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INVARIANT FUNCTIONS FOR AMENABLE SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^1

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1. Introduction.

Let $(X, \mathcal{F}, m)$ be a σ-finite measure space and Σ be an amenable semigroup of positive contractions on $L^1=L^1(X, \mathcal{F}, m)$. In this paper, we are interested in finding necessary and sufficient conditions for the existence of a strictly positive element which is invariant under every element T in an amenable semigroup of positive contractions on L^1 and obtaining a generalization of well known ergodic theorem proved for the case when Σ is the semigroup generated by a single positive contraction T on L^1. So far various necessary and sufficient conditions for the existence of invariant measure equivalent to m have been obtained by several authors for the case when Σ is the semigroup generated by a single positive contraction T on L^1; that is, there are several conditions obtained by Hopf [14], Dowker [5] [6], Calderón [2], Hajian-Kakutani [12] and Sucheston [19] for the case of an operator which arises from a measurable transformation, and by Ito [16] and Hajian-Ito [13] for the case of an operator which arises from a Markov process. Furthermore, these have been extended elegantly by Neveu [18] for the case of a positive contraction on L^1. In this paper, we extend some results obtained by Neveu [18] to an amenable semigroup of positive contractions on L^1. On the other hand, the ergodic theorem also has been obtained by several authors for the case when Σ is the semigroup generated by a single operator. It was first proved by Birkhoff [1] for point transformations with an invariant σ-finite measure. For Markov processes, Kakutani [17] proved it for a finite invariant measure and for bounded functions. Hopf [15] extended it to a finite invariant measure and functions in L^1. Dunford-Schwartz [7] proved it for a σ-finite invariant measure and functions in L^1.

Main results in this paper are the following; at first, we find necessary and sifficient conditions for the existence of a strictly positive element which is invariant under every T in Σ; see Theorem 1. Secondly, we find several equivalent conditions for no existence of non-trivial and non-negative element in L^1 which is invariant under every T in Σ; see Theorem 2. Finally, we extend the well known ergodic theorem to an arbitrary amenable semigroup which has been proved for a single positive contraction T with a strictly positive invariant function in L^1; see Theorem 3. It is interesting to note that essentially same results (Theorem 1) were performed by Y. Ito in a recent conference of the Mathematical Society of Japan.

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2. Preliminaries.

Let Σ be an abstract semigroup and $m(\Sigma)$ be the Banach space of all bounded real valued functions on Σ with the supremum norm. For each $s\in\Sigma$ and $f\in m(\Sigma)$, we define elements f_s and f^s in $m(\Sigma)$ given by $f_s(t)=f(st)$ and $f^s(t)=f(ts)$ for all $t\in\Sigma$. An element $\mu\in m(\Sigma)^*$ (the dual space of $m(\Sigma)$) is called a *mean* on $m(\Sigma)$ if $\|\mu\|=\mu(1)=1$. A mean μ is called *left* [*right*] *invariant* if $\mu(f_s)=\mu(f)$ [$\mu(f^s)=\mu(f)$] for all $f\in m(\Sigma)$ and $s\in\Sigma$. An *invariant mean* is a left and right invariant mean. A semigroup which has a left invariant mean [right invariant mean] is called *left amenable* [*right amenable*]. A semigroup which has an invariant mean is called *amenable*. Let Σ be an amenable semigroup, then $\Sigma s\cap\Sigma t\neq\phi$ and $s\Sigma\cap t\Sigma\neq\phi$ for all $s, t\in\Sigma$ [10] [11]. So, if we define an order $t\geqq s$ by $t\in\Sigma s\cup\{s\}$, Σ is a directed set.

LEMMA 1. *Let Σ be a semigroup and M be the closed linear span of the subset $\{f_s-f, f^s-f: f\in m(\Sigma)$ and $s\in\Sigma\}$ of $m(\Sigma)$. Then, Σ is amenable if and only if 1 is not contained in M. If Σ is an amenable semigroup with the order defined above and f is an element of $m(\Sigma)$, then we have*

$$\sup_s \inf_{s\leqq t} f(t) \leqq \mu(f) \leqq \inf_s \sup_{s\leqq t} f(t)$$

for any invariant mean μ on $m(\Sigma)$.

