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Invariant Measures for Markov Maps of the Interval*

Rufus Bowen†

Abstract. There is a theorem in ergodic theory which gives three conditions sufficient for a piecewise smooth mapping on the interval to admit a finite invariant ergodic measure equivalent to Lebesgue. When the hypotheses fail in certain ways, this work shows that the same conclusion can still be gotten by applying the theorem mentioned to another transformation related to the original one by the method of inducing.

It is often difficult to decide whether a given map $f: I \to I$ of an interval admits an invariant measure equivalent to Lebesgue measure. Piecewise differentiable f which are expanding [i.e., $\inf|(f^n)'(x)|>1$ for some n] have such measures under mild additional hypotheses [1, 11, 8, 13, 16]. This paper gives sufficient conditions for certain nonexpandings maps to have invariant measures. This result unifies a number of examples and its conditions are quite computable.

A map $f: I \to I$ of the interval $I=[a,b]$ is *Markov* if one can find a finite or countable collection $\{I_k\}$ of disjoint open intervals such that

a) f is defined on $\cup I_k$ and $I \backslash \cup I_k$ has measure zero.

b) $f|I_k$ is strictly monotonic and extends to a C^2 function on $\bar{I}_k$ for each k,

c) if $f(I_k) \cap I_j \neq \emptyset$, then $f(I_k) \supset I_j$, and

d) there is an R so that $\bigcup_{n=1}^{R} f^n(I_k) \supset I_j$ for every k and j.

A measure μ on I which is equivalent to Lebesgue has the form $\mu(E)=\int_E p(x)dx$ where $p(x)$ is a positive measurable function. We will be trying to understand and apply the following result of Adler [1, 2][1].

Adler's Theorem. *Let* $f: I \to I$ *be Markov,* $M=\sup_{I_k} \sup_{y,z \in I_k} \left|\frac{f''(z)}{f'(y)^2}\right| < +\infty$ *and* $\lambda_n = \inf_x |(f^n)'(x)| > 1$ *for some n. Then f admits an invariant finite measure* $d\mu = p(x)dx$ *with* $p(x)$ *bounded away from* 0 *and* $+\infty$.

Let us recall two well-known Markov maps which have invariant measures but are not expanding. The first one is $f_1(x)=4x(1-x)$ on $[0,1]$. This is not expanding because $x=\frac{1}{2}$ is a critical point. The second example is on $I=[0,\infty]: f_2(x)=1/x$ on $[0,1]$ and $f_2(x)=x-1$ on $[1,\infty]$. This map is not expanding because $x=\infty$ is a fixed point with slope 1 $\left(\text{to see this change the variable to } u=\frac{x}{1+x}\right)$. This example is of course related to continued fractions: $x=[a_0,a_1,a_2,\ldots]$ where the a_i's are the number of iterates $f^k x$ of x that are in $[1,\infty]$ between visits to $[0,1]$. It is customary in studying continued fractions to use instead of f_2 the map f_3 of $[0,1]$ defined by $f_3(x)=$ fractional part of $1/x$. One reason for this is that f_3 is expanding while f_2 is not; another is that f_3 admits the finite invariant measure $d\mu=\frac{1}{\ln 2}\frac{dx}{1+x}$ while f_2's invariant measure is infinite.

The relation between f_2 and f_3 is that f_2 *induces* f_3; namely, for $x\in[0,1]$, $f_3(x)$ is the first iterate $f_2^n(x)$ which lies in $[0,1)$. Adler invented the hypotheses of his theorem to apply to f_3. The main thrust of our result is that his conditions in fact hold for f_3 because of the way it is derived from f_2. Furthermore, not only the fixed point $+\infty$ with $f_2'(+\infty)=1$, but also the critical point $x=\frac{1}{2}$ for $f_1(x)$ can be "induced away". The shift to infinite measure accompanies sources of slope 1 but not critical points (answering a question of Adler [1]).

Before stating the theorem some acknowledgements are in order. Adler and Weiss [3] studied Boole's mapping $f(x)=x-\frac{1}{x}$ but didn't get the condition $M<\infty$ above because of the interval they induced on. Ruelle's paper [12] giving a value $R\neq 4$ for which $f_R(x)=Rx(1-x)$ has an invariant measure led me to see if inducing would work for critical points. After formulating this paper, I learned that Bunimovich [for $f(x)=q\pi\sin(x)(\operatorname{mod}\pi)$] [5] and Jakobson and Sinai [7] [for $f_R(x)=Rx(1-x)$ for certain R's] had already induced near a critical point. Jakobson and Sinai certainly know all the ideas we present, and Jakobson has now gone much further than we have. Our paper is mostly calculus and takes a computational approach.

1. Statement of Theorem

Throughout the rest of the paper f *will always be a map of* $I=[a,b]$ *which is Markov with regard to a finite set of open intervals* $I_1,\ldots,I_d$. The function f^n is then continuous on each open interval $I_{i_0}\cap f^{-1}I_{i_1}\cap\ldots\cap f^{-n+1}I_{i_{n-1}}$ and extends to a C^2 function on the closure of such an interval. If x is not in such an open interval, then x is an endpoint of two such intervals (unless $x=a$ or b). $f^n(x)$ then has two values, which we denote by $f^n(x-)$ and $f^n(x+)$. One naturally thinks of such an x as being two points, $x+$ and $x-$. Number the $I_k=(a_k,b_k)$ so that $b_k=a_{k+1}$, for $1\leq k<d$. We will consistently think of a_{k+1} and b_k as distinct points, so that $f(p)$ makes sense for any $p\in S=\{a_1,b_1,a_2,b_2,\ldots,a_r,b_r\}$ and $f:S\to S$ because f is Markov. By this convention notice that if $f^n p=p$ with $p\in S$, then one must have $(f^n)'(p)\geqq 0$. The *height* $H(p)$ of $p\in S$ is the smallest $n\geqq 0$ so that $f^n p$ is periodic under $f:S\to S$.

To formulate the theorem one needs a couple of calculus lemmas. The function $f:[x_0,x_1]\to\mathbb{R}$ is *not flat* at x_0 if for some $r\geqq 1$, $f^{(r)}(x_0)\neq 0$ and f is C^{r+1} on $[x_0,x_1]$.

