PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

INVARIANT MANIFOLDS OF NON-LINEAR OPERATORS

JOHN C. WELLS

In this paper we generalize the classical stable manifold theorem at a point as well as a recent result of M. Hirsch, C. Pugh and M. Shub. We deduce the existence of the invariant manifolds, their smoothness and their continuity under small perturbations of the underlying endomorphism entirely from the inverse function theorem and an easy proposition about smoothness of maps on $c_0(E)$. The constructive nature of our proof has the advantage of ready adaptation to numerical methods.

Introduction. The following classes of maps between Banach spaces will be used (all derivatives are Frechet derivatives). Let Lip(f) denote the Lipschitz constant of f and let $\mathrm{Lip}(E,F)=\{f \mid \mathrm{Lip}(f)<\infty\}$. For $p \geqq 1$ and $0<\alpha \leqq 1$ let $\mathscr{K}(E, E)$ denote the classes

$$C^p=\{f \mid f \text{ has } p \text{ continuous derivatives}\}$$

$$B^p_\alpha=\{f \mid f \in C^p, \|D^pf(x+h)-D^pf(x)\| \leqq M\|h\|^\alpha \text{ for some } M\}$$

$$C^p_u=\{f \mid f \in C^p \text{ and } D^p \text{ is uniformly continuous}\}$$

$$C^p_B=\{f \mid f \in C^p \text{ and } D^p \text{ is bounded}\}$$

$$C^\infty=\{f \mid f \in C^p \text{ for all } p\}$$

$$B^\infty=\{f \mid f \in B^p_1 \text{ for all } p\}$$

We will also use the following norms (or pseudo-norms)

$$\|f\|_0=\sup_x \|f(x)\|$$

$$\|f\|_p=\max(\|f\|_0, \mathrm{Lip}\, f, \cdots, \mathrm{Lip}\, D^{p-1}f)$$

If $f \in C^p$ then $\|f\|_p=\max_{0 \leqq i \leqq p} \sup_x \|D^if(x)\|$.

For maps in these classes we have an inverse function theorem. We start with a Lipschitz inverse function theorem. A stronger version of this theorem is given in Hirsch-Pugh [3]. We provide a proof along their lines for completeness.

LIPSCHITZ INVERSE FUNCTION THEOREM. *Let T be a linear invertible map fron E to E. Suppose f: $U \to E$, U an open nbhd of 0 in E, $f(0)=0$ and $\mathrm{Lip}(f)\cdot\|T^{-1}\|=\lambda<1$. Then $T+f$ is a homeomorphism of U onto an open subset V of E, $(T+f)^{-1}$ is Lipschitz and* $\mathrm{Lip}(T+f)^{-1} \leqq$

$\|T^{-1}\|/(1-\lambda)$. *If U contains the ball* B_r *of radius r and center* 0, *then V contains the ball* $B_{r'}$ *of radius* $r'=r(1-\lambda)/\|T^{-1}\|$ *and center* 0. *The map* $f\to f^{-1}$ *from Lip to Lip is continuous in the* $\|\ \|_0$ *topology on the range and domain of* $\to$.

Proof. Consider the set $\mathscr{L}$ of maps $g: B_{r'}\to E$ for which $\operatorname{Lip} g \leqq \lambda\cdot\|T^{-1}\|/(1-\lambda)$ and $g(0)=0$. If $g\in\mathscr{L}$ then $(T^{-1}+g)(B'_r)\subset B_r$ so the map $g'=-T^{-1}\circ f\circ(T^{-1}+g)$ is defined. Furthermore $\operatorname{Lip} g'\leqq \lambda\cdot\|T^{-1}\|/(1-\lambda)$, so $g'\in\mathscr{L}$. If $h'=-T^{-1}\circ f\circ(T^{-1}+h)$ with $h\in\mathscr{L}$ then $\|h'-g'\|_0\leqq\lambda\cdot\|h-g\|_0$. Thus the map $g\to -T^{-1}\circ f\circ(T^{-1}+g)$ is a contraction of the complete space $\mathscr{L}$ in the topology $\|\ \|_0$. Thus there is a unique fixed point $g\in\mathscr{L}$ satisfying

