

$$\mathcal{A} = \left(\, \mathcal{A}^{\alpha}_{\beta} \,\right), \qquad\qquad \mathcal{B}^{j}_{i} = \left(\, \mathcal{B}^{j}_{i,\beta} \,\right), \qquad\qquad i,j = 1, \ldots, p \tag{9.21}$$

denote, respectively, the *Eulerian operator*, which is an $m \times q$ matrix of invariant differential operators and the *Hamiltonian operator complex*, which is a collection of p^2 row vectors whose entries are invariant differential operators arising in (9.20). This allows us to write (9.18) in the vectorial form

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L})\,\mathcal{A}(\boldsymbol{\vartheta}) \wedge \boldsymbol{\varpi} - \sum_{i,j=1}^{p} \mathcal{H}^i_j(\widetilde{L})\,\mathcal{B}^j_i(\boldsymbol{\vartheta}) \wedge \boldsymbol{\varpi}. \tag{9.22}$$

We now apply Lemma 9.2 to integrate both terms by parts. The final result is written in terms of the twisted invariant adjoints of the Eulerian and Hamiltonian operators (9.21), so

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \left[\,\mathcal{A}^{\dagger}\,\mathcal{E}(\widetilde{L}) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^{\dagger}\,\mathcal{H}^i_j(\widetilde{L})\right] \boldsymbol{\vartheta} \wedge \boldsymbol{\varpi}. \tag{9.23}$$

We have thus proved our final result.

Proposition 9.7. *The Euler-Lagrange expressions of an invariant Lagrangian form* $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\,\boldsymbol{\varpi}$ *are equivalent to the invariant system of differential equations*

$$\mathcal{A}^{\dagger}\,\mathcal{E}(\widetilde{L}) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^{\dagger}\,\mathcal{H}^i_j(\widetilde{L}) = 0. \tag{9.24}$$

Indeed,

$$\widetilde{\pi}_{p,1}(\boldsymbol{\vartheta} \wedge \boldsymbol{\varpi}) = W \cdot \boldsymbol{\theta} \wedge d\mathbf{x}, \qquad \text{or} \qquad \vartheta^{\alpha} \wedge \boldsymbol{\varpi} \equiv \sum_{\beta=1}^{q} W^{\alpha}_{\beta}\,\theta^{\beta} \wedge d\mathbf{x}, \tag{9.25}$$

where $W = \big(\,W^{\alpha}_{\beta}\,\big)$ is a certain matrix-valued relative invariant. Thus, equating (9.22) to the standard Euler-Lagrange expression $d_V\,\lambda \equiv \mathbf{E}(L)\,\boldsymbol{\theta} \wedge d\mathbf{x}$, results in the explicit formula

$$\mathbf{E}(\lambda) = W\left[\,\mathcal{A}^{\dagger}\,\mathcal{E}(\widetilde{L}) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^{\dagger}\,\mathcal{H}^i_j(\widetilde{L})\right] \tag{9.26}$$

connecting the ordinary and invariant versions of the Euler-Lagrange equations. As before, the matrix relative invariant W is invertible on the domain of definition of the moving frame, and hence only comes into play at singular extremals.

Example 9.8. Consider the intransitive action

$$y^1 = x^1 \cos\phi - x^2 \sin\phi + a, \qquad y^2 = x^1 \sin\phi + x^2 \cos\phi + b, \qquad v = u, \tag{9.27}$$

of the Euclidean group SE(2) on $\mathbb{R}^3$. It arises, among other places, as a symmetry group of the planar Laplace equation, cf. [**16**, **29**]. We shall use the moving frame to construct differential invariants for surfaces $u = f(x^1, x^2)$ and then determine the invariant Euler-Lagrange equations. We pursue this simple example in some detail so as to illustrate the required computations in more complicated cases.

