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Invariant Measures for Homeomorphisms with Weak Specification

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Introduction

In this paper one considers the space of measures provided with the weak topology. In [7, 8], K. Sigmund discussed some categories in the space of invariant measures for homeomorphisms satisfying specification. The ingredient of his proofs is in the densely periodic property of homeomorphisms with specification. It is known that weak specification for homeomorphisms is strictly weaker than specification.

Our aim is to show that the results of K. Sigmund hold for homeomorphisms satisfying weak specification (Theorems 1 and 3). The idea of proofs is in constructing the property "smallest sets" (See § 2.) that is found in the weak specification property.

§ 1. Definitions and main results.

Let X be a compact metric space with metric d and $\mathfrak{M}(X)$ be the space of Borel probability measures of X with metric $\bar{d}$ which is compatible with the weak topology, where $\bar{d}$ is defined by

$$\bar{d}(\mu, \nu)=\inf\{\varepsilon;\ \mu(B)\leqq\nu(\{x\in X;\ d(x, B)\leqq\varepsilon\})+\varepsilon \quad \text{and} \quad \nu(B)\leqq\mu(\{x\in X;\ d(x, B)\leqq\varepsilon\})+\varepsilon \ \text{for all Borel sets } B\}$$

(p. 9 of [5] or p. 238 of [3]).

Define a point measure $\delta(x)$ by $\delta(x)(B)=1$ if $x\in B$ and $\delta(x)(B)=0$ if $x\notin B$ (Borel sets B), and denote by $B(x, \varepsilon)$ an ε-closed ball about x in X. For arbitrary finite points $x_i\in X$ and $\mu_i\in\mathfrak{M}(X)$ $(1\leqq i\leqq n)$ with $\mathrm{card}\{1\leqq i\leqq n;\ \mu_i(B(x_i, \varepsilon))<1\}/n<\varepsilon$, we get easily $\bar{d}(1/n\sum_{i=1}^{n}\delta(x_i), 1/n\sum_{i=1}^{n}\mu_i)<\varepsilon$. It is clear that the map $x\to\delta(x)$ $(x\in X)$ is a homeomorphism from X onto a subset of $\mathfrak{M}(X)$.

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Let σ be a self-homeomorphism of X. Then σ induces a homeomorphism $\sigma\colon \mathfrak{M}(X)\to\mathfrak{M}(X)$ by $\sigma\mu(B)=\mu(\sigma^{-1}B)$ (Borel sets B and $\mu\in\mathfrak{M}(X)$) such that $\delta(\sigma x)=\sigma\delta(x)$ for all $x\in X$. Hence we can consider that (X, σ) is a subsystem of $(\mathfrak{M}(X), \sigma)$. It is known (p. 17 of [5]) that the set $\mathfrak{M}_\sigma(X)$ of σ-invariant measures is a compact convex set.

Let $\mathscr{E}(X)$ denote the set of ergodic measures in $\mathfrak{M}_\sigma(X)$. Then $\mathscr{E}(X)$ is a nonempty G_δ-set in $\mathfrak{M}_\sigma(X)$ (p. 25 of [5]). Let $\mathscr{S}(X)$ denote the set of strongly mixing measures in $\mathfrak{M}_\sigma(X)$, $\mathscr{D}(X)$ denote the set of measures positive on all nonempty open sets in $\mathfrak{M}_\sigma(X)$, and $\mathscr{N}(X)$ denote the set of non-atomic measures in $\mathfrak{M}_\sigma(X)$. We denote by $V_\sigma(x)$ the set of ω-limits of the sequence $\{1/n \sum_{j=0}^{n-1} \delta(\sigma^j x)\}_{n=1}^{\infty}$ for $x\in X$. Then we know (p. 18 of [5]) that for every $x\in X$, $V_\sigma(x)$ is a nonempty compact connected subset of $\mathfrak{M}_\sigma(X)$.

Let X and σ be as above. Then (X, σ) is said to satisfy *weak specification* if for $\varepsilon>0$, there exists $M(\varepsilon)>0$ such that for every $k\geqq 1$, k points $x_1, \cdots, x_k\in X$ and for every set of integers $a_1\leqq b_1<a_2\leqq b_2<\cdots<a_k\leqq b_k$ with $a_i-b_{i-1}\geqq M(\varepsilon)$ $(2\leqq i\leqq k)$, the set $\hat{B}=\bigcap_{i=1}^{k}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j x_i, \varepsilon)$ is nonempty. Since $\emptyset\neq\bigcap_{r=1}^{\infty}\bigcap_{n=-r}^{r}\bigcap_{i=1}^{k}\bigcap_{j=a_i+nq}^{b_i+nq}\sigma^{-j}B(\sigma^{j-nq}x_i, \varepsilon)\subset\hat{B}$ for all $q\geqq b_k-a_1+M(\varepsilon)$, we get easily that $\hat{B}$ contains a σ^q-invarant subset. When (X, σ) obeys weak specification and has the following additional condition; for every $q\geqq b_k-a_1+M(\varepsilon)$ there is an $x\in B$ with $\sigma^q x=x$, we say (X, σ) to satisfy *specification*.