$$g=-T^{-1}\circ f\circ(T^{-1}+g)$$

This last equation implies $\mathrm{Id}+T\circ g+f\circ(T^{-1}+g)=\mathrm{Id}$ so that $(T+f)\circ(T^{-1}+g)=\mathrm{Id}$. Observe that

$$\|(T+f)(x)-(T+f)(y)\|\geqq\|T\|\cdot\|x+T^{-1}\circ f(x)-(y+T^{-1}\circ f(y))\|$$
$$\geqq\|T\|\cdot\{\|x-y\|-\|T^{-1}\|\cdot\operatorname{Lip}(f)\cdot\|x-y\|\}\geqq\|T\|\cdot(1-\lambda)\cdot\|x-y\|.$$

Thus $T+f$ is 1–1 on all of U and $T^{-1}+g$ is the inverse of $T+f$ on $B_{r'}$. Also $\operatorname{Lip}(T^{-1}+g)\leqq\|T^{-1}\|+\lambda\cdot\|T^{-1}\|/(1-\lambda)\leqq\|T^{-1}\|/(1-\lambda)$. By translating coordinates so that $x_0\to 0$ and $(T+f)(x_0)\to 0$, the above reasoning show that $\operatorname{Lip}(T+f)^{-1}\leqq\|T^{-1}\|/(1-\lambda)$ on all of V. The openness of V is also obtained. Finally if f and f' are invertible maps with f^{-1} Lipschitz then

$$\|f^{-1}-f'^{-1}\|_0\leqq|f^{-1}\circ f\circ f'^{-1}-f^{-1}\circ f'\circ f'^{-1}\|\leqq\operatorname{Lip}(f^{-1})\cdot\|f-f'\|_0.$$

This proves the last statement.

INVERSE FUNCTION THEOREM. *Suppose U is an open subset of E,* $f: U\to E$ *is Lip or is one of the Classes* $\mathscr{K}(E,E)$. *Suppose* $T: E\to E$ *is a linear invertible map from E to E such that* $\operatorname{Lip}(f-T)\cdot\|T^{-1}\|\leqq\lambda$ *for some* $\lambda<1$. *Then f is a homeomorphism of U onto an open subset of E and* f^{-1} *is in the same class as f. The map* $f\to f^{-1}$ *from* $\{f\mid\operatorname{Lip}(f-T)\cdot\|T^{-1}\|\leqq\lambda\}$ *to* f^{-1} *is continuous in the following way* (the indicated topologies apply to both f and f^{-1}).

TABLE 1

Classes	Pseudo-norm
Lip or C^p, $1\leqq p\leqq\infty$	$\|\ \|_0$
C^p_B	$\|\ \|_0, \|\ \|_1, \cdots \quad \|\ \|_{p-1}$
B^p_α, C^p_U	$\|\ \|_0, \cdots$ or $\|\ \|_p$
B^∞	$\|\ \|_p \quad p\geqq 0$

REMARK. If E is finite dimensional then $C^p=C^p_U$.

Proof. The case $f \in \text{Lip}$ is a restatement of the Lipschitz Inverse Function Theorem. In all other cases we can conclude from the LIFT that f is a homeomorphism of U onto an open subset and that f^{-1} is Lipschitz.

Observe that f, g in any of the classes with domain (g) bounded implies $f \circ g$ is of the same class. Now if f is at least C^1 then f^{-1} is differentiable by the usual argument which we give for completeness: It suffices to assume $f(0) = 0$ and show f^{-1} differentiable at 0.