Lemma 1. *Suppose $f:[x_0,x_1]\to\mathbb{R}$ is not flat at x_0. Then for $U=(x_0,x_0+\varepsilon)$ with ε small*

$$A(U)=\inf_{x\in U}\left|\frac{f'(x)(x-x_0)}{f(x)-f(x_0)}\right|>0 \quad \textit{and} \quad \sup_{x\in U}\left|\frac{f''(x)(f(x)-f(x_0))}{f'(x)^2}\right|<\infty\,.$$

Proof. By Taylor's formula there are $\xi_1,\xi_2,\xi_3\in(x_0,x]$ with

$$f(x)-f(x_0)=f^{(r)}(x_0)(x-x_0)^r+\frac{f^{(r+1)}(\xi_1)}{(r+1)!}(x-x_0)^{r+1}\,,$$

$$f'(x)=rf^{(r)}(x_0)(x-x_0)^{r-1}+\frac{f^{(r+1)}(\xi_2)}{r!}(x-x_0)^r\,,$$

$$f''(x)=f^{(r)}(x_0)r(r-1)(x-x_0)^{r-2}+\frac{f^{(r+1)}(\xi_3)}{(r-1)!}(x-x_0)^{r-1}\,,$$

where $r\geqq 1$ is minimal subject to $f^{(r)}(x_0)\neq 0$. Letting

$$C_{r+1}(\xi)=\sup_{\xi\in U}\left|\frac{f^{(r+1)}(\xi)}{(r+1)!}\right|$$

one has

$$\begin{aligned}\left|\frac{f'(x)(x-x_0)}{f(x)-f(x_0)}\right| &\geqq \frac{r|f^{(r)}(x_0)|-(r+1)C_{r+1}\varepsilon}{|f^{(r)}(x_0)|+\varepsilon C_{r+1}}\\ &\geqq r\left(1-\frac{\varepsilon C_{r+1}+\dfrac{r+1}{r}\varepsilon C_{r+1}}{|f^{(r)}(x_0)|+\varepsilon C_{r+1}}\right)\\ &\geqq r\left(1-\frac{3C_{r+1}\varepsilon}{|f^{(r)}(x_0)|}\right). \qquad (*)\end{aligned}$$

The sup in the lemma is finite provided

$$\inf\left|f^{(r)}(x_0)r+\frac{f^{(r+1)}(\xi_2)}{r!}(x-x_0)\right|>0$$

or

$$|f^{(r)}(x_0)r|>(r+1)\varepsilon C_{r+1}(\varepsilon)\,.$$

This holds for small ε and in particular whenever the expression $(*)$ above is positive. □

x_0 is a *source* for $f:[x_0,x_1]\to[x_0,x_2]$ provided $f(x_0)=x_0$ and $\lim_{n\to\infty}f^{-n}x=x_0$ for x near x_0. This implies that $f'(x_0)\geqq 1$. The source x_0 is called *regular* if either (a) $f'(x_0)>1$ or (b) $f'(x)$ decreases monotonically to 1 as $x\to x_0$. Taylor's formula shows that (b) holds for a source x_0 with $f'(x_0)=1$ and $f^{(r)}(x_0)\neq 0$ for some $r>1$. For $U=(x_0,x_0+\varepsilon)$ and $x\in U$, let $m_U(x)$ denote the smallest $m>0$ with $f^m x\notin U$.

Lemma 2. *Let x_0 be a regular source for f and $U=(x_0,x_0+\varepsilon)$ with ε small. There is a constant $B(U)>0$ so that*

$$(f^{m_U(x)})'(x)>\frac{B(U)}{|x-x_0|}\quad \textit{for all}\quad x\in U\,.$$

Proof. First assume $f'(x_0)=1$ and choose ε so that $f'(x)$ is decreasing as $x\to x_0$ with $x\in U$. Then $(f^m)'(t)\leqq (f^m)'(x)$ for $x_0\leqq t\leqq x$ and $m=m_U(x)$. So

$$(f^m)'(x)(x-x_0)\geqq \int_{x_0}^{x}(f^m)'(t)dt=f^m(x)-f^m(x_0)\geqq\varepsilon\,.$$

Take $B(U)=\varepsilon$.

Suppose $f'(x_0)>1$. Choose ε so that $\lambda=\inf_{s\in U} f'(s)>1$. Define $g(x)=\log f'(x)$. Then $g'(s)=\frac{f''(s)}{f'(s)}$ and so

$$|g(x)-g(y)|\leqq\frac{c}{\lambda}|x-y|\quad\text{for}\quad x,y\in U\,,$$

where $c=\sup_{t\in U}|f''(s)|$. For $t\in[x_0,x]$ one has $[f^k t, f^k x]\subset U$ for all $0\leqq k<m=m_U(x)$. Since $f|U$ expands distances by at least λ, and $m-k-1$ iterates of $[f^k t, f^k x]$ lie in U, $|f^k x-f^k t|\leqq\varepsilon\lambda^{-m+k+1}$. Then

$$\begin{aligned}|\log(f^m)'(x)-\log(f^m)'(t)&\leqq\sum_{k=0}^{m-1}|g(f^k x)-g(f^k t)|\\&\leqq\frac{c}{\lambda}\sum_{k=0}^{m-1}\varepsilon\lambda^{-m+k+1}\leqq\frac{c\varepsilon}{\lambda-1}\,.\end{aligned}$$

Therefore $(f^m)'(t)\leqq(f^m)'(x)\exp\left(\frac{c\varepsilon}{\lambda-1}\right)$ and

$$\varepsilon\leqq f^m(x)-f^m(x_0)=\int_{x_0}^{x}(f^m)'(t)dt\leqq|x-x_0|(f^m)'(x)\exp\frac{c\varepsilon}{\lambda-1}\,.$$

Take $B(U)=\varepsilon\exp\left(\frac{-c\varepsilon}{\lambda-1}\right)$. □

Points $p\in S$ with $H(p)>0$ will be assumed not flat for f. A *standard interval* $U=U_p$ is one satisfying the conditions of Lemma 1 [$x_0=p$ and $U=(x_0,x_0+\varepsilon)$ or $U=(x_0-\varepsilon,x_0)$ depending on whether p is an a_k or a b_k]. For $p\in S$ with $H(p)=0$ and having period r, we will assume p is a regular source for f^r. A *standard interval* $U=U_p$ for such a p is one with $f^r|U$ continuous,

$$(f^r)'>1\quad\text{on}\quad \bar U_p\backslash\{p\}\,,\qquad \lim_{k\to\infty}(f^{-r})^k x=p\quad\text{for}\quad x\in U$$

and such that Lemma 2 holds for U_p [here $x_0=p$, f^r is used in place of f, and $U=(x_0,x_0+\varepsilon)$ or $(x_0-\varepsilon,x_0)$].

Theorem. *Suppose that f is not flat at points $p\in S$ where $H(p)>0$ and all the points $p\in S$ with $H(p)=0$ are regular periodic sources. Suppose standard intervals U_p, $p\in S$, are given with*

a) $fU_p\subset U_{fp}$ *when* $H(p)>0$ *and*

b) length $U_p<A(U_p)A(U_{fp})\dots A(U_{f^{H(p)-1}p})B(U_{f^{H(p)}p})$

when $H(p)>0$.

