INVARIANT POINTS IN FUNCTION SPACE*

BY

G. D. BIRKHOFF AND O. D. KELLOGG

Existence theorems in analysis deal with functional transformations. This suggests that such existence theorems may be obtained from known theorems on point transformations in space of two or of three dimensions by generalization, first to space of n dimensions, and then to function space by a limiting process. This direction of attack has been followed out and has resulted in the theorems given below. For instance it is found that theorems on invariant points for the sphere or for its surface yield respectively by generalization existence theorems in analysis of non-homogeneous and of homogeneous type.

The treatment is here confined to the case of real functions of a real variable, although extensions to real functions of several real variables are indicated. Only the case of a single unknown function is considered. In many cases, of course, apparently more general problems can be reduced to this case by a process which is familiar in the theory of integral equations, namely the juxtaposition of intervals.

The applications include the classical existence theorems for differential and integral equations, linear and non-linear.

Incidentally, it is proved that an algebraic manifold $f_1 = c_1, f_2 = c_2, \ldots, f_m = c_m$, where $f_1, f_2, \ldots, f_m$ are real polynomials in the real variables $x_1, x_2, \ldots, x_n$, has no singularity for general values of the real constants $c_1, c_2, \ldots, c_m$. The authors have not been able to find any earlier proof of this simple and important theorem.

The literature on the subject of invariant points does not appear to be extensive. For a geometric treatment of one-valued transformations with one-valued inverses, we may refer to L. E. J. Brouwer.† Some existence theorems of un-

* Presented to the Society, Dec. 30, 1920 and Feb. 25, 1922.

† *Ueber eindeutige stetige Transformationen von Flächen in sich*, Mathematische Annalen, vol. 69 (1910), p. 176. See his references there to the Proceedings of the Section of Sciences of the Royal Academy at Amsterdam, and in addition, ibid., vol. 13 (1911), pp. 771, 777.

Since the writing of this paper (see, however, footnote on p. 111), we have learned from Professor J. W. Alexander that he has obtained results on the existence of invariant points, which, had we known of them, would have sufficed as the basis in n-space for a considerable portion of our theory. The importance of the subject, however, and the difference in method appear to us to warrant the inclusion here of our proofs of the needed theorems. Professor Alexander's paper will be found in these Transactions, vol. 23, 1922, pp. 89–95.

usual generality have been developed previously. Thus Mason has given a simple theorem for the linear non-homogeneous problem.* Evans has treated certain extensions of the integral equation of Volterra type.† Mrs. Pell‡ has considered linear transformations of general type.

1. A lemma on algebraic equations

Our proofs of the theorems on the existence of invariant points in n-space will be based on the method of analytic continuation. In this paragraph we shall establish a lemma affirming the validity of this method under certain conditions by means of the following theorem, a corollary of which is that the general algebraic manifold is non-singular.

THEOREM. *Let $f_1, f_2, \ldots, f_m$ be m polynomials in the n variables $x_1, x_2, \ldots, x_n$, where $m < n$. If c_1 is chosen not to have one of a finite number of values, after which c_2 is chosen not to have one of a finite number of values, and so on, until c_m is chosen, then the equations $f_1 = c_1, f_2 = c_2, \ldots, f_m = c_m$ will have no solution for which the rank of the m-rowed matrix*

$$M \equiv \left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_1}, & \dfrac{\partial f_1}{\partial x_2}, & \cdots, & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1}, & \dfrac{\partial f_2}{\partial x_2}, & \cdots, & \dfrac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial f_m}{\partial x_1}, & \dfrac{\partial f_m}{\partial x_2}, & \cdots, & \dfrac{\partial f_m}{\partial x_n} \end{matrix} \right\|$$

is less than m.

This may be proved by induction. Let us first consider the case $m = 1$. It is known that any set of algebraic equations is satisfied at a finite number of isolated points and on a finite number of analytic manifolds of various dimensions.§ On any manifold for which all the partial derivatives of f_1 vanish, it is clear that $df_1 = 0$, and hence $f_1 =$ const. The value of c_1 will be

* *Selected topics in the theory of boundary value problems of differential equations, New Haven Colloquium Lectures,* Yale University Press, 1910, pp. 174 ff.

† Proceedings of the Fifth International Congress of Mathematicians at Cambridge (1912), p. 387. See also *Functionals and their applications, Cambridge Colloquium Lectures,* New York, 1918.

‡ *Biorthogonal systems of functions,* these Transactions, vol. 12 (1911), p. 135, and *Applications of biorthogonal systems of functions to the theory of integral equations,* ibid., p. 165.

§ The facts are due to Weierstrass. Complete proofs of them will be given in the second volume of Osgood's *Funktionentheorie.* They can also be established by means of Kronecker's theory of elimination. See his *Grundzüge einer arithmetischen Theorie der algebraischen Grössen,* Journal für Mathematik, vol. 92 (1889), p. 28 ff.

taken distinct from any of the finite number of values of f_1 on these manifolds or at the isolated points referred to. Thus the surface $f_1 = c_1$ will be non-singular, and the theorem holds for $m=1$.

We now assume the theorem for $m-1$ polynomials. Let the values of $c_1, c_2, \ldots, c_{m-1}$ be so chosen that the equations

$$f_1 = c_1, f_2 = c_2, \ldots, f_{m-1} = c_{m-1} \tag{1}$$

define non-singular manifolds, i. e., such that th ematrix M' obtained from M by omitting the last row is of rank $m-1$ at all points of the manifolds.

