

# Invariant Geometric Evolutions of Surfaces and Volumetric Smoothing

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## Abstract

The study of geometric flows for smoothing, multiscale representation, and analysis of two and three dimensional objects has received much attention in the past few years. In this paper, we first present results mainly related to Euclidean invariant geometric smoothing of three dimensional surfaces. We describe results concerning the smoothing of graphs (images) via level sets of geometric heat-type flows. Then we deal with proper three dimensional flows. These flows are governed by functions of the principal curvatures of the surface, such as the mean and Gaussian curvatures. Then, given a transformation group  $G$  acting on  $\mathbf{R}^n$ , we write down a general expression for any  $G$ -invariant hypersurface geometric evolution in  $\mathbf{R}^n$ . As an application, we derive the simplest affine invariant flow for surfaces.

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# 1 Introduction

Geometric smoothing, multiscale representation, and analysis of two dimensional (2D) and three dimensional (3D) objects are of extreme importance in different applications of computer graphics, CAGD, and image analysis. These can be used for smoothing out noise or for the representation of objects at different levels of detail. When one is interested in the geometry of the given object, it is important to perform these operations in an intrinsic geometric manner. Thus image processing via geometric driven diffusion-type flows has become a major topic of research in the last few years [54]. In our work, the object is deformed via a partial differential equation which is invariant with respect to a given symmetry group.

The smoothing and multiscale representation of planar objects was originally performed by filtering their boundary with a Gaussian filter [9, 37, 68]. This process is equivalent to deforming the curve via the classical heat flow which is of course an extrinsic process unrelated to the geometry of the given image. As we will see in Section 2, this and other problems of the classical heat flow can be effectively solved by replacing it with geometric heat flows that were developed during the last few years [26, 27, 51, 55, 57, 59, 60].

The first question with which we want to deal in this paper is the problem of finding analogous flows for smoothing and multiscale representation of 3D objects. We first present results on geometric smoothing of graphs (images), via geometric smoothing of their level sets. We will see that, based on the theory of planar geometric heat flows, useful results may be obtained. We then discuss the smoothing of surfaces via proper three dimensional flows. In this case, the surface deforms with velocity given by functions of its principal curvatures. In order to make the paper accessible to the largest possible audience, many of the background results are presented in a informal way, i.e., without the mathematical details which may be found in the relevant references. The main goal of this part is to review the literature on surface evolution relevant to volumetric smoothing.

In the second part of the paper, we extend the results first reported in [51] for planar curves, to any dimension and any Lie group. We present the most general form of an invariant geometric flow for hypersurfaces. We show that the invariant flows can be formulated as functions of the invariant metric and invariant curvature, which are the basic differential invariant descriptors, together with the variational (Euler-Lagrange) derivative corresponding to this metric. We also show that if the transformation group is volume preserving, this variational derivative is invariant as well. Then, as an example, we derive the simplest affine invariant geometric flow for 3D surfaces.

This paper is organized as follows: In Section 2, we describe some of the key results related to planar curve geometric smoothing, which will be helpful to motivate and understand the surface theory. Basic concepts of 3D surface differential geometry are given in Section 3. Then, in Section 4 we deal with a “ $2\frac{1}{2}$ ”D theory of geometric flows of surfaces which is related to smoothing via level sets. Section 5 deals with proper 3D geometric smoothing. In Section 6, we define the variational derivative, which we will need in order to formulate and prove our result on the general form of an invariant hypersurface geometric evolution in Section 7. Then in Section 8, we discuss affine invariant flows of surfaces, and some concluding remarks are given in Section 9.

## 2 Planar curve smoothing

In this section, we review some results on geometric smoothing of planar curves, which we wish to extend to surfaces and in general to any dimension. As we will see in the sequel, some of the desirable results that hold for planar curves, do not hold for surfaces. The results described in this section will also be helpful in Section 4, where we describe the possibility of smoothing graphs via level-sets smoothing.

We consider now planar curves deforming in time, where “time” represents “scale.” Let  $\mathcal{C}(p, t) : S^1 \times [0, \tau) \rightarrow \mathbf{R}^2$  denote a family of closed embedded curves, where  $t$  parametrizes the family, and  $p$ , independent of  $t$ , parametrizes each curve. Originally, the classical heat flow was used for smoothing curves [9, 37, 38, 39, 40, 42, 44, 54, 68, 70]. In this case, the curves deform according to the following flow:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial p^2}, \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p). \end{cases} \quad (1)$$

It is well-known that  $\mathcal{C}(p, t) = [x(p, t), y(p, t)]^T$ , satisfying (1), can be obtained from the convolution of  $x(p, 0), y(p, 0)$  with the Gaussian  $\mathcal{G}(p, t)$  defined by

$$\mathcal{G}(p, t) := \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-p^2}{4t}\right\}. \quad (2)$$

In order to separate the geometric concept of a planar curve from its formal algebraic description, it is useful to refer to the planar curve described by  $\mathcal{C}(p, t)$  as the image (trace) of  $\mathcal{C}(p, t)$ , denoted by  $\text{Img}[\mathcal{C}(p, t)]$  [57]. Therefore, if the curve  $\mathcal{C}(p, t)$  is parametrized by a new parameter  $w$  such that  $w = w(p, t)$ ,  $\frac{\partial w}{\partial p} > 0$ , the two images agree:

$$\text{Img}[\mathcal{C}(p, t)] = \text{Img}[\mathcal{C}(w, t)].$$

We see that different parametrizations of the curve will give different results in (1), i.e., different Gaussian multi-scale representations. This is an undesirable property, since parametrizations are in general arbitrary, and may not be connected with the geometry of the curve. We can attempt to solve this problem choosing a parametrization which is intrinsic to the curve, i.e., that can be computed when only  $\text{Img}[\mathcal{C}]$  is given. A natural parametrization, for Euclidean invariant smoothing, is the *Euclidean arc-length* defined by

$$v(p) := \int_0^p \left\| \frac{\partial \mathcal{C}(\xi)}{\partial \xi} \right\| d\xi,$$

and the re-parametrization is obtained via  $\mathcal{C} \circ v$ . This parametrization means that the curve is traveled with constant velocity,  $\|\mathcal{C}_v\| \equiv 1$ . The initial curve  $\mathcal{C}_0(p)$  can be re-parametrized as  $\mathcal{C}_0(v)$ , and the Gaussian filter  $\mathcal{G}(v, t)$ , or the corresponding heat flow, is applied using this parameter. The problem is that the arc-length is a time-dependent parametrization, i.e.,  $v(p)$  depends on time. Also, with this kind of re-parametrization, some of the basic properties of scale-spaces are violated. For example, the order is not preserved, i.e., if  $\mathcal{C}_0$  and  $\hat{\mathcal{C}}_0$  are two initial curves, boundaries of planar shapes, such that  $\mathcal{C}_0 \subset \hat{\mathcal{C}}_0$ , it is not guaranteed that this

order is preserved in time. Also, the semi-group property, which means that  $\mathcal{C}(t_1)$  can be obtained from  $\mathcal{C}(t_2)$  for any  $0 \leq t_2 < t_1$ , can be violated with this kind of re-parametrization. The theory described below solves these problems.