Finally suppose that

$$\lambda_N^*=\inf\left\{\max_{1\le n\le N}|(f^n)'(x)|:x\notin\bigcup_{p\in S}\bar{U}_p\right\}>1$$

for some $N>1$. Then f admits an invariant measure μ equivalent to Lebesque, μ is ergodic for f, and μ is finite iff all the periodic points in S are expanding (i.e., $|(f^r)'(p)|>1$ where $f^rp=p$).

2. Examples

A number of Markov maps have the property that for some $n>0$

$$|(f^n)'(x)|>1 \quad \text{for all} \quad x\notin S\,.$$

For the theorem to apply here it is enough to see that the periodic points of S are not flat. The condition above implies they are all sources, Lemmas 1 and 2 guarantee the existence of standard intervals, a) and b) hold by using small intervals, and $\lambda_n^*>1$ by an obvious compactness argument.

A. Continued Fractions

Using the variable $u=\dfrac{x}{1+x}\in[0,1]$ the map f_2 of the introduction changes to

$$\tilde{f}_2(u)=\begin{cases}1-u & \text{for} \quad u\in[0,\frac{1}{2}]\\ 2-u^{-1} & \text{for} \quad u\in[\frac{1}{2},1]\,.\end{cases}$$

Here $S=\{0,\frac{1}{2}-,\frac{1}{2}+,1\}$, $|(f^2)'(u)|>1$ except for $u=0,1$, and $u=1$ is the only periodic point of S. The point $u=1$ is a regular source since $f'(u)=\dfrac{1}{u^2}$ decreases to 1 as $u\to 1$. Thus the theorem applies. f_2 is closely connected with the action of $\mathrm{GL}(2,\mathbb{Z})$ on $\mathbb{R}$ as linear fractional transformations. It seems likely that to any Fuchsian group of the first kind one can associate a natural Markov map of the real line[2].

B. Renyi's Example

Define f_4 on $[0,\infty]$ by

$$f_4(x)=\begin{cases}\dfrac{x}{1-x} & \text{for} \quad x\in[0,1]\\ x-1 & \text{for} \quad x\in[1,\infty]\,.\end{cases}$$

[2] This is in fact the case; see Comment 2

Using again the variable $u=\frac{x}{1+x}$, f_4 becomes

$$\tilde{f}_4(u)=\begin{cases}\frac{u}{1-u} & \text{for } u\in[0,\tfrac{1}{2}] \\ 2-u^{-1} & \text{for } u\in[\tfrac{1}{2},1]\,.\end{cases}$$

The periodic points in $S=\{0,\frac{1}{2}-,\frac{1}{2}+,1\}$ are the fixed points 0, 1. They are both regular sources with slope 1 and $|\tilde{f}_4'(u)|>1$ for $u\neq 0,1$. The theorem applies, the invariant measure being infinite in neighborhoods of 0 and 1 (this is not stated in the theorem but is in the proof). Now f_4 induces on $[0,1]$ the map

$$x\to \text{fractional part of } \frac{x}{1-x}\,.$$

It follows that this map has an infinite ergodic invariant measure. This fact is due to Renyi (see [1, 11]).

C. Boole Mappings

Adler and Weiss [3] showed that $f_5(x)=x-\frac{1}{x}$ is ergodic on $\mathbb{R}$. Here we use the variable $u=\arctan x\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Then $\tilde{f}_5(u)=\arctan\left(\tan u-\frac{1}{\tan u}\right)$ and one computes $\tilde{f}_5'(u)=\frac{1}{3\sin^4 u-3\sin^2 u+1}$. This shows $\tilde{f}_5'(u)>1$ for $u\notin S=\{-\frac{\pi}{2},0-,0+,\frac{\pi}{2}\}$. The periodic points in S are $\pm\frac{\pi}{2}$, both regular sources of slope 1. Thus $\tilde{f}_5$ (and so f_5) admits an infinite ergodic measure equivalent to Lebesgue.

The generalized Boole mappings are

$$f(x)=x-\sum_{i=1}^{N} p_i(x-\eta_i)^{-1}\,,$$

where the p_i are positive and $\eta_1<\eta_2<\ldots<\eta_N$. These examples have been studied by Schweiger [14], Lie and Schweiger [9] and Adler and Flatto. Here $f'(x)=1+\sum p_i(x-\eta_i)^{-2}>1$ and one sees that each of the intervals from the partition $-\infty$, $\eta_1,\ldots,\eta_N$, $+\infty$ are mapped monotonically onto $(-\infty,+\infty)$ by f. Choose $J=[-A,A]\supset[\eta_1-1,\eta_N+1]$ so that $|f(x)|\leqq|x|$ for $x\notin J$ and choose the constant $c>0$ so that

$$x\in J\,,\quad f(x)\notin J\Rightarrow|f'(x)\geqq c|f(x)|^2\,.$$

For B large choose a C^∞ diffeomorphism h of $\mathbb{R}$ onto a finite interval I so that

a) $h(x)=x$ for $|x|\leqq B$,

b) $h'(x)=\frac{B}{x^2}$ for $|x|\geqq 2B$,

c) $h'(x)\geqq\frac{B}{x^2}$ for $|x|\geqq B$, and

d) $|x|\geqq|y|\Rightarrow h'(x)\leqq h'(y)$.

Choose B large enough so $Bc>1$ and $J\subset[-B,B]$; set $\tilde{f}=h\circ f\circ h^{-1}$. For $u\in J$ one has

$$|\tilde{f}'(u)|=|f'(u)|>1 \quad \text{if} \quad |f(u)|\leqq B$$

and

$$|\tilde{f}'(u)|=|f'(u)|\,|h'(f(u))|\geqq c|f(x)|^2\frac{B}{|f(x)|^2}>1 \quad \text{if} \quad |f(u)|\geqq B\,.$$

For $u\in I\backslash J$ one has $h^{-1}(u)\notin J$, $|f(h^{-1}(u))|\leqq|h^{-1}(u)|$ and

$$\begin{aligned}|\tilde{f}'(u)|&=|h'(f(h^{-1}u))|\,|f'(h^{-1}u)|\,|h'(h^{-1}u)|^{-1}\\&\geqq|f'(h^{-1}u)|>1 \quad \text{by} \quad \text{d)}\,.\end{aligned}$$

Condition b) says that $h(x)$ differs from $-\frac{B}{x}$ by a constant on each of the intervals $(-\infty,-2B)$ and $(2B,+\infty)$. Plugging this into the expression for $f(x)$ one sees that $\tilde{f}'=1$ at each end point of I. These endpoints are sources for $\tilde{f}$ because $\pm\infty$ are for f; they are regular because $\tilde{f}$ is analytic and not constant near them. Thus the theorem applies and we recover the results of [9].