In order to solve whether every zero-dimensional ergodic automorphism satisfies specification, in [2] N. Aoki constructs a zero-dimensional ergodic automorphism without densely periodic property. This implies that such an automorphism obeys weak specification, but not specification. For the class of all solenoidal automorphisms, it is proved in [1] that the class of automorphisms with weak specification is wider than the class of automorphisms with specification.

In this paper, the following theorems are proved for the class of homeomorphisms with weak specification of compact metric spaces.

THEOREM 1. *Let X be a compact metric space* (card$(X)>1$), *and σ be a self-homeomorphism of X. If (X, σ) satisfies weak specification, then $\mathscr{E}(X)$, $\mathscr{D}(X)$, and $\mathscr{N}(X)$ are dense G_δ-sets of $\mathfrak{M}_\sigma(X)$, and $\mathscr{S}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.*

THEOREM 2. *Let X and σ be as in Theorem 1. If (X, σ) satisfies weak specification, then $(\mathfrak{M}(X), \sigma)$ has the specification property.*

THEOREM 3. *Let X and σ be as in Theorem 1. If (X, σ) satisfies* *weak specification, then for every nonempty compact connected subset V of $\mathfrak{M}_\sigma(X)$, there is an $x \in X$ such that $V_{\sigma^r}(x)=V$ for all $r \geqq 1$ and the set of such points x is a dense set in X.*

§ 2. Auxiliary results.

In this section we show two results which are used in the proof of the theorems. Hereafter X is a compact metric space with metric d and σ is a self-homeomorphism of X.

A nonempty closed subset Δ is said to be a *smallest set* if there is an integer $q \geqq 1$ such that $\sigma^q \Delta = \Delta$ and Δ contains no completely σ^q-invariant closed proper subsets. We call the least positive integer in the set of such $q \geqq 1$ the *period* of Δ, and we denote it by per(Δ). Obviously, $\sigma^i \Delta \cap \Delta = \emptyset$ for i with $1 \leqq i \leqq \text{per}(\Delta)-1$. Let Δ be a smallest set. Then $\tilde{\Delta} = \bigcup_{i=0}^{\text{per}(\Delta)-1} \sigma^i \Delta$ is a minimal set under σ; i.e., $\tilde{\Delta}$ contains no completely σ-invariant closed proper subsets. Since $\tilde{\Delta}$ is compact and $\sigma\tilde{\Delta}=\tilde{\Delta}$, as before we can consider the space $\mathfrak{M}_\sigma(\tilde{\Delta})$ of σ-invariant Borel probability measures of $\tilde{\Delta}$. Then every $\mu \in \mathfrak{M}(\tilde{\Delta})$ defines a measure $\bar{\mu} \in \mathfrak{M}_\sigma(X)$ by $\bar{\mu}(B)=\mu(B \cap \tilde{\Delta})$ for Borel sets B of X. It is clear that if $\mu \in \mathfrak{M}_\sigma(\tilde{\Delta})$ is ergodic, then $\bar{\mu} \in \mathscr{E}(X)$. We remark that $\bar{\mu}(\sigma^j \Delta)=1/\text{per}(\Delta)$ $(0 \leqq j \leqq \text{per}(\Delta)-1)$ for all $\mu \in \mathfrak{M}_\sigma(\tilde{\Delta})$. Define $\bar{\mu}_j \in \mathfrak{M}(X)$ $(j \geqq 0)$ by $\bar{\mu}_j(B)=\text{per}(\Delta)\bar{\mu}(B \cap \sigma^j \Delta)$ for Borel sets B of X. Then we have $\bar{\mu}=(1/\text{per}(\Delta)) \sum_{j=0}^{\text{per}(\Delta)-1} \bar{\mu}_j$. We say that $x \in X$ is a *generic point* for $\mu \in \mathfrak{M}_\sigma(X)$ if $V_\sigma(x)=\{\mu\}$. Every $\mu \in \mathscr{E}(X)$ has generic points and the set of generic points for μ has μ-measure one (c.f. see p. 25 of [5]).