We begin by prolonging to J^2, of which only the first order formulae will be displayed:

$$\frac{\partial v}{\partial y^1} = v_1 = u_1 \cos\phi - u_2 \sin\phi, \qquad \frac{\partial v}{\partial y^2} = v_2 = u_1 \sin\phi + u_2 \cos\phi,$$

where we abbreviate $u_i = D_i u = D_{x^i} u$. Higher order lifted invariants are obtained by repeatedly applying the implicit differentiations

$$D_{y^1} = \cos\phi\, D_1 - \sin\phi\, D_2, \qquad D_{y^2} = \sin\phi\, D_1 + \cos\phi\, D_2. \tag{9.28}$$

Choosing the cross-section $x^1 = x^2 = u_1 = 0$, we are led to the normalization equations

$$y^1 = 0, \qquad y^2 = 0, \qquad v_1 = 0. \tag{9.29}$$

Solving for the group parameters produces the (right) moving frame†

$$a = \frac{x^2 u_1 - x^1 u_2}{I}, \qquad b = -\,\frac{x^1 u_1 + x^2 u_2}{I}, \qquad \phi = \tan^{-1} \frac{u_1}{u_2},$$

and the first two differential invariants

$$v \longmapsto u = \iota(u), \qquad v_2 \longmapsto I = \|\,\nabla u\,\| = \sqrt{u_1^2 + u_2^2} = \iota(u_2). \tag{9.30}$$

Higher order differential invariants are obtained by normalizing the higher order lifted variables, and are denoted by $I_J = \iota(u_J)$. For $n \geq 2$, there are $n+1$ strictly independent n^{th} order invariants. For instance, the second order ones are

$$\iota(u_{11}) = I_{11} = I^{-2} J_{11}, \qquad \iota(u_{12}) = I_{12} = I^{-2} J_{12}, \qquad \iota(u_{22}) = I_{22} = I^{-2} J_{22},$$

where

$$J_{11} = u_2^2 u_{11} - 2 u_1 u_2 u_{12} + u_1^2 u_{22}, \qquad J_{12} = u_1 u_2 (u_{11} - u_{22}) + (u_2^2 - u_1^2) u_{12},$$
$$J_{22} = u_1^2 u_{11} + 2 u_1 u_2 u_{12} + u_2^2 u_{22},$$

are also differential invariants. The invariant differentiations

$$\mathcal{D}_1 = \frac{1}{I}\,\big(\, u_2 D_1 - u_1 D_2 \,\big), \qquad \mathcal{D}_2 = \frac{1}{I}\,\big(\, u_1 D_1 + u_2 D_2 \,\big), \tag{9.31}$$

are dual to the invariant horizontal coframe

$$\varpi^1 = \omega^1 = \iota(dx^1) = \frac{1}{I}\,\big(\, u_2\, dx^1 - u_1\, dx^2 \,\big),$$
$$\varpi^2 = \omega^2 = \iota(dx^2) = \frac{1}{I}\,\big(\, u_1\, dx^1 + u_2\, dx^2 \,\big) = \frac{d_H u}{\|\,\nabla u\,\|}\,.$$

The invariant volume form is particularly simple in this case: $\boldsymbol{\varpi} = \varpi^1 \wedge \varpi^2 = dx^1 \wedge dx^2$, since Euclidean transformations are measure-preserving.

† There is a remaining sign ambiguity in the definition of the angular variable ϕ which we ignore in order to not overly complicate the discussion.

The prolonged infinitesimal generators are

$$\mathbf{v}_1 = \partial_{x^1}, \qquad \mathbf{v}_2 = \partial_{x^2},$$
$$\mathbf{v}_3 = -x^2\,\partial_{x^1} + x^1\,\partial_{x^2} - u_2\,\partial_{u_1} + u_1\,\partial_{u_2} - 2u_{12}\,\partial_{u_{11}} + (u_{11}-u_{22})\,\partial_{u_{12}} + 2u_{12}\,\partial_{u_{22}} + \cdots .$$

First, the horizontal differentials of the normalized invariants are given by (5.19). The phantom invariants can be used to determine the one-forms γ^ℓ, namely

$$0 = \varpi^1 + \gamma^1, \qquad 0 = \varpi^2 + \gamma^2, \qquad 0 = I_{11}\,\varpi^1 + I_{12}\,\varpi^2 - I\,\gamma^3.$$