Consider the locus defined by the equations (1) together with those obtained by equating to 0 all the determinants of order m of the matrix M. As before, this locus consists in a finite number of isolated points and of analytic manifolds On any such manifold, f_m = const. For, differentiating (1), we have

$$df_1 = \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \cdots + \frac{\partial f_1}{\partial x_n} dx_n = 0,$$

$$df_2 = \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \cdots + \frac{\partial f_2}{\partial x_n} dx_n = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$df_{m-1} = \frac{\partial f_{m-1}}{\partial x_1} dx_1 + \frac{\partial f_{m-1}}{\partial x_2} dx_2 + \cdots + \frac{\partial f_{m-1}}{\partial x_n} dx_n = 0,$$

and since the matrices M and M' are both of rank $m-1$, these equations have as consequence

$$df_m = \frac{\partial f_m}{\partial x_1} dx_1 + \frac{\partial f_m}{\partial x_2} dx_2 + \cdots + \frac{\partial f_m}{\partial x_n} dx_n = 0,$$

or, f_m = const. If, therefore, the constant c_m is distinct from the values of f_m at the isolated points and on those manifolds for which the equations (1) hold and all the determinants of order m of the matrix M vanish, it will follow that the locus $f_1 = c_1, f_2 = c_2, \ldots, f_m = c_m$ is non-singular, as was to be proved.

Of course the precise conditions for singularities consist in certain algebraic relations between $c_1, c_2, \ldots, c_m$. In the non-singular case the manifolds are of dimensions $n-m$. It is not difficult to extend the theorem to the case where the functions f_i are merely restricted to be analytic.

We are now in a position to prove the desired lemma.

LEMMA. *Let $G_1, G_2, \ldots, G_{n-1}$ denote $n-1$ polynomials with real coefficients in the n variables $x_1, x_2, \ldots, x_n$, and let C denote a bounded open continuum of the real space of these variables. If there exists a point A on the boundary of C at which the polynomials G_i all vanish, while one of their functional determinants with respect to $n-1$ of the variables x_i does not vanish, and if there exists in every* *neighborhood of A both points in C and points without C at which the G_i all vanish, then there exists a point B on the boundary of C. distinct from A, at which they all vanish.*

From the theorem just proved, we know that, given any positive ϵ, there exist sets c_i less in absolute value than ϵ such that any curve, K_c, satisfying the $n-1$ equations in $x_1, x_2, \ldots, x_n$

$$(2) \qquad G_1 = c_1, G_2 = c_2, \ldots , G_{n-1} = c_{n-1}$$

is non-singular. But for $c_1 = c_2 = \cdots c_{n-1} = 0$ there is, according to the theory of implicit functions, a curve branch K_0 through A satisfying the corresponding equations (2) and passing into C.

As the constants c_i are varied, the coördinates and direction cosines of K_c vary analytically. It is possible to describe a small sphere S about A, which K_0 will cut twice, and but twice, one of these times in a point interior to C. The number ϵ can be taken so small that for all c_i, less in absolute value than ϵ, the curve K_c will, on the one hand, cut the sphere twice and only twice, one of these times in the interior of C, and, on the other hand, will contain points exterior to C and to S.

Now consider an infinite sequence of these curves, corresponding to sets of values of the c_i, for each of which the curves K_c defined by (2) are non-singular, while $\lim c_i = 0$. Each such curve can be continued analytically until it leaves C at some point B' outside of the sphere S. The set of points B' corresponding to the infinite sequence of curves K_c is bounded, and hence must have at least one limit point B, outside of S. Such a point B is a boundary point of C because the points B' are boundary points of C. Finally, since the functions G_i are continuous, the functions G_i vanish at B. Thus the lemma is established.

2. Invariant points in n-space

We proceed to the proof of the following theorem:

Theorem I. *Let R_n denote a bounded connected region of real n-space containing an interior point O (the origin for a set of rectangular coördinates $x_1, x_2, \ldots x_n$) such that every half-ray originating in O contains but one boundary point of R. Let T denote a one-valued and continuous transformation*

$$(3) \qquad x'_1 = f_1(x_1, x_2, \ldots, x_n),\ x'_2 = f_2(x_1, x_2, \ldots, x_n), \ldots, \\ x'_n = f_n(x_1, x_2, \ldots, x_n), \text{ or briefly, } x' = f(x),^*$$

which transforms each point of R_n into a point of R_n.

* A symbol without index will, in the following, be understood as standing for the totality of the corresponding symbols with indices.

Then there exists a point, a, of R_n, *which is invariant under* T, *i. e., such that* $a = f(a)$.*

We will prove the theorem first under the assumption that the functions $f(x)$ are polynomials, and will then pass to the general case by means of Weierstrass's theorem on the approximation to continuous functions by polynomials.

Consider the transformation T_λ

$$x' = \lambda f(x) \tag{4}$$

in which the parameter λ lies in the closed interval (0, 1). For $\lambda = 0$, this transformation has the single invariant point $x = 0$. Our object is to show that it has an invariant point for $\lambda = 1$. We are thus led to study the set of n algebraic equations

$$x' - \lambda f(x) = 0 \tag{5}$$

in the cylindrical region $R_{n\lambda}$ of $(n + 1)$-space: x in R_n and λ in (0, 1).

For $x = 0$, $\lambda = 0$, the functional determinant of the functions $x - \lambda f(x)$, with respect to the n variables x, is 1. Hence, in the neighborhood of this point, the values x are determined by (5) as real analytic functions of λ, and, for small negative and positive values of λ, these equations have solutions without and within $R_{n\lambda}$, respectively. The hypotheses of our lemma are thus satisfied, and there exists a second point, B, on the boundary of $R_{\lambda n}$, whose coördinates satisfy (5). For this point λ is not 0, since $x = \lambda = 0$ is the only point on $\lambda = 0$ whose coördinates satisfy the equations (5). Furthermore B does not lie on the boundary of $R_{n\lambda}$ for $0 < \lambda < 1$, since T_λ carries all such points into the *interior* of R_n. Hence B must lie on the boundary $\lambda = 1$; and, if $(a, 1)$ denote the coördinates of B, we have $a = f(a)$, as was to be proved.

