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INVARIANT MANIFOLDS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS[1]

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Invariant manifolds provide the geometric structures for describing and understanding dynamics of nonlinear systems. The theory of invariant manifolds for both finite- and infinite-dimensional autonomous deterministic systems and for stochastic ordinary differential equations is relatively mature. In this paper, we present a unified theory of invariant manifolds for infinite-dimensional *random* dynamical systems generated by *stochastic* partial differential equations. We first introduce a random graph transform and a fixed point theorem for nonautonomous systems. Then we show the existence of generalized fixed points which give the desired invariant manifolds.

1. Introduction. Invariant manifolds are essential for describing and understanding dynamical behavior of nonlinear and random systems. Stable, unstable and center manifolds have been widely used in the investigation of infinite-dimensional *deterministic* dynamical systems. In this paper, we are concerned with invariant manifolds for *stochastic* partial differential equations.

The theory of invariant manifolds for deterministic dynamical systems has a long and rich history. It was first studied by Hadamard [14], then, by Liapunov [18] and Perron [21] using a different approach. Hadamard's graph transform method is a geometric approach, while Liapunov–Perron method is analytic in nature. Since then, there is an extensive literature on the stable, unstable, center, center-stable and center-unstable manifolds for both finite- and infinite-dimensional deterministic *autonomous* dynamical systems (see [2] or [3] and the references therein). The theory of invariant manifolds for *nonautonomous* abstract semilinear parabolic equations may be found in [15]. Invariant manifolds with invariant foliations for more general infinite-dimensional *nonautonomous* dynamical systems was studied in [8]. Center manifolds for infinite-dimensional *nonautonomous* dynamical systems was considered in [7].

Recently, there are some works on invariant manifolds for stochastic or random ordinary differential equations (finite-dimensional systems) by Wanner [28],

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Arnold [1], Mohammed and Scheutzow [19] and Schmalfuss [24]. Wanner's method is based on the Banach fixed point theorem on some Banach space containing functions with particular exponential growth conditions, which is essentially the Liapunov–Perron approach. A similar technique has been used by Arnold. In contrast to this method, Mohammed and Scheutzow have applied a classical technique due to Ruelle [22] to stochastic differential equations driven by semimartingals. In [5] an invariant manifold for a stochastic reaction diffusion equation of pitchfork type has been considered. This manifold connects different stationary solutions of the stochastic differential equation. In [16] the pullback convergence has been used to construct an inertial manifold for nonautonomous dynamical systems.

In this paper, we will prove the existence of an invariant manifold for a nonlinear stochastic evolution equation with a multiplicative white noise,

$$\frac{d\phi}{dt} = A\phi + F(\phi) + \phi\dot{W}, \tag{1}$$

where A is a generator of a C_0-semigroup satisfying an exponential dichotomy condition, $F(\phi)$ is a Lipschitz continuous operator with $F(0) = 0$ and $\phi\,\dot{W}$ is the noise. The precise conditions on them will be given in the next section. Some physical systems or fluid systems with noisy perturbations proportional to the state of the system may be modeled by this equation.

A similar object, inertial manifolds, has been considered by Bensoussan and Flandoli [4], Chueshov and Girya [13] or Da Prato and Debussche [9] for the equations with pure white noises. Their approaches [9] and [13] are based on properties of Itô stochastic differential equations like Itô's formula, martingales and Itô integrals.

Here, we consider the stochastic partial differential equations with multiplicative noises and our method is based on the theory of random dynamical systems. In particular, we are able to formulate conditions such that a general random evolution equation

$$\frac{d\phi}{dt} = A\phi + G(\theta_t\omega, \phi) \tag{2}$$

has an invariant manifold under a condition on the spectral gap and the Lipschitz constant of G in ϕ. The random dynamical systems generated by (1) and (2) are conjugated, which allows us to determinate the manifold for (1) by the manifold for (2).

Our method showing the existence of an invariant manifold is different from the methods mentioned above, and it is an extension of the result by Schmalfuss [23].

However, this latter article only deals with a finite-dimensional equation which is semicoupled. We will introduce a *random* graph transform. In contrast to [4, 9, 13] this graph transform defines a new and lifted random dynamical system on the space of appropriate graphs. One ingredient of a random dynamical system is a cocycle (see the next section). An invariant graph of this graph transform is a generalized fixed point for a cocycle. This generalized fixed point defines an entire trajectory for the cocycle. Applying this fixed point theorem to the graph transform dynamical system, we can find under a gap condition a fixed point contained in the set of Lipschitz continuous graphs which represents the invariant manifold.

The main assumption is the gap condition formulated by a linear two-dimensional random equation. This equation allows us to calculate a priori estimate for the fixed point theorem. We note that this linear random differential equation has a nontrivial invariant manifold if and only if the gap condition is satisfied. Hence, our results are optimal in this sense.

We believe that our technique can be applied to other cases that are treated in [3].

We also note that we do not need to use the semigroup given by the skew product flow.

In Section 2, we recall some basic concepts for random dynamical systems and show that the stochastic partial differential equation (1) generates a random dynamical system. We introduce a random graph transform in Section 3. A generalized fixed point theorem is presented in Section 4. Finally, we present the main theorem on invariant manifolds in Section 5.

2. Random dynamical systems. We recall some basic concepts in random dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *flow* θ of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ is defined on the sample space Ω such that

$$\theta : \mathbb{R} \times \Omega \to \Omega, \qquad \theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}, \tag{3}$$

for t_1, $t_2 \in \mathbb{R}$. This flow is supposed to be $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, where $\mathcal{B}(\mathbb{R})$ is the collection of Borel sets on the real line $\mathbb{R}$. To have this measurability, it is not allowed to replace $\mathcal{F}$ by its $\mathbb{P}$-completion $\mathcal{F}^{\mathbb{P}}$; see [1], page 547. In addition, the measure $\mathbb{P}$ is assumed to be ergodic with respect to $\{\theta_t\}_{t\in\mathbb{R}}$. Then $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}, \theta)$ is called a metric dynamical system.

For our applications, we will consider a special but very important metric dynamical system induced by the *Brownian motion*. Let $W(t)$ be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$, taking zero value at $t = 0$. This set is equipped with the compact open topology. On this set we consider the measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$, defined by $\theta_t\omega = \omega(\cdot + t) - \omega(t)$. The distribution of this process generates a measure on $\mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ which is called the *Wiener measure*. Note that this measure is ergodic with respect to the above flow; see the Appendix in [1]. Later on we will consider, instead of the whole $C_0(\mathbb{R}, \mathbb{R})$, a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega \subset C_0(\mathbb{R}, \mathbb{R})$ of $\mathbb{P}$-measure one and the trace σ-algebra $\mathcal{F}$ of $\mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ with respect to Ω. A set Ω is called $\{\theta_t\}_{t\in\mathbb{R}}$-invariant if $\theta_t\Omega = \Omega$ for $t \in \mathbb{R}$. On $\mathcal{F}$ we consider the restriction of the Wiener measure also denoted by $\mathbb{P}$.

The dynamics of the system on the state space H over the flow θ is described by a cocycle. For our applications it is sufficient to assume that (H, d_H) is a complete metric space. A cocycle ϕ is a mapping:

$$\phi : \mathbb{R}^+ \times \Omega \times H \to H,$$

which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))$-measurable such that

$$\phi(0, \omega, x) = x \in H,$$
$$\phi(t_1 + t_2, \omega, x) = \phi\big(t_2, \theta_{t_1}\omega, \phi(t_1, \omega, x)\big),$$

for $t_1, t_2 \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in H$. Then ϕ together with the metric dynamical system forms a *random dynamical system*.

Random dynamical systems are usually generated by differential equations with random coefficients

$$\phi' = f(\theta_t\omega, \phi), \qquad \phi(0) = x \in \mathbb{R}^d,$$

or finite-dimensional stochastic differential equations

$$d\phi = f(\phi)dt + g(\phi)\, dW, \qquad \phi(0) = x \in \mathbb{R}^d,$$

provided that the global existence and uniqueness can be ensured. For details see [1]. We call a random dynamical system *continuous* if the mapping

$$x \to \phi(t, \omega, x)$$

is continuous for $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Now we start our investigation on the following stochastic partial differential equation

$$\frac{d\phi}{dt} = A\phi + F(\phi) + \phi\dot{W} \tag{4}$$

on a separable Banach space $(H, \|\cdot\|_H)$. Here A is a linear partial differential operator, $W(t)$ is a one-dimensional standard Wiener process and $\dot{W}$ describes formally a *white noise*. Note that $\phi\dot{W}$ is interpreted as a Stratonovich differential. However, the existence theory for stochastic evolution equations is usually formulated for Itô equations as in [10], Chapter 7. The equivalent Itô equation for (4) is given by

$$d\phi = A\phi\, dt + F(\phi)\, dt + \frac{\phi}{2}\, dt + \phi\, dW. \tag{5}$$

In the following, we assume that the linear (unbounded) operator $A : D(A) \to H$ generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on H. Furthermore, we assume that $S(t)$ satisfies the exponential dichotomy with exponents $\hat{\lambda} > \check{\lambda}$ and bound M, that is, there exists a continuous projection π^+ on H such that:

(i) $\pi^+ S(t) = S(t)\pi^+$.

(ii) The restriction $S(t)|_{R(\pi^+)}$, $t \geq 0$, is an isomorphism of $R(\pi^+)$ onto itself, and we define $S(t)$ for $t < 0$ as the inverse map.

(iii)

$$\begin{aligned} &\|\pi^+ S(t)\pi^+\|_{H,H} \leq Me^{\hat{\lambda}t}, \qquad t \leq 0, \\ &\|\pi^- S(t)\pi^-\|_{H,H} \leq Me^{\check{\lambda}t}, \qquad t \geq 0, \end{aligned} \tag{6}$$

where $\pi^- = I - \pi^+$.