Proof. If Σ is amenable, by definition there exists an invariant mean on $m(\Sigma)$. Since $\mu(f)=0$ for all $f\in M$, it is obvious that 1 is not contained in M. On the other hand, if 1 is not contained in M, then there exists an element $\mu\in m(\Sigma)^*$ such that $\|\mu\|=\mu(1)=1$ and $\mu(f)=0$ for all $f\in M$. Therefore, Σ has an invariant mean. Let c be a real number satisfying

$$c<\sup_s \inf_{s\leqq t} f(t).$$

Then there exists an element u such that $c<f(t)$ for all $t\geqq u$. Since $f^u(t)=f(tu)>c$ for all $t\in\Sigma$ and μ is an invariant mean on $m(\Sigma)$, we have $\mu(f)=\mu(f^u)\geqq\mu(c)=c$. Therefore,

$$\sup_s \inf_{s\leqq t} f(t) \leqq \mu(f).$$

Similarly, we obtain

$$\mu(f) \leqq \inf_s \sup_{s\leqq t} f(t).$$

Throughout this paper, let $(X, \mathcal{F}, m)$ be a finite or σ-finite measure space and let $L^1=L^1(X, \mathcal{F}, m)$ and $L^\infty=L^\infty(X, \mathcal{F}, m)$ be Banach spaces with their respective norms defined as usual. Since L^∞ is the dual space of L^1, we use this duality to write $\langle f, h\rangle$ for $\int f\cdot h dm$, where $f\in L^1$ and $h\in L^\infty$. A notation 1_F is the characteristic function of a measurable set F. Let T be a linear operator on L^1, then we denote the adjoint operator by T^*. If T is a positive contraction on L^1, then T^* is also a positive contraction on L^∞. The following Lemma was proved by Neveu [18].

LEMMA 2. *Let λ be a positive linear form defined on L^∞, that is, let $\lambda\in(L^\infty)^*_+$. Then there exists the largest element $g\in L^1$ such that the form induced by it on L^∞ verifies $g\leqq\lambda$. Moreover, the complement $G=\{x: g(x)=0\}$ of the support of g is the largest set in $\mathcal{F}$ for which there exists $h\in L^\infty_+$ with $h>0$ on G and $\lambda(h)=0$. In particular, if $g>0$, then $\lambda(h)=0$ for every $h\in L^\infty_+$, $h\neq 0$. If $g=0$, then $\lambda(h)=0$ for at least one $h\in L^\infty$ such that $h>0$.*

3. Invariant functions.

The main part of the following Theorem was proved by Neveu [18] for the case when Σ is the semigroup generated by a single positive contraction T on L^1. The proof is similar to that of Neveu.

THEOREM 1. *Let Σ be an amenable semigroup of positive contractions on L^1, and let $f\in L^1$ be arbitrary but fixed and $f>0$. Then, the following conditions are equivalent:*

(1) *there exists $g\in L^1$ such that $g>0$ and $Tg=g$ for all T in Σ;*
(2) *if $h\in L^\infty_+$ and $\inf_T\langle Tf, h\rangle=0$, then $h=0$;*
(3) *if $h\in L^\infty_+$ and $\sup_S\inf_{S\leqq T}\langle Tf, h\rangle=0$, then $h=0$;*
(4) *if $h\in L^\infty_+$ and $\mu_T\langle Tf, h\rangle=0$ for an invariant mean μ on $m(\Sigma)$, then $h=0$;*
(5) *if $h\in L^\infty_+$ and $0\in\overline{co}\{T^*h: T\in\Sigma\}$, then $h=0$ (here $\overline{co}B$ is the closed convex hull of $B\subset L^\infty$ in the sense of L^∞-norm);*
(6) *if $h\in L^\infty_+$ and $\sum_{i=0}^\infty T_i^*h<2$ for some sequence $\{T_i\}$ in Σ, then $h=0$;*
(7) *if $h\in L^\infty_+$ and $\sum_{i=0}^\infty T_i^*h<\infty$ for some sequence $\{T_i\}$ in Σ, then $h=0$;*
(8) *$\sum_{i=0}^\infty T_i f=\infty$ for any sequence $\{T_i\}$ in Σ;*
(9) *if $F\in\mathcal{F}$ and $\sum_{i=0}^\infty T_i^* 1_F\leqq 1+\varepsilon$ for some sequence $\{T_i\}$ in Σ, then $F=\phi$ (here $\varepsilon>0$ denotes an arbitrary but fixed number).*

Proof. (1)$\Rightarrow$(2). Take $f, f_0\in L^1$ such that $f>0$ and $f_0>0$. Since $f_0\leqq af+(f_0-af)^+$ for any real number a,

$$\langle Tf_0, h\rangle\leqq a\langle Tf, h\rangle+\|(f_0-af)^+\|_1\cdot\|h\|_\infty$$

and hence if $\inf\langle Tf, h\rangle=0$, it is seen that

$$\inf_T\langle Tf_0, h\rangle=0.$$

If we take $f_0=g$, we obtain $\langle g, h\rangle=0$ and hence $h=0$.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(4) is obvious from Lemma 1