Suppose the functions $f(x)$ are not polynomials. We shall find it convenient to develop first a property of the region R_n. Let α stand for the direction cosines of a half-ray originating in O, and let $r(\alpha)$ denote the distance from O to the single boundary point of R_n on this half-ray. Then $r(\alpha)$ is a continuous function of the variables α. For, if discontinuous, say at a point α_0, there would exist a positive number η, and an infinite sequence $[\alpha_i]$ of directions with α_0 as limit, such that $|r(\alpha_i) - r(\alpha_0)| \geqq \eta$. Since R_n is bounded, the boundary points corresponding to $[\alpha_i]$ would have at least one limit point on the half-ray α_0 with distance from O different from $r(\alpha_0)$. Such a limit point would be a boundary point of R_n, a conclusion in contradiction with the hypothesis that each half-

* The theorem is stated and proved in a degree of generality sufficient for our later purposes. It will be seen that the proof holds if merely the boundary points of R_n are transformed into points of R_n, and the theorem admits the obvious extension to any region susceptible of continuous one to one mapping on a region R_n of the kind described. See also Alexander, loc. cit.

ray from O contains but one boundary point of R_n. Hence $r(\alpha)$ is continuous, and, because of the closure of the domain of the variables α, uniformly continuous. There exists, therefore, a function, $\varphi(\Delta\theta)$, approaching 0 with $\Delta\theta$, but positive for $\Delta\theta > 0$, such that $|r(\alpha')-r(\alpha)| \leqq \varphi(\Delta\theta)$ for any two directions α and α' making an angle $\Delta\theta$ or less with each other. This function may be so chosen as to exceed in magnitude the corresponding chord also. This is the needed property of R_n.

Now for any small positive ϵ there exist n real polynomials, $p(x)$, such that

$$|p_i(x) - f_i(x)| \leqq \epsilon \tag{6}$$

for $i = 1, 2, \ldots, n$ and for all x in R_n. The transformation $\Pi: x' = p(x)$ would be of the type for which we have proved the existence of an invariant point, if it transformed points of R_n into points of R_n. In this event the theorem would follow immediately inasmuch as a limit point of the invariant points under Π as ϵ approaches 0 is clearly an invariant point under T.

In case Π does not transform all points of R_n into points of R_n, we shall modify Π slightly to a suitable new polynomial transformation Π_k: $x' = kp(x)$ where k is slightly less than 1. Let A be any point of R_n, and B the point into which the transformation $x' = p(x)$ carries A. We are only interested in the possible case that B is not in R_n, since otherwise the point B_k given by $x' = kp(x)$ lies in R_n also. On this assumption, let C be the nearest point of R_n to B. Then $BC \leqq \sqrt{n}\epsilon$, because T carries A into a point of R_n and because of the inequalities (6). If m denote the positive minimum of $r(\alpha)$, the angle subtended by BC at 0 is not greater than $2\sin^{-1}(\sqrt{n}\epsilon/2m)$. Then, if D is the boundary point of R_n on the half-ray OB, $CD \leqq \phi(2\sin^{-1}\sqrt{n}\epsilon/2m$. Hence

$$BD \leqq BC + CD \leqq \sqrt{n}\epsilon + \phi(2\sin^{-1}(\sqrt{n}\epsilon/2m)),$$

and thus there is a small upper limit for the distance from B to the nearest point D of R_n on the half-ray OB.

If now k is taken not greater than the least value of OD/OB, all points B will be brought within R_n by Π_k. If the upper limit for BD be denoted by δ, we have

$$OD/OB \leqq OD/(OD + \delta) = r(\alpha)/(r(\alpha) + \delta)$$

which is least when $r(\alpha)$ has its least value, m. Hence if $k = m/(m + \delta)$, the transformation Π_k will carry each point of R_n into a point of R_n.

The degree of approximation of Π_k to T follows from the identity

$$f_i(x) - kp_i(x) \equiv k[f_i(x) - p_i(x)] + (1 - k)f_i(x)$$

whence, if M is the maximum of $r(\alpha)$,

(7) $$|f_i(x) - kp_i(x)| \leqq \epsilon + (1 - k)M.$$

Since k approaches 1 as ϵ approaches 0, the approximation may be made arbitrarily close.

Thus we infer as before the existence of an invariant point under T as a limit of the invariant points of Π_k for $\lim \epsilon = 0$.

3. The transition to function space; first method

The transition from n-space to function space may be made in a variety of ways. First we give one based on interpolation; in §5 we shall give one based on the mediation of a Hilbert space.

It will be convenient to define a few terms in advance. We shall be concerned with real functions, $f(s)$, defined on some interval, say (0, 1). A set of such functions will be *bounded*, B, provided a constant B exists such that $|f| \leqq B$ for every function of the set, and for every s in (0, 1). The set will be said to be *equicontinuous*, $\eta(\epsilon)$, provided there exists a function, $\eta(\epsilon)$, defined and bounded for $0 \leqq \epsilon \leqq 1$, and approaching 0 with ϵ, such that $|f(s+h) - f(s)| \leqq \eta(\epsilon)$, whenever $|h| \leqq \epsilon$, and s and $s + h$ are in (0, 1), for every function of the set.* A single function of such a set will be said to be *continuous*, $\eta(\epsilon)$. An infinite sequence of bounded equicontinuous functions has the fundamental property that a subsequence can be so chosen as to approach a function of the set uniformly.