Another example with nonexpanding sources was handled by Bogoyavlenski [18, p. 21]. The second class of examples are those with critical points. Bunimovich [5] verified the hypotheses of our theorem when

$$f_n(x)=n\pi\sin x\,(\mathrm{mod}\,\pi)$$

on $[0,\pi]$ and $n\geqq 2$. The proof given in the next section has much in common with his construction of an invariant μ for these f_n's. The best known example with a critical point is $f(x)=4x(1-x)$ on $[0,1]$. This f has an invariant measure because it is conjugate by an absolutely continuous homeomorphism to a certain piecewise linear map [15].

Let us see how the theorem applies to $f(x)=4x(1-x)$. This is purely computational. Here $S=\{0,\frac{1}{2}\pm,1\}$. Set $U_{1/2-}=[\frac{3}{8},\frac{1}{2}]$, $U_{1/2+}=[\frac{1}{2},\frac{5}{8}]$, $U_1=[\frac{15}{16},1]$ and $U_0=[0,\frac{1}{4}]$. Then $f(U_{1/2\pm})\subset U_1$, $f(U_1)\subset U_0$ and the formulas of Lemmas 1 and 2 give

$$A(U_{1/2})\geqq 2\,,\qquad A(U_1)\geqq\tfrac{13}{16},\qquad B(U_0)\geqq 0.09\,.$$

Conditions a) and b) of the theorem are easily checked, as is $\lambda_2^*>1$.

A computer program was written to check the conditions of the theorem for smooth $f(x)$ with a single critical point x_0 and $f^n(x_0)$ periodic for some $n>0$. The conditions were checked for $f_R(x)=Rx(1-x)$ with the value $R\sim 3.67857$ found by Ruelle [12] as well as over 120 other values of $R\in[3.5,4]$ with $f^n(\frac{1}{2})=f^m(\frac{1}{2})$, $n<m\leqq 10$. Among these values of R, almost all possibilities $n<m\leqq 10$ occurred; we did not worry about roundoff errors, but these were probably not significant since $\lambda_n^*>1$ with $n\leqq 6$ for most all but three of these R's. Jakobson and Sinai [7] and Pianigiani [17] have shown that f_R has an absolutely continuous invariant measure for countably many values of R. Jacobson has now shown this is true for uncountably many R's.

The program was also used to check the conditions of the theorem for $f(x)=1-2|x|^k$ on $[-1,1]$ for $2\leqq k\leqq 11$; here $\lambda_n^*>1$ with $n\leqq 7$.

3. Proof of Theorem

Throughout, f satisfies the hypotheses of the theorem. For $K \subset [a,b]$ and $x \in K$ we let $n_K(x)$ denote the smallest $n>0$ with $f^n x \in K$, and $f_K(x) = f^{n_K(x)}x$. The domain of f_K is a subset of K.

Lemma 3. *If $S \cap \bar{K} = \emptyset$, there are M and $\lambda > 1$ so that $|(f_K^M)'(x)| > \lambda$ whenever $f_K^M = (f_K)^M x$ is defined.*

Proof. First consider $x \in \mathcal{U} = \bigcup_{p \in S} U_p$; say $x \in U_p$ and $q = f^{H(p)}p$ has period r. Then $f^{H(p)}x \in U_q$ and let m be the smallest positive integer such that $f^{rm}(f^{H(p)}x) \notin U_q$. Then, by Lemmas 1 and 2 and condition a) of the theorem

$$\begin{aligned} |(f^{rm+H(p)})'(x)| &= |(f^{rm})'(f^{H(p)}x)| \prod_{k=0}^{H(p)-1} |f'(f^k x)| \\ &\geqq \frac{B(U_q)}{|f^{H(p)}x - q|} \prod_{k=0}^{H(p)-1} \frac{A(U_{f^k p})|f^{k+1}x - f^{k+1}p|}{|f^k x - f^k p|} \\ &\geqq \frac{B(U_q)}{|x-p|} \prod_{k=0}^{H(p)-1} A(U_{f^k p}) . \end{aligned}$$

When $H(p)>0$, condition b) of the theorem says that this quantity is bigger than some constant $v_1 > 1$. When $H(p)=0$, then $q=p$ and $|(f^{rm})'(x)| > v_2$ for some constant $v_2 > 1$ because $(f^r)' > 1$ on $\bar{U}_q - \{q\}$. For $x \in \mathcal{U}$, let $g(x)$ denote $f^{rm+H(p)}x$. There is an integer N_1 independent of x so that the following is true: if $f^k x \in K$ is on the f-orbit between x and $g(x)$, then $rm + H(p) < k + N_1$. This follows from the fact that $\bar{K} \cap S = \emptyset$ and implies that the f-orbit between x and $g(x)$ hits K at most N_1 times.

For $x \notin \mathcal{U} \cup S$ one lets $g(x)$ be $f^n x$ with $n \in [1, N]$ minimal subject to $|(f^n)'(x)| \geqq \lambda_N^*$. For any $x \notin S$, $g(x)$ on the forward f-orbit of x is defined; $|g'(x)| \geqq \mu = \min\{v_1, v_2, \lambda_N^*\} > 1$ and the f-orbit between x and $g(x)$ hits K at most $N_2 = \max\{N, N_1\}$ times.

Now suppose $x \in K$ and $(f_K)^M x$ is defined. The f-orbit between x and $(f_K)^M x$ does not hit S since $f(S) \subset S$ and $S \cap K = \emptyset$. Thus there are defined $g(x), g^2(x) = g(g(x)), \ldots, g^j(x), g^{j+1}(x)$ with $f_K^M x$ on the f-orbit between $g^j(x)$ and $g^{j+1}(x)$. Now $M \leqq N_2(j+1)$ and the number of points on the f-orbit from $f_K^M x$ to $g^{j+1}(x)$ is at most N_2; hence, if $\alpha = \sup |f'(y)|$,

$$|(f_K^M)'(x)| \geqq \mu^{(j+1)} \alpha^{-N_2} \geqq (\mu^{1/N_2})^M \alpha^{-N_2} .$$

For M large this is bigger than 1. □

Lemma 4. *Let V be a small open interval with periodic source $p \in S$ as an endpoint. Then V contains a point of $\tilde{S} = \bigcup_{k=0}^{\infty} f^{-k} S$.*