Assume now that the family  $\mathcal{C}(p, t)$  evolves (changes) according to the following general evolution equation

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{T} + \beta \vec{N}, \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p), \end{cases} \quad (3)$$

where  $\vec{N}$  is the inward Euclidean unit normal,  $\vec{T}$  is the unit tangent [64], and  $\alpha$  and  $\beta$  are the tangential and normal components of the evolution velocity  $\vec{v}$ , respectively.

The following lemma shows that under certain conditions, the tangential velocity does not affect  $\text{Img}[\cdot]$ .

**Lemma 1** ([21]) *Let  $\beta$  be a geometric quantity for a curve, i.e., a function whose definition is independent of a particular parametrization. Then a family of curves which evolves according to*

$$\mathcal{C}_t = \alpha \vec{T} + \beta \vec{N}$$

*can be converted into the solution of*

$$\mathcal{C}_t = \bar{\alpha} \vec{T} + \bar{\beta} \vec{N}$$

*for any continuous function  $\bar{\alpha}$ , by changing the space parametrization of the original solution. Since  $\beta$  is a geometric function,  $\beta = \bar{\beta}$  when the same point in the (geometric) curve is considered.*

In particular, this result shows that  $\text{Img}[\mathcal{C}(p, t)] = \text{Img}[\hat{\mathcal{C}}(w, t)]$ , where  $\mathcal{C}(p, t)$  and  $\hat{\mathcal{C}}(w, t)$  are the solutions of

$$\mathcal{C}_t = \alpha \vec{T} + \beta \vec{N}$$

and

$$\hat{\mathcal{C}}_t = \bar{\beta} \vec{N},$$

respectively. For proofs of the lemma, see [21] and [57].

In other words, Lemma 1 means that if the normal component of the velocity is a geometric function of the curve, then  $\text{Img}[\cdot]$  (which represents the “geometry” of the curve) is only affected by this normal component. The tangential component affects only the parametrization, and not  $\text{Img}[\cdot]$  (which is independent of the parametrization by definition). Therefore, assuming that the normal component  $\beta$  of  $\vec{v}$  (the curve evolution velocity) in (3) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \vec{N}, \quad (4)$$

where  $\beta = \vec{v} \cdot \vec{\mathcal{N}}$ , i.e., the projection of the velocity vector on the normal direction.

The evolution (4) was studied by different researchers for different functions  $\beta$ . A key evolution equation is the one obtained for  $\beta = \kappa$ , where  $\kappa$  is the Euclidean curvature defined by [64]

$$\kappa := \left\| \frac{\partial^2 C}{\partial v^2} \right\|.$$

In this case, the flow is given by

$$\frac{\partial C}{\partial t} = \kappa \vec{\mathcal{N}}. \quad (5)$$

Equation (5) has its origins in physical phenomena [6, 24, 28]. It is called the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (5) [28]. Gage and Hamilton [26] proved that a planar embedded convex curve converges to a round point when evolving according to (5). (The term “round point” has the following meaning: Let  $C(t)$  be the curve at time  $t$ , which shrinks to a point as  $t \rightarrow T^*$ . Dilate  $C(t)$  to get a new curve  $\tilde{C}(t)$  centered at the origin and enclosing area  $\pi$ . Then we say that  $C(t)$  converges to a round point provided the dilated curves converge to the unit circle as  $t \rightarrow T^*$ .) Grayson [27] proved that a planar embedded smooth non-convex curve remains smooth and simple, and converges to a convex one, and from there to a round point via the Gage and Hamilton result. Note that in spite of the local character of the evolution, global properties are obtained, which is a very interesting feature of this flow. For other results related to the Euclidean shortening flow, see [1, 6, 21, 26, 27, 28, 35].

Next note that if  $v$  denotes the Euclidean arc-length, then [64]

$$\kappa \vec{\mathcal{N}} = \frac{\partial^2 C}{\partial v^2}.$$

Therefore, equation (5) can be written as

$$C_t = C_{vv}. \quad (6)$$

Note that equation (6) is not linear, since  $v$  is a function of time (the arc-length gives a time-dependent parametrization). Equation (6) is also called the (*Euclidean*) *geometric heat flow* (compare it with the classical heat equation (1)).

Equation (6) (or (5)) has been proposed by different researchers for defining a multi-scale representation of closed curves [36, 42, 69] (see [42] for extended analysis). Note that in contrast with the classical heat flow, the Euclidean geometric one defines an intrinsic, geometric, multi-scale representation. Of course, in order to complete the theory, we must prove that all the basic properties required for a multi-scale smoothing hold for the flow (6). This can be found in [42, 60], and in general are straightforward consequences of the results in [6, 26, 27].

Note that equation (5) (or its analogue (6)) is only Euclidean invariant, since it is based on Euclidean differential geometry. We have extended this theory to the affine group in [55, 56, 57, 58, 59] using affine differential geometry, and also presented a general approach for the formulation of geometric flows for any Lie group in [51, 59]. In general, let  $r$  denote

the invariant arc-length of a given Lie group, i.e., its simplest invariant parameterization. The *geometric heat flow* of the group is obtained via

$$\begin{cases} \frac{\partial \mathcal{C}(p, t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p, t)}{\partial r^2}, \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p). \end{cases} \quad (7)$$

For linear groups, it is easy to prove that, since  $r$  is an invariant of the group, so are  $\mathcal{C}_{rr}$  and the flow (7). The flow is invariant for non-linear groups as well, since  $\frac{\partial}{\partial r}$  is the unique invariant derivative of the group (see [51, 59]). More general invariant flows are obtained if the group curvature  $\chi$  is incorporated into the flow:

$$\begin{cases} \frac{\partial \mathcal{C}(p, t)}{\partial t} = \Psi(\chi, \chi_r, \dots) \frac{\partial^2 \mathcal{C}(p, t)}{\partial r^2}, \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p), \end{cases} \quad (8)$$

where  $\Psi(\cdot)$  is a given function. Since the group arc-length and group curvature are the basic invariants of the group transformations, it is natural to formulate (8) as the most general geometric invariant flow. In [51] we proved that (8) is indeed the most general geometric invariant flow for subgroups of the projective group, and the geometric heat flow is the simplest possible one for a number of important groups. These results we extend here for higher dimensions and general Lie groups.