A function $\eta(\epsilon)$ will be said to be *convex*, provided that, for every a, b, and θ in (0, 1),

$$\eta(a + \theta(b - a)) \geqq \eta(a) + \theta(\eta(b) - \eta(a)).$$

If a set of functions is equicontinuous, $\xi(\epsilon)$, there exists a convex function, $\eta(\epsilon) \geqq \xi(\epsilon)$, approaching 0 with ϵ. Of course the same set of functions is equicontinuous, $\eta(\epsilon)$.

A primary property of functions continuous, $\eta(\epsilon)$, $\eta(\epsilon)$ convex, is that any function $g(s)$, which meets the continuity requirement for $s + h$, s on the set $s_1 = 0, s_2, \ldots, s_n = 1$ and which is *linear* between these values of s, will be continuous, $\eta(\epsilon)$, throughout.

This is self-evident when $\eta(\epsilon) = c\epsilon$, when the continuity requirement is merely that every chord of the given curve $y = f(s)$ has a slope which does not exceed c; for this is true of the polygonal curve $y = g(s)$.

Moreover, in any case, we have by hypothesis,

$$|g(s_j) - g(s_i)| \leqq \eta(s_j - s_i)$$

* Cf. Ascoli, *Le curve limite di una varietà data di curve*, Atti della Reale Accademia dei Lincei, Memorie, ser. 3, vol. 18 (1882–83), pp. 545–549.

for $s_j > s_i$, say. Since the curve $y = \eta(s - s_i)$ is convex in the positive y direction for $s > s_i$, it follows that for any $s > s_i$

$$|g(s) - g(s_i)| \leqq \eta(s - s_i).$$

If now $s_i \leqq s' \leqq s_{i+1} \leqq s$, and we write

$$\lambda = \frac{s' - s_i}{s_{i+1} - s_i}, \; \mu = \frac{s_{i+1} - s'}{s_{i+1} - s_i},$$

so that $\lambda + \mu = 1$, the polygonal character of $g(s)$ yields

$$|g(s) - g(s')| = |\lambda(g(s) - g(s_{i+1})) + \mu(g(s) - g(s_i))| \leqq \lambda\eta(s - s_{i+1}) + \mu\eta(s - s_i).$$

But the right hand side of this last inequality represents the ordinate of the chord of $y = \eta(x)$ at $s - s'$, with end points at $s - s_{i+1}$ and $s - s_i$, so that

$$|g(s) - g(s')| \leqq \eta(s - s').$$

This same inequality obviously holds for s, s' lying on one and the same interval (s_i, s_{i+1}). Hence $g(s)$ is continuous, $\eta(\epsilon)$.

It may be observed that the proofs of the paper become somewhat simpler for the important case $\eta(\epsilon) = c\epsilon$, referred to above.

A transformation $f' = Sf$ will be said to be *continuous* in a region R_f of function space provided for every $\epsilon > 0$, and for every function $g(s)$ in R_f, there exists a positive number δ, such that $|Sf - Sg| \leqq \epsilon$ for all s in $(0, 1)$ whenever $|f - g| \leqq \delta$, uniformly with respect to s, f being in R_f.

THEOREM II. *Let R_f denote the totality of real functions $f(s)$, defined on the closed interval* $(0, 1)$, *which are bounded,* B, *and equicontinuous,* $\eta(\epsilon)$, $\eta(\epsilon)$ *being convex. Let $f' = Sf$ denote a one-valued, continuous transformation which carries each point of R_f into a point of R_f.*

Then there exists a point of R_f which is invariant under this transformation.

The "distance," $\delta_f = \sqrt{\int_0^1 (f(s) - Sf(s))^2 ds}$, by which a point f is moved by the transformation S, has a lower limit $\delta_0 \geqq 0$ in R_f. This limit is attained at some point, say f_0, of R_f. For if $[f_i]$ is an infinite sequence of points for which δ_f approaches δ_0, there exists a subsequence which approaches uniformly f_0 in R_f so that $\delta_{f_0} = \delta_0$.

If $\delta_0 = 0$, it is clear that f_0 is an invariant point and the theorem follows.

To show that $\delta_0 = 0$ we choose a positive n and define a region R_n of n-space by the inequalities

$$|x_i| \leqq B, \; |x_{i+j} - x_i| \leqq \eta(j/(n-1)) \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n - i).$$

This n-dimensional region satisfies the requirements of Theorem I, being convex toward the interior point O. Next we define the transformation, T, of R_n, as follows. For x in R_n, we construct the continuous polygonal function of s, $\pi(s, x)$, equal to x_i for $s = s_i = (i-1)/(n-1)$, $(i = 1, 2, \ldots, n)$, and linear for intermediate values of S. This function is in R_f, for it is evidently bounded B, and is continuous, $\eta(\epsilon)$, as has been seen (p. 102.)

Let $\pi'(s, x)$ denote the function of R_f into which $\pi(s, x)$ is transformed by S.* Then, if x' be defined by the equations $x'_i = \pi'(s_i, x)$, the transformation T is determined. This transformation is easily seen to have the properties required in Theorem I, and so has an invariant point, a, in R_n.

The function $\pi(s, a)$, formed for this point, is transformed by S into a function $\pi'(s, a)$ which coincides with it at the n points s_i. Between these points, the variation of either function is not greater than $\eta(1/(n-1))$. Hence, throughout the interval $(0, 1)$, $|\pi(s, a) - S\pi(s, a)| \leqq 2\eta(1/(n-1))$. Consequently, $\delta_\pi \leqq 2\eta(1/(n-1))$, a quantity which approaches 0 as n increases indefinitely; and $\delta_0 = 0$, as was to be proved.