Observe $f(0 + h) - f(0) - Df(0)[h] = f(h) - Df(0)[h] = o(\|h\|)$. So $f(f^{-1}(h')) - Df(0)[f^{-1}(h')] = o(\|f^{-1}(h')\|)$ which gives $Df^{-1}(0) \cdot [f(f^{-1}(h')) - f^{-1}(h')] = Df^{-1}(0) \cdot o(\|f^{-1}(h)\|)$ so $f^{-1}(h') - Df^{-1}(0)[h'] = o(\|h'\|)$. Thus f^{-1} is differentiable and $Df^{-1}(y) = (Df^{-1}(y)))^{-1}$. Now suppose f is in one of C^p, B^p_α, C^p_U or C^p_B, $p \geqq 1$ and that f^{-1} has been shown to be of class C^{k-1}, B^{k-1}_α, C^{k-1}_U or C^{k-1}_B, respectively, where $k \leqq p$. The map

$$L \to L^{-1}: \{L \mid L \in L(E, E), \text{Lip}(L - T) \cdot \|T^{-1}\| \leqq \lambda < 1\} \to L(E, E)$$

is of class B^∞. Thus Df^{-1} which is the composition: Inverse $\circ\, Df \circ f^{-1}$ is of class C^{k-1}, B^{k-1}_α, C^{k-1}_u or C^{k-1}_B respectively and so f^{-1} is of class C^k, B^k_α, C^k_u or C^k_B respectively. Repeating the argument gives $f^{-1} \in C^p$, B^p_{α}, C^p_u or C^p_B respectively.

Now we prove the continuity table for $f \to f^{-1}$. The continuity in $\| \;\|_0$ for C^p functions, $1 \leqq p \leqq \infty$ is implied by the continuity in $\| \;\|_0$ for Lipschitz functions. Since $B^p_\alpha \subset C^p_U$, $p \geqq 1$ and $C^p_B \subset C^{p-1}_U$, to prove the continuity results of the table, it suffices to show that $f \to f^{-1}$ is continuous in $\| \;\|_p$ for f in C^p_U. Suppose it has been shown for the pseudonorm $\| \;\|_{k-1}$, $k \leqq p$. We have

$$D^k f^{-1}(y) = D^{k-1}(Df(f^{-1}(y))^{-1} = [(Df(f^{-1}(y))^{-1}]^k \cdot P_k(Df(f^{-1}(y)) \cdots Df^k(f^{-1}(y))\, Df^{-1}(y) \cdots D^{k-1}f^{-1}(y)),$$

where P_k is a polynomial. Now $Df^{-1}, \cdots, D^{k-1}f^{-1}$ vary in $\| \;\|_0$ continuously as f varies in $\| \;\|_k$ by assumption. Also $Df, \cdots, Df^k$ are uniformly continuous by assumption so $Df(f^{-1}) \cdots Df^k(f^{-1})$ vary in $\| \;\|_0$ as f varies in $\| \;\|_0$. Finally $[Df(f^{-1})]^{-1}$ varies in $\| \;\|_0$ as f varies in $\| \;\|_0$ by an earlier statement. Thus $D^k f^{-1}$ varies in $\| \;\|_0$ as f varies in $\| \;\|_k$. A repetition of this argument implies that f^{-1} varies in $\| \;\|_p$ as f varies in $\| \;\|_p$.

DEFINITION. $c_0(E) = \{(x_0, x_1, \cdots) \mid x_i \in E \text{ for } i \geqq 0 \text{ and } \text{Lim}_i \|x_i\| = 0\}$ $c_0(E)$ is a Banach space with norm $\|x\| = \sup_i \|x_i\|$.

DEFINITION. If f is at least in $\text{Lip}(E, E)$ and $f(0) = 0$ and $r \geqq 1$ let $C_r f$ be the map from $c_0(E)$ to $c_0(E)$ defined by $[C_r f(x)]_i = r^i f(x_i / r^i)$, $i = 0, 1, \cdots$.