(4)$\Rightarrow$(5). Let $h\in L_+^\infty$ and $0\in\overline{co}\{T^*h: T\in\Sigma\}$. Then, for $\varepsilon>0$, there exists an element $\sum_{i=1}^n \alpha_i T_i^* h$ ($\sum_{i=1}^n \alpha_i=1$ and $\alpha_i\geqq 0$ for each i) such that

$$\left\|\sum_{i=1}^n \alpha_i T_i^* h\right\|_\infty \|f\|_1<\varepsilon.$$

Now, we have

$$\varepsilon>\sup_T \|Tf\|_1\cdot\left\|\sum_{i=1}^n \alpha_i T_i^* h\right\|_\infty \geqq \sup_T\left\langle Tf, \sum_{i=1}^n \alpha_i T_i^* h\right\rangle$$

$$\geqq \mu_T\left\langle Tf, \sum_{i=1}^n \alpha_i T_i^* h\right\rangle = \sum_{i=1}^n \alpha_i \mu_T\langle T_i Tf, h\rangle$$

$$=\sum_{i=1}^n \alpha_i \mu_T\langle Tf, h\rangle = \mu_T\langle Tf, h\rangle.$$

Therefore, $\mu_T\langle Tf, h\rangle=0$. We obtain $h=0$ by (4).

(5)$\Rightarrow$(6). Let $h\in L_+^\infty$ and $\sum_{i=0}^\infty T_i^* h<2$ for some sequence $\{T_i\}$ in Σ. From the inequalities

$$\frac{1}{n}\sum_{i=0}^{n-1} T_i^* h \leqq \frac{1}{n}\sum_{i=0}^{\infty} T_i^* h \leqq \frac{2}{n}$$

for $n=1, 2, \cdots$, it follows that $0\in\overline{co}\{T^*h: T\in\Sigma\}$. Therefore, we obtain $h=0$ by (5).

(4)$\Rightarrow$(1). We define λ by $\lambda(h)=\mu_T\langle Tf, h\rangle$ for all $h\in L^\infty$. It is obvious that λ is a positive linear form satisfying

$$|\lambda(h)|\leqq \|f\|_1\cdot\|h\|_\infty.$$

Also, for any $h\in L^\infty$ and $T_0\in\Sigma$,

$$\lambda(T_0^* h)=\mu_T\langle Tf, T_0^* h\rangle=\mu_T\langle T_0 Tf, h\rangle$$

$$=\mu_T\langle Tf, h\rangle=\lambda(h).$$

Since $\lambda\in(L^\infty)_+^*$, there exists the largest element g in L^1 with $g\leqq\lambda$ by Lemma 2. This element g is invariant under each T in Σ. In fact,

$$\langle Tg, h\rangle=\langle g, T^*h\rangle\leqq\lambda(T^*h)=\lambda(h)$$

for $h\in L_+^\infty$ and hence $Tg\leqq\lambda$. This implies $Tg\leqq g$. On the other hand, since $T^*1\leqq 1$ and $\lambda(T^*1)=1$, we have $(\lambda-g)(T^*1)\leqq(\lambda-g)(1)$ and hence $\langle Tg, 1\rangle\geqq\langle g, 1\rangle$. Therefore, $Tg=g$ for all T in Σ. Now, we show $g>0$. By (4), if $h\in L_+^\infty$ and $h\neq 0$,

$$0<\mu_T\langle Tf, h\rangle=\lambda(h).$$

Besides, suppose that $G=\{x: g(x)=0\}$ is nonempty. Then, by Lemma 2, there exists $h\in L_+^\infty$ such that $h>0$ on G and $\lambda(h)=0$. This is a contradiction. Therefore, $G=\phi$ and hence we have $g>0$.

The proofs of the remained parts have need of the following Lemma which is a generalization of lemma 3 in [18].

LEMMA 3. *Take* $h\in L^\infty$ *with* $0\leqq h\leqq 1$ *and* $f>0$, $f\in L^1$ *with*

$$\sup_S \inf_{S\leqq T} \langle Tf, h\rangle = 0,$$

then, there exists for each $\delta>0$ *an element* $h_\delta\in L_+^\infty$ *such that* $h_\delta\leqq h$, $\langle f, h-h_\delta\rangle<\delta$ *and* $\sum_{i=0}^\infty T_i^* h_\delta\leqq 1$ *for some sequence* $\{T_i : I=T_0\leqq T_1\leqq\cdots\}$ *in* Σ.

