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INVARIANT HYPERSURFACES OF A MANIFOLD WITH (f, g, u, v, λ)-STRUCTURE

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Introduction.

We have studied in [2] a differentiable manifold with (f, U, V, u, v, λ)-structure, that is, a differentiable manifold with tensor field f of type (1, 1), two vector fields U and V, two 1-forms u and v and a function λ satisfying

$$f^2X=-X+u(X)U+v(X)V,$$

$$u(fX)=+\lambda v(X), \qquad fU=-\lambda V,$$

$$v(fX)=-\lambda u(X), \qquad fV=\lambda U,$$

$$u(U)=1-\lambda^2, \qquad u(V)=0,$$

$$v(U)=0, \qquad v(V)=1-\lambda^2,$$

for any vector field X.

An (f, U, V, u, v, λ)-structure is said to be normal if it satisfies

$$N(X, Y)+du(X, Y)U+dv(X, Y)V=0,$$

where $N(X, Y)$ is the Nijenhuis tensor of f defined by

$$N(X, Y)=[fX, fY]-f[fX, Y]-f[X, fY]+f^2[X, Y]$$

for any vector fields X and Y.

If there exists a positive definite Riemannian metric g such that

$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y),$$

$$g(U, X)=u(X), \qquad g(V, X)=v(X)$$

for any vector fields X and Y, then we call the structure an (f, g, u, v, λ)-structure. In this case

$$\omega(X, Y)=g(fX, Y)$$

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is a 2-form.

A submanifold of codimension 2 of an almost Hermitian manifold admits an (f, g, u, v, λ)-structure and a hypersurface of an almost contact metric manifold admits the same kind of the structure.

In [2], we have proved

THEOREM A. *Let M be a complete manifold with normal (f, g, u, v, λ)-structure satisfying*

$$du=\phi\omega, \qquad dv=\omega,$$

ϕ being a function. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then the manifold M is isometric with a sphere.

In [3], we have also studied normal (f, g, u, v, λ)-structures on submanifolds of codimension 2 in a Euclidean space and proved

THEOREM B. *Let a complete differentiable submanifold M of codimension 2 of an even-dimensional Euclidean space be such that the connection induced in the normal bundle of M has zero curvature. If the (f, g, u, v, λ)-structure induced on M is normal, then M is a sphere, a plane, or a product of a sphere and a plane.*

The main purpose of the present paper is to study invariant hypersurfaces of a manifold with (f, g, u, v, λ)-structure. A hypersurface is said to be invariant if the tangent hyperplane is invariant by the action of f.

After stating some preliminaries in §1, we study in §2 general hypersurfaces of a manifold with (f, g, u, v, λ)-structure and obtain some general formulas valid for these general hypersurfaces.

In §3, we specialize these general formulas and obtain formulas valid for invariant hypersurfaces. We prove that an invariant hypersurface of a manifold with (f, g, u, v, λ)-structure admits an almost contact metric structure.

In §4, we study invariant hypersurfaces of M with normal (f, g, u, v, λ)-structure and prove that an invariant hypersurface of a manifold with normal (f, g, u, v, λ)-structure satisfying $dv=\omega$ is a Sasakian manifold.

§1. The (f, g, u, v, λ)-structure.

Let M be a $(2n+2)$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and throughout the paper the indices $h, i, j, k, \cdots$ run over the range $1, 2, \cdots, 2n+2$. Let there be given in M a tensor field f of type $(1, 1)$, a Riemannian metric g, two 1-forms u and v and a function λ, satisfying

$$(1)\qquad \begin{cases} f_j{}^i f_i{}^h = -\delta_j^h + u_j u^h + v_j v^h \\ f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\ f_j{}^i u_i = +\lambda v_j, \qquad f_j{}^i v_i = -\lambda u_j, \\ f_i{}^h u^i = -\lambda v^h, \qquad f_i{}^h v^i = +\lambda u^h, \\ u_i u^i = v_i v^i = 1-\lambda^2, \qquad u_i v^i = 0, \end{cases}$$

where

$$u^h = u_i g^{ih} \quad \text{and} \quad v^h = v_i g^{ih},$$

g^{ih} being contravariant components of the metric tensor. We can easily prove that

$$f_{ji} = f_j{}^t g_{ti}$$

is a skew-symmetric tensor.