The group normal  $\mathcal{C}_{rr}$  is in general not perpendicular to the curve, i.e., it is not parallel to the Euclidean unit normal  $\vec{\mathcal{N}}$ . Based on Lemma 1, we know that the effective velocity is obtained by the projection of the group normal onto the Euclidean normal. Using this result, the flows (7) and (8) can be expressed in Euclidean terms by projecting the group normal onto the Euclidean normal, and expressing the group curvature via the Euclidean one and its derivatives. For example, in the affine case, where  $r$  is replaced by the affine arc-length  $s$  given by [12, 55]

$$s(p) := \int_0^p [C_\xi, C_{\xi\xi}]^{1/3} d\xi,$$

the Euclidean-type geometric flow analogue to (7) is given by [55, 56, 57, 59]

$$\mathcal{C}_t = \kappa^{1/3} \vec{\mathcal{N}}. \quad (9)$$

We proved that as in the Euclidean case, any non-convex curve converges to a convex one, and from there to an ellipse, when evolving according the affine heat flow. We also showed that the curve remains smooth, and all the properties of scale-spaces hold [57]. This flow was also discussed by Alvarez *et al.* in [2]. Using the theory of viscosity solutions and evolution of graphs, they also proved the uniqueness of the flow under a number of conditions, which are natural for image processing. In [51] we proved that this flow is unique solely under the requirement of being “simplest flow with the affine group as symmetry group.” This flow was also used in [2, 58] for image enhancement (see next section). For results on other interesting groups, as the similarity and projective ones, see [51, 59, 60]. It is important to note that for example in the similarity and projective case, in contrast with the Euclidean and affine ones, the evolving curve can develop singularities [51].

Before concluding this section, let us point out another of the undesirable properties of Gaussian filtering that is also solved using geometric heat flows. A curve deforming according to the classical heat flow shrinks in a non-computable form. This is due to the fact that the Gaussian filter also affects low frequencies of the curve coordinate functions [44]. Different authors proposed different solution to this problem, while always remaining in the area of Gaussian or linear filtering, i.e., non-geometric smoothers [31, 40, 44]. When a curve evolves according to a geometric heat flow, the shrinking factor can be computed, since the rate of change of area, length, or any other geometric quantity can be computed exactly. Based on this, in [60] we showed how to replace the geometric heat flow (7) by an analogous one, which keeps the area (length) constant. The approach is based on formulating a new geometric flow which deforms the curve according to the flow (7) while simultaneously expanding the plane in order to preserve area (length). This way, a geometric smoother without shrinking is obtained.

### 3 Basic 3D differential geometry

In this section we present basic concepts on surface differential geometry. For details see for example [13, 30, 64]. We first define a *regular surface*:

**Definition 1** *A subset  $S \subset \mathbf{R}^3$  is a regular surface if, for each  $p \in S$ , there exists a neighborhood  $V$  and a map  $\mathbf{x} : U \rightarrow V \cap S$  of an open set  $U \subset \mathbf{R}^2$  such that*

1.  $\mathbf{x}$  is differentiable. This means that if we write

$$\mathbf{x}(u, v) = [x(u, v), y(u, v), z(u, v)], \quad (u, v) \in U,$$

*the functions  $x, y, z$  have continuous partial derivatives of all orders in  $U$ .*

2.  $\mathbf{x}$  is a homeomorphism. This means, using the previous condition, that  $\mathbf{x}$  has an inverse which is continuous.
3. For each  $q \in U$ , the differential  $d\mathbf{x}_q : \mathbf{R}^2 \rightarrow \mathbf{R}^3$  is one-to-one.

The following definitions present two special kinds of regular surfaces which will be analyzed in detail in following sections: *graphs* and *star-shaped* surfaces.

**Definition 2** *If  $\Phi : U \subset \mathbf{R}^2 \rightarrow \mathbf{R}$  is a differentiable function, then the surface given by  $(x, y, \Phi(x, y))$ , for  $(x, y) \in U$ , is a regular surface and is called a graph.*

**Definition 3** *A regular surface that can be represented as a map from  $S^2$  to  $\mathbf{R}^3$  is called a star-shaped surface with respect to a point  $x_0$  in its interior if every ray starting at  $x_0$  intersects the surface only once.*

**Definition 4** *The tangent plane  $T_p(S)$  to a regular surface  $S$  at a point  $p \in S$  is the set of tangent vectors to all parametrized curves of  $S$  passing through  $p$ . The regularity of the surface guarantees the existence of such a plane.*



Given a parametrization of  $\mathbf{x} : U \subset \mathbf{R}^2 \rightarrow S$  of a regular surface  $S$  at a point  $p \in S$ , we can obtain the unit normal vector to the surface at each point  $q$  of  $\mathbf{x}(U)$  by the rule

$$\vec{\mathbf{N}}(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}(q).$$

Based on this normal vector, we can define the *Gauss map*:

**Definition 5** Let  $S \subset \mathbf{R}^3$  be a surface with an orientation given by  $\vec{\mathbf{N}}$ . The map  $\vec{\mathbf{N}} : S \rightarrow \mathbf{R}^3$  takes its values in the unit sphere  $S^2$ , and is called the *Gauss map* of  $S$ .

The following definition presents the *normal curvature* of a curve on a regular surface.

**Definition 6** Let  $C$  be a regular curve in  $S$  passing through  $p \in S$ ,  $\kappa$  the Euclidean curvature of  $C$  at  $p$ , and  $\cos \theta := \langle \vec{\mathcal{N}}, \vec{\mathbf{N}} \rangle$ , where  $\vec{\mathcal{N}}$  is the normal vector to the curve and  $\vec{\mathbf{N}}$  the normal vector to  $S$  at  $p$ . The number  $\kappa_n := \kappa \cos \theta$  is called the *normal curvature* of  $C \subset S$  at  $p$ .

Therefore, the normal curvature is the length of the projection of the curvature vector  $\kappa \vec{\mathcal{N}}$  over the normal to the surface at  $p$ . See Figure 1. An important result [64] guarantees that all curves lying on a surface  $S$  and having at a given point  $p \in S$  the same tangent line, have at this point the same normal curvature. Therefore, the normal curvature is an intrinsic property of the surface and the given direction on it, and not of the selected curve. Given a unit vector  $v \in T_p(S)$ , the intersection of the plane containing  $v$  and  $\vec{\mathbf{N}}$  is called the *normal section* of  $S$  at  $p$  along  $v$ . In a neighborhood of  $p$ , this normal section is a regular curve, whose normal vector at  $p$  is in the direction of  $\vec{\mathbf{N}}$ , and its curvature is therefore equal to the normal curvature along  $v$  at  $p$ .

We are ready to define the *principal curvatures* of  $S$  at a point  $p$ :

**Definition 7** The maxima ( $\kappa_1$ ) and minimal ( $\kappa_2$ ) normal curvatures at  $p \in S$ , for all directions  $v \in T_p(S)$ , are called the *principal curvatures* at  $p$ . The corresponding directions  $e_1$  and  $e_2$  are called the *principal directions* at  $p$ .

**Definition 8** If a regular connected curve  $C$  on  $S$  is such that all its tangent lines are principal directions, then  $C$  is said to be a *line of curvature*.