4. Extensions

It is, of course, evident that any other finite interval than $(0, 1)$ might have been used. Even infinite intervals may be used if the equicontinuity hypothesis holds for some new variable, such as $1/s$.

Moreover, the reasoning requires no essential modification in order to be applicable to functions of two or more variables. For instance a set of functions $f(s, t)$ defined on the square $0 \leqq s, t \leqq 1$ can be defined as equicontinuous, $\eta(\epsilon)$, if equicontinuous, $\eta(\epsilon)$, in each variable separately. We should compare the function space thus defined with a space of n^2 dimensions. To the point $f(s, t)$ of function space, we can make correspond the point x of n^2-space defined by coordinates $x_{i,j} = f((i-1)/(n-1), (j-1)/(n-1))$, the subscripts of x corresponding to a network of vertices of squares of sides $1/(n-1)$ in the original square $0 \leqq s, t \leqq 1$. Given a point in this n^2-space, the intermediate coordinates of points of function space can be defined as the value of the bilinear function $ast + bs + ct + d$ which has the same values at the vertices of the corresponding square.

Another definition of equicontinuity, $\eta(\epsilon)$, of functions $f(s,t)$ might be employed namely that $|f(s_2, t_2) - f(s_1, t_1)| \leqq \eta(\epsilon)$ for $\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} \leqq \epsilon$. It is readily seen that this type of equicontinuity involves the preceding, and that conversely if the preceding type holds so will this second type, provided that $\eta(\epsilon)$ is replaced by $2\eta(\epsilon)$

* To avoid possible confusion attention is called to the fact that the prime is used to denote the result of the given transformation, S.

Theorem II may, in fact, be regarded as a type of a large body of theorems that can be established. The region R_f may be modified in a variety of ways. Important for the present method of proof are the closure and convexity of R_f, and the continuity of T. One direction of modification of the theorem is, however, of sufficient importance to warrant more detailed attention. We formulate it in the following theorem.

THEOREM III *Let R_f denote the totality of real functions f defined on* $(0, 1)$, *which are bounded, B_0, and whose first derivatives exist and are bounded, B_1, and are equicontinuous $\eta(\epsilon)$, $\eta(\epsilon)$ being convex. Let $f' = Sf$ denote a transformation which is one-valued, carries each point of R_f into a point of R_f, and which is continuous in such wise that, given any function, g, of R_f, and any positive number ϵ, there exists a positive number δ such that $|Sf-Sg| \leqq \epsilon$ for all s, whenever f is any function of R_f for which $|f-g| \leqq \delta$ and $|df/ds - dg/ds| \leqq \delta$ for all s.*

Then there exists an invariant point under S in R_f.

It will be noted that the hypothesis on the continuity of S is weaker than in Theorem II, while the region R_f is more restricted in the present theorem. Theorem III is given with applications to differential equations in view.

The kernel of the proof of the theorem lies, as before, in the choice of a suitable region R_{n+1}, and a transformation, T.

In this case we shall define R_{n+1} by the inequalities

$$|x_i| \leqq B_1, \; |x_{i+j} - x_i| \leqq \eta(j/(n-1)) \; (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n-i),$$
$$|x_0| \leqq B_0, \quad \left| x_0 + \sum_{i=1}^{k} \frac{x_{i+1} + x_i}{2(n-1)} \right| \leqq B_0 \qquad (k = 1, \ldots, n-1).$$

This $(n + 1)$-dimensional region meets the requirements of Theorem I with O as an interior point.

An appropriate determination of T may be made as follows. Given x in R_{n+1}, as defined in the above inequalities, we form the polygonal function $\pi(s, x)$, assuming the values x_i for $s = s_i = (i-1)/(n-1)$. We then write

$$\psi(s, x) = l(x_0 + \int_0^s \pi(s, x)ds) \qquad (0 < l < 1).$$

The absolute value of the ordinate of the curve $y = \psi(s, x)$ for $s = s_k$ is l times the left hand member of the last inequality defining R_{n+1}. Hence it is clear that $\psi(s, x)$, for $s = s_k$, does not exceed lB_0 in numerical value. Since $|d\psi/ds| \leqq lB_1$, we infer that

$$|\psi(s, x)| \leqq l(B_0 + B_1/(n-1)) \leqq B_0$$

for n large, while $d\psi/ds$ is bounded, B_1, and continuous, $\eta(\epsilon)$. Thus $\psi(s, x)$

lies in R_f, and consequently so does $S\psi(s, x)$. The definition of T is then given by the equations

$$x_0' = \psi'(0, x),\ x_i' = \frac{d}{ds}\psi'(s_i, x) \qquad (i = 1, 2, \ldots, n,)$$

where

$$\psi'(s, x) = l\, S\psi(s, x).$$

It is immediately evident that T is a one-valued continuous transformation throughout R_{n+1} because of the corresponding restriction on S. In order to prove that T has an invariant point, we need only show further that the point x' lies in R_{n+1}. Since ψ' is bounded, lB_0, we have

$$\left| \psi'(0, x) + \int_0^s \frac{d\psi'}{ds}\, ds \right| \leqq l\, B_0,$$

whence, for $s = s_k$, we infer

$$\cdot \left| x_0' + \sum_{i=1}^{k} \frac{x_i' + x_{i+1}'}{2(n-1)} \right| \leqq l\left(B_0 + \eta\left(\frac{1}{n-1}\right)\right)$$

inasmuch as $d\psi'/ds$ is continuous, $l\eta(\epsilon)$. Consequently, for any fixed positive l, n may be chosen so large that the last inequality defining R_{n+1} is satisfied by x'. And since $d\psi'/ds$ is bounded, B_1, and continuous, $\eta(\epsilon)$, it is clear at once that the other requirements are satisfied by x'. Hence an invariant point, a, exists for any such l, if n is large enough.