Proof. Take $f, f_0\in L^1$ such that $f>0$ and $f_0>0$. Then, since the condition $\sup\inf\langle Tf, h\rangle=0$ implies

$$\sup_S \inf_{S\leqq T} \langle Tf_0, h\rangle = 0,$$

we can choose $U_j\in\Sigma$ inductively such that $\langle f, U_1^* h\rangle<\delta/2$ and

$$\langle (U_{j-1}U_{j-2}\cdots U_1 + U_{j-1}\cdots U_2 + \cdots + U_{j-1}U_{j-2} + U_{j-1} + I)f,\ U_j^* h\rangle < 2^{-j}\delta.$$

Define

$$h_0 = \sum_{j=0}^\infty (U_{j+1}U_j\cdots U_1 + U_{j+1}U_j\cdots U_2 + \cdots + U_{j+1}U_j + U_{j+1})^* h$$

$$= \sum_{j=0}^\infty (U_jU_{j-1}\cdots U_1 + U_jU_{j-1}\cdots U_2 + \cdots + U_j + I)^* U_{j+1}^* h$$

and then $h_\delta=(h-h_0)^+$. Obviously, $0\leqq h_\delta\leqq h$ and $h_\delta\geqq h-h_0$. We will show that $\langle f, h-h_\delta\rangle<\delta$. In fact,

$$\langle f, h-h_\delta\rangle \leqq \left\langle f, \sum_{j=0}^\infty (U_jU_{j-1}\cdots U_1 + U_j\cdots U_2 + \cdots + U_j + I)^* U_{j+1}^* h \right\rangle$$

$$= \sum_{j=0}^\infty \langle (U_jU_{j-1}\cdots U_1 + U_j\cdots U_2 + \cdots + U_j + I)f,\ U_{j+1}^* h\rangle$$

$$< \sum_{j=0}^\infty 2^{-(j+1)}\delta = \delta.$$

To finish the proof of Lemma 3, it suffices to show that

$$F_{i,k} = h_\delta + U_{i+1}^* h_\delta + (U_{i+2}U_{i+1})^* h_\delta + \cdots + (U_{i+k}\cdots U_{i+1})^* h_\delta \leqq 1$$

for all nonnegative integers i, k. The sufficiency of the above inequality is clear by taking $i=0$, $I=T_0$, $T_1=U_1$, $T_2=U_2U_1$, $\cdots$, $T_j=U_jU_{j-1}\cdots U_1$, $\cdots$, and letting $k\to\infty$.

It is obvious that $F_{i,0}\leqq 1$ for all i. Assume that the inequality is true for all i and for the value $k-1$. From

$$F_{i,k}=h_\delta+U_{i+1}^*(h_\delta+U_{i+2}^*h_\delta+\cdots+(U_{i+k}\cdots U_{i+2})^*h_\delta)$$
$$=h_\delta+U_{i+1}^*(h_\delta+U_{i+1+1}^*h_\delta+\cdots+(U_{i+1+k-1}\cdots U_{i+1+1})^*h_\delta)$$
$$=h_\delta+U_{i+1}^*F_{i+1,k-1},$$

we obtain that $F_{i,k}\leqq 1$ on $\{x:h_\delta(x)=0\}$. On the other hand, we have that on $\{x:h_\delta(x)>0\}$, $h_\delta=h-h_0$ and hence

$$F_{i,k}=h_\delta+U_{i+1}^*h_\delta+\cdots+(U_{i+k}\cdots U_{i+1})^*h_\delta$$
$$\leqq h_\delta+U_{i+1}^*h+\cdots+(U_{i+k}\cdots U_{i+1})^*h$$
$$\leqq h_\delta+h_0=h\leqq 1.$$

This completes the proof of Lemma 3.

We prove that (6)$\Rightarrow$(3). Let $h\in L_+^\infty$ such that

$$\sup_S \inf_{S\leqq T}\langle Tf, h\rangle=0.$$

We can assume without loss of generality that $0\leqq h\leqq 1$ and

$$\sup_S \inf_{S\leqq T}\langle Tf, h\rangle=0.$$

By Lemma 3, there exists $h_\delta\in L_+^\infty$ such that $h_\delta\leqq h$, $\langle f, h-h_\delta\rangle<\delta$ and $\sum_{i=0}^\infty T_i^*h_\delta\leqq 1$ for some sequence $\{T_i\}$ in Σ. Since (6) implies $h_\delta=0$, we obtain $\langle f, h\rangle<\delta$ for all $\delta>0$. Therefore, we have $h=0$.

(2)$\Rightarrow$(8). Let $\{T_i\}$ be a sequence in Σ and $f_0\in L^1\cap L^\infty$ with $f_0\geqq 0$. If we define $h\in L_+^\infty$ by $h=f_0(1+\sum_{i=0}^\infty T_if)^{-1}$ with the convention $(+\infty)^{-1}=0$, then obviously $h(\sum_{i=0}^\infty T_if)\leqq f_0$ with the convention $0\cdot\infty=0$ and hence

$$\int h\cdot\sum_{i=0}^\infty T_if\,dm=\sum_{i=0}^\infty\int h\cdot T_if\,dm=\sum_{i=0}^\infty\langle T_if, h\rangle<\infty.$$

Therefore, $\inf_i\langle T_if, h\rangle=0$ and hence we obtain $h=0$ by (2).

