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INVARIANT METRICS ON THE TANGENT BUNDLE OF A HOMOGENEOUS SPACE

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Brockett and Sussmann [1] proved that the tangent bundles of homogeneous spaces are homogeneous. More precisely, if H is a closed subgroup of a Lie group G, then the tangent bundle TM of the homogeneous space $M = G/H$ is the homogeneous space $\tilde{G}/\tilde{H}$, where $\tilde{G} = G \times \mathfrak{g}$, $\tilde{H} = H \times \mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, and the group structure on $\tilde{G}$ is defined by

$$(a, X)\cdot(a', X') = \big(aa', X+\mathrm{ad}(a)X'\big).$$

In the present paper, invariant metrics on the space $\tilde{G}/\tilde{H}$ are studied.

Throughout the paper the Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ of $\tilde{G}$ and $\tilde{H}$ are identified with the products $\mathfrak{g}\times\mathfrak{g}$ and $\mathfrak{h}\times\mathfrak{h}$, respectively. The group G (respectively, $\tilde{G}$) is considered as the group of diffeomorphisms of M or TM (respectively, TM or TTM). The reader can distinguish, without difficulties, different meanings of the symbols a, $\tilde{a}$ etc.

THEOREM 1. *If* $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$, *then any two G-invariant indefinite Riemannian metrics on M induce a $\tilde{G}$-invariant metric on TM. If* $[\mathfrak{h}, \mathfrak{g}] \not\subset \mathfrak{h}$, *then TM admits no $\tilde{G}$-invariant positive-definite metric.*

Proof. In view of the natural correspondence between G-invariant indefinite Riemannian metrics on M and $\mathrm{ad}(H)$-invariant non-degenerate symmetric bilinear forms on $\mathfrak{g}/\mathfrak{h}$ (see [3], p. 200) we have to prove that

(a) *if* $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ *and B, B' are such forms on* $\mathfrak{g}/\mathfrak{h}$, *then the form B on* $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}} = \mathfrak{g}/\mathfrak{h}\times\mathfrak{g}/\mathfrak{h}$ *defined by*

$$B\big((\bar{X}, \bar{X}'), (\bar{Y}, \bar{Y}')\big) = B(\bar{X}, \bar{Y}) + B'(\bar{X}', \bar{Y}'),$$

where $\bar{X}$, $\bar{X}'$ etc. are elements of $\mathfrak{g}/\mathfrak{h}$ *represented by $X, X' \in \mathfrak{g}$, is* $\mathrm{ad}(\tilde{H})$-*invariant*;

(b) *if* $[\mathfrak{h}, \mathfrak{g}] \not\subset \mathfrak{h}$, *then there is no* $\mathrm{ad}(\tilde{H})$-*invariant positive-definite symmetric bilinear form on* $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}$.

Let us take an arbitrary element $\tilde{a} = (a, X)$ of $\tilde{H}$ and two left-invariant vector fields Y and Z on G. Denote by (a_t) and (b_t) the 1-parameter subgroups of G generated by Y and Z, respectively. Then

$$\begin{aligned}\operatorname{ad}(\tilde{a})(Y, Z) &= \frac{d}{dt}\,\tilde{a}(a_t, \dot{b}_t)\tilde{a}^{-1}|_{t=0} = \frac{d}{dt}\,(a, X)(a_t, \dot{b}_t)\big(a^{-1}, -\operatorname{ad}(a^{-1})X\big)\big|_{t=0}\\ &= \frac{d}{dt}\big(\operatorname{ad}(a)\dot{a}_t, X + \operatorname{ad}(a)\dot{b}_t - \operatorname{ad}\big(\operatorname{ad}(\dot{a})a_t\big)X\big)\big|_{t=0}\\ &= \big(\operatorname{ad}(a)Y, \operatorname{ad}(a)Z - [X, \operatorname{ad}(a)Y]\big).\end{aligned}$$

From this formula it follows that if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$, then the equality

$$\operatorname{ad}(\tilde{a})(\overline{Y}, \overline{Z}) = \big(\overline{\operatorname{ad}(a)Y}, \overline{\operatorname{ad}(a)Z}\big)$$

holds for any $\tilde{a} = (a, X)$ of $\tilde{H}$ and Y, Z of $\mathfrak{g}$, where, as previously, $\overline{Y}, \overline{Z}$ etc. are elements of $\mathfrak{g}/\mathfrak{h}$ represented by Y, Z etc. Thus (a) follows immediately from that equality.

Now let us assume that $\tilde{B}$ is an $\operatorname{ad}(\tilde{H})$-invariant positive-definite symmetric bilinear form on $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h}$. That form defines an $\operatorname{ad}(\tilde{H})$-invariant symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ by

$$B\big((Y, Y'), (Z, Z')\big) = \tilde{B}\big((\overline{Y}, \overline{Y}'), (\overline{Z}, \overline{Z}')\big).$$