Denote $H^- = \pi^- H$ and $H^+ = \pi^+ H$. Then $H = H^+ \oplus H^-$.

For simplicity we set $M = 1$. For instance, if the operator $-A$ is a strongly elliptic and symmetric differential operator on a smooth domain $\bar{D}$ of order 2 under the homogeneous Dirichlet boundary condition, then the above assumptions are satisfied with $H = L^2(D)$. In this case A has the spectrum

$$\lambda_1 > \cdots > \lambda_u > \lambda_{u+1} > \lambda_{u+2} > \cdots,$$

where the space spanned by the associated eigenvectors is equal to H. For any λ_u the associated eigenspace is finite dimensional. The space H^+ is spanned by the associated eigenvectors for $\lambda_1, \lambda_2, \ldots, \lambda_u$ and $\hat{\lambda} = \lambda_u > \lambda_{u+1} = \check{\lambda}$.

We assume that F is Lipschitz continuous on H

$$\big\|\pi^{\pm}\big(F(x_1) - F(x_2)\big)\big\|_H \leq L\|x_1 - x_2\|_H$$

with the Lipschitz constant $L > 0$. Then, for any initial data $x \in H$, there exists a unique solution of (5). For details about the properties of this solution see [10], Chapter 7. We also assume that $F(0) = 0$.

The stochastic evolution equation (5) can be written in the following *mild* integral form:

$$\begin{aligned} \phi(t) = S(t)x &+ \int_0^t S(t-\tau)\left(F(\phi(\tau)) + \frac{\phi(\tau)}{2}\right) d\tau \\ &+ \int_0^t S(t-\tau)\phi(\tau)\, dW, \qquad x \in H, \end{aligned}$$

almost surely for any $x \in H$. Note that the theory in [10] requires that the associated probability space is complete.

In order to apply the random dynamical systems techniques, we introduce a coordinate transform converting conjugately a stochastic partial differential equation into an infinite-dimensional random dynamical system. Although it is well known that a large class of partial differential equations with stationary random coefficients and Itô stochastic ordinary differential equations generate random dynamical systems (for details see [1], Chapter 1), this problem is still unsolved for stochastic partial differential equations with a general noise term $C(u)\, dW$. The reasons are: (i) The stochastic integral is only defined almost surely where the exceptional set may depend on the initial state x; and (ii) Kolmogorov's theorem, as cited in [17], Theorem 1.4.1, is only true for finite-dimensional random fields. Moreover, the cocycle has to be defined for *any* $\omega \in \Omega$.

However, for the noise term $\phi\, dW$ considered here, we can show that (5) generates a random dynamical system. To prove this property, we need the following preparation.

We consider the linear stochastic differential equation:

$$dz + z\, dt = dW. \tag{7}$$

A solution of this equation is called an Ornstein–Uhlenbeck process.

LEMMA 2.1. (i) *There exists a* $\{\theta_t\}_{t\in\mathbb{R}}$*-invariant set* $\Omega \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ *of full measure with sublinear growth*,

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \qquad \omega \in \Omega,$$

of $\mathbb{P}$*-measure one*.

(ii) *For* $\omega \in \Omega$ *the random variable*

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau}\omega(\tau)\, d\tau$$

exists and generates a unique stationary solution of (7) *given by*

$$\Omega \times \mathbb{R} \ni (\omega, t) \to z(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}\theta_t\omega(\tau)\, d\tau = -\int_{-\infty}^{0} e^{\tau}\omega(\tau + t)\, d\tau + \omega(t).$$

The mapping $t \to z(\theta_t\omega)$ *is continuous*.

(iii) *In particular, we have*

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0 \qquad \textit{for } \omega \in \Omega.$$

(iv) *In addition*,

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_\tau\omega)\, d\tau = 0$$

for $\omega \in \Omega$.

PROOF. (i) It follows from the law of iterated logarithm that there exists a set $\Omega_1 \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$, $\mathbb{P}(\Omega_1) = 1$, such that

$$\limsup_{t\to\pm\infty} \frac{|\omega(t)|}{\sqrt{2|t|\log\log|t|}} = 1$$

for $\omega \in \Omega_1$. The set of these ω's is $\{\theta_t\}_{t\in\mathbb{R}}$-invariant.

(ii) This can be proven as in [20], page 35. The existence of the integral on the right-hand side for $\omega \in \Omega_1$ follows from the law of iterated logarithm. Using the law of iterated logarithm again, the function

$$\tau \to e^{\tau} \sup_{[t_0-1,t_0+1]} |\omega(\tau + t_0)|$$

is an integrable majorant for $e^{\tau}\omega(\tau + t)$ for $t \in [t_0 - 1, t_0 + 1]$ and $\tau \in (-\infty, 0]$. Hence the continuity at $t_0 \in \mathbb{R}$ follows straightforwardly from Lebesgue's theorem of dominated convergence.

(iii) By the law of iterated logarithm, for $1/2 < \delta < 1$ and $\omega \in \Omega_1$ there exists a constant $C_{\delta,\omega} > 0$ such that

$$|\omega(\tau + t)| \le C_{\delta,\omega} + |\tau + t|^{\delta} \le C_{\delta,\omega} + |\tau|^{\delta} + |t|^{\delta}, \qquad \tau \le 0.$$

Hence

$$\lim_{t\to\pm\infty} \left| \frac{1}{t} \int_{-\infty}^{0} e^{\tau}\omega(\tau + t)\, d\tau \right| \le \lim_{t\to\pm\infty} \frac{1}{|t|} \int_{-\infty}^{0} e^{\tau}(C_{\delta,\omega} + |\tau|^{\delta} + |t|^{\delta})\, d\tau = 0,$$

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0,$$

which gives the convergence relation in (iii). Hence, these convergence relations always define a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set which has a full measure.

(iv) Clearly, $\mathbb{E}z = 0$ from (ii). Hence by the ergodic theorem we obtain (iv) for $\omega \in \Omega_2 \in \mathscr{B}(C_0(\mathbb{R}, \mathbb{R}))$. This set Ω_2 is also $\{\theta_t\}_{t\in\mathbb{R}}$-invariant. Then we set

$$\Omega := \Omega_1 \cap \Omega_2.$$

The proof is complete. $\square$

We now replace $\mathscr{B}(C_0(\mathbb{R}, \mathbb{R}))$ by

$$\mathcal{F} = \{\Omega \cap A,\ A \in \mathscr{B}(C_0(\mathbb{R}, \mathbb{R}))\}$$

for Ω given in Lemma 2.1. The probability measure is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. In the following we will consider the metric dynamical system

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}, \theta).$$

We now show that the solution of (5) defines a random dynamical system. To see this, we consider the random partial differential equation

$$\frac{d\phi}{dt} = A\phi + G(\theta_t\omega, \phi) + z(\theta_t\omega)\phi, \qquad \phi(0) = x \in H, \tag{8}$$

where $G(\omega, u) := e^{-z(\omega)}F(e^{z(\omega)}u)$. It is easy to see that for any $\omega \in \Omega$ the function G has the same global Lipschitz constant L as F. In contrast to the original stochastic differential equation, no stochastic integral appears here. The solution can be interpreted in a mild sense:

$$
\begin{aligned}
\phi(t) &= \exp\left\{\int_0^t z(\theta_\tau\omega)\,d\tau\right\}S(t)x \\
&\quad + \int_0^t \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}S(t-\tau)G(\theta_\tau\omega,\phi(\tau))\,d\tau.
\end{aligned}
\tag{9}
$$

We note that this equation has a unique solution for every $\omega \in \Omega$. No exceptional sets appear. Hence the solution mapping

$$(t,\omega,x) \to \phi(t,\omega,x)$$

generates a random dynamical system. Indeed, the mapping ϕ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))$-measurable.

Let $\hat{\phi}(t,\omega,x)$ be the solution mapping of (5) which is defined for $\omega \in \Omega \in \mathcal{F}^{\mathbb{P}}$, $\mathbb{P}(\Omega) = 1$. We now introduce the transform

$$T(\omega,x) = xe^{-z(\omega)} \tag{10}$$

and its inverse transform

$$T^{-1}(\omega,x) = xe^{z(\omega)} \tag{11}$$

for $x \in H$ and $\omega \in \Omega$.

LEMMA 2.2. *Suppose that ϕ is the random dynamical system generated by* (8). *Then*

$$(t,\omega,x) \to T^{-1}(\theta_t\omega,\cdot) \circ \phi(t,\omega,T(\omega,x)) =: \hat{\phi}(t,\omega,x) \tag{12}$$

is a random dynamical system. For any $x \in H$ this process is a solution version of (5).

PROOF. Applying the Itô formula to $T(\theta_t\omega, \hat{\phi}(t,\omega,T^{-1}(\omega,x)))$ gives a solution of (8). The converse is also true, since $T^{-1}(\theta_t\omega,x)$ and $\phi(t,\omega,x)$ are defined for *any* $\omega \in \Omega$ and T^{-1} is the inverse of T, and thus

$$(t,\omega,x) \to T^{-1}\big(\theta_t\omega, \phi(t,\omega,T(\omega,x))\big)$$

gives a solution of (5) for each $\omega \in \Omega$. It is easy to check that (12) defines a random dynamical system. Since ϕ is measurable with respect to $\mathcal{F}$ so is this $\hat{\phi}$. $\square$

Similar transformations have been used by Caraballo, Langa and Robinson [5] and Schmalfuss [23]. Note that our transform has the advantage that the solution of (8) generates a random dynamical system for the ω-wise differential equation.

