KŌDAI MATH. SEM. REP.
23 (1971), 121—126

INVARIANT IDEALS FOR AMENABLE SEMIGROUPS OF MARKOV OPERATORS

BY WATARU TAKAHASHI

1. Introduction.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of continuous real or complex valued functions on X, with supremum norm. We denote by $C(X)^*$ the strong dual of Banach space $C(X)$. A *Markov operator* on $C(X)$ is a continuous linear mapping of $C(X)$ into itself such that $Te=e$ and $Tf\geqq 0$ whenever $f\geqq 0$, where e denotes the constant 1 function on X. Let Σ be an amenable semigroup of Markov operators T of $C(X)$ into itself. Some properties on invariant ideals have been investigated by Schaefer [5], [6] and Sine [7] for the case when Σ is the semigroup generated by a single Markov operator T. These results can be extended in obvious way to an amenable semigroup of Markov operators on $C(X)$. For example, we can extend the notion of ergodicity of Markov operator T on $C(X)$, defined for the case of the semigroup generated by T in [5]; that is, an amenable semigroup $\Sigma=\{T\}$ is *ergodic* if and only if for each $f\in C(X)$, the convex closure $\overline{co}\{Tf: T\in\Sigma\}$ of $\{Tf: T\in\Sigma\}$ contains an invariant function g for all $T\in\Sigma$. In fact, this invariant function is unique in $\overline{co}\{Tf: T\in\Sigma\}$. Thus, in this paper, we generalize main results in [5] and [7] by a modification of their methods; that is, in § 2 we give a representation theorem for maximal ideals invariant under each element T of an amenable semigroup Σ. In § 3 we prove that an amenable semigroup $\Sigma=\{T\}$ is ergodic if and only if invariant functions under every T in $\Sigma=\{T\}$ separate invariant probabirities under every adjoint operator T^* of T in Σ and then prove the bijective correspondence, for ergodic amenable semigroup $\Sigma=\{T\}$, between the family of maximal ideals invariant under every T in Σ and the set of the extreme points of the set of probabilities on X invariant under every T^*.

The author wishes to express his hearty thanks to Professor H. Umegaki and Professor T. Shimogaki for many kind suggestions and advices in the course of preparing the present paper.

2. Representation theorem.

Let Σ be an abstract semigroup and $m(\Sigma)$ be the space of all bounded real

Received June 4, 1970.

valued functions of Σ, with supremum norm. An element $\mu \in m(\Sigma)^*$ (the dual space of $m(\Sigma)$) is said to be a *mean* on $m(\Sigma)$ if $\mu(e)=\|\mu\|=1$. A mean μ is *left* [*right*] *invariant* if $\mu(l_s f)=\mu(f)$ $[\mu(r_s f)=\mu(f)]$ for all $f \in m(\Sigma)$ and $s \in \Sigma$, where the left [right] translation l_s $[r_s]$ of $m(\Sigma)$ by s given by $(l_s f)(s')=f(ss')$ $[(r_s f)(s')=f(s's)]$. An *invariant mean* is a left and right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called *left amenable* [*right amenable*]. A semigroup that has a invariant mean is called *amenable*. The following Lemma is essentially contained in Day's fixed point theorem [2]:

LEMMA 1. *Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$, then there exists $\phi \in C(X)^*$ such that $\|\phi\|=1$, $\phi \geqq 0$ and $T^*\phi=\phi$ for each $T \in \Sigma$, where T^* is the adjoint of T.*

Proof. $K=\{\psi \in C(X)^* : \|\psi\|=1, \psi \geqq 0\}$ is a compact and convex set. Since $\Sigma^*=\{T^*\}$ is an amenable semigroup of affine weakly*-continuous mappings of K into itself, from Day's fixed point theorem, there exists $\phi \in C(X)^*$ such that $\|\phi\|=1$, $\phi \geqq 0$ and $T^*\phi=\phi$ for all $T \in \Sigma$.

Under a modification of Schaefer [5], we obtain the following two Definitions and two Lemmas.