Substituting these into the remaining formulae, we find

$$\begin{aligned}
d_{\mathcal{H}}\,u &= I\,\varpi^2, \qquad\qquad d_{\mathcal{H}}\,I = I_{12}\,\varpi^1 + I_{22}\,\varpi^2,\\
d_{\mathcal{H}}\,I_{11} &= I_{111}\,\varpi^1 + I_{112}\,\varpi^2 - 2I_{12}\,\gamma^3\\
&= \left(I_{111} - \frac{2\,I_{11}I_{12}}{I} \right)\varpi^1 + \left(I_{112} - \frac{2\,I_{12}^2}{I} \right)\varpi^2,\\
d_{\mathcal{H}}\,I_{12} &= I_{112}\,\varpi^1 + I_{12}\,\varpi^2 + (I_{11}-I_{22})\,\gamma^3\\
&= \left(I_{111} + \frac{I_{11}(I_{11}-I_{22})}{I} \right)\varpi^1 + \left(I_{112} + \frac{I_{12}(I_{11}-I_{22})}{I} \right)\varpi^2,\\
d_{\mathcal{H}}\,I_{22} &= I_{122}\,\varpi^1 + I_{222}\,\varpi^2 + 2I_{12}\,\gamma^3\\
&= \left(I_{111} + \frac{2\,I_{11}I_{12}}{I} \right)\varpi^1 + \left(I_{112} + \frac{2\,I_{12}^2}{I} \right)\varpi^2,
\end{aligned}$$

which imply the invariant differentiation formulae

$$\begin{aligned}
\mathcal{D}_1 u &= I_1 = 0, & \mathcal{D}_2 u &= I_2 = I,\\
\mathcal{D}_1 I &= I_{12}, & \mathcal{D}_2 I &= I_{22},\\
\mathcal{D}_1 I_{11} &= I_{111} - \frac{2\,I_{11}I_{12}}{I}\,, & \mathcal{D}_2 I_{11} &= I_{112} - \frac{2\,I_{12}^2}{I}\,,\\
\mathcal{D}_1 I_{12} &= I_{112} + \frac{I_{11}(I_{11}-I_{22})}{I}\,, & \mathcal{D}_2 I_{12} &= I_{122} + \frac{I_{12}(I_{11}-I_{22})}{I}\,,\\
\mathcal{D}_1 I_{22} &= I_{122} + \frac{2\,I_{11}I_{12}}{I}\,, & \mathcal{D}_2 I_{22} &= I_{222} + \frac{2\,I_{12}^2}{I}\,.
\end{aligned}$$

A key remark is that *all* differential invariants can be obtained by invariantly differentiating the simplest one, namely u. The only tricky one is to produce the second order invariant I_{11}, which is not obtained by taking one derivative of I. However, subtracting the equations for $\mathcal{D}_2 I_{12}$ and $\mathcal{D}_1 I_{22}$ and rearranging terms will produce the desired formulae:

$$I = \mathcal{D}_2 u, \qquad I_{12} = \mathcal{D}_1 I = \mathcal{D}_1\mathcal{D}_2 u, \qquad I_{22} = \mathcal{D}_2 I = \mathcal{D}_2^2 u,$$
$$I_{11} = -I_{22} + \frac{I}{I_{12}}\big(\mathcal{D}_2 I_{12} - \mathcal{D}_1 I_{22}\big) = -\mathcal{D}_2^2 u + \frac{\mathcal{D}_2 u}{\mathcal{D}_1\mathcal{D}_2 u}\big(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_2 u - \mathcal{D}_1\mathcal{D}_2^2 u\big),$$

which are valid away from singular points where $I_{12} = 0$. Therefore, any differential invariant can be written as a function depending upon u and its invariant derivatives $u_{,K} = \mathcal{D}_{\widetilde{K}} u$.

Next we compute the first few vertical differentiation formulae from (5.25). Again, the phantom invariants (9.29) give the formulae for the one-forms ε^ℓ, namely

$$0 = \varepsilon^1, \qquad 0 = \varepsilon^2, \qquad 0 = \vartheta_1 - I\,\varepsilon^3.$$

Substituting these into the remaining formulae, we have

$$d_{\mathcal{V}}\, u = \vartheta, \qquad d_{\mathcal{V}}\, I = \vartheta_2, \qquad d_{\mathcal{V}}\, I_{12} = \vartheta_{12} + (I_{11} - I_{22})\,\varepsilon^3 = \vartheta_{12} + \frac{I_{11} - I_{22}}{I}\,\vartheta_1,$$
$$d_{\mathcal{V}}\, I_{11} = \vartheta_{11} - 2 I_{12}\,\varepsilon^3 = \vartheta_{11} - 2\,\frac{I_{12}}{I}\,\vartheta_1, \qquad d_{\mathcal{V}}\, I_{22} = \vartheta_{22} + 2 I_{12}\,\varepsilon^3 = \vartheta_{22} + 2\,\frac{I_{12}}{I}\,\vartheta_1.$$