In Section 5 we will prove the existence of invariant manifolds generated by (8). These invariant manifolds can be transformed into invariant manifolds for (4).

3. Random graph transform. In this section, we construct a random graph transform. The fixed point of this transform gives the desired invariant manifold for the random dynamical system ϕ generated by (8).

We first recall that a multifunction $M = \{M(\omega)\}_{\omega\in\Omega}$ of nonempty closed sets $M(\omega)$, $\omega \in \Omega$, contained in a complete separable metric space (H, d_H) is called a *random set* if

$$\omega \to \inf_{y\in M(\omega)} d_H(x, y)$$

is a random variable for any $x \in H$.

DEFINITION 3.1. A random set $M(\omega)$ is called an invariant set if

$$\phi(t, \omega, M(\omega)) \subset M(\theta_t\omega).$$

If we can represent M by a graph of a Lipschitz mapping

$$\gamma^*(\omega, \cdot) : H^+ \to H^-, \qquad H^+ \oplus H^- = H$$

such that

$$M(\omega) = \{x^+ + \gamma^*(\omega, x^+), x^+ \in H^+\},$$

then $M(\omega)$ is called a Lipschitz continuous invariant manifold.

Let $\gamma(\cdot) : H^+ \to H^-$ be a Lipschitz continuous function with Lipschitz constant $L_\gamma \geq 0$ and also let $\gamma(0) = 0$. We consider the system of equations

$$\begin{aligned} w(t) &= \exp\left\{\int_T^t z(\theta_\tau\omega)\, d\tau\right\}\pi^+ S(t-T)y^+ \\ &\quad - \int_t^T \exp\left\{\int_\tau^t z(\theta_{\tau'}\omega)\, d\tau'\right\}\pi^+ S(t-\tau)\pi^+ G\bigl(\theta_\tau\omega, w(\tau)+v(\tau)\bigr)\, d\tau, \\ v(t) &= \exp\left\{\int_0^t z(\theta_\tau\omega)\, d\tau\right\}\pi^- S(t)\gamma(w(0)) \\ &\quad + \int_0^t \exp\left\{\int_\tau^t z(\theta_{\tau'}\omega)\, d\tau'\right\}\pi^- S(t-\tau)\pi^- G\bigl(\theta_\tau\omega, w(\tau)+v(\tau)\bigr)\, d\tau \end{aligned} \tag{13}$$

on some interval $[0, T]$. Note that if (13) has a solution (w, v) on $[0, T]$ then $w(0)$ defines a mapping $\gamma \to \Psi(T, \theta_T\omega, \gamma)(y^+)$ and $v(T)$ defines another mapping

$$\gamma \to \Phi(T, \omega, \gamma)(y^+). \tag{14}$$

This latter mapping Φ will serve as the random graph transform.

Recall that a random variable $\omega \to \gamma^*(\omega)$ is a generalized fixed point of the mapping Φ if

$$\Phi\bigl(T, \omega, \gamma^*(\omega)\bigr) = \gamma^*(\theta_T\omega) \tag{15}$$

for $\omega \in \Omega$, $T \geq 0$. We assume that $\gamma^*(\omega)$ is a Lipschitz continuous mapping from H^+ to H^- and it takes zero value at zero. Conditions for the existence of a generalized fixed point are derived in the next section in the case of Φ a random dynamical system. The following theorem describes the relation between generalized fixed points and invariant manifolds.

THEOREM 3.2. *Suppose that γ^* is the generalized fixed point of the mapping Φ. Then the graph of γ^* is the invariant manifold $M(\omega)$ of the random dynamical system ϕ generated by* (8).

PROOF. Let $M(\omega)$ be the graph of $\gamma^*(\omega)$ such that $(x^+, \gamma^*(x^+, \omega)) \in M(\omega)$. Then for x^+, $y^+ \in H^+$, we obtain

$$\begin{aligned}
&\phi(T, \omega, x^+ + \gamma^*(\omega, x^+)) \\
&\quad = \pi^+\phi(T, \omega, x^+ + \gamma^*(\omega, x^+)) + \pi^-\phi(T, \omega, x^+ + \gamma^*(\omega, x^+)) \\
&\quad = y^+ + \pi^-\phi\Big(T, \omega, \Psi(T, \theta_T\omega, \gamma^*(\omega))(y^+) \\
&\qquad\qquad + \gamma^*(\omega, \Psi(T, \theta_T\omega, \gamma^*(\omega))(y^+))\Big) \\
&\quad = y^+ + \Phi(T, \omega, \gamma^*(\omega))(y^+) = y^+ + \gamma^*(\theta_T\omega)(y^+) \in M(\theta_T\omega)
\end{aligned}$$

by the definition of Ψ:

$$x^+ = \Psi(T, \theta_T\omega, \gamma^*(\omega))(y^+) \quad \text{if and only if } y^+ = \pi^+\phi(T, \omega, x^+ + \gamma^*(\omega, x^+)).$$

For the measurability statement see Section 5. $\square$

By this theorem, we can find invariant manifolds of the random dynamical system ϕ generated by (8) by finding generalized fixed points of the mapping Φ defined in (14). To do so, we will use a generalized fixed point theorem for *cocycles* and thus we need to show that the above mapping Φ is in fact a random dynamical system. For the remainder of this section we will show that Φ defines a random dynamical system. We will achieve this in a few lemmas.

In the following we denote by $C_0^{0,1}(H^+; B)$ the Banach space of Lipschitz continuous functions from H^+, with value zero at zero, into a Banach space B with the usual (Lipschitz) norm

$$\|u\|_{C_0^{0,1}} = \sup_{y_1^+ \neq y_2^+ \in H^+} \frac{\|u(y_1^+) - u(y_2^+)\|_B}{\|y_1^+ - y_2^+\|_H}.$$

Moreover, $C_0^G(H^+; B)$ denotes the Banach space of bounded continuous functions, with value zero at zero and with linear growth. The norm in this space is defined as

$$\|u\|_{C_0^G} = \sup_{0 \neq y^+ \in H^+} \frac{\|u(y^+)\|_B}{\|y^+\|_H}.$$

We first present a result about the existence of a solution of the integral system (13). The proof is quite technical and is given in the Appendix.

LEMMA 3.3. *Let L be the Lipschitz constant of the nonlinear term $G(\omega,\cdot)$ in the random partial differential equation* (8). *Then for any $\gamma \in C_0^{0,1}(H^+;H^-)$, $\omega \in \Omega$, there exists a $T = T(\gamma,\omega) > 0$ such that on $[0,T]$ the integral system* (13) *has a unique solution $(w(\cdot), v(\cdot)) \in C([0,T]; C_0^G(H^+;H^+) \times C_0^G(H^+;H^-))$.*

Let $C([0,T];B)$ be the space of continuous mappings from $[0,T]$ into B. Note that for some $T > 0$ and $\gamma \in C_0^{0,1}(H^+;H^-)$, the fixed point problem defined by the integral system (13) has a contraction constant less than one. Then for $T' < T$ and some Lipschitz continuous function $\gamma' \in C_0^{0,1}(H^+;H^-)$ such that $\|\gamma'\|_{C_0^{0,1}} \le \|\gamma\|_{C_0^{0,1}}$ the same contraction constant can be chosen. This follows from the structure of the contraction constant; see (30).

We would like to calculate a priori estimates for the solution of (13). To do this we need the following lemma and its conclusion on monotonicity will also be used later on.

LEMMA 3.4. *We consider the differential equations*

$$W' = \hat{\lambda} W + z(\theta_t\omega)W - LW - LV,$$
$$V' = \check{\lambda} V + z(\theta_t\omega)V + LW + LV, \tag{16}$$

with generalized initial conditions

$$W(T) = Y \ge 0, \qquad V(0) = \Gamma W(0) + C, \qquad \Gamma, C \ge 0. \tag{17}$$

Then this system has a unique solution on $[0,T]$ for some $T = T(\Gamma, C, \omega) > 0$. This interval is independent of C. Let $\hat{W}$, $\hat{V}$ be solutions of (16) *but with the generalized initial conditions*

$$\hat{W}(T) = Y \ge 0, \qquad \hat{V}(0) = \hat{\Gamma}\hat{W}(0) + \hat{C}, \qquad 0 \le \hat{\Gamma} \le \Gamma,\ 0 \le \hat{C} \le C.$$

Then we have $0 \le \hat{V}(t) \le V(t)$ and $0 \le \hat{W}(t) \le W(t)$ for $t \in [0,T]$.

The proof is given in the Appendix.

Now we can compare the norms for the solution of (13) and that of (16) and (17).

LEMMA 3.5. *Let $[0,T]$ be an interval on which the assumptions of the Banach fixed point theorem* (*see the proofs of Lemmas* 3.3 *and* 3.4) *are satisfied for* (13) *and* (16) *and* (17) *for some $\gamma \in C_0^{0,1}(H^+;H^-)$. Then the norm of the solution of* (13) *is bounded by the solution of* (16) *and* (17) *with $Y = 1$, $C = 0$ and $\Gamma = L_\gamma$ being the Lipschitz norm of γ. That is,*

$$\|w(t)\|_{C_0^G} \le W(t), \qquad \|v(t)\|_{C_0^G} \le V(t).$$

The proof is given in the Appendix.