PROPOSITION 1. *If (X, σ) satisfies weak specification, then $\mathscr{E}(X)$ is dense in $\mathfrak{M}_\sigma(X)$.*

PROOF. It is clear that $\mathscr{E}(X) \neq \emptyset$. First we prove that for every $\mu_1, \mu_2 \in \mathscr{E}(X)$, every $t \in [0, 1]$ and every $\varepsilon > 0$, there exists $\nu \in \mathscr{E}(X)$ with $\bar{d}(\nu, t\mu_1+(1-t)\mu_2) < \varepsilon$.

Take an integer $m > 4/\varepsilon$, then there exists an integer m_1 with $1 \leqq m_1 \leqq m-1$ such that $|m_1/m - t| \leqq 1/m$. It follows from the definition of $\bar{d}$ that

$$\bar{d}\left(t\mu_1+(1-t)\mu_2,\ \frac{m_1}{m}\mu_1+\frac{m-m_1}{m}\mu_2\right) < \varepsilon/2 .$$

Let x_1 and x_2 be generic points for μ_1 and μ_2, respectively and choose $M=M(\varepsilon/4)$ as in the definition of weak specification. Since x_i is a generic point for μ_i $(i=1, 2)$, we can find an $N_0 \geqq 4M/\varepsilon$ such that for all $n \geqq N_0$, $\bar{d}(\mu_i, (1/n) \sum_{j=0}^{n-1} \delta(\sigma^j x_i)) < \varepsilon/4$ $(i=1, 2)$.

Put $N_1=m_1N_0-M$ and $N_2=(m-m_1)N_0-M$. Then we can calculate easily that

$$\begin{aligned}&\bar{d}\Big(\frac{m_1}{m}\mu_1+\frac{m-m_1}{m}\mu_2,\ (N_1+N_2+2M)^{-1}\sum_{i=1}^{2}\sum_{j=0}^{N_i+M-1}\delta(\sigma^j x_i)\Big)\\&=\bar{d}\Big(\frac{m_1}{m}\mu_1+\frac{m-m_1}{m}\mu_2,\ \frac{m_1}{m}\Big(\frac{1}{N_1+M}\sum_{j=0}^{N_1+M-1}\delta(\sigma^j x_1)\Big)\\&\quad+\frac{m-m_1}{m}\Big(\frac{1}{N_2+M}\sum_{j=0}^{N_2+M-1}\delta(\sigma^j x_2)\Big)\Big)\\&<\varepsilon/4\,.\end{aligned}$$

To use the weak specification property, we put $a_1=0$, $b_1=N_1$, $a_2=b_1+M$, $b_2=a_2+N_2$, $q=b_2+M$, $y_1=x_1$ and $y_2=\sigma^{-a_2}x_2$. Since X is compact, it follows that there is a smallest set $\varDelta$ such that

$$\sigma^q\varDelta=\varDelta\subset\bigcap_{i=1}^{2}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j y_i,\ \varepsilon/4)\,.$$

Take an ergodic measure $\nu\in\mathfrak{M}_\sigma(\tilde{\varDelta})$, then $\bar{\nu}_j(B(\sigma^j y_i,\ \varepsilon/4))=1$ $(a_i\leqq j\leqq b_i,$ $i=1,2)$, and so $\sum_{i=1}^{2}\mathrm{card}\,\{a_i\leqq j\leqq b_i+M-1;\ \bar{\nu}_j(B(\sigma^j y_i,\ \varepsilon/4))<1\}/q<2M/q<\varepsilon/4$. We remark that $\bar{\nu}=(1/q)\sum_{j=0}^{q-1}\bar{\nu}_j$ since q is divided by per$(\varDelta)$. Then

$$\begin{aligned}&\bar{d}\Big(\bar{\nu},\ \frac{1}{q}\Big(\sum_{i=1}^{2}\sum_{j=0}^{N_i+M-1}\delta(\sigma^j x_i)\Big)\Big)\\&=\bar{d}\Big(\frac{1}{q}\sum_{j=0}^{q-1}\bar{\nu}_j,\ \frac{1}{q}\sum_{i=1}^{2}\sum_{j=a_1}^{b_i+M-1}\delta(\sigma^j y_i)\Big)\leqq\varepsilon/4\,.\end{aligned}$$

Hence

$$\begin{aligned}&\bar{d}(\bar{\nu},\ t\mu_1+(1-t)\mu_2)\\&\leqq\bar{d}\Big(\bar{\nu},\ \frac{1}{q}\sum_{i=1}^{2}\sum_{j=0}^{N_i+M-1}\delta(\sigma^j x_i)\Big)\\&\quad+\bar{d}\Big(\frac{1}{q}\sum_{i=1}^{2}\sum_{j=0}^{N_i+M-1}\delta(\sigma^j x_i),\ \frac{m_1}{m}\mu_1+\frac{m-m_1}{m}\mu_2\Big)\\&\quad+\bar{d}\Big(\frac{m_1}{m}\mu_1+\frac{m-m_1}{m}\mu_2,\ t\mu_1+(1-t)\mu_2\Big)<\varepsilon\\&\qquad\qquad(\text{since } q=N_1+N_2+2M)\,.\end{aligned}$$