DEFINITION 1. Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$. A $\{T\}$-*ideal* is a closed proper ideal in $C(X)$ which is invariant under each $T \in \Sigma$. A $\{T\}$-ideal is said to be *maximal* if it is not properly contained in any other $\{T\}$-ideal.

DEFINITION 2. $\Sigma=\{T\}$ is said to be *irreducible* if there exist no $\{T\}$-ideals distinct from (O).

LEMMA 2. *$\Sigma=\{T\}$ possesses at least one maximal $\{T\}$-ideal, and each $\{T\}$-ideal is contained in some maximal $\{T\}$-ideal.*

LEMMA 3. *Let J be a $\{T\}$-ideal and denote by q the canonical mapping of $C(X)$ onto $C(X)/J$. Then $I \to q(I)$ is a bijective map of the set of all $\{T\}$-ideals containing J onto the set of all $\{T_J\}$-ideals, where T_J is an operator induced by T on $C(X)/J$. A $\{T\}$-ideal I is maximal if and only if $\{T_J\}$ is irreducible.*

Since the above two Lemmas are clear, we do not give the proofs. If ϕ is an element of $C(X)^*$, I_ϕ denotes the ideal $\{f \in C(X) : \phi(|f|)=0\}$, where $|f|(x)=|f(x)|$ for $x \in X$.

THEOREM 1. *Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$ and let I be a maximal $\{T\}$-ideal, then there exists a nomalized positive* *measure* $\phi \in C(X)^*$ *such that* $I=I_\phi$ *and* $T^*\phi=\phi$ *for all* $T \in \Sigma$.

Proof. Since $\{T_I\}$ is the amenable semigroup of Markov operators on $C(S_I)$ where S_I denotes the support of I, it follows from Lemma 1 that there exists a nomalized positive measure $\hat{\phi} \in C(S_I)^*$ such that $T_I^*\hat{\phi}=\hat{\phi}$ for $T \in \Sigma$. Now, since I is maximal and hence $\{T_I\}$ is irreducible, $I_{\hat{\phi}}=(O)$. Therefore, if q denotes the canonical mapping of $C(X)$ onto $C(X)/I$, $\phi=\hat{\phi}\circ q$ is a positive measure on X such that $I=I_\phi$. The fact $T^*\phi=\phi$ for $T \in \Sigma$ follows from that for all $f \in C(X)$

$$T^*\phi(f)=\phi(Tf)=\hat{\phi}\circ q(f)=\hat{\phi}(Tf+I)$$

$$=\hat{\phi}(T_J(f+I))=T_J^*\hat{\phi}(f+I)$$

$$=\hat{\phi}(f+I)=\hat{\phi}\circ q(f)=\phi(f).$$

3. Ergodic amenable semigroup of Markov operators.

Schaefer in [5] defined that a bounded operator T on a Banach space E is called *ergodic* if for each $x \in E$, the convex closure $K(x)$ of the orbit $(x, Tx, T^2x, \cdots)$ contains a fixed vector x_0 of T. We extend this and give the following Definition.

DEFINITION 3. Let $\Sigma=\{T\}$ be a semigroup of bounded operators on $C(X)$. $\Sigma=\{T\}$ is said to be *ergodic* if for each $f \in C(X)$, the convex closure $\overline{co}\{Tf: T \in \Sigma\}$ of $\{Tf: T \in \Sigma\}$ contains an invariant function g for all $T \in \Sigma$.

It is known (e.g. [2]) that if a semigroup $\Sigma=\{T\}$ is amenable and $\{Tf: T \in \Sigma\}$ weakly compact then $\Sigma=\{T\}$ is ergodic.

THEOREM 2. *Let* $\Sigma=\{T\}$ *be an ergodic and amenable semigroup of Markov operators on* $C(X)$. *Then there exists a positive projection* P *from* $C(X)$ *onto closed subspace*

$$F=\{f \in C(X): Tf=f \text{ for each } T \in \Sigma\}$$

such that $Pe=e$ *and* $PT=TP=P$ *for all* $T \in \Sigma$.