We obtain from Lemma 3.3 that $w(t, y^+)$, $v(t, y^+)$ exist for any $y^+ \in H^+$ on some interval $[0, T]$. We also have $\|w(T)\|_{C_0^{0,1}} = 1$ and

$$\begin{aligned}
&\frac{\|\gamma(w(0, y_1^+)) - \gamma(w(0, y_2^+))\|_H}{\|y_1^+ - y_2^+\|_H} \\
&\quad = \frac{\|\gamma(w(0, y_1^+)) - \gamma(w(0, y_2^+))\|_H}{\|w(0, y_1^+) - w(0, y_2^+)\|_H} \frac{\|w(0, y_1^+) - w(0, y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} \\
&\quad \le L_\gamma \|w(0)\|_{C_0^{0,1}}
\end{aligned}$$

for $y_1^+ \neq y_2^+$ and $w(0, y_1^+) \neq w(0, y_2^+)$. Hence $\|v(0)\|_{C_0^{0,1}} \le L_\gamma \|w(0)\|_{C_0^{0,1}}$. We have that $w(0, y_1^+) \neq w(0, y_2^+)$ because $\Psi(T, \theta_T\omega, \gamma)(\cdot)$ is a bijection. Indeed this mapping is the inverse of $x^+ \to \pi^+\phi(T, \omega, x^+ + \gamma(x^+))$ on H^+. One can see this if we plug in $x^+ = \Psi(T, \theta_T\omega, \gamma)(\cdot)$, which is given by $w(0)$, the right-hand side of (13) at zero into the π^+-projection of the right-hand side of (9) for $t = T$, and vice versa if we plug in this expression into the right-hand side of the first equation of (13). On the other hand, we have

$$\begin{aligned}
&\frac{\|\pi^\pm G(\omega, w(y_1^+) + v(y_1^+)) - \pi^\pm G(\omega, w(y_2^+) + v(y_2^+))\|_H}{\|y_1^+ - y_2^+\|_H} \\
&\quad \le L \frac{\|w(y_1^+) + v(y_1^+) - (w(y_2^+) + v(y_2^+))\|_H}{\|y_1^+ - y_2^+\|_H} \\
&\quad \le L \frac{\|w(y_1^+) - w(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} + L \frac{\|v(y_1^+) - v(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H}.
\end{aligned}$$

Repeating the arguments of Lemma 3.5 we obtain

$$\frac{\|w(y_1^+) - w(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} \le W(t), \qquad \frac{\|v(y_1^+) - v(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} \le V(t)$$

for any $y_1^+ \neq y_2^+$. Hence, we have the following result.

LEMMA 3.6. *The solution of the integral system* (13) *has the following regularity*: *$w(t) \in C_0^{0,1}(H^+; H^+)$ and $v(t) \in C_0^{0,1}(H^+; H^-)$. In particular, $\Phi(T, \omega, \gamma) \in C_0^{0,1}(H^+; H^-)$ for sufficiently small T. Moreover, the comparison result in Lemma* 3.5 *remains true*.

Note that by the fixed point argument, $\Phi(T, \omega, \gamma)$ and $\Psi(T, \theta_T\omega, \gamma)$ exist only for small T. We would like to extent these definitions to $T \in \mathbb{R}^+$. To see this, we are going to show that if the Lipschitz constant of γ is bounded by a particular value, then the Lipschitz constant of $\mu = \Phi(T, \omega, \gamma)$ has the same bound.

As a preparation we consider the matrix

$$B := \begin{pmatrix} \hat{\lambda} - L & -L \\ L & \check{\lambda} + L \end{pmatrix},$$

which has the eigenvalues λ_+, λ_-. These eigenvalues are real and distinct if and only if

$$\hat{\lambda} - \check{\lambda} > 4L. \tag{18}$$

Then the associated eigenvectors can be written as

$$(e_+, 1), \qquad (e_-, 1).$$

We order λ_+, λ_- as $\lambda_+ > \lambda_-$. The elements e_+, e_- are positive.

LEMMA 3.7. *Let $T = T(\Gamma, 0, \omega) > 0$ be chosen such that* (16) *and* (17) *have a solution on $[0, T]$ given by the fixed point argument for $\Gamma = e_+^{-1} =: \kappa$, $Y = 1$ and $C = 0$. Then the closed ball $B_{C_0^{0,1}}(0, \kappa)$ in $C_0^{0,1}(H^+; H^-)$ will be mapped into itself*: $\Phi(T, \omega, B_{C_0^{0,1}}(0, \kappa)) \subset B_{C_0^{0,1}}(0, \kappa)$.

PROOF. Let $Q_1(t)\vec{x}_0$ be the solution of the linear initial value problem

$$\vec{x}' = B\vec{x}, \qquad \vec{x}(0) = \vec{x}_0$$

and let

$$Q_2(t) = \begin{pmatrix} \exp\left\{\int_0^t z(\theta_\tau\omega)\, d\tau\right\} & 0 \\ 0 & \exp\left\{\int_0^t z(\theta_\tau\omega)\, d\tau\right\} \end{pmatrix}$$

be the solution operator of

$$\begin{aligned} \psi' &= z(\theta_t\omega)\psi, \qquad & \psi(0) &= \psi_0, \\ \eta' &= z(\theta_t\omega)\eta, \qquad & \eta(0) &= \eta_0. \end{aligned}$$

Note that $Q_2(t)$ and $Q_1(t)$ commute. Hence $Q_2(t)Q_1(t)$ is a solution operator of the linear differential equation (16). Since

$$Q_1(t)\begin{pmatrix} e_+ \\ 1 \end{pmatrix} = e^{\lambda_+ t}\begin{pmatrix} e_+ \\ 1 \end{pmatrix},$$

we obtain that

$$Q_2(t)Q_1(t)\begin{pmatrix} e_+ \\ 1 \end{pmatrix} = \exp\left\{\lambda_+ t + \int_0^t z(\theta_\tau\omega)\, d\tau\right\}\begin{pmatrix} e_+ \\ 1 \end{pmatrix}.$$

For the initial conditions $Y = 1$, $\Gamma = e_+^{-1}$ we can calculate explicitly for the solution of (16) and (17)

$$\begin{aligned} W(0) &= \exp\left\{-\lambda_+ T - \int_0^T z(\theta_\tau\omega)\, d\tau\right\}, \\ c_1 &= e_+^{-1} \exp\left\{-\lambda_+ T - \int_0^T z(\theta_\tau\omega)\, d\tau\right\} \end{aligned}$$

and $c_2 = 0$. Hence $V(T) = e_+^{-1}$. By the comparison results from Lemmas 3.5 and 3.6, we find that $\|w(0)\|_{C_0^{0,1}} \le W(0)$ and $\|v(T)\|_{C_0^{0,1}} = \|\Phi(T,\omega,\gamma)\|_{C_0^{0,1}} \le V(T) = e_+^{-1}$ for small T depending on ω such that

$$\Phi\big(T,\omega,B_{C_0^{0,1}}(0,\kappa)\big) \subset B_{C_0^{0,1}}(0,\kappa). \qquad \square$$

Since we will equip $B_{C_0^{0,1}}(0,\kappa)$ with the C_0^G-norm in Section 5, in the following we will choose the state space $\mathcal{H} = B_{C_0^{0,1}}(0,\kappa)$ with the metric $d_{\mathcal{H}}(x,y) := \|x-y\|_{C_0^G}$.

Now we show that the random graph transform Φ defines a random dynamical system.

THEOREM 3.8. *Suppose that the gap condition* (18) *is satisfied. Then* Φ *is well defined by* (14) *for* any $T \ge 0$, $\omega \in \Omega$ *and* $\gamma \in \mathcal{H}$. *In addition*, Φ *together with the metric dynamical system* θ *induced by the Brownian motion defines a random dynamical system. In particular, the following measurability for the operators of the cocycle holds*:

$$\Omega \ni \omega \to \Phi(T,\omega,\gamma)(y^+) \in H^-$$

is $(\mathcal{F},\mathcal{B}(H^-))$*-measurable for any* $y^+ \in H^+$, $T \ge 0$.

PROOF. By Lemma 3.3, the mapping $\Phi(T,\omega,\gamma)$ is defined for small T. So we first have to extend this definition for any $T > 0$.

To this end we introduce random variables $T_\kappa(\omega) > 0$ by

$$T_\kappa(\omega) := \tfrac{1}{2}\inf\{T > 0 : K(\omega,T,\kappa) \ge 1\},$$

where K is defined in (30) below. Since $T \to K(\omega,T,\kappa)$ is continuous in T this is a random variable. Hence, $K(\omega,T_\kappa(\omega),\kappa) < 1$ and (13) has a unique solution on $[0,T_\kappa(\omega)]$ for $\gamma \in \mathcal{H}$. We define a sequence by $T_1 = T_1(\omega) = T_\kappa(\omega)$, $T_2 = T_2(\omega) = T_\kappa(\theta_{T_1(\omega)}\omega)$ and so on. Suppose that for some $\omega \in \Omega$ we have that $\sum_{i=1}^\infty T_i(\omega) = T_0 < \infty$. Then the definition of K in (30) implies that $\int_0^{T_0} |z(\theta_\tau\omega)|\,d\tau = \infty$. This is a contradiction, because by Lemma 2.1 the mapping $t \to z(\theta_t\omega)$ is continuous. Hence for any $T > 0$ and $\omega \in \Omega$ there exists an $i = i(T,\omega)$ such that

$$T = T_1 + T_2 + \cdots + T_{i-1} + \hat{T}_i, \qquad 0 < \hat{T}_i \le T_i.$$

We can now define

$$\Phi(T,\omega,\gamma) = \Phi\big(\hat{T}_i,\theta_{T_{i-1}}\omega,\cdot\big) \circ \Phi\big(T_{i-1},\theta_{T_{i-2}}\omega,\cdot\big) \circ \cdots \circ \Phi(T_1,\omega,\gamma). \tag{19}$$

We show that the right-hand side satisfies (13).