Proof. If $x \notin \tilde{S}$, then $g^j x$ above is defined for all $j \geqq 1$ and so

$$\sup_{n>0} |(f^n)'(x)| = +\infty .$$

If the present lemma were false, by induction one would have that $f|f^m V$ is one-to-one and continuous and that $f^m V$ is an interval for all $m>0$ (f is a homeomorphism on any interval disjoint from S). Since $f^r V \supsetneq V$ (here $f^r p = p$), one gets that $V \subset f^r V \subset f^{2r} V \subset \ldots$ is a strictly increasing sequence of open intervals. $f^{mr} V$ has endpoints p and $q_m = f^{mr} q_0$. The sequence q_m is strictly monotonic; let $q = \lim q_m$. Then $f^r q = q$ by continuity and $f^r : [p, q] \to [p, q]$ is a homeomorphism. Since $q = \lim_{m \to \infty} f^{mr} q_0$, q is a sink for f^r and $(f^r)'(q) \leqq 1$. Then

$$\sup_{n>0} |(f^n)'(q)| = \sup_{0<n<r} |(f^n)'(q)| < \infty$$

and so $q \in \tilde{S}$, i.e., $f^k q \in S$ for some $k \geqq 0$. Then $q \in S$ because q is periodic and $fS \subset S$. This is a contradiction since all periodic points of S are sources. $\square$

Let s denote the number of periodic orbits in S. Let $r_1, \ldots, r_s$ be their periods and choose points $p_1, \ldots, p_s$ on them. By Lemma 4 one can find points $y_i \in \tilde{S} \cap U_{p_i}$ arbitrarily close to p_i. By making the y_i very close to the p_i one may assume that

$$z(f^k p_i) = f^{k - r_i} y_i \in U_{f^k p_i} \quad \text{for} \quad 0 \leqq k < r_i \tag{1}$$

and

$$z(p) = f^{-H(p)} z(f^{H(p)} p) \in U_p \quad \text{for} \quad p \in S,\ H(p) > 0 \tag{2}$$

are well defined: For (2) we are using that $f^{H(p)} : U_p \to U_{f^{H(p)} p}$ is one-to-one. Define $W_p = (z(p), p]$ [or $[p, z(p))$] for $p \in S$.

Let $j(i)$ be the largest $j \geqq 0$ with $f^j y_i \notin S$ and let

$$T = \{ f^k y_i : a \leqq k \leqq j(i),\ 1 \leqq i \leqq s \} .$$

Then S, T and $Z = \{ z(p) : p \in S \}$ are pairwise disjoint. We may assume that $T \cap \bigcup_{p \in S} W_p = \emptyset$ by using $f^{-Mr_i} y_i$ (M large) in place of y_i if necessary. Let $S' = S \cup T \cup Z$ partition $[a, b]$ into the intervals $\{J_1, \ldots, J_t\}$. We claim that f is Markov with respect to these intervals. That $f(S') \subset S'$ and f is Markov using $\{I_1, \ldots, I_d\}$ imply conditions a)–c) in the definition of Markov. There is Q so that $f^Q S' \subset S$; then $\bigcup_{n=1}^{R+Q} f^n(J_k) \supset J_j$ where R is from condition d) for the I_k's.

The W_p's ($p \in S$) are among the J_k's, as are the intervals $V_i = (y_i, z(p_i)]$. Setting $q_i = f^{r_i - 1} p_i$, one has $f(W_{q_i}) = V_i \cup W_{p_i}$ and $f(W_p) = W_{fp}$ for $p \in S - \{q_1, \ldots, q_s\}$. Let $K = [a, b] \setminus \bigcup_{p \in S} W_p$ and f_K be the map induced on K by f, as defined earlier. K is the union of certain J_j's.

Lemma 5. *$f_K : K \to K$ is Markov.*

Proof. Notice first that f_K is defined on $K \setminus f^{-1} S$. For $1 \leqq i \leqq s$ let $\tilde{W}_i = \bigcup_{k=0}^{r_i - 1} W_{f^k p_i}$. Then $f : \tilde{W}_i \to \tilde{W}_i \cup V_i$ is a homeomorphism; use the branch of f^{-1} here to define $L_{i,j}$

$=f^{-j}V_i\cap\tilde{W}_i$ for $j\geqq 1$. For $0\leqq k<r_i$ one has

$$W_{f^k p_i}=\{f^k p_i\}\cup\bigcup\{L_{i,j}:j+k\equiv 0\,(\mathrm{mod}\,r_i)\}.$$

Since f is Markov with respect to the intervals $\{J_1,\dots,J_t\}$, it is Markov also with respect to $\{J_{u,v}=J_u\cap f^{-1}J_v:1\leqq u,v\leqq t\}$; $fJ_{u,v}=J_v$ when $J_{u,v}\neq\emptyset$. When J_u is some $W_p(p\in S)$, then $J_{u,v}\neq\emptyset$ for only one v and that $J_{u,v}=J_u=W_p$.

For $p\in S$ let $i(p)$ be the i so that $f^{H(p)}p$ is on the orbit of p_i. If $J_{u,v}\neq\emptyset$ and $J_v=W_p(p\in S)$, define $J_{u,v,j}=J_{u,v}\cap f^{-1}f^{-H(p)}L_{i(p),j}$. This interval will be nonempty for those j's congruent mod $r_{i(p)}$ to some fixed integer $e(p)$. We claim that f_K is Markov using the intervals

$$\mathscr{J}=\{J_{u,v}:\text{neither } J_u \text{ nor } J_v \text{ is a } W_p\}$$
$$\cup\{J_{u,v,j}:J_u \text{ not a } W_p, J_v \text{ is a } W_p\}.$$

Notice first that $\mathscr{J}$ covers K except for at most countably many points. Also $n_K=1$ on an interval $J_{u,v}$ of the first type in $\mathscr{J}$ and then $f_K(J_{u,v})=f(J_{u,v})=J_v$. On an interval $J_{u,v,j}\in\mathscr{J}$ one has $n_K=j+1+H(p)$ and $f_K(J_{u,v,j})=V_{i(p)}$.

For any $J\in\mathscr{J}$, $f_K|J=f^n|J$ with f^kJ an interval and $f|f^kJ$ monotonic and C^2 for each $0\leqq k<n$. It follows that $f_K|J$ is monotonic and C^2. Finally, $f_K(J)$ contains some J_u and so

$$\bigcup_{n=1}^{R+Q+1} f_K^n(J)\supset\left(\bigcup_{m=1}^{R+Q} f^m(J_u)\right)\cap K=K\,.$$ [3] □

Lemma 6. *Let U be a standard neighborhood of a regular source x_0 for f. There are constants $C(U)$ and $D(U)$ so that if $x,y\in U$ with $m_U(x)=m_U(y)$, then*

$$|x-x_0|\leqq C(U)|y-x_0|$$

and

$$|(f^k)'(x)|\leqq D(U)|(f^k)'(y)| \quad \textit{for} \quad 1\leqq k\leqq m_U(x),$$

(recall $m_U(x)=\inf\{m>0:f^m(x)\notin U\}$.)

Proof. Since $m_U(x)=m_U(y)$ either $y<x<f(y)$ or $f^{-1}y<x<y$. In the first case $|x-x_0|\leqq|y-x_0|\sup_{z\in U}|f'(x)|$; in the second $|x_0-x|\leqq|y-x_0|$. Here we are assuming x_0 is the left endpoint of U; the changes are obvious for the other case.