We use induction to get the conclusion. Take $\mu\in\mathfrak{M}_\sigma(X)$, then for every $\varepsilon>0$ there exist $k\geqq1$, $\mu_1,\cdots,\mu_k\in\mathscr{E}(X)$ and $t_1,\cdots,t_k\geqq0$ with $t_1+t_2+\cdots+t_k=1$ such that $\bar{d}(\mu,\ \sum_{i=1}^{k}t_i\mu_i)<\varepsilon/2$ (p. 25 of [5]). By the first part of the proof, there is a $\nu_1\in\mathscr{E}(X)$ such that $\bar{d}(t_1/(t_1+t_2)\mu+_1$

$t_2/(t_1+t_2)\mu_2, \nu_1)<\varepsilon/4$. Also there is a $\nu_2 \in \mathscr{E}(X)$ such that $\bar{d}((t_1+t_2)/(t_1+t_2+t_3)\nu_1+t_3/(t_1+t_2+t_3)\mu_3, \nu_2)<\varepsilon/8$. Put $t^{(i)}=\sum_{j=1}^{i} t_j$ for $1\leqq i\leqq k$, then it follows from definition of $\bar{d}$ that

$$\begin{aligned}
&\bar{d}\Big(\sum_{j=1}^{3}\frac{t_j}{t^{(3)}}\mu_j,\ \nu_2\Big)\\
&\leqq \bar{d}\Big(\frac{t^{(2)}}{t^{(3)}}\Big(\frac{t_1}{t^{(2)}}\mu_1+\frac{t_2}{t^{(2)}}\mu_2\Big)+\frac{t_3}{t^{(3)}}\mu_3,\quad \frac{t^{(2)}}{t^{(3)}}\nu_1+\frac{t_3}{t^{(3)}}\mu_3\Big)\\
&\quad +\bar{d}\Big(\frac{t^{(2)}}{t^{(3)}}\nu_1+\frac{t_3}{t^{(3)}}\mu_3,\ \nu_2\Big)\\
&<\varepsilon/4+\varepsilon/8 .
\end{aligned}$$

When $\nu_i \in \mathscr{E}(X)$ $(2\leqq i\leqq k-2)$ is already defined, by the above way we can find $\nu_{i+1} \in \mathscr{E}(X)$ such that

$$\bar{d}\Big(\frac{t^{(i+1)}}{t^{(i+2)}}\nu_i+\frac{t_{i+2}}{t^{(i+2)}}\mu_{i+2},\ \nu_{i+1}\Big)<\varepsilon/2^{i+1} .$$

Since $\nu_{k-1} \in \mathscr{E}(X)$ and $\bar{d}(\sum_{i=1}^{k} t_i\mu_i,\ \nu_{k-1})\leqq \sum_{i=1}^{k-1} 1/2^{i+1}<\varepsilon/2$, the proof is completed.

Let us put $Z(\Delta, \delta)=\{0\leqq j<\operatorname{per}(\Delta);\ \operatorname{diam}(\sigma^j\Delta)<\delta\}$ for a smallest set Δ and $\delta>0$. Denote by $A(\delta)$ the collection of smallest sets Δ with prime period satisfying the conditions;

$$\operatorname{per}(\Delta)>\delta^{-1} \text{ and } \operatorname{card}(Z(\Delta, \delta))/\operatorname{per}(\Delta)>1-\delta .$$

It is easy to check that $A(\delta_1)\subset A(\delta_2)$ when $\delta_1\leqq\delta_2$.

Proposition 2. *If (X, σ) $(\operatorname{card}(X)>1)$ satisfies weak specification, for every $\delta>0$ with $\delta<\operatorname{diam}(X)/4$ and for every $\mu \in \mathfrak{M}_\sigma(X)$ there exists a $\Delta \in A(\delta)$ such that every measure ν in $\mathfrak{M}_\sigma(\tilde{\Delta})$ holds $\bar{d}(\mu, \bar{\nu})<\delta$. Consequently the set $\bigcup_{\Delta \in A(\delta)}\{\bar{\nu} \in \mathfrak{M}_\sigma(X);\ \nu \in \mathfrak{M}_\sigma(\tilde{\Delta})\}$ is dense in $\mathfrak{M}_\sigma(X)$ for all $\delta>0$.*