Suppose that $(w^1,v^1) = (w^1(t,\omega,\gamma,y^+), v^1(t,\omega,\gamma,y^+))$ is given by (13) on some interval $[0,t_1]$, $t_1 \le T_1$ for $\gamma \in \mathcal{H}$. We have

$$\mu(\cdot) := v^1(t_1,\omega,\gamma,\cdot) = \Phi(t_1,\omega,\gamma)(\cdot) \in \mathcal{H};$$

see Lemma 3.7. Similarly,

$$(w^2, v^2) = \big(w^2(t, \theta_{t_1}\omega, \mu, z^+), v^2(t, \theta_{t_1}\omega, \mu, z^+)\big)$$

is given by (13) on some interval $[0, t_2]$, $t_2 \le T_2$. We set

$$w(t, \omega, \gamma, z^+) = \begin{cases} w^1\big(t, \omega, \gamma, w^2(0, \theta_{t_1}\omega, \mu, z^+)\big), & t \in [0, t_1], \\ w^2(t - t_1, \theta_{t_1}\omega, \mu, z^+), & t \in (t_1, t_1 + t_2]. \end{cases}$$

By the variation of constants formula on w, we have for $t \in [0, t_1]$,

$$\begin{aligned}
&\exp\biggl\{\int_{t_1}^{t} z(\theta_\tau\omega)\,d\tau\biggr\}\pi^+ S(t - t_1)\exp\biggl\{\int_{t_2}^{0} z(\theta_{\tau+t_1}\omega)\,d\tau\biggr\}\pi^+ S(-t_2)z^+ \\
&\quad - \pi^+ S(t - t_1)\exp\biggl\{\int_{t_1}^{t} z(\theta_\tau\omega)\,d\tau\biggr\}\int_0^{t_2}\pi^+ S(-\tau)\exp\biggl\{\int_\tau^0 z(\theta_{r+t_1}\omega)\,dr\biggr\} \\
&\qquad\qquad \times \pi^+ G(\theta_{\tau+t_1}\omega, w^2 + v^2)\,d\tau \\
&\quad - \int_t^{t_1}\pi^+ S(t - \tau)\exp\biggl\{\int_\tau^t z(\theta_r\omega)\,dr\biggr\}\pi^+ G(\omega, w^1 + v^1)\,d\tau \\
&\quad = \exp\biggl\{\int_{t_1+t_2}^{t} z(\theta_\tau\omega)\,d\tau\biggr\}\pi^+ S(t - t_1 - t_2)z^+ \\
&\qquad - \int_t^{t_1+t_2}\pi^+ S(t - \tau)\exp\biggl\{\int_\tau^t z(\theta_r\omega)\,dr\biggr\}\pi^+ G(\theta_\tau\omega, w + v)\,d\tau \\
&\quad = w(t).
\end{aligned}\tag{20}$$

Now we consider the second equation of (13) with initial condition

$$\gamma(w(0)) = \gamma\big(w^1\big(0, \omega, \gamma, w^2(0, \theta_{t_1}\omega, \mu, z^+)\big)\big).$$

Then at t_1 we have for the solution of the second equation

$$v^1\big(t_1, \omega, \gamma, w^2(0, \theta_{t_1}\omega, \mu, z^+)\big) = \mu\big(w^2(0, \theta_{t_1}\omega, \mu, z^+)\big),$$

which is equal to $v^2(0, \theta_{t_1}\omega, \mu, z^+)$. Hence for

$$v(t, \omega, \gamma, z^+) = \begin{cases} v^1\big(t, \omega, \gamma, w^2(0, \theta_{t_1}\omega, \mu, z^+)\big), & t \in [0, t_1], \\ v^2\big(t - t_1, \theta_{t_1}\omega, \mu, z^+\big), & t \in (t_1, t_1 + t_2], \end{cases}$$

we can find

$$\begin{aligned}
v(t_1 + t_2) &= \exp\biggl\{\int_0^{t_1+t_2} z(\theta_\tau\omega)\,d\tau\biggr\}\pi^- S(t_1 + t_2)\gamma(w(0)) \\
&\quad + \int_0^{t_1+t_2}\exp\biggl\{\int_\tau^{t_1+t_2} z(\theta_{\tau'}\omega)\,d\tau'\biggr\} \\
&\qquad\qquad \times \pi^- S(t_1 + t_2 - \tau)\pi^- G\big(\theta_\tau\omega, w(\tau) + v(\tau)\big)\,d\tau,
\end{aligned}$$

which gives us together with (20) that (w, v) solves (13) on $[0, t_1 + t_2]$ and $v(t_1 + t_2) = \Phi(t_1 + t_2, \omega, \gamma)(z^+)$. Since $\mu \in \mathcal{H}$ so is $\Phi(t_1 + t_2, \omega, \gamma)(z^+)$ by Lemma (3.7). The extension of the definition of Φ is correct since we obtain the same value for different $t_1 \in [0, T_1]$, $t_2 \in [0, T_1]$ whenever $t_1 + t_2 = \text{const}$. For this uniqueness we note that $z \to w(0, \omega, \gamma, z^+)$ given by the above formula is the inverse of $x^+ \to \pi^+\phi(t_1 + t_2, \omega, x^+ + \gamma(x^+))$ which is independent of the choice of t_1 and t_2. This implies the independence of $v(t_1 + t_2)$ on $t_1 + t_2 = \text{const}$. By a special choice of t_1, t_2 (e.g., $t_1 = T_1$, $t_2 = T_2$) and continuing the above iteration procedure we get (19). By this iteration we also obtain that $\Phi(T, \omega, \gamma) \in \mathcal{H}$.

For the measurability, we note that

$$\Psi\big(T \wedge T_\kappa(\omega), \theta_{T\wedge T_\kappa(\omega)}, \gamma\big)(y^+), \qquad \Phi\big(T \wedge T_\kappa(\omega), \theta_{T\wedge T_\kappa(\omega)}, \gamma\big)(y^+)$$

are $\mathcal{F}, H^\pm$-measurable because these expressions are given as an ω-wise limit of the iteration of the Banach fixed point theorem starting with a measurable expression. On the other hand,

$$y^+ \to \Psi\big(T \wedge T_\kappa(\omega), \theta_{T\wedge T_\kappa(\omega)}, \gamma\big)(y^+),$$
$$y^+ \to \Phi\big(T \wedge T_\kappa(\omega), \theta_{T\wedge T_\kappa(\omega)}, \gamma\big)(y^+)$$

is continuous. Hence by Castaing and Valadier ([6], Lemma III.14), the above terms are measurable with respect to (ω, y^+). The measurability follows now by the composition formula (19). $\square$

REMARK 3.9. (i) Note that the solution of (16) and (17) can be extended to any time interval $[0, T]$. Then Lemmas 3.5 and 3.6 remain true for *any* $T > 0$.

(ii) Similar to the extension procedure we can show that $\Psi(T, \omega, \gamma)$ is defined for any $T > 0$, $\omega \in \Omega$ and $\gamma \in \mathcal{H}$.

4. Existence of generalized fixed points. By Theorem 3.2, the problem of finding invariant manifolds for a cocycle is equivalent to finding generalized fixed points for a related (but different) cocycle. In this section, we present a generalized fixed point theorem for cocycles.

Let Ω and θ be as in Section 2, except that, in this section, we do not need any measurability assumptions. Namely, Ω is an invariant set (of full measure) under the metric dynamical system θ. Let Φ be a cocycle on a complete metric space $(\mathcal{G}, d_{\mathcal{G}})$.

Recall that a mapping $\gamma^* : \Omega \to \mathcal{G}$ is called a *generalized fixed point* of the cocycle Φ if

$$\Phi\big(t, \omega, \gamma^*(\omega)\big) = \gamma^*(\theta_t\omega) \qquad \text{for } t \in \mathbb{R}.$$

Note that by the invariance of Ω with respect to $\{\theta_t\}_{t\in\mathbb{R}}$, the trajectory $\mathbb{R} \ni t \to \gamma^*(\theta_t\omega) \in \mathcal{G}$ forms an *entire trajectory* for Φ.

The following generalized fixed point theorem for cocycles is similar to the third author's earlier work [24].

THEOREM 4.1. *Let* $(\mathcal{G}, d_{\mathcal{G}})$ *be a complete metric space with bounded metric. Suppose that*

$$\Phi(t, \omega, \mathcal{G}) \subset \mathcal{G}$$

for $\omega \in \Omega$, $t \geq 0$, *and that* $x \to \Phi(t, \omega, x)$ *is continuous. In addition, we assume the contraction condition: There exists a constant* $k < 0$ *such that, for* $\omega \in \Omega$,

$$\sup_{x \neq y \in \mathcal{G}} \log \frac{d_{\mathcal{G}}(\Phi(1, \omega, x), \Phi(1, \omega, y))}{d_{\mathcal{G}}(x, y)} \leq k.$$

Then Φ *has a unique generalized fixed point* γ^* *in* $\mathcal{G}$. *Moreover, the following convergence property holds*:

$$\lim_{t \to \infty} \Phi(t, \theta_{-t}\omega, x) = \gamma^*(\omega)$$

for any $\omega \in \Omega$ *and* $x \in \mathcal{G}$.