In the case $f'(x_0)>1$, the second statement follows from the proof of Lemma 2. Suppose x_0 is a regular source with $f'(x_0)=1$. If $y>x$, then for $0\leqq j\leqq k$ one has $f^jy>f^jx$, $f'(f^jy)\geqq f'(f^jx)$ and $(f^k)'(y)=\prod_{j=0}^{k-1}f'(f^jy)\geqq(f^k)'(x)$. If $y<x$, then $f(y)>x$ since $m_U(y)=m_U(x)$ and

$$|(f^k)'(y)|\geqq\prod_{j=0}^{k-1}f'(f^{j-1}x)=\frac{f'(f^{-1}x)(f^k)'(x)}{f'(f^kx)}.$$

Let

$$D(U)=\frac{\sup_{z\in U}f'(z)}{\inf_{z\in U}f'(z)}.$$ □

[3] One checks easily that condition (d'') for Markov maps also holds

Lemma 7. *Let x_0 be a regular source with standard neighborhood U. Then*

$$\sup\left\{\frac{|(f^n)''(x)|}{|(f^n)'(x)|^2} : x\in U, n=m_U(x)\right\} < \infty .$$

Proof. Consider $y\in U$ with $m_U(y)=m_U(x)=n$. If $g(x)=\log f'(x)$ as in Lemma 2 and $d=\sup_{\xi\in U}|g'(\xi)|<\infty$, then

$$|\log(f^n)'(x)-\log(f^n)'(y)|\leqq d\sum_{k=0}^{n-1}|f^k y-f^k x| .$$

Let $U=[x_0,\beta]$. Then $x,y\in(f^{-n}\beta, f^{-n+1}\beta]$ and

$$\begin{aligned}|f^k y-f^k x| &= \int_x^y |(f^k)'(t)|dt \leqq |y-x| \sup_{t\in[x,y]} |(f^k)'(t)| \\ &\leqq |y-x|D(U)\frac{1}{(f^{-n+1}\beta-f^{-n}\beta)}\int_{f^{-n}\beta}^{f^{-n+1}\beta}|(f^k)'(s)|ds \\ &\leqq \frac{D(U)|y-x|}{f^{-n+1}\beta-f^{-n}\beta}(f^{k-n+1}\beta-f^{k-n}\beta) .\end{aligned}$$

Therefore

$$|\log(f^n)'(x)-\log(f^n)'(y)|\leqq\frac{dD(U)|y-x|(\beta-f^{-n}\beta)}{f^{-n+1}\beta-f^{-n}\beta} .$$

Now $\beta-f^{-1}\beta=(f^{-n+1}\beta-f^{-n}\beta)(f^{(n-1)})'(w)$ for some w with $n_U(w)=n$. Now, for some $\tilde{w}$ with $m_U(\tilde{w})=n$ one has

$$\begin{aligned}|(f^n)'(x)-(f^n)'(y)| &= |e^{\log(f^n)'(x)}-e^{\log(f^n)'(y)}| \\ &\leqq |(f^n)'(\tilde{w})|\,|\log(f^n)'(x)-\log(f^n)'(y)| .\end{aligned}$$

Putting together inequalities

$$\left|\frac{(f^n)'(x)-(f^n)'(y)}{x-y}\right|\leqq dD(U)|(f^n)'(\tilde{w})(f^{n-1})'(w)|\left(\frac{\beta_1-f^{-n}\beta}{\beta-f^{-1}\beta}\right).$$

Now $(f^{n-1})'(w)\leqq(f^n)'(w)$ since $f'\geqq 1$ on U; letting $y\to x$ we get (sung Lemma 6) $|(f^n)''(x)|\leqq K(U)dD(U)^3|(f^n)'(x)|^2$, where $K(U)$ is a constant depending only on β and U.

Lemma 8. *$h=f_K: K\to K$ satisfies the hypotheses of Adler's theorem.*

Proof. The proof of Lemma 3 shows that $\inf|h'(x)|>0$. Since h is C^2 on any $\bar{J}(J\in\mathscr{J})$, one has $\beta(J)=\sup_{y,z\in J}\left|\frac{h''(z)}{h'(y)^2}\right|<\infty$ for any $J\in\mathscr{J}$. By Lemmas 3 and 5 we only need to show $\sup_{J\in\mathscr{J}}\beta(J)<\infty$. If f is not flat at x_0 [i.e., $f^{(r)}(x_0)\neq 0$ some r with f locally C^{r+1} at x_0], then Taylor's formula shows that for any constant $C_2>0$ there is a constant C_1 so that

$$|fx-fx_0|\leqq C_2|fy-fx_0|\Rightarrow|x-x_0|\leqq C_1|y-x_0|$$

for x, y near x_0. If $x, y\in J_{u,v,j}$, then this remark plus Lemma 6 gives a finite sequence of constants $C_1, C_2, \ldots$ so that

$$|f^k(fx)-f^kp|\leqq C_k|f^k(fy)-f^kp| \quad \text{for} \quad 0\leqq k\leqq H(p)\,. \tag{$*$}$$

In Lemma 1 we saw that $\left|\frac{f'(x)(x-x_0)}{f(x)-f(x_0)}\right|$ is bounded away from 0 for x near a non-flat point x_0 by using Taylor's formula. The same type of argument shows this quantity is bounded away from ∞. This means that $|f'(x)|$ differs from $\left|\frac{f(x)-f(x_0)}{x-x_0}\right|$ by a multiplicative factor bounded away from 0 and ∞. Lemma 6 and the inequalities $(*)$ above now show that

$$\left|\frac{(f^k)'(x)}{(f^k)'(y)}\right|\leqq E \quad \text{for all} \quad 1\leqq k\leqq n_K(x), \qquad x, y\in J_{u,v,j}\,,$$

where E is a constant independent of x, y, j, u, v. It is therefore enough to bound $\tilde\beta(J)=\sup_{y\in J}\left|\frac{h''(y)}{h'(y)^2}\right|$.