Proof. Since $\mathscr{E}(X)$ is dense in $\mathfrak{M}_\sigma(X)$ by Proposition 1, there is an $\mu_1 \in \mathscr{E}(X)$ such that $\bar{d}(\mu, \mu_1)<\delta/3$. Choose $M=M(\delta/3)$ as in the definition of weak specification. Let x_1 be a generic point for μ_1. Then there is an $N_0>6M/\delta$ such that $\bar{d}((1/n)\sum_{j=1}^{n-1}\delta(\sigma^j x_1),\ \mu_1)<\delta/3$ $(n\geqq N_0)$. Take a prime p with $p>N_0+2M$ and put $N=p-2M$. For $x_2 \in X$ with $d(\sigma^{N+M}x_2, x_1)>2\delta$, putting $a_1=0$, $b_1=N$ and $a_2=b_2=N+M$. As before we have that there is a smallest set Δ such that $\sigma^p\Delta=\Delta\subset\bigcap_{i=1}^{2}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j x, \delta/3)$.

Since $\Delta\cap\sigma^{N+M}\Delta\subset B(x_1, \delta/3)\cap B(\sigma^{N+M}x_2, \delta/3)=\emptyset$, we get $\mathrm{per}(\Delta)\neq 1$ and $\mathrm{per}(\Delta)$ divides p. But p is prime so that $\mathrm{per}(\Delta)=p>\delta^{-1}$. Since $\{0, 1, \cdots, N\}\subset Z(\Delta, \delta)$ and $\mathrm{card}(Z(\Delta, \delta))/p>1-2M/p>1-\delta/3$, we get $\Delta\in A(\delta)$. Since $\bar{\nu}_j(B(\sigma^j x_1, \delta/3))=1$ $(0\leqq j\leqq N)$ for all $\nu\in\mathfrak{M}_\sigma(\tilde{\Delta})$, it follows that

$$\mathrm{card}\{0\leqq j\leqq p;\ \bar{\nu}_j(B(\sigma^j x_1, \delta/3))<1\}<\frac{p-(N+1)}{p}<2M/p<\delta/3 .$$

Since $\bar{\nu}=(1/p)\sum_{j=0}^{p-1}\bar{\nu}_j$, we get easily that $\bar{d}((1/p)\sum_{j=0}^{p-1}\delta(\sigma^j x_1), \bar{\nu})=\bar{d}((1/p)\sum_{j=0}^{p-1}\delta(\sigma^j x_1), (1/p)\sum_{j=0}^{p-1}\bar{\nu}_j)<\delta/3$. Therefore

$$\bar{d}(\mu_1, \bar{\nu})\leqq\bar{d}\Big(\mu_1, \frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1)\Big)+\bar{d}\Big(\frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1), \bar{\nu}\Big)<2\delta/3 \qquad (\nu\in\mathfrak{M}_\sigma(\tilde{\Delta}))$$

and the proof is completed.

§ 3. Proof of theorems.

In this section we prove Theorems 1, 2, and 3 that are mentioned in § 1.

PROOF OF THEOREM 1. Since $\mathscr{E}(X)$ is dense in $\mathfrak{M}_\sigma(X)$ by Proposition 1, $\mathscr{E}(X)$ is a dense G_δ-subset of $\mathfrak{M}_\sigma(X)$. Let $\mathscr{U}=\{U_i\}_{i=1}^\infty$ be a countable open basis of X. Since (X, σ) satisfies weak specification, we can find a smallest set Δ_i with $\Delta_i\subset U_i$ for $U_i\in\mathscr{U}$. For every $i\geqq 1$, take $\mu_i\in\mathfrak{M}_\sigma(\tilde{\Delta}_i)$, then $\mu_i(U_i)\geqq\mathrm{per}(\Delta_i)^{-1}>0$. Hence $\mu=\sum_{i=1}^\infty(1/2^i)\mu_i$ is a measure positive on all nonempty open sets; i.e., $\mu\in\mathscr{D}(X)$. It follows from Proposition 21.11 of [5] that $\mathscr{D}(X)$ is a dense G_δ-subset of $\mathfrak{M}_\sigma(X)$ unless $\mathscr{D}(X)$ is empty. For every integer $r>0$, $K_r=\{\mu\in\mathfrak{M}_\sigma(X);\ \mu(x)<1/r$ for all $x\in X\}$ is open in $\mathfrak{M}_\sigma(X)$. Using Proposition 2, we have that K_r is a dense in $\mathfrak{M}_\sigma(X)$ for all $r\geqq 1$. Since $\mathscr{N}(X)=\bigcap_{r=1}^\infty K_r$, $\mathscr{N}(X)$ is a dense G_δ-subset of $\mathfrak{M}_\sigma(X)$.