The invariant horizontal and vertical differentials of the invariant coframe elements are based on (5.27), with

$$\begin{aligned} d_{\mathcal{H}}\,\varpi^1 &= -\gamma^3 \wedge \varpi^2 = -\,\frac{I_{11}}{I}\,\boldsymbol{\varpi}, & d_{\mathcal{H}}\,\varpi^2 &= \gamma^3 \wedge \varpi^1 = -\,\frac{I_{12}}{I}\,\boldsymbol{\varpi}, \\ d_{\mathcal{V}}\,\varpi^1 &= -\varepsilon^3 \wedge \varpi^2 = -\,\frac{1}{I}\,\vartheta_1 \wedge \varpi^2, & d_{\mathcal{V}}\,\varpi^2 &= \varepsilon^3 \wedge \varpi^1 = \frac{1}{I}\,\vartheta_1 \wedge \varpi^1. \end{aligned} \tag{9.32}$$

Since

$$\boldsymbol{\varpi}_{(1)} = \mathcal{D}_1 \,\lrcorner\, \boldsymbol{\varpi} = \varpi^2, \qquad \boldsymbol{\varpi}_{(2)} = \mathcal{D}_2 \,\lrcorner\, \boldsymbol{\varpi} = -\,\varpi^1, \tag{9.33}$$

(9.32) yields the formulae for the twist invariants:

$$\begin{aligned} d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(1)} &= d_{\mathcal{H}}\,\varpi^2 = -\,\frac{I_{12}}{I}\,\boldsymbol{\varpi}, & & & Z_1 &= Y^2_{12} = -\,\frac{I_{12}}{I}\,, \\ d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(2)} &= -\,d_{\mathcal{H}}\,\varpi^1 = \frac{I_{11}}{I}\,\boldsymbol{\varpi}, & &\text{so} & Z_2 &= -Y^1_{12} = \frac{I_{11}}{I}\,. \end{aligned} \tag{9.34}$$

Indeed, the same terms appear in the commutation formula for the invariant differential operators, which takes the following equivalent forms:

$$[\mathcal{D}_1, \mathcal{D}_2] = Z_2\,\mathcal{D}_1 - Z_1\,\mathcal{D}_2, \quad \text{or} \quad (\mathcal{D}_1 + Z_1)\,\mathcal{D}_2 = (\mathcal{D}_2 + Z_2)\,\mathcal{D}_1 \quad \text{or} \quad \mathcal{D}_1^\dagger\,\mathcal{D}_2 = \mathcal{D}_2^\dagger\,\mathcal{D}_1. \tag{9.35}$$

Remark: The particular commutation formulae (9.35) are universally valid for arbitrary transformation groups acting on surfaces (two-dimensional submanifolds) in any higher dimensional manifold.

Finally, we use (5.33) to compute

$$d_{\mathcal{H}}\,\vartheta = \varpi^1 \wedge \vartheta_1 + \varpi^2 \wedge \vartheta_2, \qquad \mathcal{D}_1 \vartheta = \vartheta_1, \qquad \mathcal{D}_2 \vartheta = \vartheta_2. \tag{9.36}$$

Since $d_{\mathcal{V}}\, u = \vartheta$, the Eulerian operator is $\mathcal{A} = 1$. Furthermore, (9.32), (9.36) yield the Hamiltonian operators

$$\mathcal{B}^1_1 = \mathcal{B}^2_2 = 0, \qquad -\,\mathcal{B}^1_2 = \mathcal{B}^2_1 = \frac{1}{I}\,\mathcal{D}_1.$$

Note finally that $\vartheta \wedge \boldsymbol{\varpi} = \theta \wedge d\mathbf{x}$, and so the relative invariant in (9.25) is trivial: $W = 1$. Therefore, according to our fundamental formula (9.24), the Euler-Lagrange equations for an invariant Lagrangian $\widetilde{\lambda} = \widetilde{L}\,\boldsymbol{\varpi} = L\,d\mathbf{x}$ depending on the fundamental differential invariant u and its invariant derivatives are

$$\mathbf{E}(L) = \mathcal{E}(\widetilde{L}) - (\mathcal{D}_1 + Z_1)\left(\frac{\mathcal{H}_2^1(\widetilde{L}) - \mathcal{H}_1^2(\widetilde{L})}{I}\right) = 0, \tag{9.37}$$