PROPOSITION. *If $f(0)=0$ and $f \in \mathrm{Lip}$ or f is in one of the classes $\mathcal{K}$ and $r \geqq 1$ then $C_r f$ is of the same class as f.* $\mathrm{Lip}\, C_r f = \mathrm{Lip}\, f$ *and the map $f \to C_r f$ is continuous according to the following table.*

TABLE 2

Classes	Pseudo-norms
Lip	$\| \ \|_0$ or $\| \ \|_1$
C^p, C^p_U, B^p_α, C^p_B	$\| \ \|_0, \cdots$ or $\| \ \|_{p+1}$
C^∞, B^∞	$\| \ \|, \ p \geqq 0$

Proof. The conclusion of the theorem for Lipschitz functions is obvious. The validity of the general case results from showing that $f \in \mathrm{Lip}$ and C^k implies $C_r f \in C^k$ and

$$(1) \qquad D^k C_r f(x)[h] = (D^k f(x_0)[h_0], \cdots, r^{(1-k)n} D^k f(x_n/r^n)[h_n], \cdots)$$

For $k=0$, (1) is the definition of $C_r f$. Suppose it has been proved for $k = 0, \cdots, k-1$. Then

$$\begin{aligned}\sup_n \| r^n f((x_n + h_n)/r^n) - r^n f(x_n/r^n) - Df(x_n/r^n)[h_n] \\ - \cdots r^{(1-k)n} D^k f(x_n/r^n)[h_n] \| \\ \leqq \sup_{n, 0<\theta_n<1} \|^{(1-k)n} (D^k f((x_n + \theta_n h_n)/r^n) - D^k f(x_n/r^n))[h_n])\|,\end{aligned}$$

by the mean value theorem.

Given $\epsilon > 0$ choose δ s.t. $\|x\| < 2\delta$ implies $\|D^k f(x) - D^k f(0)\| < \epsilon/2$. Choose N so large that $\|x_i/r^i\| < \delta$ when $i > n$ and then choose $\delta' < \delta$ such that when $\|h\| < \delta'$

$$\| r^{(1-k)i} (D^k f((x_i + h_i)/r^i) - D^k f(x_i/r_i))\| < \epsilon/2 \text{ for } 0 \leqq i \leqq N.$$

Then whenever $\|h\| < \delta'$, $0 < \theta_i < 1$,

$$\sup_i \| r^{(1-k)i} \cdot (D^k f((x_i + \theta_i h_i)/r^i) - D^k f(x_i/r_i))[h_i]\| \leqq \epsilon \cdot \|h\|^k.$$

By the inverse mean value theorem (see Abraham and Robbin [**1**]) $C_r f \in C^k$, equation (1) is true and hence the proposition is true.

THEOREM 1. *Suppose $E = E_1 \oplus E_2$ is a direct sum decomposition of a Banach space E into two Banach spaces. Suppose that E has been renormalized (isomorphically) if necessary so that* $\|(x,y)\|_E =$

$\max(\|x\|_{E_1}, \|y\|_{E_2})$. *Suppose* $L_1: E_1 \to E_1$ *and* $L_2: E_2 \to E_2$ *are bounded linear maps such that* L_1 *has a right inverse* L_1' *and* $\|L_2\| < \|L_1'\|^{-1}$. *Suppose that* U_1 *and* U_2 *are open spheres with center zero in* E_1, E_2 *and* $\mathcal{U} = U_1 \times U_2$. *Suppose* $f: \mathcal{U} \to E$ *is* Lip *or is in one of the classes* $\mathcal{K}$. *In addition suppose* $f(0) = 0$, *and that* Lip $\tilde{f} \leqq \sigma$ *for some* $\|L_2\| < r < \|L_1'\|^{-1}$ *and* $\sigma < \min(r^{-1} \cdot (1 - r \cdot \|L_1'\|), \ r - \|L_2\|)$ *where* $\tilde{f} = f - \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$. *Then in*

(Case 1) $r \geqq 1$. *The set* $W_{E_1} = \{x^0 \in E \mid \exists\, x^1, x^2, x^3 \cdots$ *with* $f(x^{n+1}) = x^n$ *for* $n \geqq 0$ *and* $x^n = o(r^{-n})\}$ *is invariant under* f *and there exists* $g_1: U_1 \to E_2$ *of the same class as* f *such that* $W_{E_1} = \{(x, g_1(x)) \mid x \in U_1\}$.