The set (f, g, u, v, λ) satisfying (1) is called an (f, g, u, v, λ)-structure on M. An M with (f, g, u, v, λ)-structure is orientable.

A typical example of an M with (f, g, u, v, λ)-structure is an even dimensional sphere in a Euclidean space.

The (f, g, u, v, λ)-structure is said to be normal if it satisfies

$$(2)\qquad S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j)u^h + (\nabla_j v_i - \nabla_i v_j)v^h = 0,$$

where

$$N_{ji}{}^h = f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h$$

is the Nijenhuis tensor formed with $f_i{}^h$ and ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols $\{_j{}^h{}_i\}$ formed with g_{ji}.

In [2], we have proved

THEOREM C. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying*

$$(3)\qquad \nabla_j v_i - \nabla_i v_j = 2 f_{ji},$$

then we have

$$(4)\qquad f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} = u_j(\nabla_i u_h) - u_i(\nabla_j u_h) + v_j(\nabla_i v_h) - v_i(\nabla_j v_h).$$

§2 Hypersurfaces of M with (f, g, u, v, λ)-structure.

We consider a $(2n+1)$-dimensional differentiable manifold V covered by a system of coordinate neighborhoods $\{U'; y^a\}$ where here and in the sequel the indices a, b, c, d, e run over the range $1, 2, \cdots, 2n+1$. We assume that the manifold V is immersed in M by the immersion $i: V \to M$ as a hypersurface $i(V)$ of M and that the equation of $i(V)$ of M are

$$x^h = x^h(y^a).$$

If we put

$$B_a{}^h = \partial_a x^h \qquad (\partial_a = \partial/\partial y^a),$$

the Riemannian metric induced on $i(V)$ from that of M is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

We denote by N^h the unit normal to $i(V)$ such that the vectors $B_1{}^h, B_2{}^h, \cdots, B_{2n+1}{}^h, N^h$ form the positive orientation of M and by ∇_c the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_c{}^a{}_b\}$ formed with g_{cb}. Then we have equations of Gauss

$$\nabla_c B_b{}^h = h_{cb} N^h,$$

where

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - \left\{ {a \atop c\ b} \right\} B_a{}^h$$

is the so-called van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ and h_{cb} is the second fundamental tensor and equation of Weingarten

$$\nabla_c N^h = -h_c{}^a B_a{}^h,$$

where

$$h_c{}^a = h_{cb} g^{ba},$$

g^{ba} being contravariant components of the induced Riemannian metric tensor.

Now the transform $f_i{}^h B_b{}^i$ of $B_b{}^i$ by $f_i{}^h$ and the transform $f_i{}^h N^i$ of N^i by $f_i{}^h$ are respectively given by

$$f_i{}^h B_b{}^i = \varphi_b{}^a B_a{}^h + w_b N^h, \tag{5}$$

where $\varphi_b{}^a$ is a tensor field of type (1, 1) and w_b is a 1-form in V, and

$$f_i{}^h N^i = -w^a B_a{}^h, \tag{6}$$

where

$$w^a = w_b g^{ba}.$$

The vector fields u^h and v^h are respectively written as

$$u^h = B_a{}^h u^a + \alpha N^h \tag{7}$$

and

$$v^h = B_a{}^h v^a + \beta N^h \tag{8}$$

along $i(V)$, where u^a and v^a are vector fields of V and α and β are functions of V.