For the function $\psi(s, a)$ of R_f corresponding to this invariant point, it follows from the invariance of a under T that

$$l\frac{d}{ds}S\psi(s, a) = \frac{d}{ds}\psi(s, a), \text{ for } s = s_i \qquad (i = 1, 2, \ldots, n),$$

$$l\, S\psi(0, a) = \psi(0, a).$$

From the first of these equations results the inequality

$$\left| \frac{d}{ds}S\psi(s, a) - \frac{d}{ds}\psi(s, a) \right| \leqq (1 - l)B$$

for the same values of s, so that for any s the left hand member is less than

$$(1 - l)\, B_1 + \eta(1/(n - 1)).$$

From the second equation follows

$$| S\psi(0, a) - \psi(0, a) | \leqq (1 - l)B_0.$$

Hence

$$| S\psi(s, a) - \psi(s, a) | \leqq (1 - l)\,(B_0 + B_1) + \eta\left(\frac{1}{n-1}\right)$$

so that the "distance," δ_ψ, between ψ and $S\psi$ in R_f, can be made arbitrarily small by taking l near enough to 1 and n large enough.

By the argument used in the proof of Theorem II it follows that there exists an invariant function in R_f.

We close this paragraph with a formulation of a generalization of Theorem III which is useful in applications, and which may be proved by a simple extension of the reasoning used to establish that theorem.

THEOREM IV. *Let R_f denote the totality of real functions defined on* (0, 1) *which, with their first $n-1$ derivatives are bounded, $B_0, B_1, B_2, \ldots, B_{n-1}$, respectively, and whose nth derivatives are bounded, B_n, and equicontinuous, $\eta(\epsilon)$, $\eta(\epsilon)$ being convex. Let $f' = Sf$ denote a transformation which is one-valued, carries each point of R_f into a point of R_f, and is continuous in such wise that given any function g of R_f and any positive number ϵ, there exists a positive number δ such that $|Sf-Sg| \leqq \epsilon$ for all s, whenever f is any function of R_f for which $f-g$ and its first $n-1$ derivatives nowhere exceed δ in absolute value.*

Then there exists an invariant point under S in R_f.

5. THE TRANSITION TO FUNCTION SPACE; A SECOND METHOD

A second method of establishing the existence of an invariant point in a function space consists in setting up a correspondence with a certain Hilbert space, i. e., a space of countably infinitely many dimensions such that the sum of the squares of the coördinates of each point converges. The correspondence is based on a closed set of continuous bounded orthogonal functions $[\phi]$, such as the set appearing in Fourier's series, so that no function with summable square is orthogonal to the set unless it vanishes except on a set of zero measure. We shall denote the generalized Fourier coefficients of a function f by a_i, i. e., $a_i = \int_0^1 f\phi_i\, ds$. We may then state

THEOREM V. *Let R_f denote the totality of summable functions with summable squares, $f(s)$,* $(0 \leqq s \leqq 1)$, *such that*

(*a*) *there exists a constant B such that $\int_0^1 f^2 ds \leqq B^2$, and*

(*b*) *there exists a function $\eta(m)$, approaching* 0 *with* $1/m$, *such that $\sum_m^\infty a_i^2 \leqq \eta(m)$. Let $f' = Sf$ denote a transformation which is one-valued, carries each point of R_f into a point of R_f, and is continuous in such wise that given g in R_f and $\epsilon > 0$, there exists a $\delta > 0$ such that $\int_0^1 (Sf - Sg)^2 ds \leqq \epsilon$ for all points f of R_f for which $\int_0^1 (f-g)^2 ds \leqq \delta$.*

Then there exists an invariant point under S in R_f, where it is understood that two points are to be regarded as identical provided their coordinates differ at most on a set of zero measure.

We first establish the closure of the space R_f. Given an infinite set $[f]$ of points of R_f, an infinite sequence $[f_j]$, can be selected from this set which converges in the mean to a function f of R_f. To show this, we consider the first Fourier coefficients of the set $[f_j]$. As these first coefficients are bounded, B, and infinitely numerous, an infinite sequence can be selected from the set $[f_j]$, whose first Fourier coefficients approach a limit, a_1. From this sequence can be selected an infinite sub-sequence whose second Fourier coefficients approach a limit a_2'. And from this subsequence, in turn, another, whose third Fourier coefficients approach a limit a_3', and so on.

These constants $(a_1', a_2', a_3', \ldots)$, are the Fourier constants of a function f of R_f. For, the sum of the squares of the constants a_i' converges to a limit not exceeding B^2. But this, together with the properties assumed for the set $[\phi_i]$ at the outset, is sufficient, according to the Riesz-Fischer theorem,* for the existence of a summable function with summable square, whose Fourier coefficients are the a_i'. This is the function f whose existence was asserted, for it is easily verified that the hypotheses (a) and (b) are fulfilled.

If now we select the first function of the first sequence, the second function of the second sequence, and so on, we have a single sequence $[f_j]$ which converges in the mean to f, as may be seen by use of hypothesis (b) and the equation $\int_0^1 (f_j - f)^2 ds = \sum_1^\infty (a_{ji} - a_i')^2$. Thus the region R_f is closed in the sense that every infinite set of points in it contains a sequence which converges in the mean to a point of R_f.