(Case 2) $r \leqq 1$. *The set* $W_{E_2} = \{x \mid f^n(x) = o(r^n)\}$ *is invariant under* f *and there exists* $g_2: U_2 \to E_1$ *of the same class as* f *such that* $W_{E_2} = \{(y, g_2(y)) \mid y \in U_2\}$. W_{E_1}, W_{E_2} *and* g_1 *and* g_2 *are independent of* r *satisfying the above conditions. In both cases* g_1 *and* g_2 *vary topologically with* $f \in \{f \mid \text{Lip}\, \tilde{f} \leqq \sigma\}$ *according to the following table.*

TABLE 3

Class	Pseudo-norm on f and g_1 or g_2
Lip, C^p, $p \geqq 1$ and C^∞	$\|\ \|_0$
C^p_B	$\|\ \|_0, \cdots$ or $\|\ \|_{p-1}$
C^p_u, B^p_α	$\|\ \|_0, \cdots$ or $\|\ \|_p$
B^∞	$\|\ \|_p$ for all p

REMARKS. The case $r = 1$ with L_1 and L_2 invertible is the classic stable manifold and unstable manifold theorem (see A. Kelley **[6]** or Hirsch-Pugh **[3]**). The case of arbitrary r, L_1 invertible and $f \in C^p$, $p \geqq 1$ has been proved in Hirsch-Pugh-Shub **[4]** using other methods. Refer to M. Irwin **[5]** for a proof similar to ours in the $r = 1$, L_1 and L_2 invertible case.

Proof. In either case the sets W_{E_1} and W_{E_2} are clearly invariant under f. Let $\Pi_i : E_1 \oplus E_2 \to E_i$ be the projections. We consider first case 1.

Let $\mathcal{X} = ((x_0, y_0), (x_1, y_1), \cdots)$ be an element of $c_0(E)$, Define $g: \mathcal{U}_0 \to c_0(E)$ by $g = \mathcal{L} + h$ on $\mathcal{U}_0 = \{\mathcal{X} \mid (x_i, y_i) \in \mathcal{U}$ for all $i\}$ where

$$\mathcal{L}(\mathcal{X})_i = (r \cdot L_1'(x_{i-1}),\ L_2(y_{i+1})/r), \qquad i \geqq 1$$

$$\mathcal{L}(\mathcal{X})_0 = (0, L_2(y_1)/r)$$

and

$$h(\mathscr{X})_i = (-r^i \cdot L_i' \circ \Pi_1 \circ \tilde{f}(x_i/r^i, y_i/r^i), \qquad r^i \cdot \Pi_2 \circ \tilde{f}(x_{i+1}/r^{i+1}, y_{i+1}/r^{i+1})) \qquad i \geqq 1$$