If $H=F\circ G$, then

$$\frac{H''(x)(Hx-Hx_0)}{H'(x)^2}=\frac{F''(Gx)(F(Gx)-F(Gx_0))}{F'(Gx)^2}+\frac{F(Gx)-F(Gx_0)}{(Gx-Gx_0)F(Gx)}\;\frac{G''(x)(Gx-Gx_0)}{G'(x)^2}\,.$$

Provided one stays near non-flat points, Lemma 1 gives a bound on

$$\frac{F(Gx)-f(Gx_0)}{(Gx-Gx_0)F'(Gx)}$$

and thus a bound for the expression on the left for $H=F\circ G$ in terms of those for F and G. Lemma 1 says this type of expression is bounded near a non-flat point and Lemma 7 says it is for $f^{m_u(x)}$ near a regular source. These combine to give a universal bound on

$$\left|\frac{h''(x)(h(x)-f^{j+H(p)+1}(p)}{h'(x)^2}\right|$$

for $x\in J_{u,v,j}$, $J_v=W_p$. As $|h(x)-f^{j+H(p)+1}(p)|\geqq \inf_{p\in S}|z(p)-p|>0$, we get that $\tilde\beta(J)$ has a uniform bound over all $J_{u,v,j}$'s. This is enough as there are only finitely many other J's in $\mathscr{J}$. □

Adler's theorem gives a measure $d\tilde\mu=p(x)dx$ on K, invariant and ergodic under $h=f_K$, with $c_1\leqq p(x)\leqq c_2$ for some positive constants c_1, c_2. For $E\subset[a,b]\backslash K$ define

$$\tilde E=\{x\in K: f^n x\in E \quad \text{for some} \quad 0<n<n_K(x)\}\,.$$

Call E *singly visited* if

$$x \in E, \; f^m x \in E, \; m > 0 \Rightarrow f^k x \in K \quad \text{some} \quad 0 < k < m \, .$$

The sets $L_{i,j}$ and W_p with $H(p) > 0$ are all singly visited. There is a unique measure μ on $[a,b]$ so that

$$\mu | K = \tilde{\mu}, \qquad \mu(E) = \tilde{\mu}(\tilde{E})$$

for singly visited sets

$$E \subset [a,b] \backslash K, \quad \text{and} \quad \mu\left([a,b] \Big\backslash \bigcup_{n=0}^{\infty} f^n K\right) = 0 \, .$$

This measure is seen to be σ-finite, equivalent to Lebesgue, invariant and ergodic for f, and finite on every singly visited set.

Finally, note that μ is finite iff $\sum_j \mu(L_{i,j}) < \infty$ for each i. Now $\tilde{L}_{i,j}$ is the union over certain u, v of the interval $J^*_{u,v,j} = \cup \{J_{u,v,k} : k \geqq j, \, k \equiv j \, (\text{mod} \, r_i)\}$. If $J_v = W_p$, then $J^*_{u,v,j}$ is mapped by $f^{H(p)+1}$ onto $[f^{-j} y_i, f^\alpha p_i)$ where $0 \leqq \alpha < r_i$ satisfies $j + \alpha \equiv 0 \, (\text{mod} \, r_i)$. Since $f^{H(p)+1}$ is not flat at the endpoint q of $J_{u,v}$ with $fq = p$, there are positive constants d_1, d_2 and an integer n so that

$$\frac{|f^{H(p)+1} x - f^\alpha p_i|}{|x - q|^n} \in [d_1, d_2]$$

for x near q. Hence $J^*_{u,v,j}$ has length in the interval

$$|f^{-j} y_i - f^\alpha p_i|^{1/n} [d_2^{-1/n}, d_1^{-1/n}]$$

and $\tilde{\mu}(J^*_{u,v,j})$ differs from $|f^{-j} y_i - f^\alpha p_i|^{1/n}$ by a factor in $[d_2^{-1/n} c_1, d_1^{-1/n} c_2]$. Hence $\mu(L_{i,j}) = \tilde{\mu}(\tilde{L}_{i,j})$ is a linear combination $\sum_{u,v} c_{u,v} |f^{-j} y_i - f^\alpha p_i|^{1/n_{u,v}}$ where the nonzero $c_{u,v}$ are bounded away from 0 and $+\infty$. That $\sum_j \mu(L_{i,j}) < \infty$ iff the periodic source p_i is expanding follows now from

Lemma 9. *Let* x_0 *be a source for* f, y *near* x_0 *and* $n \geqq 1$. *Then*

$$\sum_{j=0}^{\infty} |f^{-j} y - x_0|^{1/n} < \infty \quad \textit{iff} \quad |f'(x_0)| > 1 \, .$$

Proof. We are assuming f is C^2 near x_0. If $|f'(x_0)| > 1$, then $|f'(x)| \geqq \lambda > 1$ for x near x_0 and $|f^{-k} y_0 - x_0| \leqq \lambda^{-k}$. The result holds since $\sum_{j=0}^{\infty} (\lambda^{-1/n})^j$ converges.

Suppose $|f'(x_0)| = 1$. We may assume $n = 1$, $f'(x_0) = 1$ and $x_0 = 0$. Then

$$f^{-1}(x) = x + \varepsilon(x) x^2 \quad \text{where} \quad \lim_{x \to 0} \varepsilon(x) < \infty \, .$$

Let $y_j = f^{-j} y$. Then

$$y_{j+1} = y_j + \varepsilon(y_j) y_j^2$$

and so $y_k = y_0 \prod_{j=0}^{k-1} (1+\varepsilon(y_j)y_j)$. Since $y_k \to 0$ as $k \to \infty$, we must have $\sum_{j=0}^{\infty} |\varepsilon(y_j)y_j| = \infty$ and so $\sum_{j=0}^{\infty} |y_j| = \infty$. □

4. Final Remarks

Adler showed that maps satisfying his conditions are much more than ergodic, namely their natural extensions are Bernoulli [1]. The maps in our theorem are therefore loose Bernoulli [6]. Ratner [10] and this author [4] have shown that expanding maps of the interval are Bernoulli when they are ergodic, under some mild hypotheses.

Problem 1. Suppose f on $[a,b]$ is Markov with a finite number of intervals and has the following property: for every nonempty subinterval J, $[a,b]-f^nJ$ is finite for some $n=n(J)$. Does f admit an invariant measure equivalent to Lebesgue?

Problem 2. Suppose f on $[a,b]$ admits an ergodic μ equivalent to Lebesgue. Is μ loose Bernoulli?

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Afterword

Roy L. Adler
IBM T. J. Watson Research Center, Yorktown Heights, NY 10598, USA

This paper was submitted by Rufus Bowen. When he learned that it overlapped some unpublished work of Leopold Flatto and myself, it was Rufus's wish that it be revised as a joint work. But his sudden death intervened, and the prospective co-authors decided that the original work should be published.

First, I must say something about the history, so far as I know it, of the so-called Adler's theorem which considers the question: under what conditions does a mapping on the interval admit a finite invariant ergodic measure (by measure we shall mean one equivalent to Lebesgue). The theorem gives three conditions which persist under iteration and which are sufficient: a Markov condition, an expansive condition, and a second derivative condition. Although it appears in an article of mine in a 1972 conference proceedings (ibid), I would never claim it as my own. I first learned it from Flatto in the fall of 1969. He learned it during the previous summer from Benjamin Weiss. So perhaps it should be called the Weiss theorem, but then Weiss believes that he may have extracted it from the work of Sinai [Construction of Markov partitions. Funct. Anal. Appl. **2**, 245–283 (1968)]. Although Krzyzewski and Szlenk who also give credit to Sinai [On invariant measures for expanding differentiable mappings. Studia Math. **33**, 83–92 (1969)] come close, I have not yet discovered to my satisfaction the theorem explicitly stated before 1972. Since attribution remains difficult and I believe many people have independently discovered some version of it, I shall henceforth refer to it as the folklore theorem.