(8)$\Rightarrow$(7). Take $h\in L_+^\infty$ such as $\sum_{i=0}^\infty T_i^*h<\infty$ for some sequence $\{T_i\}$ in Σ and take $f_0\in L^1\cap L^\infty$ with $f_0>0$. If we define $f'\in L_+^1$ by

$$f'=f_0\left(1+\sum_{i=0}^\infty T_i^*h\right)^{-1},$$

then $f'>0$ and $f'(\sum_{i=0}^\infty T_i^*h)\leqq f_0$. Therefore, we obtain

$$\int f'\left(\sum_{i=0}^\infty T_i^*h\right)dm=\int\left(\sum_{i=0}^\infty T_if'\right)\cdot h\,dm<\infty$$

and hence $h=0$ by (8).

(7)$\Rightarrow$(6) and (7)$\Rightarrow$(9) are obvious.

(9)$\Rightarrow$(3). Take $h\in L_+^\infty$ such as

$$\sup_S \inf_{S \leqq T} \langle Tf, h\rangle = 0.$$

If $F = \{x : h(x) > a\}$ where $a > 0$, then we obtain

$$\sup_S \inf_{S \leqq T} \langle Tf, 1_F\rangle = 0.$$

Let $\varepsilon, \varepsilon' > 0$ and $\delta = \varepsilon\varepsilon'/(1+\varepsilon)$. From Lemma 3, there exists $h_\delta \in L_+^\infty$ such that $h_\delta \leqq 1_F$, $\langle f, 1_F - h_\delta\rangle < \delta$ and $\sum_{i=0}^\infty T_i^* h_\delta \leqq 1$ for some sequence $\{T_i\}$ in Σ. If $F_{\varepsilon,\varepsilon'} = \{x : h_\delta(x) > 1/(1+\varepsilon)\}$, then $F_{\varepsilon,\varepsilon'}$ is a subset of F satisfying $\langle f, 1_F - 1_{F_{\varepsilon,\varepsilon'}}\rangle < \varepsilon'$ and

$$\sum_{i=1}^\infty T_i^* 1_{F_{\varepsilon,\varepsilon'}} \leqq 1 + \varepsilon.$$

Therefore, we obtain $F_{\varepsilon,\varepsilon'} = \phi$ from (9) and hence $F = \phi$. Finally, since a is arbitrary, we obtain $h = 0$.

We obtain necessary and sufficient conditions for no existence of invariant measure weaker than m.

THEOREM 2. *Let Σ be an amenable semigroup of positive contractions on L^1 and $f \in L^1$ be arbitrary but fixed and $f > 0$. Then, the following conditions are equivalent*:

(1) *if $g \in L_+^1$ and $Tg = g$ for all T in Σ, then $g = 0$;*

(2) *there exists $h \in L^\infty$ such that $h > 0$ and*

$$\inf_T \langle Tf, h\rangle = 0;$$

(3) *there exists $h \in L^\infty$ such that $h > 0$ and*

$$\sup_S \inf_{S \leqq T} \langle Tf, h\rangle = 0;$$

(4) *there exists $h \in L^\infty$ such that $h > 0$ and*

$$\mu_T \langle Tf, h\rangle = 0,$$

where μ is an invariant mean on $m(\Sigma)$;

(5) *there exists $h \in L^\infty$ such that $h > 0$ and*

$$0 \in \overline{co}\{T^*h : T \in \Sigma\};$$

(6) *there exists $h \in L^\infty$ such that $h > 0$ and $\sum_{i=0}^\infty T_i^* h < 2$ for some sequence $\{T_i\}$ in Σ;*

(7) *there exists $h \in L^\infty$ such that $h > 0$ and $\sum_{i=0}^\infty T_i^* h < \infty$ for some sequence $\{T_i\}$ in Σ;*

(8) *$\sum_{i=0}^\infty T_i f < \infty$ for some sequence $\{T_i\}$ in Σ;*

(9) *there exist positive real numbers $M_n \uparrow \infty$ and elements $F_n \uparrow X$ in $\mathfrak{F}$ such that $\sum_{i=0}^\infty T_i^* 1_{F_n} \leqq 1 + M_n$ for some sequence $\{T_i\}$ in Σ.*

Proof. (1)$\Rightarrow$(2), (3) or (4). Let $\lambda(h)=\mu_T\langle Tf, h\rangle$ for all $h\in L^\infty$. Then $\lambda\in(L^\infty)^*_+$. From Lemma 2, we can find $g\in L^1_+$ such that $g\leqq\lambda$ and $Tg=g$ for all $T\in\Sigma$. Since (1) is valid, $g=0$. Therefore, by Lemma 2, there exists $h\in L^\infty$ such that $h>0$ on $X=\{x: g(x)=0\}$ and

$$0=\lambda(h)=\mu_T\langle Tf, h\rangle\geqq\sup_S\inf_{S\leqq T}\langle Tf, h\rangle$$

$$\geqq\inf\langle Tf, h\rangle\geqq 0.$$

(4)$\Rightarrow$(3)$\Rightarrow$(2) is obvious.