Proof. Since each $T \in \Sigma$ transforms real function in $C(X)$ to real one, for the present purpose, we can restrict the domain of each $T \in \Sigma$ to real function in $C(X)$. Since $\Sigma=\{T\}$ is ergodic, there exists $g \in \overline{co}\{Tf: T \in \Sigma\}$ such that $Tg=g$ for all $T \in \Sigma$. For $\varepsilon>0$, there exists $T_1, T_2, \cdots, T_n \in \Sigma$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\alpha_i>0$, $\Sigma\alpha_i=1$ such that

$$||g-\Sigma\alpha_i T_i f||<\varepsilon$$

and we have

$$|g(x)-\Sigma\alpha_i TT_i f(x)|<\varepsilon$$

for $x\in X$ and $T\in\Sigma$.

If μ is an invariant mean on $m(\Sigma)$ and we denote $\mu_T(h(T))=\mu(h)$ where $h\in m(\Sigma)$, we obtain

$$\begin{aligned}\varepsilon &> \sup_T |g(x)-\Sigma\alpha_i TT_i f(x)|\\ &\geqq |\mu_T(g(x)-\Sigma\alpha_i TT_i f(x))|\\ &=|g(x)-\Sigma\alpha_i\mu_T(TT_i f(x))|\\ &=|g(x)-\Sigma\alpha_i\mu_T(Tf(x))|\\ &=|g(x)-\mu_T(Tf(x))|.\end{aligned}$$

Therefore, $g(x)=\mu_T(Tf(x))$ for each $x\in X$.

Defining $(Pf)(x)=\mu_T(Tf(x))$, we obtain Theorem. In fact, for $T_0\in\Sigma$,

$$PT_0 f(x)=\mu_T(TT_0 f(x))=\mu_T(Tf(x))=Pf(x)$$

and hence $PT_0=P$.

We call the above projection $\{T\}$-*projection.*

COROLLARY 1. *Let P be $\{T\}$-projection, then P^* is the mapping of $C(X)^*$ onto $\{\phi\in C(X)^*: T^*\phi=\phi$ for each $T\in\Sigma\}$. Moreover, $\phi\in C(X)^*$ is invariant under each $T^*\in\Sigma^*$ if and only if it is invariant under P^*.*

Proof. Since $PT=TP=P$ for each $T\in\Sigma$ and $Pf\in\overline{co}\{Tf: T\in\Sigma\}$ for each $f\in C(X)$, we obtain Corollary.

We recall that K is the set of positive nomalized elements of $C(X)^*$ and F is the set of functions invariant under each $T\in\Sigma$. An element of K is called *probability measure.*

COROLLARY 2. *Let P be $\{T\}$-projection and let*

$$\Phi=\{\phi\in K: T^*\phi=\phi \text{ for each } T\in\Sigma\}.$$

Then, for distinct elements $\phi,\psi\in\Phi$, there exists an invariant function $g\in F$ such that $\phi(g)\neq\psi(g)$.

Proof. If $\phi \neq \psi$, there exists $f \in C(X)$ such that $\phi(f) \neq \psi(f)$. From

$$\phi(Pf)=P^*\phi(f)=\phi(f)\neq\psi(f)=P^*\psi(f)=\psi(Pf),$$

if $Pf=g$, we obtain Corollary.

We can prove the converse of Corollary 2.

THEOREM 3. *Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators $C(X)$. Suppose that for distinct elements $\phi, \psi \in \Phi$, there exists an invariant function $f \in F$ such that $\phi(f) \neq \psi(f)$. Then $\Sigma=\{T\}$ is ergodic.*