$$h(\mathscr{X})_0 = (0, \Pi_2 \circ \tilde{f}((x_1, y_1) \cdot r^{-1})).$$

By the proposition, $\mathscr{L}$, h and hence g is of the same class as f and furthermore $\mathrm{Lip}(g) \leqq \max(r \cdot \|L_1'\| + \|L_1'\| \cdot \mathrm{Lip}\tilde{f}, \ (\|L_2\| + \mathrm{Lip}\tilde{f})/r) > 1$. Thus by the inverse function theorem and the proposition, $G = \mathrm{Id} - g$ is a homeomorphism of $\mathscr{U}_0$ onto $G(\mathscr{U}_0)$, an open subset of $c_0(E)$, and G^{-1} is of the same class of f and varies with f according to Table 3. Let $I_1\colon U_1 \to c_0(E)$ be defined by $I_1(x_0) = ((x_0, 0), (0,0), \cdots)$. Observe that if we let $\mathscr{X}_0 = I_1(x_0)$ and $\mathscr{X}_{n+1} = \mathscr{X}_0 + g(\mathscr{X}_0)$ then $\mathscr{X}_n \in \mathscr{U}_0$ for all n and $\lim_n \mathscr{X}_n = \mathscr{X}$ exists and satisfies $G \circ \mathscr{X} = \mathscr{X}_0$. Thus $I_1(x_0) \in \mathrm{Range}\,(G)$ so we can define $w(x) = G^{-1} \circ I_1(x)$ on U_1. The equation $G \circ G^{-1}(I_1(x)) = I_1(x)$ is equivalent to $w(x) = I_1(x) + g(w(x))$ and writing this out gives

$$\Pi_1 \circ w_0(x) = x + 0$$

$$\Pi_2 \circ w_0(x) = r^{-1} L_2(\Pi_2 \circ w_1(x)) + \Pi_2 \circ \tilde{f}(w_1(x)/r)$$

(2) $$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Pi_1 \circ w_i(x) = r \cdot L_1'(\Pi_1 \circ w_{i-1}(x)) - r^i \cdot L_1' \circ \Pi_1 \circ \tilde{f}(w_i(x)/r^i)$$

$$\Pi_2 \circ w_i(x) = (1/r) \cdot (L_2(\Pi_2 \circ w_{i+1}(x)) + r^i \cdot \Pi_2 \circ \tilde{f}(w_{i+1}(x)/r^{i+1})$$

Multiply the equations with Π_1 and $i \geqq 1$ by L_1 and then move the 2nd term on the right to the left to get

(3) $$r^i \cdot \Pi_1 \circ f(w_i(x)/r^i) = r \cdot \Pi_1 \circ w_{i-1}(x)$$

The terms involving Π_2 give

(3′) $$\Pi_2 \cdot w_i(x) = r^i \cdot \Pi_2 \circ f(w_{i+1}(x)/r^{i+1})$$

Thus

(4) $$f(w_{i-1}(x)/r^{i-1}) = w_i(x)/r^i, \qquad i \geqq 1$$

Since $w_i(x) \in c_0(E)$, $w_i(x)/r^i = o(r^{-i})$. Therefore letting $g_1(x) = \Pi_2 \circ w_0(x)$ we have $(x, g_1(x)) = w_0(x) \in W_{E_1}$ and g_1 is of the same class as f and varies with f as in the Table 3. On the other hand if $(x, y) \in W_{E_1}$ then there exists $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \cdots) \in c_0(E)$ s.t. $\tilde{w}_0 = (x, y)$ and $f(\tilde{w}_{i-1}/r^{i-1}) = \tilde{w}_i/r^i \cdot \tilde{w}$ satisfies equations (4) and hence equations (3) and by the 1–1 ness of L_1 equation (2). But G is 1–1 so $\tilde{w} = w(x)$ and $y = g_1(x)$. Thus $W_{E_1} = \{(x, g_1(x)) \mid x \in U_1\}$. If r' also satisfies the conditions and $r < r'$ then defining $\bar{w}_i(x) = (r^i/r'^i)\bar{w}_i(x)$ we have that $\bar{w}(x)$ satisfies equations 4, 3, 2 with r replaced by r'. By the uniqueness of G, $\bar{w}(x) = w(x)$(using r'). Thus g_1 is independent of r.