(2)$\Rightarrow$(1). Take $f_0\in L^1$ with $f_0>0$. Since the condition $\inf\langle Tf, h\rangle=0$ implies

$$\inf_T\langle Tf_0, h\rangle=0,$$

by taking $f_0=g$, we obtain

$$\inf_T\langle Tg, h\rangle=\langle g, h\rangle=0.$$

Therefore, we have $g=0$.

(5)$\Rightarrow$(4) and (6)$\Rightarrow$(5) are obvious from (4)$\Rightarrow$(5) and (5)$\Rightarrow$(6) in Theorem 1.

(3)$\Rightarrow$(6). The proofs of the rest have need of the following Lemma.

LEMMA 4. *For $h\in L^\infty$ with $0<h\leqq 1$ and $f>0$, $f\in L^1$ with*

$$\sup_S\inf_{S\leqq T}\langle Tf, h\rangle=0,$$

there exists an element $h'\in L^\infty$ such that $0<h'\leqq h$ and $\sum_{i=0}^\infty T_i^*h'\leqq 1$ for some sequence $\{T_i: I=T_0\leqq T_1\leqq\cdots\}$ in Σ.

Proof. Take $f_0\in L^1$ with $f_0>0$. Then, since the condition $\sup\inf\langle Tf, h\rangle=0$ implies

$$\sup_S\inf_{S\leqq T}\langle Tf_0, h\rangle=0,$$

we can find $U_j\in\Sigma$ inductively such that $\langle f, U_1^*h\rangle<1/2^2$ and

$$\langle(U_{j-1}U_{j-2}\cdots U_1+U_{j-1}\cdots U_2+\cdots+U_{j-1}U_{j-2}+U_{j-1}+I)f, U_j^*h\rangle<2^{-(j+1)}.$$

For $i=1, 2, \cdots$, define

$$h_i=\sum_{j=i}^\infty(U_{j+1}U_j\cdots U_1+U_{j+1}\cdots U_2+\cdots+U_{j+1}U_j+U_{j+1})^*h$$

$$=\sum_{j=i}^\infty(U_jU_{j-1}\cdots U_1+U_j\cdots U_2+\cdots+U_j+I)^*U_{j+1}^*h$$

and $h_{2^{-i}}=(h-h_i)^+$. Obviously, $0\leqq h_{2^{-i}}\leqq h$ and $h_{2^{-i}}\geqq h-h_i$ for all i. We can show by the method of Lemma 3 that $\langle f, h-h_{2^{-i}}\rangle \leqq 2^{-i}$ for all i. Define $h' \in L_+^\infty$ by

$$h' = \sum_{i=0}^{\infty} \frac{1}{2^i} h_{2^{-i}}.$$

Then, we have $\{x : h'(x) > 0\} = \cup_i \{x : h_{2^{-i}}(x) > 0\}$ and

$$\int_{\{h_{2^{-i}}=0\}} f \cdot h \, dm \leqq \int f \cdot (h - h_{2^{-i}}) dm \leqq \frac{1}{2^i}.$$

Therefore, we obtain $h' > 0$. Also, we can show by the method of Lemma 3 that

$$F_{i,k,p} = h_{2^{-p}} + U_{i+1}^* h_{2^{-p}} (U_{i+2} U_{i+1})^* h_{2^{-p}} + \cdots + (U_{i+k} \cdots U_{i+1})^* h_{2^{-p}} \leqq 1$$

for all nonnegative integers i, k and a fixed nonnegative integer p. Taking $i=0$ and $T_0 = I$, $T_1 = U_1, \cdots, T_j = U_i U_{i-1} \cdots U_1, \cdots$ and letting $k \to \infty$, we obtain $\sum_{i=0}^{\infty} T_i^* h_{2^{-p}} \leqq 1$ for all p. Therefore, $\sum_{i=0}^{\infty} T_i^* h' \leqq 1$. This completes the proof of Lemma 4.

The proof of (3)⇒(6) is obvious from Lemma 4. (6)⇒(7) is also clear. (7)⇒(8) is obvious from (8)⇒(7) in Theorem 1.