It is important to know that the knowledge of the principal curvatures and directions allows one to compute the normal curvature at any other direction. In particular, if  $\theta$  is the angle between  $v \in T_p(S)$  and  $e_1$ , then

$$\kappa_n(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta.$$

Finally, we can present the definitions of *Gaussian* and *mean* curvatures:

**Definition 9** The *Gaussian curvature* is the determinant of the differential of the Gauss map, and thus is given by

$$\mathbf{K} = \kappa_1 \kappa_2. \tag{10}$$

**Definition 10** The *mean curvature* is the negative of one half the trace of the differential of the Gauss map, and thus is given by

$$\mathbf{H} = \frac{\kappa_1 + \kappa_2}{2}. \tag{11}$$

## 4 Smoothing graphs via level-sets

In this section we consider graphs, i.e., maps from  $U \subset \mathbf{R}^2$  to  $\mathbf{R}^+$ . The results here described can be applied to smoothing images, which provide particular examples of graphs. One way of smoothing a graph is to smooth its level-sets according to one of the geometric heat flows described in Section 2. This topic has been studied in different works [2, 3, 15, 22, 23, 52, 58], and we wish to present some of the basic results here.

Let  $\Phi : \mathbf{R} \times \mathbf{R} \times [0, \tau) \rightarrow \mathbf{R}^+$  be a graph. In the case of an image,  $\Phi(x, y, t)$  represents the gray-value at the point  $(x, y)$  at time (scale)  $t$ . Define the level set  $\mathcal{X}_c(t)$  of  $\Phi$  as

$$\mathcal{X}_c(t) := \{(x, y) : \Phi(x, y, t) = c\}, \quad (12)$$

and assume that this level set evolves according to

$$\frac{\partial \mathcal{X}_c}{\partial t} = \beta \vec{\mathcal{N}}, \quad (13)$$

where  $\beta$  is, as before, a geometric function of the level set, and  $\vec{\mathcal{N}}$  its normal. We are interested now in studying the behavior of  $\Phi$  when the level sets evolves according to (13). In the following we assume that  $\Phi$  is negative in the interior and positive in the exterior of the zero level set. By differentiation (12) with respect to  $t$  we obtain:

$$\nabla \Phi(\vec{\mathcal{X}}, t) \cdot \vec{\mathcal{X}}_t + \Phi_t(\vec{\mathcal{X}}, t) = 0. \quad (14)$$

(This equation holds for all level sets. Therefore, the subindex is removed from  $\mathcal{X}$ .) Note that for the level sets, the following relation holds:

$$\frac{\nabla \Phi}{\|\nabla \Phi\|} = \vec{\mathcal{N}}. \quad (15)$$

In this equation, the left side is written in terms of the surface  $\Phi$ , while the right side depends just on the curve  $\mathcal{X}$ . The combination of equations (13) to (15) gives

$$\Phi_t = -\beta \|\nabla \Phi\|, \quad (16)$$

which gives the evolution equation of the graph when its level sets evolves according to (13).

Alvarez *et al.* [3] recently proposed an algorithm for image selective smoothing and edge detection which is based on the flow (16) for  $\beta = \kappa$ , i.e., for the Euclidean heat flow of the level sets. In this case, the image (graph) evolves according to

$$\Phi_t = f(\|G * \nabla \Phi\|) \|\nabla \Phi\| \operatorname{div} \left( \frac{\nabla \Phi}{\|\nabla \Phi\|} \right), \quad (17)$$

where  $G$  is a smoothing kernel (for example, a Gaussian), and  $f(p)$  is a nonincreasing function which tends to zero as  $p \rightarrow \infty$ . Equation (17) can be interpreted as follows [3]:

1. The term

$$\|\nabla\Phi\| \operatorname{div} \left( \frac{\nabla\Phi}{\|\nabla\Phi\|} \right),$$

which is equal to  $\Phi_{\xi\xi}$ , where  $\xi$  is the direction normal to  $\nabla\Phi$ , diffuses  $\Phi$  in the direction orthogonal to the gradient  $\nabla\Phi$ , and does not diffuse in the direction of  $\nabla\Phi$ . Thus, the image is being smoothed on both sides of the edge, with minimal smoothing at the edge itself. It can be shown that the evolution

$$\Phi_t = \|\nabla\Phi\| \operatorname{div} \left( \frac{\nabla\Phi}{\|\nabla\Phi\|} \right)$$

is identical to

$$\Phi_t = \frac{\Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + \Phi_x^2 \Phi_{yy}}{\Phi_x^2 + \Phi_y^2}, \quad (18)$$

which means that the level sets of  $\Phi$  move according to the Euclidean heat flow [3, 52]. For general results concerning the evolution of level sets, see [15, 22, 52, 62].

2. The term

$$f(\|G * \nabla\Phi\|)$$

is used for the enhancement of the edges. If  $\|\nabla\Phi\|$  is “small”, then the diffusion is strong. If  $\|\nabla\Phi\|$  is “large” at a certain point  $(x, y)$ , this point is considered as an edge point, and the diffusion is weak.

Recapping, equation (17) gives an anisotropic diffusion, extending the ideas first proposed by Perona and Malik [53]. The equation looks like the level sets of  $\Phi$  are moving according to Euclidean heat flow, with the velocity value “altered” by the function  $f(\cdot)$ .

This flow was extended to the affine case in [58] (see also [2]). In this case, the level sets evolve according to the affine heat flow, and therefore, the graph evolves according to

$$\Phi_t = (\Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + \Phi_x^2 \Phi_{yy})^{1/3}. \quad (19)$$

If we compare equation (18) with equation (19), we observe that the denominator is eliminated in the latest one. As pointed out in [58], this makes the numerical implementation of the affine image smoothing more stable than the Euclidean one. The affine flow was compared to the Euclidean one and to the classical heat flow in [43] for MRI images, and produced much better results, as expected.

In real applications, like image smoothing, the original surface, and its level sets, are non-smooth. Therefore, the previous theory should be extended to non-smooth curves. In [2, 15, 22, 23], the authors studied the evolution of surfaces via level-sets flows, and extended this type of flows to non-smooth curves using the theory of viscosity solutions [18]. They

also proved the existence of a unique “physical” weak solution to the flow, which can be interpreted as an extension of curvature flows for singular curves. The existence of a unique solution for Lipschitz initial curves, was studied for the affine heat flow in [8] with a different approach as well. Therefore, we conclude that the theory of level-sets flows is well developed for non-smooth initial curves as well, allowing the practical implementation of this kind of smoothing process in real applications like image smoothing. Note that the algorithm for curve evolution proposed by Osher and Sethian [52] is based precisely on level set evolutions, making this work one of the first in the area. Figure 2 presents an example of the use of the affine smoother for edge detection.

## 5 Geometric surface evolution

In this section, we summarize some of the main results on the evolution of surfaces according to functions of their principal curvatures. This topic was first investigated by Brakke [11] for mean curvature flows, and by many others since then. In contrast to the results presented in Section 4, where the surface flow was driven by 2D evolutions of level sets, the flows analyzed now will be governed by proper 3D equations. We will analyze both graph flows and pure surface flows. As we will see, in contrast to the planar case, different constraints must be imposed to the initial surface in order to the evolving surface remain smooth. See also Section 9. In Sections 7 and 8, we will write down a general expression for an invariant evolution of a given hypersurface with respect to a transformation group acting on  $\mathbf{R}^n$ .