Now applying the operator $f_h{}^k$ to both members of (5), we find

$$(-\delta_i^k+u_iu^k+v_iv^k)B_b{}^i=\varphi_b{}^c(\varphi_c{}^aB_a{}^h+w_cN^h)-w_bw^aB_a{}^h,$$

from which

$$-B_b{}^k+u_b(B_a{}^ku^a+\alpha N^k)+v_b(B_a{}^kv^a+\beta N^k)=\varphi_b{}^c(\varphi_c{}^aB_a{}^k+w_cN^k)-w_bw^aB_a{}^k,$$

by virtue of (7) and (8) and consequently, comparing the tangential part and the normal part of both members, we find

$$\varphi_b{}^c\varphi_c{}^a=-\delta_b^a+u_bu^a+v_bv^a+w_bw^a \tag{9}$$

and

$$\alpha u_b+\beta v_b=\varphi_b{}^cw_c. \tag{10}$$

Applying the operator $f_h{}^k$ to both members of (6), we find

$$(-\delta_i^k+u_iu^k+v_iv^k)N^i=-w^c(\varphi_c{}^aB_a{}^k+w_cN^k),$$

from which

$$-N^k+\alpha(B_a{}^ku^a+\alpha N^k)+\beta(B_a{}^kv^a+\beta N^k)=-w^c(\varphi_c{}^aB_a{}^k+w_cN^k)$$

by virtue of (7) and (8) and consequently

$$\alpha u^a+\beta v^a=-\varphi_c{}^aw^c \tag{11}$$

and

$$\alpha^2+\beta^2=1-w_cw^c. \tag{12}$$

Applying the operator $f_h{}^k$ to both members of (7), we find

$$-\lambda v^k=(\varphi_a{}^cB_c{}^k+w_aN^k)u^a-\alpha w^cB_c{}^k,$$

from which

$$-\lambda(B_c{}^kv^c+\beta N^k)=(\varphi_a{}^cB_c{}^k+w_aN^k)u^a-\alpha w^cB_c{}^k$$

by virtue of (8), and consequently

$$-\lambda v^a=\varphi_c{}^au^c-\alpha w^a \tag{13}$$

and

$$-\lambda\beta=w_au^a. \tag{14}$$

Applying also the operator $f_h{}^k$ to both members of (8), we find

$$\lambda u^k=(\varphi_a{}^cB_c{}^k+w_aN^k)v^a-\beta w^cB_c{}^k,$$

from which

$$\lambda(B_c{}^k u^c+\alpha N^k)=(\varphi_a{}^c B_c{}^k+w_a N^k)v^a-\beta w^c B_c{}^k,$$

by virtue of (7), and consequently

(15) $$\lambda u^a=\varphi_c{}^a v^c-\beta w^a$$

and

(16) $$\lambda\alpha=w_a v^a.$$

Considering the lengths of u^h and v^h and the inner product of u^h and v^h, we have

(17) $$1-\lambda^2=u_a u^a+\alpha^2,$$

(18) $$1-\lambda^2=v_a v^a+\beta^2,$$

(19) $$0=u_a v^a+\alpha\beta.$$

Summing up, we have

(20) $$\begin{cases} \varphi_b{}^c\varphi_c{}^a=-\delta_b^a+u_b u^a+v_b v^a+w_b w^a, \\ \alpha u^a+\beta v^a=-\varphi_c{}^a w^c \\ \varphi_c{}^a u^c=-\lambda v^a+\alpha w^a, \qquad \varphi_c{}^a v^c=\lambda u^a+\beta w^a, \\ \alpha^2+\beta^2=1-w_a w^a, \\ \lambda\alpha=w_a v^a, \qquad \lambda\beta=-w_a u^a, \\ u_a u^a=1-\alpha^2-\lambda^2, \qquad v_a v^a=1-\beta^2-\lambda^2, \\ u_a v^a=-\alpha\beta. \end{cases}$$

§ 3. Invariant hypersurfaces of *M* with (*f, g, u, v*, λ)-structure.

We now assume that the hypersurface $i(V)$ is invariant, that is, the tangent hyperplane of $i(V)$ is invariant by the linear transformation $f_i{}^h$. Then we have

(21) $$f_i{}^h B_b{}^i=\varphi_b{}^a B_a{}^h,$$

that is,

(22) $$w_b=0.$$

Thus (20) becomes

$$(23)\qquad \begin{cases} \varphi_b{}^c\varphi_c{}^a=-\delta_b^a+u_bu^a+v_bv^a, \\ \alpha u^a+\beta v^a=0, \\ \varphi_c{}^au^c=-\lambda v^a, \qquad \varphi_c{}^av^c=+\lambda u^a, \\ \alpha^2+\beta^2=1, \\ \lambda\alpha=0, \qquad \lambda\beta=0, \\ u_au^a=1-\alpha^2-\lambda^2, \qquad v_av^a=1-\beta^2-\lambda^2, \\ u_av^a=-\alpha\beta. \end{cases}$$