Since $\mathscr{D}(X)$ is a dense G_δ-subset of $\mathfrak{M}_\sigma(X)$, it is enough to show that $\mathscr{S}(X)\cap\mathscr{D}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

Since $\mathrm{card}(X)>1$, there is two nonempty disjoint closed neighborhoods F_1 and F_2 in X. For $n\geqq 2$, put $S(n)=\{\mu\in\mathscr{S}(X);\ \mu(F_1)\geqq 1/n$ and $\mu(F_2)\geqq 1/n\}$, then $\mathscr{S}(X)\cap\mathscr{D}(X)\subset\bigcup_{n=2}^\infty S(n)$. Let V_m be an $1/m$ open neighbourhood of F_1 for every $m\geqq 1$, then $S(n)\subset\bigcup_{m=1}^\infty\bigcup_{r=1}^\infty E[m, r]$ where $E[m, r]=\bigcap_{j=r}^\infty\{\mu\in\mathfrak{M}_\sigma(X);\ \mu(V_m\cap\sigma^j V_m)-\mu(F_1)^2\leqq 1/2r^2,\ \mu(F_1)\geqq 1/n$ and $\mu(F_2)\geqq 1/n\}$. Since V_m $(m\geqq 1)$ is open and F_1 and F_2 are closed, it is easy to check that each $E[m, r]$ is closed.

We show that for every $m\geqq 1$ and $r\geqq 1$, $E[m, r]$ is a nowhere dense subset of $\mathfrak{M}_\sigma(X)$. For fixed m, take $r \geqq n$ such that $m \leqq 2r^2$. For every $\Delta \in A(1/2r^2)$, define a set $Z = \{0 \leqq j < \mathrm{per}(\Delta);\ \sigma^j \Delta \cap F_1 \neq \emptyset$ and $\sigma^j \Delta \not\subset V_m\}$. Then by the definition of $A(1/2r^2)$, we have $\mathrm{card}(Z)/\mathrm{per}(\Delta) < 1/2r^2$. For every $\nu \in \mathfrak{M}_\sigma(\tilde{\Delta})$, $\bar{\nu}(V_m \cap \sigma^{j\,\mathrm{per}(\Delta)} V_m) > \bar{\nu}(F_1) - 1/2r^2$ $(j \geqq 1)$, and so $\bar{\nu}(V_m \cap \sigma^{j\,\mathrm{per}(\Delta)} V_m) - \bar{\nu}(F_1)^2 > \bar{\nu}(F_1)(1 - \bar{\nu}(F_1)) - 1/2r^2$. This shows that $\bar{\nu} \notin E[m, r]$. By Proposition 2, $\bigcup_{\Delta \in A(1/2r^2)} \{\bar{\nu} \in \mathfrak{M}_\sigma(X);\ \nu \in \mathfrak{M}_\sigma(\tilde{\Delta})\}$ is dense in $\mathfrak{M}_\sigma(X)$. Hence $\mathscr{S}(X) \cap \mathscr{D}(X)$ contained in a countable union of nowhere dense closed sets, and so $\mathscr{S}(X) \cap \mathscr{D}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

PROOF OF THEOREM 2. Let $\varepsilon > 0$ be given and $M(\varepsilon/2)$ be as in the definition of weak specification. Let $\mu_1, \cdots, \mu_k \in \mathfrak{M}(X)$ be given, as well as integers $a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots < a_k \leqq b_k$ and q with $a_i - b_{i-1} \geqq M(\varepsilon/2)$ and $q \geqq M(\varepsilon/2) + b_k - a_1$. Since $\sigma: \mathfrak{M}(X) \to \mathfrak{M}(X)$ is uniformly continuous, there exists an $\eta > 0$ such that $\bar{d}(\mu, \nu) < \eta$ implies $\bar{d}(\sigma^j \mu, \sigma^j \nu) < \varepsilon/2$ for $a_1 \leqq j \leqq b_k$. For some integer $n > 0$ there exist $x_r^i \in X$ $(1 \leqq r \leqq n,\ 1 \leqq i \leqq k)$ such that putting $\nu_i = 1/n \sum_{r=1}^n \delta(x_r^i)$ $(1 \leqq i \leqq k)$, $\bar{d}(\mu_i, \nu_i) < \eta$ holds for $1 \leqq i \leqq k$ (c.f. p. 11 of [5]). Since $\sigma: X \to X$ satisfies weak specification, there exist smallest sets Δ_r with $\sigma^q \Delta_r = \Delta_r$ and $\Delta_r \subset \bigcap_{i=1}^k \bigcap_{j=a_i}^{b_i} \sigma^{-j} B(\sigma^j x_r^i, \varepsilon/2)$ for $1 \leqq r \leqq n$. Take $\rho^r \in \mathfrak{M}_\sigma(\tilde{\Delta}_r)$ and put $\rho = (1/n) \sum_{r=1}^n \bar{\rho}_0^r$ where $\bar{\rho}_0^r(B) = \mathrm{per}(\Delta_r) \bar{\rho}^r(B \cap \Delta_r)$ for Borel sets B. Obviously $\sigma^q \rho = \rho$ and $\bar{d}(\sigma^j \rho, \sigma^j \nu_i) = \bar{d}((1/n) \sum_{r=1}^n \sigma^j \bar{\rho}_0^r, (1/n) \sum_{r=1}^n \delta(\sigma^j x_r^i)) \leqq \varepsilon/2$ $(a_i \leqq j \leqq b_i,\ i = 1, \cdots, k)$. Hence $\bar{d}(\sigma^j \rho, \sigma^j \mu_i) < \varepsilon$ for $a_i \leqq j \leqq b_i$, $i = 1, \cdots, k$. The proof is completed.