Huisken analyzed boundary-initial value problems of mean curvature flows in [33] (see also [34] for a nice review of some of the results). He proved that a smooth initial surface defined on a bounded domain  $\Omega$  with vertical contact angle <sup>1</sup> in its boundary  $\partial\Omega$ , remains smooth and converges to a constant value when evolving in the vertical direction (the  $z$  axes when viewed as a 2D graph), with velocity equal to the mean curvature. Note that the problem has both initial and boundary conditions (imposed by the vertical contact). This result means that for a graph (or image) possessing the vertical condition property, the curvature decreases as a function of time, and the surface is smoothed. He also proved that if  $\partial\Omega$  has non-negative mean curvature, then an initial smooth surface  $S_0$  converges to the solution of the minimal surface equation when evolving according to the mean curvature flow while keeping its value equal to a given function in the boundary  $\partial\Omega$ . See also Chopp [16] for the computation of minimal surfaces using this geometric flow.

Further results on evolution of graphs via mean curvature, without boundary conditions, were obtained by Ecker and Huisken in [19]. Here, the authors proved that any polynomial growth for the height and the gradient of the initial surface is preserved during the evolution. They also proved that, for Lipschitz initial data with linear growth, the flow has a solution for all times. The asymptotic behavior of the evolving surface was also studied by the authors, proving that under certain conditions, the surface converges to self-similar solutions of the mean curvature flow. It is very important to note that this condition, which is related to the fact that the initial graph is “straight” at infinity, is necessary and sufficient. Therefore, with this result we may also conclude that not every graph will converge to a self-similar

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<sup>1</sup>The gradient is parallel to the normal.

solution, and some other conditions must be added for convergence.

The initial-boundary problem related to the evolution of surfaces by mean curvature was also studied recently by Olikar and Uraltseva in [48]. In this case, in contrast with the aforementioned problem studied by Huisken, the boundary condition is given by the graph attached to a zero value, i.e., the boundary of the surface remains fixed ( $\mathcal{S} \equiv 0$  in  $\partial\Omega$ ). The authors studied the existence of generalized solutions to the mean curvature flow for arbitrary domains. They showed that such a solution may develop singularities at the boundary at some finite time. It is precisely the possibility of the development of singularities in the boundary which makes it a generalized solution. These singularities disappear after that, and the solution becomes smooth up to the boundary. The authors also gave sufficient conditions on the domain  $\Omega$  and the initial surface  $S_0$  for this problem to have classical solutions for all time. The asymptotic behavior of the surface was studied as well, showing that a normalized solution of the mean curvature flow with fixed boundary, approaches exponentially the first eigenfunction of the Laplace operator with Dirichlet data in  $\Omega$ . The evolution also “picks up” the symmetries of the domain  $\Omega$ . For example if  $\Omega$  is a sphere, then asymptotically the solution becomes radially symmetric. From the results of Olikar and Uraltseva, we conclude again that not every surface with fixed boundaries becomes smooth in time. For the surface to become smooth, constraints on the initial data and the geometry of the boundary must be added. Some of these results were extended in [49] for functions of the mean curvature and other boundary conditions.

Olikar also studied the evolution of surfaces via the Gaussian curvature in [45, 48]. In [45], the author assumed that the domain  $\Omega$  is convex, and the surface is attached to the boundary  $\partial\Omega$ . He studied the existence of self-similar solutions, and also showed, as in the mean curvature flow, that the solution to the flow “picks up” the symmetries of the domain.

The results presented above and in previous sections, are related to graphs or surfaces with boundary conditions. Those results can be used for example for smoothing images, when those images satisfy the required properties. We deal now with the evolution of proper (closed) 3D structures.

The first results concerning the evolution of surfaces, are related to the evolution of convex ones. For convex surfaces, analogous results to those proved in the plane by Gage and Hamilton are valid. Huisken proved in 1984 [32] that a convex surface evolves into a round point<sup>2</sup> when evolving in the normal direction with velocity equal to the mean curvature, remaining smooth during the flow. Chow [17] proved the same result when the velocity is given by the square root of the Gaussian curvature. He actually proved existence of a smooth solution for any (positive) power of the Gaussian curvature. All these results were related to surfaces contracting in time, i.e., moving inward. Urbas investigated the expanding evolution of convex surfaces in [66, 67]. He studied the expansion of convex surfaces by a family of positive, symmetric, and concave functions of their principal radii of curvature, proving that a smooth initial surface remains smooth, and its normalized version converges to a sphere. See the aforementioned papers and references therein for more details about the behavior of convex surfaces deforming via geometric flows. (The above results hold more generally for convex hypersurfaces. Here Chow [17] shows convergence to a round point when evolving according to the  $n$ -th root of the Gaussian curvature, where  $n$  is the dimension of

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<sup>2</sup>The normalized surface evolves into an sphere.

the hypersurface.)

The situation for non-convex surfaces is much more complicated and still the subject of much research (see for example [7, 29, 63]). In general, a non-convex surface evolving according to the mean curvature will not remain smooth, or even connected, as we can see from the famous dum-bell example. The question is if we can ensure for certain class of non-convex surfaces that they remain smooth and connected when evolving according to some geometric flow. Gerhardt [25] considers star-shaped surfaces under an *outward* unit normal flow; similar results were also obtained in [65]. More precisely, Gerhardt studies the evolution of those surfaces in the outward normal direction, with velocity equal to a function  $k$ , where  $k = 1/f(\kappa_1, \kappa_2)$ , being  $f$  a positive, symmetric function on an open, convex and symmetric cone in  $\mathbf{R}^2$ . The function  $f$  is also assumed to be homogeneous of degree one, concave and increasing in the cone, as well as zero on its boundary. An example of this function is of course the mean curvature. For these functions, he proved that an initial star-shaped surface remains smooth and star-shaped. When the surface is normalized, it converges into an sphere. Other results, such as short-term existence for the mean curvature flow for Lipschitz initial data, can be found for example, in [20].

We conclude this section with some remarks on weak solutions of the aforementioned geometric flows. As pointed out in Section 2, in [15, 22], the geometric evolution of level sets was studied in the framework of viscosity solutions. In [22] the mean curvature flow is analyzed, while in [15] more general evolution equations are studied. In both papers the authors showed the existence of a unique weak solution for partial differential equations in which the level sets evolve in time according to the mean curvature. Short-term existence of a classical (smooth) solution is proved as well (see also [23]). Therefore, even if the initial surface does not hold one of the properties which are required for long-term existence of classical solutions—for example convexity—nevertheless, a unique weak solution can be constructed, based on the theory of viscosity solutions. These results allows one to generalize the definition of mean curvature flows also for non-smooth surfaces. Of course, the generalized definition coincides with the classical one when the surface is smooth and the flow can be defined in the framework of classical differential geometry. These generalized flows also satisfy some of the analogous properties to the planar case. For example:

1. The order is preserved. If  $S_0$  and  $\hat{S}_0$  are two initial surfaces, and  $S_t$  and  $\hat{S}_t$  are the corresponding generalized solutions of the mean curvature flow, and  $S_0 \subseteq \hat{S}_0$ , then  $S_t \subseteq \hat{S}_t$  for all  $t > 0$ .
2. The distance between two surfaces increases with time.