From equation (7), (8) and

$$\alpha u^a+\beta v^a=0, \qquad \alpha^2+\beta^2=1,$$

we find

$$N^h=\alpha u^h+\beta v^h.$$

Thus we have

THEOREM 1. *The normal to an invariant hypersurface $i(V)$ of M with (f, g, u, v, λ)-structure is in the plane spanned by two vector fields u^h and v^h.*

Since $\alpha^2+\beta^2=1$, at least one of α and β is different from zero and consequently, from $\lambda\alpha=0$, $\lambda\beta=0$, we have $\lambda=0$ and consequently (23) becomes

$$(24)\qquad \begin{cases} \varphi_b{}^c\varphi_c{}^a=-\delta_b^a+u_bu^a+v_bv^a, \\ \alpha u^a+\beta v^a=0, \\ \varphi_c{}^au^c=0, \qquad \varphi_c{}^av^c=0, \\ \alpha^2+\beta^2=1, \\ u_au^a=1-\alpha^2, \qquad v_av^a=1-\beta^2, \qquad u_av^a=-\alpha\beta. \end{cases}$$

Now we put

$$V_\alpha=\{P\in i(V)\,|\,\alpha(P)\neq 0\}$$

and

$$V_\beta=\{P\in i(V)\,|\,\beta(P)\neq 0\}.$$

Then, V_α and V_β are both open in $i(V)$ and $V_\alpha\cup V_\beta=i(V)$, because of the fact that $\alpha^2+\beta^2=1$.

In V_α, from $\alpha u^a+\beta v^a=0$, we have

$$u^a=-\frac{\beta}{\alpha}v^a,$$

and consequently

$$
\begin{aligned}
u_b u^a + v_b v^a &= \frac{\beta^2}{\alpha^2} v_b v^a + v_b v^a \\
&= \frac{1}{\alpha^2} v_b v^a
\end{aligned} \tag{25}
$$

by virtue of $\alpha^2+\beta^2=1$. Thus, putting

$$\eta_b{}^{(\alpha)} = \frac{1}{\alpha} v_b, \tag{26}$$

we have

$$u_b u^a + v_b v^a = \eta_b{}^{(\alpha)} \eta^{a(\alpha)} \tag{27}$$

and

$$\eta_a{}^{(\alpha)} \eta^{a(\alpha)} = 1, \tag{28}$$

by virtue of $u_a u^a + v_a v^a = 2-(\alpha^2+\beta^2)=1$, where

$$\eta^{a(\alpha)} = \eta_b{}^{(\alpha)} g^{ba}.$$

We also have in V_α

$$u_b u_a + v_b v_a = \eta_b{}^{(\alpha)} \eta_a{}^{(\alpha)}. \tag{29}$$

In the same way, in V_β, we put

$$\eta_b{}^{(\beta)} = -\frac{1}{\beta} u_b.$$

Then, the equations similar to (27), (28) and (29) are valid for $\eta_b{}^{(\beta)}$ in V_β. On the other hand, in $V_\alpha \cap V_\beta$, we have

$$
\begin{aligned}
\eta_b{}^{(\beta)} &= -\frac{1}{\beta} u_b = -\frac{1}{\beta}\left(-\frac{\beta}{\alpha} v_b\right) \\
&= \frac{1}{\alpha} v_b = \eta_b{}^{(\alpha)},
\end{aligned}
$$

which shows that if we define a 1-form η by

$$
\eta = \begin{cases} \eta_b{}^{(\alpha)} dy^b & \text{in} \quad V_\alpha \\ \eta_b{}^{(\beta)} dy^b & \text{in} \quad V_\beta, \end{cases}
$$

then η is well defined on $i(V)$.