PROOF OF THEOREM 3. Since V is compact and connected, by Proposition 2, there exist a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers with $\varepsilon_n \searrow 0$ and a sequence $\{\Delta_n\}_{n=1}^\infty$ in $A(\varepsilon_n)$ such that for some $\mu_n \in \mathfrak{M}_\sigma(\tilde{\Delta}_n)$ the followings hold;

(a) $B_n \cap B_{n+1} \cap V \neq \emptyset$,

(b) $\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty B_n = V$

where B_n $(n \geqq 1)$ is the ε_n-closed neighborhood of $\bar{\mu}_n$ in $\mathfrak{M}(X)$. We have to show that for every $x_0 \in X$ and $\delta > 0$ there exists an $x \in B(x_0, \delta)$ such that $V_{\sigma^r}(x) = V$ for all $r \geqq 1$. For every $n \geqq 1$, take an $x_n \in \Delta_n$. Since (X, σ) satisfies weak specification, there exist positive integers M_n $(n \geqq 0)$ such that for every set of integers $a_0 \leqq b_0 < a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots$ with $a_n - b_{n-1} \geqq M_{n-1}$ $(n \geqq 1)$, there exists an $x \in X$ such that $d(\sigma^j x, \sigma^j x_n) \leqq \varepsilon_n$ $(a_n \leqq j \leqq b_n,\ n > 0)$ and $d(\sigma^j x, \sigma^j x_0) \leqq \delta$ $(a_0 \leqq j \leqq b_0)$ (c.f. see Orbit specification lemma in [8]). With the above notations, take a_n and b_n $(n \geqq 0)$ as follows;

(i) $a_0 = b_0 = 0$,

(ii) a_n is divided by $n!$ and
$b_{n-1} + M_{n-1} \leqq a_n < b_n + M_{n-1} + n!$ $(n \geqq 1)$ and

(iii) $b_n = a_n + (n+1)!\ (a_n + M_n)\,\mathrm{per}\,(\Delta_n)\,\mathrm{per}\,(\Delta_{n-1})\ (n \geqq 1)$.
Then, we have an $x \in B(x_0, \sigma)$ with $d(\sigma^j x, \sigma^j x_n) \leqq \varepsilon_n\ (a_n \leqq j \leqq b_n,\ n \geqq 1)$.

We have to show that $V_{\sigma^r}(x) = V$ for all $r \geqq 1$. Though the proof is similar to that in [8], we sketch it for completeness.

It is clear that for $r \geqq 1$ there is $N_0 \geqq r$ such that $\mathrm{per}\,(\Delta_n) > r$ for all $n \geqq N_0$. Now we fix the integers r, n with $n \geqq N_0$ and k with $b_n/r < k \leqq b_{n+1}/r$, and write

$$A_1 = A \cap \left[\frac{a_n}{r},\ \frac{b_n}{r}\right)$$

where $A = \{0 \leqq j \leqq k;\ j \text{ is an integer}\}$. Take k' with $k - \mathrm{per}\,(\Delta_{n+1}) < k' \leqq k$ such that $k' - a_{n+1}/r$ is divided by $\mathrm{per}\,(\Delta_{n+1})$.

Then it is easy to see that $A_2 = A \cap [a_{n+1}/r,\ k')$ is nonempty when $k \geqq a_{n+1}/r + \mathrm{per}\,(\Delta_{n+1})$ and A_2 is empty when $k < a_{n+1}/r + \mathrm{per}\,(\Delta_{n+1})$.