It is a consequence of this closure, and of the continuity of the transformation, that the distance $\delta_f = \sqrt{\int_0^1 (Sf - f)^2 ds}$, by which the transformation S displaces the point f, attains its lower limit δ_0 in R_f. That this lower limit cannot be different from 0 is seen by a comparison with a region R_n of n-space, namely the region defined by $\sum_1^n x_i^2 \leqq B^2, \sum_m^n x_i^2 \leqq \eta(m), (m = 1, 2, \ldots, n)$. To a point x of this region corresponds the point $\phi(s, x) = \sum_1^n x_i \phi_i$ of R_f. Let $\phi'(s, x)$ denote the function into which S transforms $\phi(s, x)$. Then the transformation T is defined by the equations $x_i' = \int_0^1 \phi'(s, x)\phi_i(s) ds$, $(i = 1, 2, \ldots, n)$. The

* Cf. E. Fischer, *Sur le convergence en moyenne*, Paris Comptes Rendus, vol. 144 (1907), pp. 1022–24.

hypotheses of Theorem I are fulfilled, and an invariant point, a, exists. Moreover, δ_ϕ approaches 0 with $1/n$, so that $\delta_0 = 0$. But the point for which $\delta_0 = 0$ is an invariant point, in the specified sense.

6. Applications

There are evidently numerous applications of the above theorems to the existence problems of analysis. We give two instances in this paragraph.

In the first place, let us consider a differential equation

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \tag{8}$$

for which a solution is desired which satisfies n linear conditions on the interval $(0, a)$

$$\int_0^a \sum_{j=0}^{n-1} p_{ij}(x) y^{(j)}(x)dx + \sum_{j=0}^{n-1}\sum_{k=1}^{m} q_{ijk} y^{(j)}(x_k) = c_i \tag{9}$$
$$(i = 1, 2, \ldots, n; 0 \leqq x_1 \leqq x_2, \ldots, x_m \leqq a),$$

where the functions $p_{ij}(x)$ are continuous and the conditions are such as to determine uniquely a polynomial y of degree $n-1$. The problem of proving the existence of a solution of the differential equation (8) with auxiliary conditions (9) is identical with the problem of proving the existence of an invariant point of the transformation $z = Sy$;

$$z = \int_0^x \int_0^x \cdots \int_0^x F(x, y, y', \ldots, y^{(n-1)})dxdx\ldots dx$$
$$+ a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},$$

the coefficients a_i being explicit functionals of y determined by the demand that z satisfy the conditions (9). But here Theorem IV gives information, so that we may state the following corollary:

If there exists a set of n constants, $B_0, B_1, \ldots, B_{n-1}$, such that when $|y| \leqq B_0$, $|y'| \leqq B_1, \ldots, |y^{(n-1)}| \leqq B_{n-1}$, the same inequalities hold for z and its derivatives, and if F is a one-valued and continuous function of its arguments thus restricted and with $0 \leqq x \leqq a$, then there exists in R_f a solution of the differential equation and the auxiliary conditions.

The region R_f will be determined by the given inequalities on $y, y', \ldots, y^{(n-1)}$ together with the requirement that $y^{(n-1)}$ is continuous, $\eta(\epsilon) = M\epsilon$, where M is the maximum numerical value of F for $y, \ldots, y^{(n-1)}$ restricted as in the theorem. It is apparent that $z^{(n-1)}$ is also continuous, $\eta(\epsilon)$.

For a sufficiently small value of a, the conditions stated will be satisfied, so that the above includes the classical existence theorem for the solution of a differential equation with initial values assigned to this solution and its first $n-1$ derivatives at a single point.

Secondly, let us consider an integral equation:*

$$f(s) = y(s) - \lambda \int_0^1 F(y(t))dt \tag{10}$$

in which $f(s)$ is known, and $F(y(t))$ is a given functional of $y(t)$, which may also depend on s and t. We suppose for simplicity that F does not contain derivatives of $y(t)$. If, in this integral equation, $y(s)$ be replaced, outside of the integral sign, by $z(s)$, the equation defines a transformation $z = Sy$, an invariant point of which is a solution of the integral equation. We may therefore conclude from Theorem II, *that, if for continuous functions $y(t)$ such that $|y(t)| \leqq B$, B greater than the maximum of $|f(s)|$, the set of functions of s, $\int_0^1 F(y(t))dt$, is bounded, B, and equicontinuous, $\eta(\epsilon)$, and if $f(s)$ is continuous, $\eta(\epsilon)$, then, for sufficiently small values of λ, the integral equation has a continuous solution, $y(s)$.* Various extensions will occur to the reader.

7. Invariant directions in a space of an odd number of dimensions

In the above theorems and applications, the invariant point may, in certain cases, turn out to be $f = 0$. For homogeneous problems, however, this would yield only a trivial solution. We therefore now turn our attention to *invariant directions*, in which there is no essential difference between f and kf for $k \neq 0$.

We begin with the case of n-space. It will be convenient to think of the transformations of directions as transformations of points of the hypersphere, H_n : $\sum x_i^2 = 1$, the transformation carrying these points into points other than the origin.

Theorem VI. *Let n be odd, and let T denote a transformation with the following properties:*

(*a*) *it transforms any real point of H_n into one real finite point different from the origin;*

(*b*) *it is continuous, i.e., the point x of H_n is transformed by it into the point x' whose coördinates are continuous functions of the coördinates of x on H_n.*

Then there exists a direction invariant under T on H_n, i.e., a point x of H_n such that $x' = cx$, $c \neq 0$.

The restriction that n be odd is necessary, since there exist transformations (rotations) in spaces of even dimensionality which have no invariant directions.