where $\mathcal{E}(\widetilde{L})$ and $\mathcal{H}_j^i$ are, respectively, the invariant Eulerian and Hamiltonian tensor for the invariant Lagrangian $\widetilde{L}(u^{(n)})$ based on the twist invariants (9.34). As an example, for the surface area Lagrangian

$$\widetilde{L} = I = \|\,\nabla u\,\| = \sqrt{u_1^2 + u_2^2}\; = \mathcal{D}_2 u,$$

we have

$$\mathcal{E}(I) = -\,(\mathcal{D}_2 + Z_2)1 = -Z_2 = -\,\frac{I_{11}}{I}\,, \qquad\qquad \mathcal{H}_2^1(I) = 0 = \mathcal{H}_1^2(I).$$

Thus, the Euler-Lagrange equation is

$$0 = \mathbf{E}\big(\,\|\,\nabla u\,\|\,\big) = -\,\frac{I_{11}}{I} = -\,\frac{u_2^2 u_{11} - 2u_1u_2u_{12} + u_1^2u_{22}}{(u_1^2 + u_2^2)^{3/2}}\,,$$

which expresses the minimal surface equation in Euclidean-invariant form.

Example 9.9. As a final example, we consider the standard action of the Euclidean group $(R, a) \in \mathrm{SE}(3)$ on surfaces $S \subset \mathbb{R}^3$. The computations provide a simple, direct route to the fundamental quantities of Euclidean surface geometry. It is worth re-emphasizing that all the formulae in this example follow from our infinitesimal moving frame calculus using only linear algebra and differentiation; the explicit formulae for the actual differential invariants (principal curvatures), the Frenet coframe, the dual invariant differential operators, the invariant contact forms, etc., are never required! We assume that the surface is parametrized by $z = (x, y, u(x, y))$, noting that the final formulae are, in fact, parameter-independent. The classical moving frame construction, [**19**], relies on the coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0. \tag{9.38}$$

The resulting (local) left moving frame consists of the point on the curve defining the translation component $\tilde{a} = z$, while the columns of the rotation matrix $\widetilde{R}$ contain the unit tangent vectors forming the Frenet frame along with the unit normal to the surface. The fundamental differential invariants are the principal curvatures

$$\kappa^1 = \iota(u_{xx}), \qquad\qquad \kappa^2 = \iota(u_{yy}).$$

The mean and Gaussian curvature invariants

$$H = \tfrac{1}{2}(\kappa^1 + \kappa^2), \qquad\qquad K = \kappa^1\kappa^2,$$

are often used as convenient alternative invariants, since they eliminate some of the residual discrete ambiguities in the moving frame. Higher order differential invariants are obtained by differentiation with respect to the Frenet coframe $\varpi^1 = \iota(dx^1), \varpi^2 = \iota(dx^2)$. We let $\mathcal{D}_1, \mathcal{D}_2$ denote the dual invariant differential operators. The differentiated invariants $\kappa^\alpha_{,J}$ are not functionally independent, since there is a fundamental syzygy

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0, \tag{9.39}$$

which is the *Codazzi equations*. This syzygy can, in fact, be directly deduced from the infinitesimal moving frame computations by comparing the recurrence formulae for the differentiated invariants $\kappa^1_{,22}$ and $\kappa^2_{,11}$. Note that the denominator in (9.39) vanishes at umbilic points on the surface, where the principal curvatures coincide $\kappa^1 = \kappa^2$, and the moving frame is not valid. We avoid such singular points in our subsequent computations.

Any Euclidean-invariant variational problem has the form

$$\mathcal{I}[u] = \int \widetilde{L}(\kappa^{(n)})\,\omega^1 \wedge \omega^2. \tag{9.40}$$

Here $\omega^1 \wedge \omega^2 = \pi_{2,0}(\varpi^1 \wedge \varpi^2)$ is the usual intrinsic surface area 2-form. The invariant Lagrangian $\widetilde{L}$ is an arbitrary differential invariant, and so can be rewritten in terms of the principal curvature invariants (or, equivalently, in terms of the Gaussian and mean curvatures) and their derivatives. The former representation leads to simpler formulae and will be retained. Using (9.33), we obtain the twist invariants

$$\begin{aligned} d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(1)} &= d_{\mathcal{H}}\,\varpi^2 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\,\boldsymbol{\varpi}, \\ d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(2)} &= -\,d_{\mathcal{H}}\,\varpi^1 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}\,\boldsymbol{\varpi}, \end{aligned} \qquad \text{so} \qquad \begin{aligned} Z_1 &= \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\,, \\ Z_2 &= \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}\,. \end{aligned} \tag{9.41}$$