Thus, from (24), (27), (28) and (29), we find

$$(30)\qquad \begin{cases} \varphi_b{}^c\varphi_c{}^a=-\delta_b^a+\eta_b\eta^a, \\ \varphi_b{}^a\eta_a=0, \qquad \varphi_b{}^a\eta^b=0, \\ \eta_a\eta^a=1, \end{cases}$$

where η_a is the components of the 1-form η and $\eta^b=\eta_a g^{ab}$.

From

$$f_j{}^t f_i{}^s g_{ts}=g_{ji}-u_j u_i-v_j v_i,$$

we find, by transvection with $B_c{}^j B_b{}^i$,

$$f_j{}^t f_i{}^s g_{ts} B_c{}^j B_b{}^i=g_{cb}-u_c u_b-v_c v_b,$$

from which

$$\varphi_c{}^e B_e{}^t \varphi_b{}^d B_d{}^s g_{ts}=g_{cb}-u_c u_b-v_c v_b$$

by virtue of (21), and consequently

$$(31)\qquad \varphi_c{}^e\varphi_b{}^d g_{ed}=g_{cb}-\eta_c\eta_b$$

by virtue of (27). Thus we have proved

THEOREM 2. *An invariant hypersurface of a manifold with (f, g, u, v, λ)-structure admits an almost contact metric structure.*

§4. Invariant hypersurfaces of M with normal (f, g, u, v, λ)-structure.

We now assume that the (f, g, u, v, λ)-structure of M is normal, that is,

$$(32)\qquad S_{ji}{}^h=N_{ji}{}^h+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h=0.$$

It will be easily verified that, for an invariant hypersurface $i(V)$, we have

$$(33)\qquad N_{ji}{}^h B_c{}^j B_b{}^i=n_{cb}{}^a B_a{}^h,$$

where $n_{cb}{}^a$ is the Nijenhuis tensor formed with $\varphi_b{}^a$:

$$(34)\qquad n_{cb}{}^a=\varphi_c{}^e\nabla_e\varphi_b{}^a-\varphi_b{}^e\nabla_e\varphi_c{}^a-(\nabla_c\varphi_b{}^e-\nabla_b\varphi_c{}^e)\varphi_e{}^a.$$

On the other hand, we have

$$(\nabla_j u_i-\nabla_i u_j)B_c{}^j B_b{}^i=\nabla_c(u_j B_b{}^i)-\nabla_b(u_j B_c{}^i)$$
$$=\nabla_c u_b-\nabla_b u_c$$

by virtue of

$$\nabla_c B_b{}^i=\nabla_b B_c{}^i$$

and consequently

(35) $$(\nabla_j u_i - \nabla_i u_j) B_c{}^j B_b{}^i u^h = (\nabla_c u_b - \nabla_b u_c) u^a B_a{}^h + \alpha(\nabla_c u_b - \nabla_b u_c) N^h.$$

Similarly, we have

(36) $$(\nabla_j v_i - \nabla_i v_j) B_c{}^j B_b{}^i v^h = (\nabla_c v_b - \nabla_b v_c) v^a B_a{}^h + \beta(\nabla_c v_b - \nabla_b v_c) N^h.$$

Thus, from (32), (35) and (36), we have

$$\{n_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a\} B_a{}^h$$
$$+ \{\alpha(\nabla_c u_b - \nabla_b u_c) + \beta(\nabla_c v_b - \nabla_b v_c)\} N^h = 0,$$

from which

(37) $$n_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a = 0$$

and

(38) $$\alpha(\nabla_c u_b - \nabla_b u_c) + \beta(\nabla_c v_b - \nabla_b v_c) = 0.$$

Since we have easily

$$u_b = -\beta\eta_b, \qquad v_b = \alpha\eta_b,$$

we get

$$\nabla_c u_b - \nabla_b u_c = -\beta(\nabla_c \eta_b - \nabla_b \eta_c) - (\nabla_c \beta)\eta_b + (\nabla_b \beta)\eta_c,$$
$$\nabla_c v_b - \nabla_b v_c = \alpha(\nabla_c \eta_b - \nabla_b \eta_c) + (\nabla_c \alpha)\eta_b - (\nabla_b \alpha)\eta_c,$$

from which

(39) $$(\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a = (\nabla_c \eta_b - \nabla_c \eta_b)\eta^a$$

by virtue of $\alpha^2 + \beta^2 = 1$. Thus, (37) gives

(40) $$n_{cb}{}^a + (\nabla_c \eta_b - \nabla_b \eta_c)\eta^a = 0.$$

Thus we have

THEOREM 3. *An invariant hypersurface of a manifold with normal* (f, g, u, v, λ)-*structure admits a normal almost contact metric structure.*