Obviously $\mathrm{per}\,(\Delta_{n+1})$ divides $\mathrm{card}\,(A_2)$. By (iii), $\mathrm{per}\,(\Delta_n)$ divides $\mathrm{card}\,(A_1)$. Remark that $\mathrm{per}\,(\Delta_n)$ and $\mathrm{per}\,(\Delta_{n+1})$ are prime numbers. Since $n \geqq N_0$, $\mathrm{per}\,(\Delta_n)$ and $\mathrm{per}\,(\Delta_{n+1})$ are both prime to the integer r, so that

$$\bar{d}(\mathrm{card}\,(A_1)^{-1} \sum_{j \in A_1} \delta(\sigma^{jr} x_n),\ \bar{\mu}_n) \leqq \varepsilon_n$$

and

$$\bar{d}(\mathrm{card}\,(A_2)^{-1} \sum_{j \in A_2} \delta(\sigma^{jr} x_{n+1}),\ \bar{\mu}_{n+1}) \leqq \varepsilon_{n+1}\,.$$

By the definition of metric $\bar{d}$, we get that

$$\begin{aligned} &\bar{d}\Big(\frac{1}{k} \sum_{j \in A} \delta(\sigma^{jr} x),\ \mathrm{card}\,(A_1 \cup A_2)^{-1} \sum_{j \in A_1 \cup A_2} \delta(\sigma^{jr} x)\Big) \\ &\quad < 2\,\mathrm{card}\,(A_1)^{-1}\{k - \mathrm{card}\,(A_1 \cup A_2)\} \\ &\quad \leqq \frac{4}{(n+1)!} + 2\varepsilon_n\,. \end{aligned}$$

Since $d(\sigma^{jr} x, \sigma^{jr} x_n) \leqq \varepsilon_n\ (j \in A_1)$ and $d(\sigma^{jr} x, \sigma^{jr} x_n) \leqq \varepsilon_{n+1}\ (j \in A_2)$, it is easy to check that

$$\begin{aligned} &\bar{d}\Big(\frac{1}{k} \sum_{j \in A} \delta(\sigma^{jr} x),\ \mathrm{card}\,(A_1 \cup A_2)^{-1}\big(\sum_{j \in A_1} \delta(\sigma^{jr} x_n) + \sum_{j \in A_2} \delta(\sigma^{jr} x_{n+1})\big)\Big) \\ &\quad < \frac{4}{(n+1)!} + 2\varepsilon_n + \bar{d}(\mathrm{card}\,(A_1 \cup A_2)^{-1} \sum_{j \in A_1 \cup A_2} \delta(\sigma^{jr} x)\,, \\ &\qquad\qquad \mathrm{card}\,(A_1 \cup A_2)^{-1}(\sum_{j \in A_1} \delta(\sigma^{jr} x_n) + \sum_{j \in A_2} \delta(\sigma^{jr} x_{n+1}))) \\ &\quad < \frac{4}{(n+1)!} + 3\varepsilon_n + \varepsilon_{n+1}\,. \end{aligned}$$

Thus we can compute that

$$\bar{d}\Big(\frac{1}{k}\sum_{j\in A}\delta(\sigma^{jr}x),\ \text{card}\,(A_1\cup A_2)^{-1}(\text{card}\,(A_1)\bar{\mu}_n+\text{card}\,(A_2)\bar{\mu}_{n+1})\Big)$$

$$<\frac{4}{(n+1)!}+3\varepsilon_n+\varepsilon_{n+1}$$

$$+\bar{d}(\text{card}\,(A_1\cup A_2)^{-1}(\sum_{j\in A_1}\delta(\sigma^{jr}x_n)+\sum_{j\in A_2}\delta(\sigma^{jr}x_{n+1}))\ ,$$

$$\text{card}\,(A_1\cup A_2)^{-1}(\text{card}\,(A_1)\bar{\mu}_{n+1}+\text{card}\,(A_2)\bar{\mu}_{n+1}))$$

$$<\frac{4}{(n+1)!}+4\varepsilon_n+2\varepsilon_{n+1}\ .$$

Since $\bar{d}(\bar{\mu}_n, \bar{\mu}_{n+1})\leqq\varepsilon_n+\varepsilon_{n+1}$ by (a), we have that

$$\bar{d}\Big(\frac{1}{k}\sum_{j\in A}\delta(\sigma^{jr}x),\ \bar{\mu}_n\Big)<\frac{4}{(n+1)!}+5\varepsilon_n+3\varepsilon_{n+1}\ .$$

Since $n\geqq N_0$ and $b_n/r<k\leqq b_{n+1}/r$ are arbitrary, $V_{\sigma^r}(x)$ coincides with the ω-limit set of the sequence $\{\bar{\mu}_n\}_{n=1}^{\infty}$ and so $V_{\sigma^r}(x)$ coincides with V by (b). The proof is completed.

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