These and other properties are proved for planar curves, for geometric heat flows, in [6, 26, 27, 55, 57, 59].

The evolution of surfaces as level sets of higher order ones was proposed and also studied experimentally by Osher and Sethian in [52, 62].

## 6 Variational derivatives

In order to present our results on the general form of an invariant evolution equation<sup>3</sup>, we need to recall a basic concept from the calculus of variations — the variational derivative of a functional. The full details may be found in [50]. We work in an open domain  $M$  of the Euclidean space  $X \times U$ , where  $X = \mathbf{R}^p$  has coordinates  $x = (x^1, \dots, x^p)$  representing the independent variables, and  $U = \mathbf{R}$  has coordinate  $u$  representing the dependent variable. (For simplicity, and since our applications are all of this form, we restrict our attention to the case of a single dependent variable  $u$ , although extensions to several dependent variables are straightforward.) We use the notation  $u^{(n)}$  to denote the collection of all partial derivatives  $u_J = \partial_J u$  up to order  $n$ . Here  $J = (j_1, \dots, j_k)$ ,  $1 \leq j_\nu \leq p$ , is a symmetric multi-index of order  $\#J = k \leq n$ . The variables  $(x, u^{(n)})$  provide coordinates in the  $n$ -th order jet space (or bundle) associated with  $M$ .

An  $n$ -th order *variational problem* consists of finding the extremals (maxima or minima) of a *functional*

$$\mathcal{L}[u] = \int_D L(x, u^{(n)}) dx, \quad (20)$$

over some class of functions  $u = f(x)$  defined over a domain  $D \subset X$ , subject to certain boundary conditions. We assume that the integrand  $L(x, u^{(n)})$ , which is referred to as the *Lagrangian* of the variational problem, is a smooth function of  $x$ ,  $u$  and the derivatives of  $u$ .

**Theorem 1** *The smooth extremals of the variational problem  $\mathcal{L}[u]$  must satisfy the Euler-Lagrange equation*

$$E(L) \equiv \sum_{\#J=0}^n (-D)_J \frac{\partial L}{\partial u_J} = 0, \quad \alpha = 1, \dots, q. \quad (21)$$

In (21), for each multi-index  $J = (j_1, \dots, j_k)$ , we define the total derivative  $(-D)_J := (-1)^{\#J} D_{j_1} \cdot D_{j_2} \cdots D_{j_k}$ .

The differential operator  $E = (E_1, \dots, E_q)$  giving rise to the Euler-Lagrange equation is known as the *variational derivative*. For example, in the case of one independent and one dependent variable, the Euler-Lagrange equation associated with a Lagrangian  $L(x, u^{(n)})$  is the ordinary differential equation

$$\frac{\partial L}{\partial u} - D_x \left( \frac{\partial L}{\partial u_x} \right) + D_x^2 \left( \frac{\partial L}{\partial u_{xx}} \right) - \cdots + (-1)^n D_x^n \left( \frac{\partial L}{\partial u_n} \right) = 0,$$

where  $u_n = D_x^n u$  is the  $n$ -th order derivative of  $u$ . For nondegenerate  $n$ -th order Lagrangians, the Euler-Lagrange equation has order  $2n$ .

The proof of Theorem 1 relies on the analysis of variations of the extremal  $u$ . In general, a one-parameter family of functions  $u(x, \varepsilon)$  a *family of variations* of a fixed function  $u(x) =$

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<sup>3</sup>When we deal with evolution equations, we refer to flows depending on only first order time derivatives, but, possibly, higher order space derivatives. The importance of this kind of flows in image processing was analyzed in [2].

$u(x, 0)$  provided that, outside a compact subset  $K \subset D$ , the functions are all the same:  $u(x, \varepsilon) = u(x)$  for  $x \in D \setminus K$ . An integration by parts argument shows that if  $u(x, \varepsilon)$  is any one-parameter family of variations of a fixed function  $u(x) = u(x, 0)$ , then

$$\left. \frac{d}{d\varepsilon} \mathcal{L}[u(x, \varepsilon)] \right|_{\varepsilon=0} = \int_D E(L) \cdot v \, dx, \quad (22)$$

where  $v(x) = \partial u(x, \varepsilon) / \partial \varepsilon|_{\varepsilon=0}$ . (Usually, this formula is used in the case  $u(x, \varepsilon) = u(x) + \varepsilon v(x)$ , where  $v(x)$  has compact support, but adding in higher order terms in  $\varepsilon$  has no effect.) In particular, if  $u(x)$  is an extremal of the variational problem, then, by elementary calculus, the left hand side of (22) must vanish. Since this happens for all variations  $v(x)$ , we deduce the necessary conditions in Theorem 1.  $\square$

## 7 Invariant hypersurface flows

Let  $G$  be a finite-dimensional, connected transformation group acting on an open subset  $M \subset X \times U \simeq \mathbf{R}^{p+1}$  of the space of independent and dependent variables. In this section, we write down the general form that any  $G$ -invariant evolution in  $p$  independent and one dependent variable must have. Thus for  $p = 1$ , we get the family of all possible invariant curve evolutions in the plane under a given transformation group, and for  $p = 2$  the family of all possible invariant surface evolutions a given transformation group.

We let

$$\omega = g \, dx^1 \wedge \dots \wedge dx^p = g \, dx,$$

denote a  $G$ -invariant  $p$ -form. Note that we can consider the function  $g(x, u^{(n)})$  as a Lagrangian of the  $G$ -invariant variational problem  $\mathcal{L}[u] = \int \omega$ . In applications, then, the  $p$ -form  $\omega$  represents the  $G$ -invariant element of arc length, or surface area. The Euler-Lagrange equations associated with  $\omega$ , then, describe the  $G$ -invariant minimal curves or surfaces. We will always assume that the Euler-Lagrange equations are not identically zero,  $E(g) \neq 0$ .

The infinitesimal generators of  $G$  are vector fields of the form

$$\mathbf{v} = \xi(x, u) \partial_x + \varphi(x, u) \partial_u = \xi_1(x, u) \frac{\partial}{\partial x^1} + \dots + \xi_p(x, u) \frac{\partial}{\partial x^p} + \varphi(x, u) \frac{\partial}{\partial u} \quad (23)$$

on  $M$ . The *characteristic* of the vector field (23) is the first order function

$$Q(x, u^{(1)}) = \varphi(x, u) - \sum_{i=1}^p \xi^i(x, u) \frac{\partial u}{\partial x^i}. \quad (24)$$

We let  $\text{pr } \mathbf{v}$  denote the prolongation of the vector field  $\mathbf{v}$  to the jet space. The explicit formulae for the prolongation can be found in [50].