Since

$$\begin{aligned}&B\big(\operatorname{ad}(\tilde{a})(Y, Y'), \operatorname{ad}(\tilde{a})(Z, Z')\big)\\ &\quad = B\Big(\big(\operatorname{ad}(a)Y, \operatorname{ad}(a)Y'\big), \big(\operatorname{ad}(a)Z, \operatorname{ad}(a)Z'\big)\Big) -\\ &\qquad - B\Big(\big(0, [X, \operatorname{ad}(a)Y]\big), \big(\operatorname{ad}(a)Z, \operatorname{ad}(a)Z'\big)\Big) -\\ &\qquad - B\Big(\big(\operatorname{ad}(a)Y, \operatorname{ad}(a)Y'\big), \big(0, [X, \operatorname{ad}(a)Z]\big)\Big) +\\ &\qquad + B\Big(\big(0, [X, \operatorname{ad}(a)Y]\big), \big(0, [X, \operatorname{ad}(a)Z]\big)\Big) = B\big((Y, Y'), (Z, Z')\big),\end{aligned}$$

where $\tilde{a} = (a, X) \in \tilde{H}$, we see (putting $X = 0$) that

$$B\Big(\big(\operatorname{ad}(a)Y, \operatorname{ad}(a)Y'\big), \big(\operatorname{ad}(a)Z, \operatorname{ad}(a)Z'\big)\Big) = B\big((Y, Y'), (Z, Z')\big)$$

and, consequently, that

$$\begin{aligned}B\big((0, [X, Y]), (0, [X, Z])\big) - B\big((0, [X, Y]), (Z, Z')\big) -&\\ - B\big((Y, Y'), (0, [X, Z])\big) &= 0\end{aligned}$$

for any X of $\mathfrak{h}$, and Y, Y', Z, Z' of $\mathfrak{g}$. Taking $Z = 0$ and $Z' = [X, Y]$ we obtain

$$B\big((0, [X, Y]), (0, [X, Y])\big) = 0.$$

This shows that $[X, Y] \in \mathfrak{h}$ for any $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$, which completes the proof of Theorem 1.

Recall that a positive-definite Riemannian metric g on an arbitrary manifold N determines a Riemannian metric $\tilde{g}$ on the tangent bundle TN defined by the formula

$$\tilde{g}(v, w) = g\big(d\pi(v), d\pi(w)\big) + g\big(K(v), K(w)\big),$$

where v and w are vectors tangent to TN, π is the natural projection $TN \to N$, and $K\colon TTN \to TN$ is the connection mapping corresponding to the Levi-Civita connection ∇ on the Riemannian manifold (N, g) (see [2]). The mapping K is completely determined by the equality

$$K\big(dY(v)\big) = \nabla_v Y$$

for any vector field Y on N and any vector v of TN. The metric $\tilde{g}$ was defined and investigated by Sasaki [5].

THEOREM 2. *If g is a G-invariant positive-definite Riemannian metric on a connected homogeneous space $M = G/H$ and the metric $\tilde{g}$ is $\tilde{G}$-invariant, then the Levi-Civita connection ∇ on (M, g) is flat.*

Proof. Considering, if necessary, the universal covering of M we can assume that M is simply connected.

Let us take an arbitrary element $\hat{a} = (a, X)$ of $\tilde{G}$ and denote by X' the vector field on M defined by

$$X'_x = d\sigma_x(X),$$

where σ_x is the mapping $G \ni b \mapsto bx$. We have the equalities

$$(*) \qquad K\circ\tilde{a} = a\circ K + \nabla X'\circ a\circ d\pi$$

and

$$(**) \qquad d\pi\circ\tilde{a} = a\circ d\pi.$$

In order to prove $(*)$ let us take a vector field Y on M and a vector u of TM, and put $v = dY(u)$. Then

$$a\circ K(v) = a\nabla_u Y = \nabla_{au} aY$$

and

$$K\circ\tilde{a}(v) = K\big(d(\tilde{a}\circ Y)(u)\big) = K\big(d(aY)(au) + dX'(au)\big) = \nabla_{au} aY + \nabla_{au} X'.$$

This yields $(*)$. The proof of $(**)$ is similar.

It follows from $(**)$ that

$$g\big(d\pi\circ\tilde{a}(v), d\pi\circ\tilde{a}(w)\big) = g\big(d\pi(v), d\pi(w)\big) \quad \text{for any } v, w, \tilde{a}.$$

Therefore, the metric $\tilde{g}$ is $\tilde{G}$-invariant if and only if

$$g\big(K\circ\tilde{a}(v), K\circ\tilde{a}(w)\big) = g\big(K(v), K(w)\big) \quad \text{for any } v, w, \tilde{a}.$$

Using (*) we obtain

$$g\big(K\circ\tilde{a}(v),\, K\circ\tilde{a}(w)\big)-g\big(K(v),\, K(w)\big)$$
$$= g\big(aK(v),\, \nabla_{ad\pi(w)}X'\big)+g\big(aK(w),\, \nabla_{ad\pi(v)}X'\big)+g\big(\nabla_{ad\pi(v)}X',\, \nabla_{ad\pi(w)}X'\big).$$

This shows that $\tilde{g}$ is a $\tilde{G}$-invariant metric if and only if $\nabla X' = 0$ for any X of $\mathfrak{g}$.

Vector fields X', $X \in \mathfrak{g}$, generate the module of vector fields on an open neighbourhood of the origin of M. Thus, if the metric $\tilde{g}$ is invariant, then the connection ∇ is flat.

The following fact is an immediate consequence of Theorem 2 and the result of Kowalski [4].

COROLLARY. *If the metric $\tilde{g}$ on TM is $\tilde{G}$-invariant, then the Riemannian manifold $(TM, \tilde{g})$ is flat.*

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