(8)⇒(2). Let $\{T_i\}$ be a sequence in Σ such that $\sum_{i=0}^{\infty} T_i f < \infty$. If we define $h = f_0 (1 + \sum_{i=0}^{\infty} T_i f)^{-1}$ where $f_0 \in L^1 \cap L^\infty$ with $f_0 > 0$, then $h > 0$ and

$$\inf_T \langle Tf, h \rangle = 0.$$

In fact, from $h(\sum_{i=0}^{\infty} T_i f) \leqq f_0$, we obtain

$$\int h \left(\sum_{i=0}^{\infty} T_i f \right) dm = \sum_{i=0}^{\infty} \langle T_i f, h \rangle < \infty.$$

(3)⇒(9). Let h be an element in L^∞ such that $0 < h \leqq 1$ and

$$\sup_S \inf_{S \leqq T} \langle Tf, h \rangle = 0.$$

Then by Lemma 4, there exists $h' \in L^\infty$ such that $0 < h' \leqq h$ and $\sum_{i=0}^{\infty} T_i^* h' \leqq 1$ for some $I = T_0 \leqq T_1 \leqq \cdots$. Since h' is strictly positive, there exist positive real numbers $M_n \uparrow \infty$ such that

$$F_n = \{x : h'(x) > 1/(1+M_n)\} \uparrow X.$$

We can also show that $\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leqq 1 + M_n$. In fact, since $(1+M_n) h' \geqq 1_{F_n}$,

$$\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leqq (1+M_n) \sum_{i=0}^{\infty} T_i^* h' \leqq 1 + M_n.$$

(9)⇒(6). Let $M_n \uparrow \infty$, $M_n \geqq 0$ and $F_n \uparrow X$, $F_n \in \mathcal{F}$ such that $\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leqq 1 + M_n$ for some sequence $\{T_i\}$ in Σ. if we choose k_n such that $1 + M_n < 2^{k_n}$ and define

$$h_0 = \sum_{n=0}^{\infty} \frac{1}{2^{k_n+n}} 1_{F_n - F_{n-1}},$$

then h_0 is strictly positive and $\sum_{i=0}^{\infty} T_i^* h_0 < 2$. In fact,

$$\sum_{i=0}^{\infty} T_i^* h_0 = \sum_{i=0}^{\infty} T_i^* \left(\sum_n \frac{1}{2^{k_n+n}} 1_{F_n - F_{n-1}} \right)$$

$$= \sum_n \frac{1}{2^{k_n+n}} \left(\sum_i T_i^* 1_{F_n - F_{n-1}} \right)$$

$$\leqq \sum_n \frac{1}{2^{k_n+n}} \left(\sum_i T_i^* 1_{F_n} \right) \leqq \sum_n \frac{1}{2^{k_n+n}} (1+M_n)$$

$$< \sum_n \frac{1}{2^n} = 2.$$

This completes the proof of Theorem 2.

4. Ergodic theorem.

In this section, let $(X, \mathcal{F}, m)$ be a finite measure space. The following Theorem is a generalization of the well known ergodic theorem for the case when Σ is the semigroup generated by a single positive contraction T in Σ.

THEOREM 3. *Let $(X, \mathcal{F}, m)$ be a finite measure space and Σ be an amenable semigroup of positive contractions T on L^1 and suppose that $T1=1$ for all T in Σ. If $f \in L_+^1$ and $\mathcal{B} = \{A : T^*1_A = 1_A, T \in \Sigma\}$, then the conditional expectation $E(f|\mathcal{B})$ of f relative to $\mathcal{B}$ is contained in $\overline{co}\{Tf : T \in \Sigma\}$, where $\overline{co}B$ is the closed convex hull of $B \subset L^1$ in the sense of L^1-norm.*

Proof. Since $T^*1=1$ for all T in Σ, it is obvious that $X \in \mathcal{B}$. That $\mathcal{B}$ is a σ-field is obvious from that $\mathcal{B}_T = \{A : T^*1_A = 1_A\}$ for each T in Σ is a σ-field. We will show that $\{Tf : T \in \Sigma\}$ is weakly sequentially compact. To show this, it suffices to show that countable additivity of the integrals $\int_E Tf\, dm$ is uniform with respect to T in Σ. (See p. 292 in [8].)

Let $f_n = \min(f, n1)$ for all $n=1, 2, \cdots$ and $\varepsilon > 0$. Since $f_n \uparrow f$, by the Lebesque's convergence theorem, there exists an integer $n_0 > 0$ such that $\|f - f_{n_0}\|_1 < \varepsilon/2$. Fix this integer n_0 and determine a number $\delta = \varepsilon/2n_0$. If $m(E) < \delta$, we have

$$\langle Tf, 1_E \rangle \leqq \langle Tf_{n_0}, 1_E \rangle + \langle Tf - Tf_{n_0}, 1_E \rangle$$