We now assume that the normal (f, g, u, v, λ)-structure satisfies

(41) $$\nabla_j v_i - \nabla_i v_j = 2f_{ji}.$$

Transvecting (41) with $B_c{}^j B_b{}^i$, we find

$$(\nabla_c v_i) B_b{}^i - (\nabla_b v_j) B_c{}^j = 2f_{ji} B_c{}^j B_b{}^i,$$

or

$$\nabla_c (v_i B_b{}^i) - \nabla_b (v_j B_c{}^j) = 2f_{ji} B_c{}^j B_b{}^i,$$

or

$$\nabla_c v_b - \nabla_b v_c = 2\varphi_{cb} \tag{42}$$

by virtue of

$$\nabla_c B_b{}^i = \nabla_b B_c{}^i \qquad \text{and} \qquad f_{ji} B_b{}^i = \varphi_b{}^e B_{ej},$$

where

$$B_{ej} = B_e{}^k g_{kj}.$$

In [2], we have proved Theorem C, that is, *let M be a manifold with normal (f, g, u, v, λ)-structure satifying* (41), *then we have*

$$\begin{aligned} f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} &= u_j(\nabla_i u_h) - u_i(\nabla_j u_h) \\ &\quad + v_j(\nabla_i v_h) - v_i(\nabla_j v_h). \end{aligned} \tag{43}$$

Transvecting (43) with $B_c{}^j B_b{}^i B_a{}^h$ and taking account of (21), we find

$$\begin{aligned} &\varphi_c{}^e B_e{}^t(\nabla_a f_{ti}) B_b{}^i - \varphi_b{}^e B_e{}^t(\nabla_a f_{tj}) B_c{}^j \\ &= u_c(\nabla_b u_h) B_a{}^h - u_b(\nabla_c u_h) B_a{}^h + v_c(\nabla_b v_h) B_a{}^h - v_b(\nabla_c v_h) B_a{}^h, \end{aligned}$$

or

$$\begin{aligned} &\varphi_c{}^e \nabla_a(f_{ti} B_e{}^t B_b{}^i) - \varphi_b{}^e \nabla_a(f_{tj} B_e{}^t B_c{}^j) \\ &= u_c\{\nabla_b(u_h B_a{}^h) - u_h \nabla_b B_a{}^h\} - u_b\{\nabla_c(u_h B_a{}^h) - u_h \nabla_c B_a{}^h\} \\ &\quad + v_c\{\nabla_b(v_h B_a{}^h) - v_h \nabla_b B_a{}^h\} - v_b\{\nabla_c(v_h B_a{}^h) - v_h \nabla_c B_a{}^h\} \end{aligned}$$

or

$$\begin{aligned} &\varphi_c{}^e \nabla_a \varphi_{eb} - \varphi_b{}^e \nabla_a \varphi_{ec} \\ &= u_c(\nabla_b u_a - \alpha h_{ba}) - u_b(\nabla_c u_a - \alpha h_{ca}) + v_c(\nabla_b v_a - \beta h_{ba}) - v_b(\nabla_c v_a - \beta h_{ca}) \end{aligned}$$

by virtue of

$$f_{ti} B_e{}^t B_b{}^i = \varphi_{eb}$$

and equations of Gauss, or again

$$\varphi_c{}^e \nabla_a \varphi_{eb} - \varphi_b{}^e \nabla_a \varphi_{ec} = u_c(\nabla_b u_a) - u_b(\nabla_c u_a) + v_c(\nabla_b v_a) - v_b(\nabla_c v_a), \tag{44}$$

by virtue of $\alpha u_c + \beta v_c = 0$.