We now prove the following key result:

**Lemma 2** *Let  $\text{pr } \mathbf{v}$  be the prolongation of the vector field  $\mathbf{v} = \xi(x, u) \partial_x + \varphi(x, u) \partial_u$ . Let  $L \, dx$  be a (Lagrangian)  $p$ -form. Then*

$$E[\text{pr } \mathbf{v}(L) + L \text{Div } \xi] = \text{pr } \mathbf{v}(E[L]) + (Q_u + \text{Div } \xi) E[L]. \quad (25)$$



Here

$$Q_u = \frac{\partial Q}{\partial u} = \frac{\partial \varphi}{\partial u} - \sum_{i=1}^p \frac{\partial \xi^i}{\partial u} \frac{\partial}{\partial x^i}, \quad (26)$$

and  $\text{Div } \xi = \sum_{i=1}^p D_i \xi^i$  is the total divergence of the  $\xi$ 's.

**Proof.** Let  $u(x, \varepsilon) = u(x) + \varepsilon v(x) + \dots$  be a one-parameter family of variations of a fixed function  $u(x)$  (as in Section 6). Let  $u(x, \varepsilon, t) = \exp(t\mathbf{v})u(x, \varepsilon)$  be the corresponding transformed functions, as in [50]. The fact that the variations have compact support in  $D$  implies that, for  $t$  sufficiently small, the family  $u(x, t, \varepsilon)$  also satisfies the relevant boundary conditions. As we shall see, (25) is just a statement of the equality of mixed partials. We compute the derivative

$$\left. \frac{\partial^2}{\partial \varepsilon \partial t} \mathcal{L}[u(x, \varepsilon, t)] \right|_{\varepsilon=t=0}$$

in two different ways, using the variational formula (22) and the basic definition of the group action on a function. We first note that, expanding  $u(x, \varepsilon, t)$  in a Taylor series in  $\varepsilon$  and  $t$ , we have

$$u(x, \varepsilon, t) = u(x) + \varepsilon v(x) + tQ(x, u(x)) + \varepsilon t Q_u(x, u(x))v(x) + \dots \quad (27)$$

First differentiating with respect to  $\varepsilon$ , we find, as in section 6,

$$\left. \frac{\partial}{\partial \varepsilon} \mathcal{L}[u(x, \varepsilon, t)] \right|_{t=0} = \int_D E(L)[u(x, t)] \cdot v(x, t) dx$$

where  $u(x, t) = u(x, 0, t)$ ,  $v(x, t) = \partial u(x, 0, t) / \partial \varepsilon$ . Note that, by the preceding expansion (27),

$$v(x, t) = v(x) + tQ_u(x, u(x))v(x) + \dots \quad (28)$$

Therefore

$$\left. \frac{\partial^2}{\partial \varepsilon \partial t} \mathcal{L}[u(x, \varepsilon, t)] \right|_{\varepsilon=t=0} = \int_D \{ \text{pr } \mathbf{v}(E(L))v + E(L)Q_u v + E(L)v \text{Div } \xi \} dx,$$

the final term coming from the change in the  $p$ -form  $dx$  due to the group transformations.

On the other hand, if we first differentiate with respect to  $t$ , we find

$$\left. \frac{\partial}{\partial t} \mathcal{L}[u(x, \varepsilon, t)] \right|_{t=0} = \int_D \{ \text{pr } \mathbf{v}(L)[u(x, \varepsilon)] + L[u(x, \varepsilon)] \text{Div } \xi \} dx.$$

Therefore,

$$\left. \frac{\partial^2}{\partial \varepsilon \partial t} \mathcal{L}[u(x, \varepsilon, t)] \right|_{\varepsilon=t=0} = \int_D E[\text{pr } \mathbf{v}(L) + L \text{Div } \xi] dx.$$

Since these two integrals must agree for arbitrary variations  $v$ , we conclude the truth of the identity (25).  $\square$

**Remark.** In particular, if  $L = g$ , then the  $p$ -form  $g dx$  is  $G$ -invariant, so

$$\text{pr } \mathbf{v}(g) + g \text{Div } \xi = 0. \quad (29)$$

Therefore (25) implies the identity

$$\text{pr } \mathbf{v}[E(g)] + (\text{Div } \xi + Q_u)E(g) = 0, \quad (30)$$

where  $E = E(g)$ .

We can now prove the main result of this paper:

**Theorem 2** *Notation as above. Then every  $G$ -invariant evolution equation has the form*

$$u_t = \frac{g}{E(g)} I, \quad (31)$$

where  $I$  is a differential invariant of  $G$ .

**Proof.** Let

$$u_t = P \quad (32)$$

be a  $G$ -invariant flow. Then

$$\text{pr } \mathbf{v}[u_t - P] = Q_u u_t - \text{pr } \mathbf{v}[P] = 0. \quad (33)$$

Thus, the evolution (32) is invariant if and only if

$$Q_u P = \text{pr } \mathbf{v}[P]. \quad (34)$$

Next note that we get from (34, 30) that

$$\begin{aligned} \text{pr } \mathbf{v}[E(g)P] &= \text{pr } \mathbf{v}[E(g)]P + \text{pr } \mathbf{v}[P]E(g) \\ &= (-\text{Div } \xi - Q_u)E(g)P + Q_u E(g)P \\ &= (-\text{Div } \xi)E(g)P. \end{aligned} \quad (35)$$

Therefore,

$$\begin{aligned} \text{pr } \mathbf{v} \left[ \frac{E(g)P}{g} \right] &= \frac{E(g)P \text{pr } \mathbf{v}[g] - g \text{pr } \mathbf{v}[E(g)P]}{g^2} \\ &= \frac{E(g)P(-\text{Div } \xi) + E(g)P \text{Div } \xi}{g} = 0. \end{aligned}$$

This means that  $E(g)P/g$  is invariant, hence

$$P = \frac{g}{E(g)} I,$$

where  $I$  is a  $G$ -invariant function, which completes the proof.  $\square$

**Remark.** Theorem 2 and Lemma 2 also extend to several dependent variables (suitably reinterpreted, since you can't divide by  $E(g)$ ). Here you need as many independent volume forms as the number of dependent variables, and the  $1/E(g)$  becomes the matrix inverse of the variational derivatives of the volume forms.

We should also remark that an alternative proof of Theorem 2, based on the “variational bicomplex”, was communicated to us by Ian Anderson and Juha Pohjanpelto.