Examination of the fundamental paper of Renyi [ibid] on the topic reveals almost the same theorem (stated in different notation) except that another condition, called condition C, is used instead of the second derivative one. Renyi's theorem was also discovered independently in 1960 by Richard Scoville who was then a graduate student at Yale. When he learned, with great disappointment, of Renyi's prior claim, he had to change his thesis topic and abandon this nice result. Renyi's condition C states that, for x, y restricted to the same interval in the Markov partition for f^n, the quantity $|f^{n'}(x)/f^{n'}(y)|$ should be uniformly bounded independently of n and the interval chosen. This condition plays an important role in obtaining invariant measures and is the main idea in Renyi's paper. However, it is not readily checkable since it involves an infinite number of iterates of f. Renyi did not address this problem, and perhaps it was no obstacle to him. In contrast the second derivative condition seems to be more satisfactory because it does not involve higher iterates of f. Any such condition must allow for unbounded $f''(x)$ in order to handle maps with infinite Markov partitions such as the continued fraction transformation. The previously stated one can be improved to read $\sup|f''(x)/f'(x)^2| < \infty$. This fact was known to Bowen and used by him in the chain rule equation, $f^{n'}(x)=\prod_{j=0}^{n-1} f'(f^j(x))$, a calculation shows that $|f^{n''}(x)/f^{n'}(x)^2|$ can be bounded uniformly in n by a bound for $|f''(x)/f'(x)^2|$ times a convergent series of negative powers of a root of the expansive constant. It then follows, for x, y restricted to the same interval of the Markov partition for f^n (where we know $f^{n'}$ does not change sign), that

$$\begin{aligned}|\log|f^{n'}(x)|-\log|f^{n'}(y)||&=\left|\int_x^y (f^{n''}(t)/f^{n'}(t))dt\right|\\&\leqq \sup_x |f^{n''}(x)/f^{n'}(x)^2|\left|\int_x^y f^{n'}(t)dt\right|\\&\leqq \sup_{x,n} |f^{n''}(x)/f^{n'}(x)^2|\cdot|I|\end{aligned}$$

which is another form of the Renyi condition (Notice for $n=1$ we have that the improved second derivative condition implies the "Adler's theorem" of Bowen's paper). Thus we see how close the folklore theorem is to the original one of Renyi.

The present work is concerned with the method of inducing which is extremely useful for dealing with mappings which fail to satisfy all the conditions of the folklore theorem. Often the failure is due to trouble at some particular point. For instance, there might be a fixed point at which the derivative has absolute value one in which case the expansive condition cannot be satisfied; or there might be a point (not a fixed point) where the derivative vanishes or becomes infinite in which case Renyi's condition will not hold. In such cases it may be possible to find an induced transformation on an appropriate subinterval which will satisfy the folklore theorem. There is then a formula relating the finite invariant ergodic measure for the induced transformation to a σ-finite one for the original mapping. Whether the sought after measure is finite or infinite depends on properties of the original transformation like the existence of fixed points where the derivative has absolute value one.

The method of inducing was applied in the present work to the class of maps $f(x)=1-2|x|^k$ on $[-1,1]$ for $k\geqq 2$, and only partial success was achieved. However, there is another method, "change of variables", which is more effective in this case. Here the map $g=h\circ f\circ h^{-1}$, where

$$h(x)=\int_{-1}^{x}(1-t^2)^{(1-k)/k}dt\Big/\int_{-1}^{1}(1-t^2)^{(1-k)/k}dt\,,$$

can be shown to satisfy the folklore theorem for all $k\geqq 2$. If μ is the invariant measure for g then $\mu\circ h$ is the one for f.

This change of variables was devised particularly for the case $k=4$. It was suggested by the fact that for $k=2$ the map $h\circ f\circ h^{-1}$ is piecewise linear (furthermore the same transformation h "straightens" out all the Chebyshev polynomials) and has Lebesgue measure itself as the finite invariant ergodic one. The original purpose was to use the result for $k=4$ in connection with the map $f_R: x\to Rx(1-x)$ on $[0,1]$ for the specific value of R near 3.67 where $f_R^3(\frac{1}{2})$ is a fixed point for f_R. For the second iterate of this map there is an invariant subinterval (in fact two disjoint ones) upon which $f_R \circ f_R$ is a quartic. This quartic transforms to $x \to 1-2x^4$ on $[-1,1]$. Thus from the above considerations there exists a finite ergodic invariant measure μ for $f_R \circ f_R$ supported on the invariant subinterval. Therefore $(\mu + \mu \circ f_R^{-1})/2$ is the finite invariant ergodic measure for f_R which is supported on the nonwandering set. This is not the only way of doing this. For instance, there are other changes of variables which transform f_R itself to satisfy the folklore theorem, and Ruelle [ibid] has given one of them.

We have also done similar things for other values of R with about the same degree of success as in the present work. Serious difficulties developed for us in trying to use the change of variable method for a countable number of values of R. So inducing may be a better approach to the problem after all.

Additional Comments

Caroline Series

Warwick University, Coventry, England

1. Adler's Theorem

It was pointed out to us by Dennis Sullivan that the actual statement of Adler's theorem in [2] applies only to maps which satisfy the Markov conditions a), b), c) and

d′) $f(I_k) = I$ for all k.

It is not at all clear how to modify Adler's result so that d) is sufficient; indeed it seems doubtful that it is true. Exactly what Rufus had in mind, I do not know. For the purposes of this paper it is enough to replace d) by

d″) $\bigcup_{k=1}^{\infty} f_k(\partial \bar{I}_k)$ is finite,

where f_k is the extension of f to $\bar{I}_k$ and $\partial \bar{I}_k$ is the boundary of I_k.

It is possible to modify the proof of Adler's theorem in [2] to cover this situation. The details are worked out in [19].

2. Examples

The idea mentioned in Example A, that to any Fuchsian group Γ of the first kind acting on $\mathbb{R}$ is associated a Markov map of $\mathbb{R}$ is worked out in [19]. These maps f_Γ have the property that $x = gy$, $g \in \Gamma \Leftrightarrow f_\Gamma^n(x) = f_\Gamma^m(y)$ for some $n, m \geqq 0$. If Γ contains parabolic elements the Markov partition for f_Γ is nesessarily countable, and the results of this paper apply. Alternatively, in [19] we show directly by a simple computation that one can induce away from fixed points with derivative one and get maps satisfying the modified version of Adler's theorem above.