Proof. For $x \in X$, δ_x denotes the point measure at x. The set $\{T^*\delta_x : T \in \Sigma\}$ is invariant under each $T_0^* \in \Sigma^*$ and so is the weak*-closed convex hull $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$ of $\{T^*\delta_x : T \in \Sigma\}$. Since $\Sigma^*=\{T^*\}$ is amenable, by Day's fixed point theorem [2], $\Sigma^*=\{T^*\}$ has an invariant probability measure ϕ_x in $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$. That the invariant probability measure ϕ_x is unique in $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$ is clear from $T^*\delta_x(f)=\delta_x(Tf)=\delta_x(f)=f(x)$ for $f \in F$ and that invariant functions separate invariant probability measures. The weak*-continuity of the mapping $x \rightarrow \phi_x$ follows from the facts that $f(x)=T^*\delta_x(f)=\phi_x(f)$ for $f \in F$ and that invariant functions separate invariant probability measures. Defining $Pf(x)=\phi_x(f)$ for each $f \in C(X)$, we obtain $Pf \in C(X)$. Now, we show that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function and Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$. In fact, for $\phi \in C(X)^*$, let $Q\phi$ be a unique invariant measure in $w^*\overline{co}\{T^*\phi : T \in \Sigma\}$. Then we obtain $P^*\phi=Q\phi$ from that invariant functions separate invariant probability measures. On the other hand, we obtain $T^*Q=QT^*=Q$ for all $T \in \Sigma$. Hence, we have $T^*P^*=P^*T^*=P^*$. By using this, it follows that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function. If Pf is not contained in $\overline{co}\{Tf : T \in \Sigma\}$, there exists $\phi \in C(X)^*$ such that $\phi(Pf)>\sup[\phi(g) : g \in \overline{co}\{Tf : T \in \Sigma\}]$. From

$$\sup[\phi(g) : g \in \overline{co}\{Tf : T \in \Sigma\}] \geqq \phi(Pf),$$

we obtain a contradiction. Hence for each $f \in C(X)$, Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$.

The following Theorem is an extension of the theorem 2 in [5].

THEOREM 4. *Let $\Sigma=\{T\}$ be an ergodic and amenable semigroup of Markov operators and let*

$$\Phi=\{\phi \in K : T^*\phi=\phi \text{ for each } T \in \Sigma\}.$$

Then, $\phi \rightarrow I_\phi$ is a bijective mapping of the set $ex\Phi$ of extreme points of Φ onto the set of maximal $\{T\}$-ideals and also every $\{T\}$-ideal of the form I_φ ($\phi \in \Phi$) is the intersection of all maximal $\{T\}$-ideals containing it. Moreover, Φ is simplex in the sense of [4] *and $ex\Phi$ is weakly*-closed.*

Proof. To show that Φ is simplex in the sense of [4], it is sufficient that ϕ^+ is P^*-invariant whenever so is ϕ. Since $\phi^+ \geqq 0$ and $\phi^+ \geqq \phi$, $P^*\phi^+ \geqq 0$ and $P^*\phi^+ \geqq P^*\phi = \phi$. Hence $P^*\phi^+ \geqq \phi^+$. Therefore, we obtain $P^*\phi^+ = \phi^+$ from $(P^*\phi^+ - \phi^+)(e) = 0$.

Since an element of Φ is invariant under P^*, it follows that $ex\Phi$ is weakly*-compact. The remainder is obvious from Theorem 1, Corollary 1 and theorem 2 in [5].

References

[1] Day, M. M., Amenable semigroup. Illinois J. Math. **1** (1957), 509–544.

[2] Day, M. M., Fixed point theorems for compact convex sets. Illinois J. Math. **5** (1961), 585–590.

[3] Dunford, N., and J. T. Schwartz, Linear operators, Part 1. Interscience, New York (1958).

[4] Phelps, R. R., Lectures on Choquet's theorem. Van Nostrand (1966).

[5] Schaefer, H. H., Invariant ideals of positive operators in $C(X)$. Illinois J. Math. **11** (1967), 703–715.

[6] Schaefer, H. H., Invariant ideals of positive operators in $C(X)$. Illinois J. Math. **12** (1968), 525–538.

[7] Sine, R., Geometric theory of a single Markov operator. Pacific J Math. **27** (1968), 155–166.

[8] Takahashi, W., Fixed point theorem for amenable semigroups of nonexpansive mappings. Kōdai Math. Sem. Rep. **21** (1969), 383–386.

[9] Takahashi, W., Invariant functions for amenable semigroups of positive contractions on L^1. To appear in Kōdai Math. Sem. Rep. **23** (1971).

Department of Mathematics,
Tokyo Institute of Technology.