We note that these key quantities appear in Guggenheimer's proof of the fundamental existence theorem for Euclidean surfaces, [**19**; p. 234], and also Eisenhart, [**11**; p. 159]. Note that the Codazzi syzygy (9.39) can be written compactly as

$$K = \kappa^1\kappa^2 = \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2) = -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2,$$

which expresses the Gaussian curvature K as an invariant divergence, cf. (9.5). Consequently, the Gaussian curvature defines a Euclidean-invariant null Lagrangian $\lambda = K\,\boldsymbol{\omega}$, which is the source of the famous Gauss–Bonnet Theorem.

The invariant vertical derivatives of the principal curvatures are straightforwardly determined via our general methods; supressing the computational details,

$$\begin{aligned} d_{\mathcal{V}}\,\kappa^1 &= \iota(\theta_{xx}) = \left(\mathcal{D}_1^2 + Z_2\,\mathcal{D}_2 + (\kappa^1)^2\right)\vartheta, \\ d_{\mathcal{V}}\,\kappa^2 &= \iota(\theta_{yy}) = \left(\mathcal{D}_2^2 + Z_1\,\mathcal{D}_1 + (\kappa^2)^2\right)\vartheta, \end{aligned} \tag{9.42}$$

where $\vartheta = \iota(\theta) = \iota(du - u_x\,dx - u_y\,dy)$ is the fundamental invariant contact form. Therefore, the Eulerian operator is

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\,\mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\,\mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} d_{\mathcal{V}}\,\varpi^1 &= -\,\kappa^1\,\vartheta\wedge\varpi^1 + \frac{1}{\kappa^1-\kappa^2}\bigl(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,\bigr)\vartheta\wedge\varpi^2, \\ d_{\mathcal{V}}\,\varpi^2 &= \frac{1}{\kappa^2-\kappa^1}\bigl(\,\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\,\bigr)\vartheta\wedge\varpi^1 - \kappa^2\,\vartheta\wedge\varpi^2, \end{aligned} \tag{9.43}$$

which yields the Hamiltonian operator complex

$$\begin{aligned} \mathcal{B}^1_1 &= -\,\kappa^1, \\ \mathcal{B}^2_2 &= -\,\kappa^2, \end{aligned} \qquad \mathcal{B}^1_2 = \frac{1}{\kappa^1-\kappa^2}\bigl(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,\bigr) = \frac{1}{\kappa^1-\kappa^2}\bigl(\,\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\,\bigr) = -\,\mathcal{B}^2_1,$$

the equality following from the commutation formula (9.35). Therefore, according to our fundamental formula (9.24), the Euler-Lagrange equations for a Euclidean-invariant variational problem (9.40) are

$$\begin{aligned} 0 = \mathbf{E}(L) = {} & \bigl[\,(\mathcal{D}_1+Z_1)^2 - (\mathcal{D}_2+Z_2)\cdot Z_2 + (\kappa^1)^2\,\bigr]\,\mathcal{E}_1(\widetilde{L}) \\ & + \bigl[\,(\mathcal{D}_2+Z_2)^2 - (\mathcal{D}_1+Z_1)\cdot Z_1 + (\kappa^2)^2\,\bigr]\,\mathcal{E}_2(\widetilde{L}) + \kappa^1\,\mathcal{H}^1_1(\widetilde{L}) + \kappa^2\,\mathcal{H}^2_2(\widetilde{L}) \\ & + \bigl[\,(\mathcal{D}_2+Z_2)(\mathcal{D}_1+Z_1) + (\mathcal{D}_1+Z_1)\cdot Z_2\,\bigr]\cdot\left(\frac{\mathcal{H}^1_2(\widetilde{L}) - \mathcal{H}^2_1(\widetilde{L})}{\kappa^1-\kappa^2}\right). \end{aligned} \tag{9.44}$$

As before, $\mathcal{E}_\alpha(\widetilde{L})$ are the invariant Eulerians with respect to the principal curvatures κ^α, while $\mathcal{H}^i_j(\widetilde{L})$ are the invariant Hamiltonians based on (9.41).