We will call a group  $G$  acting on  $M \subset X \times U$  *volume preserving* if it leaves the  $(p + 1)$ -form  $dx \wedge du = dx^1 \wedge \dots \wedge dx^p \wedge du$  invariant. Equivalently, using (26), the infinitesimal condition reads

$$0 = \sum_{i=1}^p \frac{\partial \xi^i}{\partial x^i} + \frac{\partial \varphi}{\partial u} = \text{Div } \xi + Q_u. \quad (36)$$

**Proposition 1** *Suppose  $G$  is a connected transformation group, and  $g dx$  a  $G$ -invariant  $p$ -form such that  $E(g) \neq 0$ . Then  $E(g)$  is a differential invariant if and only if  $G$  is volume preserving.*

**Proof.** This follows trivially from the fundamental equation (30) and the infinitesimal volume preserving condition (36). Since

$$\text{pr } \mathbf{v}[E(g)] + (\text{Div } \xi + Q_u)E(g) = 0,$$

we conclude that  $E(g)$  is invariant, i.e.,  $\text{pr } \mathbf{v}[E(g)] = 0$ , if and only if (36) holds.  $\square$

**Corollary 1** *Let  $G$  be a connected volume preserving transformation group. Then the  $G$ -invariant flow of lowest order has the form*

$$u_t = cg, \quad (37)$$

where  $\omega = g dx^1 \wedge \dots \wedge dx^p$  is the invariant  $p$ -form of minimal order such that  $E(g) \neq 0$ .

**Remark.** The  $p$ -form of minimal order will be unique unless  $G$  has a differential invariant of equal or lower order than  $g$ .

## 8 Affine invariant surface flows

In this section, we describe the simplest possible affine invariant surface evolution. This gives the surface version of the affine shortening flow for curves. This equation was also derived using completely different methods by [4]. Note that besides affine invariance, a number of properties were required in [4] to obtain the flow we present below (some of these properties are related to the importance of the flow being an “evolution equation”). In our approach, after the starting point of formulation of an evolution equation, the only requirement besides

the affine invariance, is to be the simplest possible flow. That is, the only requirement is “the simplest flow which admits the affine group as its symmetry group.”

We define the (*special*) *affine group* on  $\mathbf{R}^3$  as the group of transformations generated by  $SL_3(\mathbf{R})$  (the group of  $3 \times 3$  matrices with determinant 1) and translations.

Let  $\mathcal{S}$  be a smooth strictly convex surface in  $\mathbf{R}^3$ , which we write locally as the graph  $(x, y, u)$ . The Gaussian curvature is given by

$$\kappa = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}.$$

Now from [10, 12], the affine invariant metric is given by

$$g = \kappa^{1/4} \sqrt{\det g_{ij}} = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2},$$

where  $g_{ij}$  are the coefficients of the first fundamental form.

Thus from Corollary 1, we conclude:

**Corollary 2** *Notation as above. Then*

$$u_t = c\kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2}, \tag{38}$$

(for  $c$  a constant) is the simplest affine invariant surface flow. This corresponds to the global evolution

$$\mathcal{S}_t = c\kappa^{1/4} \vec{\mathcal{N}}, \tag{39}$$

where  $\vec{\mathcal{N}}$  denotes the inward normal.

We will call the evolution

$$\mathcal{S}_t = \kappa^{1/4} \vec{\mathcal{N}}, \tag{40}$$

the *affine surface flow*. Note that it is the surface analogue of the affine heat equation (9).

**Remarks.**

1. Recently, it has been announced that a convex ( $C^2$ ) surface will converge to an ellipsoidal point under the affine surface flow (40); [5, 47]. Indeed, one must verify that the affine curvature [30] becomes constant for the corresponding normalized dilated surfaces flow. (Another possibility would be to show that the affine isoperimetric inequality converges to the right value [41].) Of course, this result generalizes in a straightforward way to convex hypersurfaces in any dimension (where one uses the  $(n + 2)$ -nd root of the Gaussian curvature for  $n$  the dimension of the hypersurface).
2. In general, Chow [17] has shown that a convex hypersurface converges smoothly to a point under the flow defined by any power  $\beta > 0$  of the Gaussian curvature. Moreover, it is shown that for  $\beta = 1/n$  where  $n$  is the dimension of the hypersurface, the point is round. Other than  $\beta = 1/n, 1/(n + 2)$ , the shape of the point is not known.

3. Finally, V. Caselles and C. Sbert have recently shown that a dumb-bell does not become singular under the flow (40) [14] (they actually take  $(\kappa^{1/4})^+$  as velocity). This is in contrast to flow via mean curvature. On the other hand, they also presented examples where the flow disconnects an initially connected non-convex surface. Several examples of this flow, as well as the mean curvature one, can be found in this paper as well.

## 9 Concluding remarks

In this work, we first reviewed basic results concerning geometric smoothing of surfaces. We considered both  $2\frac{1}{2}$ D smoothing processes, based on smoothing graphs via level set smoothing, and pure 3D processes. When dealing with 3D surfaces, we presented results related to the evolution of graphs and pure three dimensional objects. The results concerned the evolution via functions of the principal curvatures, such as the mean and Gaussian curvatures. Unfortunately, the results expected from the planar theory do not hold in the 3D case. An arbitrary regular surface can develop singularities when evolving according to the Gaussian or mean curvature, or even other more general functions as we described in this paper. Therefore, these kind of flows cannot be used for smoothing general surfaces. However, they can be used for specific graphs or surfaces, e.g. star-shaped surfaces. We are currently investigating the evolution of surfaces by other functions of their principal curvature. Our goal with these functions is to achieve surface flows with analogous behavior to those of planar geometric flows, and then to be able to perform geometric smoothing of more general surfaces.

Another topic that we are currently investigation is the possibility of smoothing 3D surfaces via geometric 2D flows applied to curves on the surface, different from the level sets. One possibility is to smooth lines of curvature, or lines of maximal slope. The main advantage of smoothing 3D objects via 2D geometric flows is the existence of a well developed theory for these kind of flows, as we saw in Section 2.

In the second part of the paper we presented a general formulation for invariant geometric flows of hypersurfaces. This result completes the theory started in [51] for planar curves. We showed that the invariant flows can be formulated as functions of the invariant metric and invariant curvature, which are the basic differential invariant descriptors, together with the variational derivative of this metric. As an example, we derived the simplest affine invariant geometric flow for 3D surfaces. We also showed that if the transformation group is volume preserving, this variational derivative is invariant as well. Note that the invariant geometric flows for planar curves are smoothing processes for both the Euclidean and special affine groups, but not for the similarity, full affine, and projective ones [51]. One of the key differences among these groups is that the first two are area preserving while the others are not. We are currently investigating if there is any connection between the lack of smoothing and the lack of invariance of the variational derivative for non-area preserving groups. For such groups, we are also investigating the use of different invariant metrics to define geometric smoothing processes. These metrics can be used either to define different “heat flows,” obtained via derivatives according to the corresponding arc-length, or to derive geometric variational problems which can define smoothing processes.

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## Figure Captions

1. Normal curvature diagram.
2. Denoising based on the affine invariant scale-space. The original image is presented first, then the noisy one, and then steps (different times) of the smoothing process.