$$\leqq \langle T(n_0 1), 1_E \rangle + \|T(f - f_{n_0})\|_1$$

$$\leqq n_0 m(E) + \|f - f_{n_0}\|_1$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all T in Σ. Therefore, it follows that the countable additivity of the integrals $\int_E Tf\, dm$ is uniform with respect to T in Σ. Since $\{Tf : T \in \Sigma\}$ is weakly sequentially compact, it follows that $\overline{co}\{Tf : T \in \Sigma\}$ is weakly compact (see p. 430 and p. 434

in [8]). On the other hand, since $\{Tf: T\in\Sigma\}$ is invariant under each T in Σ and each T in Σ is weakly continuous and linear, $\overline{co}\{Tf: T\in\Sigma\}$ is also invariant under each T in Σ. Now, by using Day's fixed point theorem [4], we can find an element $u\in\overline{co}\{Tf: T\in\Sigma\}$ such that $Tu=u$ for all T in Σ. We will show that this function u is $\mathscr{B}$-measurable. Let a be a real number, then it is obvious that $T(u-a1)=u-a1$ for all T in Σ. Therefore,

$$(u-a1)^+-(u-a1)^-=T(u-a1)^+-T(u-a1)^-.$$

By positivity of T, we obtain that

$$(u-a1)^+\leqq T(u-a1)^+ \quad \text{and} \quad (u-a1)^-\leqq T(u-a1)^-.$$

Hence, it follows by $||T||\leqq 1$ that

$$(u-a1)^+=T(u-a1)^+ \quad \text{and} \quad (u-a1)^-=T(u-a1)^-.$$

Therefore,

$$T\min(1, n(u-a1)^+)\leqq \min(1, n(u-a1)^+)$$

for all $n=1, 2, \cdots$ and hence

$$T\min(1, n(u-a1)^+)=\min(1, n(u-a1)^+).$$

Since $\min(1, n(u-a1)^+)\uparrow 1_{\{u>a\}}$ as $n\to\infty$, we obtain that $T1_{\{u>a\}}=1_{\{u>a\}}$ for all T in Σ. By using this, we shall obtain that u is $\mathscr{B}$-measurable. In fact, since $T^*1_{\{u>a\}}\leqq T^*1=1$, we obtain

$$1_{\{u>a\}}T^*1_{\{u>a\}}\leqq 1_{\{u>a\}}$$

and hence following equalities

$$\begin{aligned}&\int(1_{\{u>a\}}-1_{\{u>a\}}T^*1_{\{u>a\}})dm\\ &=m(\{u>a\})-\int 1_{\{u>a\}}T^*1_{\{u>a\}}dm\\ &=m(\{u>a\})-\int T1_{\{u>a\}}1_{\{u>a\}}dm\\ &=m(\{u>a\})-\int 1_{\{u>a\}}1_{\{u>a\}}dm=0.\end{aligned}$$

Therefore, we have

$$1_{\{u>a\}}=1_{\{u>a\}}T^*1_{\{u>a\}}.$$

Besides, since

$$\int T^*1_{\{u>a\}}dm=\int T1\cdot 1_{\{u>a\}}dm$$

$$=\int 1_{\{u>a\}}dm=m(\{u>a\}),$$

we obtain $T^*1_{\{u>a\}}=1_{\{u>a\}}$. Therefore u is $\mathscr{B}$-measurable.

Finally, we show that $u=E(f|\mathscr{B})$. Let $A\in\mathscr{B}$ and

$$\sum_{i=1}^{n}\alpha_i T_i f\in\overline{co}\{Tf: T\in\Sigma\}$$

where $\sum_{i=1}^{n}\alpha_i=1$ and $\alpha_i\geqq 0$ for $i=1,2,\cdots,n$. Then

$$\left\langle\sum_{i=1}^{n}\alpha_i T_i f, 1_A\right\rangle=\sum_{i=1}^{n}\alpha_i\langle f, T_i^*1_A\rangle$$

$$=\sum_{i=1}^{n}\alpha_i\langle f, 1_A\rangle=\langle f, 1_A\rangle.$$

If $\sum_{i=1}^{n}\alpha_i T_i f\to u$ in the sense of L^1-norm, it follows that $\langle u, 1_A\rangle=\langle f, 1_A\rangle$. On the other hand, we know that

$$\langle f, 1_A\rangle=\langle E(f|\mathscr{B}), 1_A\rangle.$$

Therefore,

$$\langle u, 1_A\rangle=\langle E(f|\mathscr{B}), 1_A\rangle$$

for all $A\in\mathscr{B}$. Since u is $\mathscr{B}$-measurable, we obtain that $u=E(f|\mathscr{B})$.

REMARK. It is obvious that u is a unique invariant function in $\overline{co}\{Tf: T\in\Sigma\}$. For the case when Σ is the semigroup generated by a single positive contraction T on L^1, $1/n\sum_{i=0}^{n-1}T^i f$ tends to $E(f|\mathscr{B})$ in the sense of L^1-norm.

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