In particular, if $\widetilde{L}(\kappa^1,\kappa^2)$ does not depend on any differentiated invariants, (9.44) reduces to

$$\mathbf{E}(L) = \bigl[\,(\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger\cdot Z_2 + (\kappa^1)^2\,\bigr]\,\frac{\partial\widetilde{L}}{\partial\kappa^1} + \bigl[\,(\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger\cdot Z_1 + (\kappa^2)^2\,\bigr]\,\frac{\partial\widetilde{L}}{\partial\kappa^2} - (\kappa^1+\kappa^2)\widetilde{L}. \tag{9.45}$$

For example, the problem of minimizing surface area has invariant Lagrangian $\widetilde{L}=1$, and so (9.45) gives the Euler-Lagrange equation

$$\mathbf{E}(L) = -\,(\kappa^1+\kappa^2) = -\,2H = 0, \tag{9.46}$$

and so we conclude that minimal surfaces have vanishing mean curvature. As noted above, the Gauss–Bonnet Lagrangian $\widetilde{L} = K = \kappa^1\kappa^2$ is an invariant divergence, and hence its the Euler-Lagrange equation is identically zero. The mean curvature Lagrangian $\widetilde{L} = H = \frac{1}{2}(\kappa^1+\kappa^2)$ has Euler-Lagrange equation

$$\tfrac{1}{2}\bigl[\,(\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1+\kappa^2)^2\,\bigr] = -\,\kappa^1\,\kappa^2 = -K = 0. \tag{9.47}$$

For the Willmore Lagrangian $\widetilde{L} = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$, [**3**, **6**], formula (9.44) immediately gives the known Euler-Lagrange equation

$$\mathbf{E}(L) = \Delta(\kappa^1+\kappa^2) + \tfrac{1}{2}(\kappa^1+\kappa^2)(\kappa^1-\kappa^2)^2 = 2\,\Delta H + 4(H^2-K)H = 0, \tag{9.48}$$

where

$$\Delta = (\mathcal{D}_1+Z_1)\mathcal{D}_1 + (\mathcal{D}_2+Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger\cdot\mathcal{D}_1 - \mathcal{D}_2^\dagger\cdot\mathcal{D}_2 \tag{9.49}$$

is the Laplace–Beltrami operator on our surface.

Remark: Anderson, [**3**], derives the Euler-Lagrange equations for Euclidean surfaces by writing the invariant Lagrangian in terms of the first and second fundamental forms on the surface, whereas, in accordance with our moving frame approach, we write it directly in terms of the intrinsic principal curvature differential invariants. Bryant, [**6**], uses conformal invariance to construct the Euler-Lagrange equations for the Willmore variational problem. Implementing our methods for the conformal moving frame will give a formula for the Euler-Lagrange equation associated with a general conformally-invariant variational problem.

If, as in the discussion at the end of Section 8, instead of the invariant horizontal coframe $\varpi^1, \ldots, \varpi^p$ provided by the moving frame, the choice of an invariant coframe $\varpi^i = dK^i$ given by the differentials of p functionally independent differential invariants $K^1, \ldots, K^p$ leads to some significant simplifications. First of all, the dual invariant differential operators $D_K = (D_{K^1}, \ldots, D_{K^p})$ all mutually commute. Secondly, $d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(j)} = 0$, and hence the twist invariants $Z_j = 0$ all vanish. Consequently, the adjoints $D_{K^i}^{\dagger} = -D_{K^i} = D_{K^i}^{*}$ mutually commute and are not twisted. Moreover, the second term in the integration by parts formula (9.13) vanishes, and the result is that there is no Hamiltonian contribution to the invariant Euler-Lagrange equations.