From (43) we find

$$\begin{aligned} &\nabla_a(\varphi_c{}^e \varphi_{eb}) - (\nabla_a \varphi_c{}^e)\varphi_{eb} - \varphi_b{}^e \nabla_a \varphi_{ec} \\ &= u_c(\nabla_b u_a) - u_b(\nabla_c u_a) + v_c(\nabla_b v_a) - v_b(\nabla_c v_a), \end{aligned}$$

or

$$(45)\qquad \begin{aligned} &\nabla_a(-g_{cb}+\eta_c\eta_b)-2(\nabla_a\varphi_c{}^e)\varphi_{eb}\\ &=u_c(\nabla_b u_a)-u_b(\nabla_c u_a)+v_c(\nabla_b v_a)-v_b(\nabla_c v_a).\end{aligned}$$

Substituting

$$u_c=-\beta\eta_c,\qquad v_c=\alpha\eta_c$$

into the right hand member of (45), we have

$$\begin{aligned} &(\nabla_a\eta_c)\eta_b+\eta_c(\nabla_a\eta_b)-2(\nabla_a\varphi_c{}^e)\varphi_{eb}\\ &=\beta^2\{\eta_c(\nabla_b\eta_a)-\eta_b(\nabla_c\eta_a)\}+\beta(\nabla_b\beta)\eta_c\eta_a-\beta(\nabla_c\beta)\eta_b\eta_a\\ &\quad+\alpha^2\{\eta_c(\nabla_b\eta_a)-\eta_b(\nabla_c\eta_a)\}+\alpha(\nabla_b\alpha)\eta_c\eta_a-\alpha(\nabla_c\alpha)\eta_b\eta_a,\end{aligned}$$

or

$$(46)\qquad 2(\nabla_a\varphi_c{}^e)\varphi_{eb}=\eta_c(\nabla_a\eta_b-\nabla_b\eta_a)+\eta_b(\nabla_a\eta_c+\nabla_c\eta_a)$$

by virtue of $\alpha^2+\beta^2=1$.

On the other hand, using (38) and (42), we find

$$\alpha(\nabla_c u_b-\nabla_b u_c)=-2\beta\varphi_{cb},$$

from which

$$(47)\qquad \alpha(\nabla_c\eta_b-\nabla_b\eta_c)\eta^a=2(\alpha v^a-\beta u^a)\varphi_{cb}$$

because of (39) and (42).

Substituting

$$u^a=-\beta\eta^a,\qquad v^a=\alpha\eta^a$$

into (47), we have

$$(48)\qquad \alpha(\nabla_c\eta_b-\nabla_b\eta_c)=2\varphi_{cb}$$

because of $\alpha^2+\beta^2=1$.

This shows that α does not vanish everywhere on $i(V)$. Thus, from (46) and (48), we have

$$(49)\qquad 2\alpha(\nabla_a\varphi_c{}^e)\varphi_{eb}=2\varphi_{ab}\eta_c+\alpha\eta_b(\nabla_a\eta_c+\nabla_c\eta_a),$$

from which

$$(50)\qquad \nabla_a\eta_c+\nabla_c\eta_a=0,$$

because α never vanishes on $i(V)$.

Consequently we have

(51) $$\alpha \nabla_b \eta_a = \varphi_{ba}$$

by virtue of (48).

Substituting (50) into (49), we find

(52) $$\alpha(\nabla_a \varphi_c{}^e)\varphi_{eb} = \varphi_{ab}\eta_c.$$

Transforming (52) with $\varphi_d{}^b$ and taking account of

$$\varphi_{eb}\varphi_d{}^b = g_{ed} - \eta_e\eta_d,$$

we find

$$\alpha(\nabla_a \varphi_c{}^e)(g_{ed} - \eta_e\eta_d) = \eta_c(g_{ad} - \eta_a\eta_d)$$

or

$$\alpha \nabla_a \varphi_{cd} + \alpha(\nabla_a \eta_e)\varphi_c{}^e\eta_d = \eta_c(g_{ad} - \eta_a\eta_d)$$

by virtue of $\varphi_c{}^e\eta_e = 0$.