(Case 2) This case follows along the same lines as Case 1. Define $g: \mathscr{U}_0 \to c_0(E)$ by $g = \mathscr{L} + h$ where $\mathscr{L}(\mathscr{X})_i = ((r \cdot L_1'(x_{i+1}), L_2(y_{i-1})/r)$, $i \geqq 1$ and $\mathscr{L}(\mathscr{X})_0 = (r \cdot L_1'(x_1), 0)$ and

$$h(\mathscr{X})_i = (1/r^i)[-L_1' \circ \Pi_1 \circ \tilde{f}((x_i, y_i) \cdot r^i)), \qquad i \geqq 1$$
$$\Pi_2 \circ \tilde{f}(r^{i-1}(x_{i-1}, y_{i-1}))]$$
$$h(\mathscr{X}) = (-L_1' \circ \Pi_1 \circ \tilde{f}(x_0, y_0), 0)$$

Then $\operatorname{Lip}(g) \leqq \max(r \cdot \|L_1'\| + \|L_1'\| \cdot \operatorname{Lip}(\tilde{f}), (\|L_2\| + \operatorname{Lip}(\tilde{f}))/r) < 1$. So $G = \mathrm{Id} - g$ and G^{-1} are of the same class as f, G is a homeomorphism of $\mathscr{U}_0$ onto $G(\mathscr{U}_0)$ and G^{-1} varies with f as in the Table 3. Let I_2: $U_2 \to c_0(E)$ be defined by $I_2(y) = ((0, y), (0, 0), \cdots)$ and let $w(y) = G^{-1} \circ I_2$. As before $w(y)$ is defined on all of U_2. The equation $G \circ G^{-1}(I_2(y)) = I_2(y)$ is equivalent to

$$w(y) = ((0, y), (0, 0), \cdots) + g(w(y))$$

which is equivalent to

$$\Pi_1 \circ w_0(y) = r \cdot L_1'(\Pi_1 \circ w_1(y)) - L_1' \circ \Pi_1 \circ \tilde{f}(w(y))$$
$$\Pi_2 \circ w_0(y) = y$$
$$\cdots\cdots$$
$$\Pi_1 \circ w_i(y) = r \cdot L_1' \circ \Pi_1 \circ w_{i+1}(y) - r^{-i} L_1' \circ \Pi_1 \circ \tilde{f}(r^i w_i(y))$$
$$\Pi_2 \circ w_i(y) = r^{-1} \cdot L_2 \circ \Pi_2 \circ w_{i-1}(y) + r^{-i} \Pi_2 \circ \tilde{f}(r^{i-1} w_{i-1}(y))$$

which is equivalent to

$$f(r^{i-1} w_{i-1}(y)) = r^i w_i(y) \qquad i \geqq 1.$$

Letting $g_2(y) = \Pi_1 \circ w_0(y)$ we have $\{(g_2(y), y) \mid y \in U_2\} = W_{E_2}$ as before and g_2 is of the same class as f and varies wih f as in Table 3.

The independence of g_2 and W_{E_2} from r follows as before.

REMARK. As noted in the proof, $w(x) = \operatorname{Lim} w^n(x)$ where $w^n(x) = I_1(\text{or } I_2)(x) + g \circ w^{n-1}(x)$, $n \geqq 1$, $w^0 = 0$. Thus

$$g_1 \text{ (or } g_2) = \operatorname*{Lim}_n \Pi_2 \circ w_0^n(x) \left(\text{or } \operatorname*{Lim}_n \Pi_2 \circ w_0^n(x)\right)$$

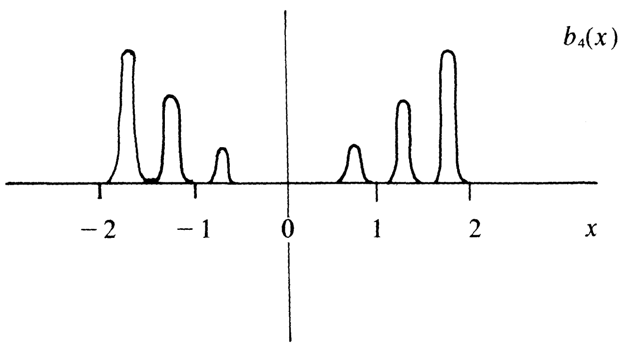
and the evaluation of the right hand side for any value of n involves $2n-1$ evaluations of f. This gives an effective iterative method for numerically determining the invariant manifolds.