PROOF. Let $x \in \mathcal{G}$. For $\omega \in \Omega$ we consider the sequence

$$n \to (\Phi(n, \theta_{-n}\omega, x)). \tag{21}$$

To see that this sequence is a Cauchy sequence, we compute by using the cocycle property:

$$\begin{aligned}
&d_{\mathcal{G}}\big(\Phi(n, \theta_{-n}\omega, x), \Phi(n+1, \theta_{-n-1}\omega, x)\big) \\
&\quad = d_{\mathcal{G}}\big(\Phi(n, \theta_{-n}\omega, x), \Phi\big(n, \theta_{-n}\omega, \Phi(1, \theta_{-n-1}\omega, x)\big)\big) \\
&\quad = d_{\mathcal{G}}\Big(\Phi\big(1, \theta_{-1}\omega, \Phi(n-1, \theta_{-n}\omega, x)\big), \\
&\qquad\qquad \Phi\big(1, \theta_{-1}\omega, \Phi\big(n-1, \theta_{-n}\omega, \Phi(1, \theta_{-n-1}\omega, x)\big)\big)\Big) \\
&\quad \leq e^k d_{\mathcal{G}}\big(\Phi(n-1, \theta_{-n}\omega, x), \Phi\big(n-1, \theta_{-n}\omega, \Phi(1, \theta_{-n-1}\omega, x)\big)\big) \\
&\quad \leq e^{kn} d_{\mathcal{G}}\big(x, \Phi(1, \theta_{-n-1}\omega, x)\big)
\end{aligned}$$

for $n \in \mathbb{N}$. We denote the limit of this Cauchy sequence by $\gamma^*(\omega)$.

If we replace x in (21) by another element $y \in \mathcal{G}$ we obtain the same limit which follows from

$$d_{\mathcal{G}}\big(\Phi(n, \theta_{-n}\omega, x), \Phi(n, \theta_{-n}\omega, y)\big) \leq e^{kn} d_{\mathcal{G}}(x, y).$$

This implies that $\gamma^*(\omega)$ is independent of choice of x.

Now we prove the convergence property

$$\lim_{t \to \infty} \Phi(t, \theta_{-t}\omega, x) = \gamma^*(\omega).$$

In fact,

$$\begin{aligned}
&d_{\mathcal{G}}\big(\Phi(t,\theta_{-t}\omega,x),\Phi([t],\theta_{-[t]}\omega,x)\big)\\
&\quad= d_{\mathcal{G}}\big(\Phi\big([t],\theta_{-[t]}\omega,\phi(t-[t],\theta_{-t}\omega,x)\big),\Phi([t],\theta_{-[t]}\omega,x)\big)\\
&\quad\le e^{k[t]}d_{\mathcal{G}}\big(\Phi(t-[t],\theta_{-t}\omega,x),x\big)\to 0 \qquad \text{for } t\to\infty,
\end{aligned}$$

where $[t]$ denotes the integer part of t. Since $\Phi(t-[t],\theta_{-t}\omega,x)\in\mathcal{G}$ the values $d_{\mathcal{G}}(\Phi(t-[t],\theta_{-t}\omega,x),x)$ are uniformly bounded for $t\in\mathbb{R}$ and $x\in\mathcal{G}$.

Next, we show that γ^* is, as a matter of fact, a generalized fixed point for Φ. Since $x\to\Phi(t,\omega,x)$ is continuous, for $t\ge 0$ we obtain

$$\begin{aligned}
\Phi\big(t,\omega,\gamma^*(\omega)\big) &= \Phi\Big(t,\omega,\lim_{n\to\infty}\Phi(n,\theta_{-n}\omega,x)\Big)\\
&= \lim_{n\to\infty}\Phi(t+n,\theta_{-n}\omega,x)\\
&= \lim_{n\to\infty}\Phi(t+n,\theta_{-n-t}\theta_t\omega,x)=\gamma^*(\theta_t\omega).
\end{aligned}$$

Finally, we prove the uniqueness of the generalized fixed point. Suppose there is another generalized fixed point $\bar\gamma^*(\omega)\in\mathcal{G}$. Let $\Gamma^*=\{\gamma^*(\theta_t\omega),\ t\in\mathbb{R},\ \omega\in\Omega\}$ and $\bar\Gamma^*=\{\bar\gamma^*(\theta_t\omega),\ t\in\mathbb{R},\ \omega\in\Omega\}$. Since Γ^* and $\bar\Gamma^*$ are bounded in $\mathcal{G}$ and

$$\begin{aligned}
d_{\mathcal{G}}\big(\gamma^*(\omega),\bar\gamma^*(\omega)\big) &= d_{\mathcal{G}}\big(\Phi\big(n,\theta_{-n}\omega,\gamma^*(\theta_{-n}\omega)\big),\Phi\big(n,\theta_{-n}\omega,\bar\gamma^*(\theta_{-n}\omega)\big)\big)\\
&\le e^{kn}\sup\{d_{\mathcal{G}}(x,y)\mid x\in\Gamma^*,\ y\in\bar\Gamma^*\},
\end{aligned}$$

letting $n\to\infty$, we have $\gamma^*(\omega)=\bar\gamma^*(\omega)$. This completes the proof. □

REMARK 4.2. The constant k in the above generalized fixed point theorem may be taken as ω-dependent, as long as the following condition is satisfied:

$$\lim_{n\to\pm\infty}\frac{1}{n}\sum_{i=0}^{n-1}k(\theta_i\omega)=c<0.$$

This latter condition is usually assumed in the situation of ergodicity. For applications see, for instance, [24] and [11].

5. Random invariant manifolds.

In this final section, we show that the random graph transform, defined in (14), has a generalized fixed point in the state space

$$\mathcal{H}=B_{C_0^{0,1}}(0,\kappa) \qquad \text{with the metric } d_{\mathcal{H}}(x,y):=\|x-y\|_{C_0^G}, \tag{22}$$

by using Theorem 4.1. Thus by Theorem 3.2, the graph of this generalized fixed point is an invariant manifold of the random dynamical system generated by (8).

We first consider the basic properties of the metric space $\mathcal{H}$.

LEMMA 5.1. *The metric space* $\mathcal{H} = (B_{C_0^{0,1}}(0,\kappa), d_{\mathcal{H}})$, $d_{\mathcal{H}}(x,y) := \|x-y\|_{C_0^G}$ *is complete and the metric* $d_{\mathcal{H}}$ *is bounded.*

PROOF. Let (x_n) be a Cauchy sequence in $\mathcal{H}$. Since $C_0^G(H^+;H^-)$ is complete we have $x_n \to x_0 \in C_0^G(H^+;H^-)$. Hence, we have for any $y^+ \in H^+$ that $x_n(y^+) \to x_0(y^+)$. Subsequently,

$$\frac{\|x_n(y_1^+) - x_n(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} \to \frac{\|x_0(y_1^+) - x_0(y_2^+)\|_H}{\|y_1^+ - y_2^+\|_H} \qquad \text{for } n \to \infty, \tag{23}$$

for $y_1^+ \neq y_2^+ \in H^+$. Since the left-hand side is uniformly bounded by κ so is the right-hand side of (23). Hence $x_0 \in B_{C_0^{0,1}}(0,\kappa)$. The boundedness assertion is easily seen. □

We now check the assumptions of the generalized fixed point Theorem 4.1. Let Φ be the random dynamical system given by the graph transform in (14).

THEOREM 5.2. *Suppose that the gap condition* (18) *is satisfied. Then the random graph transform defined in* (14) *has a unique generalized fixed point* $\gamma^*(\omega,\cdot)$ *in* $\mathcal{H}$ *where* κ *is given in Lemma* 3.7. *The graph of this generalized fixed point, namely,* $M(\omega) = \{(x^+, \gamma^*(\omega,x^+)), x^+ \in H^+\}$ *is an invariant manifold for the random dynamical system* ϕ *generated by* (8).

PROOF. By Lemma 3.7, Theorem 3.8 we know that $\Phi(T,\omega,\cdot)$ maps $\mathcal{H}$ into itself.

Before we check the contraction condition in Theorem 4.1 we calculate an estimate for $\|\Psi(1,\theta_1\omega,\gamma)\|_{C_0^G}$ for $\gamma \in \mathcal{H}$. This norm is given by $\|w(0)\|_{C_0^G}$ where (w,v) is a solution of (13) for $T=1$ and $\gamma \in \mathcal{H}$. An estimate for $\|w(0)\|_{C_0^G}$ is given by $W(0)$ defined in (16) and (17) for $T=1$, $C=0$, $Y=1$. By the monotonicity of $W(0)$ in Γ, we obtain that $W(0)$ for $\Gamma = \kappa = e_+^{-1}$ is an estimate of $\|w(0)\|_{C_0^G}$ for *any* $\gamma \in \mathcal{H}$. Now we can calculate $W(0)$ explicitly which gives us the estimate

$$\|w(0)\|_{C_0^G} \le W(0) = \exp\left\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\right\}, \qquad T = 1!. \tag{24}$$

We now check the contraction condition. To this end we consider problem (13) for two different elements $\gamma_1, \gamma_2 \in \mathcal{H}$ and we denote the solutions by w_i, v_i, $i = 1,2$. In particular, we have

$$w_1(T) - w_2(T) = 0, \qquad v_1(0) - v_2(0) = \gamma_1(w_1(0)) - \gamma_2(w_2(0)).$$

By the Lipschitz continuity of the nonlinear term G in the random partial differential equation (8), we can estimate

$$\frac{\|\pi^\pm G(w_1+v_1) - \pi^\pm G(w_2+v_2)\|_H}{\|y^+\|_H} \le L\frac{\|w_1-w_2\|_H + \|v_1-v_2\|_H}{\|y^+\|_H},$$

which implies that

$$\|\pi^{\pm} G(w_1+v_1) - \pi^{\pm} G(w_2+v_2)\|_{C_0^G} \le L\big(\|w_1-w_2\|_{C_0^G} + \|v_1-v_2\|_{C_0^G}\big).$$