In addition to the "independent variable" differential invariants $K^1, \ldots, K^p$, we require a certain number of "dependent variable" differential invariants, $I^1, \ldots, I^m$, with the property that all higher order differential invariants $I^\alpha_{,J} = \mathcal{D}_J I^\alpha$ are given by invariant differentiation of the I^α with respect to the K^i. We denote the complete system of differential invariants up to order n as $(K, I^{(n)})$. Note that there may well be nontrivial syzygies among the differentiated invariants, but these will not affect the final formulae. In view of the preceding observations, the Euler-Lagrange equations of an invariant Lagrangian of the form

$$\widehat{L}(K, I^{(n)})\, dK^1 \wedge \cdots \wedge dK^p \qquad \text{is given by} \qquad \widetilde{\mathcal{A}}^*\, \mathcal{E}\big(\,\widehat{L}\,\big) = 0. \tag{9.50}$$

The invariant Eulerian expression

$$\mathcal{E}_\alpha\big(\,\widehat{L}\,\big) = \sum_J\, (-D_K)^J\, \frac{\partial \widehat{L}}{\partial I^\alpha_{,J}} \tag{9.51}$$

is now identical to the ordinary Euler operator, treating the K's as independent variables and the I's as dependent variables. The associated Eulerian operator $\widetilde{\mathcal{A}} = \big(\,\widetilde{\mathcal{A}}^\alpha_\beta\,\big)$ is constructed from the formula for the modified vertical derivative of the dependent invariants,

$$\widetilde{d}_{\mathcal{V}}\, I^\alpha = \widetilde{\mathcal{A}}^\alpha_\beta(\vartheta^\beta), \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, m, \tag{9.52}$$

where the modified invariant bigrading based on the new invariant horizontal coframe dK^i is used to decompose $d = \widetilde{d}_{\mathcal{H}} + \widetilde{d}_{\mathcal{V}}$, in analogy with (8.15). See Itskov, [**21**] for further developments.

10. Conclusions.

In this paper, we have provided a complete, algorithmic solution to the problem of constructing the invariant form of the Euler-Lagrange equations associated with a Lagrangian which admits a finite-dimensional Lie group as a group of variational symmetries.

The algorithm relies on the moving frame method, but only requires the infinitesimal generators, differentiation and linear algebra to construct the required formula. The general construction of the invariant variational complex can now be applied to a wide range of symmetry-based investigations in the geometric theory of differential equations and variational problems. A number of interesting further research directions are inspired by these results:

(*a*) While we have treated a few of the most basic examples arising in geometrical applications, there are a wide variety of additional group actions of current interest in geometry, physics, and applied mathematics, including computer vision. Further development of the applications of our formulae for practical problems would be of immediate interest. In particular, this leads to explicit determination of the differential invariants that govern minimal submanifolds for a given group action.

(*b*) One key issue is the proper interpretation of the Eulerian and Hamiltonian operators. While we have found an constructive algorithm amenable to symbolic computation, their underlying geometrical and analytical interpretation remains obscure. For example, the formulae for equi-affine surfaces, [**19**], was computed, but proved to be too complex to include in this paper.

(*c*) Our results cover regular extremals, which are solutions to the differential invariant version of the Euler-Lagrange equations. It would be of great interest to understand the role of singular extremals, lying outside the domain of definition of the moving frame and causing the relative differential invariant $W(x, u^{(n)}) = 0$ in (7.24) or (9.26) to vanish.

(*d*) The characteristic cohomology of the invariant complex plays an important role in many applications. For instance, invariant null Lagrangians underly the curvature integral and Gauss–Bonnet type theorems in Euclidean geometry. Anderson and Pohjanpelto, [**5**], and Itskov, [**21**], identify the local invariant characteristic cohomology with the Lie algebra cohomology of the transformation group.

(*e*) In recent work, Itskov, [**21**], and Olver and Pohjanpelto, [**32, 33**], have successfully extended the moving frame method to infinite-dimensional pseudo-group actions, which arise in physics as gauge groups, in soliton theories, and in fluid mechanics as particle relabeling symmetries. Further applications to variational problems admitting infinite-dimensional pseudo-groups of symmetries are under development.

(*f*) Our computational formulae should help shed additional light on symmetry reduction of variational problems and Palais' principle of symmetric criticality, where one tries to construct the group-invariant solutions to a variational problem by solving a problem on the reduced orbit space. Anderson and Fels, [**4**], have shown that the applicability of this principle is not universal, but requires the *nonvanishing* of a certain Lie algebra cohomology class.

(*g*) Formula (9.49) and its multidimensional counterpart can be immediately generalized to any transformation group acting on submanifolds of any dimension. The result is a distinguished, group-invariant version of the Laplace-Beltrami operator. Understanding the associated G-invariant harmonic functions, differential forms and Hodge theory promise significant developments in both theory and potential applications.

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