Counterexample. The continuity in the theorem of g_1 and g_2 as functions of f for f in C^1 cannot be sharpened from $\| \ \|_0$ continuity to $\| \ \|_1$ continuity, as the following example of a C^1 map from l_2 to l_2 shows.

Let $s\colon R^1 \to R^1$ be defined by

$$s(x) = x,\ 0 \leqq x \leqq 1/4;\ = 1/2 - x,\ 1/4 \leqq x \leqq 3/4;\ = x - 1,\ 3/4 \leqq x \leqq 1.$$

$s(x)$ outside of $[0, 1]$ is defined such that s is periodic with period one. Let $a_n(x) = s(nx) \cdot \lfloor n|x| \rfloor / n$, $|x| \leqq 2$; $= 0$, $|x| > 2$ and $\tilde{a}_n(x) = a_n(x)$, $|x| \leqq 1$; $= 0$, $|x| > 1$. Then a_n, $\tilde{a}_n$ are continuous and $|a_n(x)|$, $|\tilde{a}_n(x)| \leqq |x|/4$ for all n. Let $b_n(x) = \int_0^x a_n(t)dt$ and $\tilde{b}_n(x) = \int_0^x \tilde{a}_n(t)dt$. Define A and $\tilde{A}\colon l_2 \to l_2$ by $A(x) = \Sigma\, a_n(x)e_n$ and $A(x) = \Sigma\, \tilde{a}_n(x)e_n$ where e_n is an orthonormal basis. Define B and $\tilde{B}\colon l_2 \to R^1$ by $B(x) = \Sigma\, b_n(x)$ and $\tilde{B}(x) = \Sigma\, \tilde{b}_n(x_n)$. Then it is not to hard to show that B and $\tilde{B} \in C^1(l_2, R^1)$ that $DB = A$ and $D\tilde{B} = \tilde{A}$. $b_n(x)$ is depicted in the figure.



To construct f we let $E = l_2 \oplus R$ with $\|(x, y)\|_E = \max(\|x\|, |y|)$ and $f(x, y) = (2x, (y - 1/10 \cdot \tilde{B}(x))/2 + 1/10 \cdot B(2x))$. On

$$\mathcal{U} = \{(x, y) \mid \|(x, y)\|_E \leqq 4\}, \qquad \operatorname{Lip}\left(f - \begin{pmatrix} 2\mathrm{Id} & 0 \\ 0 & 1/2 \end{pmatrix}\right) < 1/2.$$

This f will satisfy the conditions of theorem 1 and $g_f(x) = B(x)$ when $\|x\| \leqq 2$. Here $E_1 = l_2$ and $E_2 = R^1$. Now define perturbations of f,

$$f_n(x, y) = f(x, y) + 3/2n \cdot (1 - 3/4n)^{-2} \cdot (\max((x_n - 1), 0))^2.$$

Then $f_n \to f$ in $\| \quad \|_2$. On the other hand the effect of this perturbation is to shift g_f along the segment $\{x \mid x = te_n, 2 \leqq t \leqq 4\}$ in such way that $\sup_x \|Dg_{f_n} - Dg_f\|$ stays bounded away from zero.

References

1. R. Abraham and J. Robbin, *Transversal Mappings and Flows*, Benjamin, New York, 1967.
2. P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.
3. M. Hirsch and C. Pugh, *Stable Manifolds and Hyperbolic Sets*, Proc. of the Symp. in Pure Math., Vol. **14** Amer. Math. Soc., Providence, Rhode Island, 1970, pp. 133–164.
4. M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds*, to appear.
5. M. Irwin, *On the stable manifold theorem*, Bull. London Math. Soc., **2** (1970), 196–198.
6. Unstable Manifolds. An appendix in *Transversal Mappings and Flows* by R. Abraham and J. Robbin, Benjamin, New York, 1967.

Received May 23, 1975.