Similarly to Lemma 3.5 we can estimate

$$\|\Phi(1,\omega,\gamma_1) - \Phi(1,\omega,\gamma_2)\|_{C_0^G} = \|v_1(1)-v_2(1)\|_{C_0^G} \tag{25}$$

by $V(1)$ and $\|w_1(0)-w_2(0)\|_{C_0^G}$ by $W(0)$, where $V(t)$ and $W(t)$ is a solution of (16) with

$$\begin{aligned} &W(1)=0,\\ &V(0)=\|\gamma_1-\gamma_2\|_{C_0^G}\exp\left\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\right\} + \kappa W(0). \end{aligned} \tag{26}$$

Indeed, we can estimate the norm of initial condition $v_1(0)-v_2(0)$:

$$\begin{aligned} &\|v_1(0)-v_2(0)\|_{C_0^G}\\ &\quad = \|\gamma_1(w_1(0)) - \gamma_2(w_2(0))\|_{C_0^G}\\ &\quad \le \|\gamma_1(w_1(0)) - \gamma_2(w_1(0))\|_{C_0^G} + \|\gamma_2(w_1(0)) - \gamma_2(w_2(0))\|_{C_0^G}\\ &\quad \le \|\gamma_1-\gamma_2\|_{C_0^G}\|w_1(0)\|_{C_0^G} + \|\gamma_2\|_{C_0^{0,1}}\|w_1(0)-w_2(0)\|_{C_0^G}\\ &\quad \le \|\gamma_1-\gamma_2\|_{C_0^G}\exp\left\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\right\} + \kappa W(0). \end{aligned}$$

We have a bound for $\|w_1(0)\|_{C_0^G}$ from (24) and $\|\gamma\|_{C_0^{0,1}} \le \kappa$. Since $V(1)$ as a solution (16) and (17) at $T=1$ is increasing in Γ and C the value $V(1)$ for the above generalized initial conditions (26) is an estimate for (25) for any $\gamma_1,\ \gamma_2 \in \mathcal{H}$. We have chosen $C = \|\gamma_1-\gamma_2\|_{C_0^G}\exp\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\}$.

We now can calculate $V(1)$ explicitly. For these calculations we have used that the solution operator $Q(t)$ for the linear problem (16) can be written as

$$\begin{aligned} Q(t)[c_1,c_2] = c_1 &\begin{pmatrix} e_+\\ 1\end{pmatrix}\exp\left\{\lambda_+ t + \int_0^t z(\theta_\tau\omega)\,d\tau\right\}\\ &+ c_2\begin{pmatrix} e_-\\ 1\end{pmatrix}\exp\left\{\lambda_- t + \int_0^t z(\theta_\tau\omega)\,d\tau\right\}. \end{aligned}$$

These calculations of (16) yield, with the initial conditions (26),

$$c_1 = e_-\|\gamma_1-\gamma_2\|_{C_0^G}\frac{-\exp\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\}}{e_+-e_-}e^{\lambda_- -\lambda_+},$$

$$c_2 = e_+\|\gamma_1-\gamma_2\|_{C_0^G}\frac{\exp\{-\lambda_+ - \int_0^1 z(\theta_\tau\omega)\,d\tau\}}{e_+-e_-}.$$

In summary, we have for $\gamma_1, \gamma_2 \in \mathcal{H}$,

$$\begin{aligned}\|\Phi(1,\omega,\gamma_1) - \Phi(1,\omega,\gamma_2)\|_{C_0^G} \\ = \|v_1(1) - v_2(1)\|_{C_0^G} \le V(1) = \|\gamma_1 - \gamma_2\|_{C_0^G} e^{\lambda_- - \lambda_+}.\end{aligned}$$

Since $\lambda_+ > \lambda_-$, we thus obtain the contraction condition in Theorem 4.1 for $k = \lambda_- - \lambda_+ < 0$.

We obtain similar estimates if we replace $T = 1$ by $T > 0$. Then these estimates show us that

$$\gamma \to \Phi(T, \omega, \gamma)$$

is continuous at $\gamma \in \mathcal{H}$.

So we have found that all assumption of Theorem 4.1 are satisfied. Hence the dynamical system generated by the graph transform Φ has a unique generalized fixed point γ^* in $\mathcal{H}$. The graph of γ^* defines a desired invariant manifold for the random dynamical system ϕ by Theorem 3.2. $\square$

It remains to prove that this manifold is measurable.

LEMMA 5.3. *The manifold $M(\omega)$ is a random manifold.*

PROOF. The fixed point $\gamma^*(\omega, x^+)$ is the ω-wise limit of $\Phi(t, \theta_{-t}\omega, \gamma)(x^+)$ for $x^+ \in H^+$ and for some γ in $\mathcal{H}$ as $t \to \infty$; see Theorem 4.1. Hence the mapping $\omega \to \gamma^*(\omega, x^+)$ is measurable for any $x^+ \in H^+$. In order to see that M is a random set we have to verify that, for any $x \in H$,

$$\omega \to \inf_{y \in H} \|x - \pi^+ y - \gamma^*(\omega, \pi^+ y)\|_H \tag{27}$$

is measurable; see [6], Theorem III.9. Let H_c be a countable dense set of the separable space H. Then the right-hand side of (27) is equal to

$$\inf_{y \in H_c} \|x - \pi^+ y - \gamma^*(\omega, \pi^+ y)\|_H, \tag{28}$$

which follows immediately by the continuity of $\gamma^*(\omega, \cdot)$. The measurability of (28) follows since $\omega \to \gamma^*(\omega, \pi^+ y)$ is measurable for any $y \in H$. $\square$

Under the additional assumption $\hat{\lambda} > 0 > \check{\lambda}$ we can show that M is an *unstable* manifold denoted by M^+: For any $\omega \in \Omega$, $t \ge 0$ and $x \in M^+(\omega)$ there exists an $x_{-t} \in M(\theta_{-t}\omega)$ such that

$$\phi(t, \theta_{-t}\omega, x_{-t}) = x = x^+ + \gamma^*(\omega, x^+) \tag{29}$$

and x_{-t} tends to zero. We set

$$x_{-t} = \Psi(t, \omega, \gamma^*)(x^+) + \gamma^*\big(\theta_{-t}\omega, \Psi(t, \omega, \gamma^*)(x^+)\big), \qquad x^+ := \pi^+ x.$$

Equation (29) is satisfied because $x^+ \to \pi^+\phi(t, \theta_{-t}\omega, x^+ + \gamma^*(x^+))$ is the inverse of $x^+ \to \Psi(t, \omega, \gamma^*)(x^+)$ and because γ^* is the fixed point of the graph transform. The value $\|\Psi(t, \theta_t\omega, \gamma^*)(x^+)\|_{C_0^G}$ can be estimated by $W(0)$ a solution of (16) and (17) on $[0, T]$ with $\Gamma = \kappa$, $C = 0$ and $Y = 1$ and $\omega = \theta_{-t}\omega$. $W(0)$ can be calculated explicitly for any $T > 0$. Hence

$$\|\Psi(t, \omega, \gamma^*)(x^+)\|_H \le \exp\left\{-\lambda_+ t - \int_{-t}^0 z(\theta_\tau\omega)\, d\tau\right\}\|x^+\|_H.$$

(We have to replace ω by $\theta_{-t}\omega$!.) We can derive from Lemma 2.1(iv)

$$\int_{-t}^0 z(\theta_\tau\omega)\, d\tau < \varepsilon t$$

for any $\varepsilon > 0$ if t is chosen sufficiently large depending on ω and ε. Hence $\|\Psi(t, \omega, \gamma^*)(x^+)\|_{C_0^G}$ tends to zero exponentially. On the other hand, we have for $\gamma^* \in \mathcal{H}$,

$$\big\|\gamma^*\big(\theta_{-t}\omega, \Psi(t, \omega, \gamma^*)(x^+)\big)\big\|_H \le \kappa\|\Psi(t, \omega, \gamma^*)(x^+)\|_H \to 0 \qquad \text{for } t \to \infty.$$

This convergence is exponentially fast. We conclude that M^+ is the unstable manifold for (8).

However, our intention is to prove that (5) has an invariant (unstable) manifold. On account of conjugacy of (5) and (8) by (10) and (11) we will now formulate the following result.

THEOREM 5.4. *Let ϕ by the random dynamical system generated by* (8) *and $\hat{\phi}$ be the solution version of* (5) *generated by* (12). *Then $M(\omega)$ is the invariant manifold of ϕ if and only if $\hat{M}^+(\omega) = T^{-1}(\omega, M^+(\omega))$ is the invariant manifold of $\hat{\phi}$. Moreover, if M^+ is an unstable manifold, then so is $\hat{M}^+$.*

PROOF. We have the relationship between ϕ and $\hat{\phi}$ given in Lemma 2.2:

$$\begin{aligned}
&\hat{\phi}\big(t, \omega, \hat{M}^+(\omega)\big)\\
&\quad = T^{-1}\big(\theta_t\omega, \phi\big(t, \omega, T\big(\omega, \hat{M}^+(\omega)\big)\big)\big)\\
&\quad = T^{-1}\big(\theta_t\omega, \phi\big(t, \omega, M^+(\omega)\big)\big) \subset T^{-1}\big(\theta_t\omega, M^+(\theta_t\omega)\big) = \hat{M}^+(\theta_t\omega).
\end{aligned}$$

Note that $t \to z(\theta_t\omega)$ has a sublinear growth rate; see Lemma 2.1(iii). Thus the transform $T^{-1}(\theta_{-t}\omega)$ does not change the exponential convergence of $\Psi(t, \omega, \gamma^*(\omega))(x^+)$:

$$\begin{aligned}
\hat{\Psi}\big(t, \omega, \hat{\gamma}^*(\omega)\big) &= T^{-1}\big(\theta_{-t}\omega, \Psi\big(t, \omega, T\big(\omega, \hat{\gamma}^*(\omega)\big)\big)\big),\\
\hat{\gamma}^*(\omega) &:= T^{-1}\big(\omega, \gamma^*(\omega)\big).
\end{aligned}$$

It follows that $\hat{M}^+(\omega)$ is unstable. $\square$

REMARK 5.5. Note that the main Theorem 5.2 represents the best possible result in the following sense. If we consider the solution of the two-dimensional problem (16) then this differential equation generates a *nontrivial* invariant manifold *if and only if* the gap condition (18) is satisfied. Hence we can not formulate stronger *general* conditions for the existence of global manifolds. Here nontrivial means that the dimension of the manifold is less than the dimension of the space.