Substituting (51) into the last equation, we have

$$\alpha \nabla_a \varphi_{cd} + \varphi_{ae}\varphi_c{}^e\eta_d = \eta_c(g_{ad} - \eta_a\eta_d)$$

or

(53) $$\alpha \nabla_a \varphi_{cd} = \eta_c g_{ad} - \eta_d g_{ac}$$

by virtue of $\varphi_{ae}\varphi_c{}^e = g_{ac} - \eta_a\eta_c$.

Now we prove the

LEMMA. *In an invariant hypersurface of a manifold with normal (f, g, u, v, λ)-structure satisfying $\nabla_j v_i - \nabla_i v_j = 2f_{ji}$, α is constant.*

Proof. Substituting

$$v^a = \alpha \eta^a$$

into (42), we find

$$\alpha(\nabla_c \eta_b - \nabla_b \eta_c) + (\nabla_c \alpha)\eta_b - (\nabla_b \alpha)\eta_c = 2\varphi_{cb},$$

from which

(54) $$(\nabla_c \alpha)\eta_b = (\nabla_b \alpha)\eta_c$$

because of (48).

Transvecting (54) with η^b and making use of $\eta_b\eta^b = 1$, we have

$$\nabla_c \alpha = \rho \eta_c, \qquad \rho = \eta^b(\nabla_b \alpha),$$

from which

$$\nabla_d \nabla_c \alpha = (\nabla_d \rho)\eta_c + \rho \nabla_d \eta_c,$$

or

$$\alpha \nabla_d \nabla_c \alpha = \alpha(\nabla_d \rho)\eta_c + \rho\varphi_{dc}.$$

Transvecting the last equation with φ^{dc}, we have

$$\rho = 0.$$

Thus, $\nabla_c \alpha$ being zero, α is constant. This completes the proof of the lemma.

Since α is constant, we get

$$2\varphi_{ba} = \partial_b(\alpha\eta_a) - \partial_a(\alpha\eta_b).$$

This equation, together with the first two equations of (30), shows that

$$(\alpha\eta) \wedge (d(\alpha\eta))^n = \alpha(\eta \wedge \varphi^n) \neq 0,$$

because α never vanishes on $i(V)$.

Consequently we have

$$\eta \wedge (d\eta)^n \neq 0.$$

Thus η is a contact form on $i(V)$.

On the other hand, substituting (53) into

$$S_{cb}{}^a = \varphi_c{}^e(\nabla_e \varphi_b{}^a) - \varphi_b{}^e(\nabla_e \varphi_c{}^a) - (\nabla_c \varphi_b{}^e - \nabla_b \varphi_c{}^e)\varphi_e{}^a + (\nabla_c \eta_b - \nabla_b \eta_c)\eta^a,$$

we have

$$\alpha S_{cb}{}^a = \varphi_c{}^e(\eta_b \delta_e^a - \eta^a g_{eb}) - \varphi_b{}^e(\eta_c \delta_e^a - \eta^a g_{ec})$$
$$- (\eta_b \delta_c^e - \eta^e g_{cb} - \eta_c \delta_b^e + \eta^e g_{bc})\varphi_e{}^a + 2\varphi_{cb}\eta^a = 0.$$

Since, α is a non-zero valued function on $i(V)$, we have

$$S_{cb}{}^a = 0.$$

Thus, we have proved

THEOREM 4. *An invariant hypersurface of a manifold with normal* (f, g, u, v, λ)-*structure satisfying*

$$\nabla_j v_i - \nabla_i v_j = 2f_{ji}$$

is a Sasakian manifold.

Bibligraphy

[1] Sasaki, S., and Y. Hatakeyama, On differentiable manifolds with contact metric structures. J. of the Math. Soc. of Japan, **14** (1962), 249–271.
[2] Yano, K. and M. Okumura, On (f, g, u, v, λ)-structures. Kōdai Math. Sem. Rep. **22** (1970), 401–423.
[3] Yano, K., and M. Okumura, On normal (f, g, u, v, λ)-structures on submanifolds of codimension 2 in a Euclidean space. Kōdai Math. Sem. Rep. **23** (1971), 172–197.

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