APPENDIX: PROOFS OF LEMMAS 3.3–3.5

We now give the proof of the technical lemmas (Lemmas 3.3–3.5) which are based on the usual Banach fixed point theorem.

PROOF OF LEMMA 3.3. We consider the following operator:

$$\begin{aligned}\mathcal{T}_T : C\big([0,T]; C_0^G(H^+;H^+)\times C_0^G(H^+;H^-)\big)\\ \to C\big([0,T]; C_0^G(H^+;H^+)\times C_0^G(H^+;H^-)\big)\end{aligned}$$

for some $T>0$. Set $\mathcal{T}_T(w_1,v_1)=(w_2,v_2)$, where

$$\begin{aligned}w_2(t) &= \exp\left\{\int_T^t z(\theta_r\omega)\,dr\right\}\pi^+ S(t-T)y^+\\ &\qquad - \int_t^T \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\pi^+S(t-\tau)\pi^+G\big(\theta_\tau\omega, w_1(\tau)+v_1(\tau)\big)\,d\tau,\\ v_2(t) &= \exp\left\{\int_0^t z(\theta_r\omega)\,dr\right\}\pi^- S(t)\gamma(w_2(0))\\ &\qquad + \int_0^t \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\pi^-S(t-\tau)\pi^-G\big(\theta_\tau\omega, w_1(\tau)+v_1(\tau)\big)\,d\tau.\end{aligned}$$

Note that w_1, v_1 depend on y^+, t, ω and γ. A fixed point for $\mathcal{T}_T$ is a solution of (13) on $[0,T]$. It is obvious that if

$$(w_1,v_1)\in C\big([0,T]; C_0^G(H^+;H^+)\times C_0^G(H^+;H^-)\big)$$

so is (w_2,v_2). We check that the contraction condition of the Banach fixed point theorem is satisfied. We set

$$\Delta w_i = w_i - \bar{w}_i, \qquad \Delta v_i = v_i-\bar{v}_i, \qquad i=1,2.$$

By the Lipschitz continuity of γ,

$$\|\gamma(w_i(0))-\gamma(\bar{w}_i(0))\|_H \le L_\gamma\|\Delta w_i(0)\|_H, \qquad L_\gamma = \|\gamma\|_{C_0^{0,1}}.$$

Hence we obtain by (6), for $H^+ \ni y^+ \neq 0$,

$$\begin{aligned}
&\frac{\|\Delta w_2(t)\|_H}{\|y^+\|_H} \\
&\quad \le \int_t^T \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\exp\{\hat{\lambda}(t-\tau)\}L\frac{\|\Delta w_1(\tau)\|_H + \|\Delta v_1(\tau)\|_H}{\|y^+\|_H}\,d\tau \\
&\quad \le L\int_t^T \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\exp\{\hat{\lambda}(t-\tau)\}\,d\tau \\
&\qquad \times\left(\sup_{t\in[0,T]}\frac{\|\Delta w_1(t)\|_H}{\|y^+\|_H} + \sup_{t\in[0,T]}\frac{\|\Delta v_1(t)\|_H}{\|y^+\|_H}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\|\Delta v_2(t)\|_H}{\|y^+\|_H} \\
&\quad \le L_\gamma\frac{\|\Delta w_2(0)\|_H}{\|y^+\|_H}\exp\left\{\int_0^t z(\theta_r\omega)\,dr\right\}e^{\check{\lambda}t} \\
&\qquad + \int_0^t \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\exp\{\check{\lambda}(t-\tau)\}L\frac{\|\Delta w_1(\tau)\|_H + \|\Delta v_1(\tau)\|_H}{\|y^+\|_H}\,d\tau \\
&\quad \le L_\gamma\int_0^T \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\exp\{\hat{\lambda}(t-\tau)\}L\frac{\|\Delta w_1(\tau)\|_H + \|\Delta v_1(\tau)\|_H}{\|y^+\|_H}\,d\tau \\
&\qquad + \int_0^t \exp\left\{\int_\tau^t z(\theta_r\omega)\,dr\right\}\exp\{\check{\lambda}(t-\tau)\}L\frac{\|\Delta w_1(\tau)\|_H + \|\Delta v_1(\tau)\|_H}{\|y^+\|_H}\,d\tau \\
&\quad \le K(\omega,T,L_\gamma)\left(\sup_{t\in[0,T]}\frac{\|\Delta w_1(t)\|_H}{\|y^+\|_H} + \sup_{t\in[0,T]}\frac{\|\Delta v_1(t)\|_H}{\|y^+\|_H}\right).
\end{aligned}$$

Choosing T sufficiently small, we have

$$\begin{aligned}
&K(\omega,T,L_\gamma) < 1, \\
&K(\omega,T,L_\gamma) = LT\left((L_\gamma+1)\exp\left\{\int_0^T |z(\theta_r\omega)| + |\hat{\lambda}|\,dr\right\}d\tau\right. \\
&\qquad\qquad\qquad \left. + \exp\left\{\int_0^T |z(\theta_r\omega)| + |\check{\lambda}|\,dr\right\}\right).
\end{aligned} \tag{30}$$

We now can take the supremum with respect to $y^+ \neq 0$ and $t \in [0,T]$ for the left-hand side. Hence for sufficiently small $T \le 1$ the operator $\mathcal{T}_T$ is a contraction. $\square$

PROOF OF LEMMA 3.4. The proof of existence and uniqueness is similar to the proof in Lemma 3.3. The solution can be constructed by successive iterations of (16) and (17). If we start with $V_1(t) \equiv \Gamma Y + C \geq \hat{V}_1(t) \equiv \hat{\Gamma} Y + \hat{C}$, $\hat{W}_1(t) = W_1(t) \equiv Y$, we get

$$V_2(t) \geq \hat{V}_2(t), \quad W_2(t) \geq \hat{W}_2(t), \quad \dots, \quad V_i(t) \geq \hat{V}_i(t), \quad W_i(t) \geq \hat{W}_i(t), \quad \dots,$$

which gives the conclusion. These inequalities also show if $(W(t), V(t))$ exist on $[0, T]$ so do $(\hat{W}(t), \hat{V}(t))$. The inequalities for the contraction condition do not contain C. $\square$

PROOF OF LEMMA 3.5. Let (w_i, v_i), (W_i, V_i) be sequences generated by the successive iterations starting with $v_1(t) \equiv \gamma(y^+)$, $w_1(t) \equiv y^+$ and $W_1 = 1$, $V_1 = L_\gamma = \|\gamma\|_{C_0^{0,1}}$. These sequences converge to the solution of (13) and (16) and (17) provided T sufficiently small. We then have

$$\begin{aligned}
&\|w_i(t)\|_{C_0^G} \\
&\qquad \leq \exp\left\{\int_T^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \\
&\qquad\qquad + \int_t^T \exp\left\{\int_s^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \left\|\pi^+ G\bigl(\theta_s\omega, w_{i-1}(s) + v_{i-1}(s)\bigr)\right\|_{C_0^G} ds \\
&\qquad \leq \exp\left\{\int_T^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \\
&\qquad\qquad + \int_t^T \exp\left\{\int_s^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \bigl(L\|w_{i-1}(s)\|_{C_0^G} + L\|v_{i-1}(s)\|_{C_0^G}\bigr)\, ds, \\
&\|v_i(t)\|_{C_0^G} \\
&\qquad \leq \exp\left\{\int_0^t z(\theta_r\omega) + \check{\lambda}\, dr\right\} L_\gamma \|w_i(0)\|_{C_0^G} \\
&\qquad\qquad + \int_0^t \exp\left\{\int_s^t z(\theta_r\omega) + \check{\lambda}\, dr\right\} \bigl(L\|w_{i-1}(s)\|_{C_0^G} + L\|v_{i-1}(s)\|_{C_0^G}\bigr)\, ds
\end{aligned}$$

and

$$\begin{aligned}
W_i(t) &= \exp\left\{\int_T^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \\
&\qquad + \int_t^T \exp\left\{\int_s^t z(\theta_r\omega) + \hat{\lambda}\, dr\right\} \bigl(LW_{i-1}(s) + LV_{i-1}(s)\bigr)\, ds, \\
V_i(t) &= \text{Ł}_\gamma W_i(0) \exp\left\{\int_0^t z(\theta_r\omega) + \check{\lambda}\, dr\right\} \\
&\qquad + \int_0^t \exp\left\{\int_s^t z(\theta_r\omega) + \check{\lambda}\, dr\right\} \bigl(LW_{i-1}(s) + LV_{i-1}\bigr)\, ds.
\end{aligned}$$

It is easily seen that $W_1(t) = \|w_1(t)\|_{C_0^G}$, $V_1(t) \geq \|v_1(t)\|_{C_0^G}$ and that if

$$W_{i-1}(t) \geq \|w_{i-1}(t)\|_{C_0^G}, \qquad V_{i-1}(t) \geq \|v_{i-1}(t)\|_{C_0^G},$$

then

$$W_i(t) \geq \|w_i(t)\|_{C_0^G}, \qquad V_i(t) \geq \|v_i(t)\|_{C_0^G}$$

which gives the conclusion. $\square$

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