Suppose the transformation given in the form $\rho x' = f(x)$, the $f_i(x)$ being continuous, one-valued functions, such that $\rho^2 = \sum f_i^2(x)$ vanishes nowhere on H_n. If the functions $f_i(x)$ are defined only on the hypersphere H_n, their defini-

* See E. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen*, Mathematische Annalen, vol. 65 (1908), pp. 370–399.

tion can easily be extended continuously, so that the functions $f_i(x)$ are proportional to the distance $\sqrt{\sum x_i^2}$ of the point x from the origin.

If, then, the transformation have no invariant direction, a sufficiently close polynomial approximation on H_n will have no invariant direction. Assuming, therefore, that the transformation has no invariant direction, we are at liberty to suppose that the $f_i(x)$ are polynomials, and such that $\sum f_i^2(x) > 0$ on H_n. We now form the tangent vector, $t(x) = f(x) - x(\sum x_i f_i(x))$, which never vanishes on H_n since $\sum t_i^2(x) = \sum f_i^2(x) - (\sum x_i f_i(x))^2 = \sum f_i^2(x) \sin^2 \theta$, where θ is the angle between the vectors x and $f(x)$. Moreover, $\sum t_i^2(x)$ will remain positive in a finite neighborhood of H_n, and in particular in a closed region S_n bounded by $\sum x_i^2 = r^2$ and H_n, where r is sufficiently near 1.

We then form the vector $s(x) = t(x) \sqrt{\sum x_i^2 / \sum t_i^2(x)}$; it vanishes nowhere in S_n, is perpendicular to the radius x, and its components are analytic throughout S_n. Then $x' = F(x, \lambda) = x \cos \lambda + s(x) \sin \lambda$ is a transformation of S_n into itself, which is analytic for x in S_n and for all λ, and which preserves distances from the origin. As its jacobian, J, is 1 for $\lambda = 0$, the transformation has, for small λ, a one-valued inverse, and the volume S_n may be expressed, for such λ, by the integral

$$V = \int\int_{S_n} \cdots \int J \, dx_1 \, dx_2 \ldots dx_n .$$

Hence this integral is constant for small λ, and, being analytic in λ, for all λ. But $J = 1$ for $\lambda = 0$, and $J = -1$ for $\lambda = \pi$. We thus arrive at a contradiction, and an invariant direction must exist.*

8. Invariant directions in function space

From Theorem VI may be derived the following:

Theorem VII. *Let R_f denote the region of function space corresponding to real continuous functions on the interval* (0, 1), *and let R'_f denote the subregion of R_f corresponding to functions which are bounded, B, and equicontinuous, $\eta(\epsilon)$, $\eta(\epsilon)$ being convex.*

Let $f' = Sf$ denote a transformation applicable to all normalized functions (*i. e., such that* $\int_0^1 f^2 ds = 1$) *of R_f, which yields a unique function in R'_f when so applied,*

* After the authors had found proofs of a number of special cases of the above theorem, including the important symmetric case, $f(-x) = -f(x)$ (which includes the linear case), they became acquainted with the details of Professor Alexander's elegant paper, from which the theorem above can readily be inferred. They then completed the general proof above. No other changes in the text have been made since they learned of his results.

is continuous, and is such that the functions f' into which it transforms all normalized functions of R_f have norms, $\int_0^1 f'^2 ds \geqq c^2$, *a definite positive number.*

Then there exists in R_f' an invariant direction, i.e., a normalized function f such that $f' = kf$, where $k \neq 0$.

We consider the distances

$$\delta_f^{\pm} = \sqrt{\int_0^1 (f \pm kf')^2 ds}$$

between the point f of the hypersphere H_f, $\int f^2 ds = 1$, and the point kf' on H_f corresponding to f by the transformation, and the diametrically opposite point, $-kf'$, respectively. We shall show that the lower limit of either δ_f^+ or δ_f^- must be 0. The existence of an invariant direction will then follow, since the region R_f' is closed.

To show that there are points of H_f in R_f' for which one of these distances is arbitrarily small, we consider an n-space, with n odd, and subject to inequalities to be given presently. We define the transformation T as follows: to x on the hypersphere H_n: $\sum x_i^2 = n-1$, corresponds the continuous function $\pi(s, x)$, equal to x_i for $s = s_i = (i-1)/(n-1)$, $(i = 1, 2, \ldots, n)$, and linear for other values of s. This function is then normalized, giving a function $f(s, x)$. Denoting $Sf(s, x)$ by $f'(s, x)$, we define the point x' on H_n into which T transforms x by the equations

$$x_i' = f'(s_i, x) \Big/ \sqrt{\frac{1}{n-1} \sum f'^2(s_j, x)}.$$

Our first restriction will be to make n so large that T is continuous and Theorem VI applicable. This will be seen to be the case provided the denominator in the expression for x_i' does not vanish. But it is clear that

$$\left| \frac{1}{(n-1)} \sum f'^2(s_i, x) - \int_0^1 f'^2(s, x) ds \right| \leqq 2B\eta\left(\frac{1}{n-1}\right),$$

since the difference in question cannot exceed the maximum variation of $f'^2(s, x)$ in a sub-interval of length $1/(n-1)$, and in turn this variation does not exceed $2B\eta(1/(n-1))$. Inasmuch as $\int_0^1 f'^2(s, x) ds \geqq c^2$, n can be chosen so large that the denominator in the expression for x_i is greater than $c/2$. This is the first condition imposed on n.

Under these circumstances the transformation T has an invariant direction, a, for which $\sum a_i^2 = n-1$. Observing that

$$a_i = \pi(s_i, a) = f'(s_i, a) \Big/ \sqrt{\frac{1}{n-1}\sum f'^2(s_j, a)},$$

we conclude that $f(s, a) = c_n Sf(s, a)$ for $s = s_i(i = 1, 2, \ldots, n)$ where $c_n \neq 0$. It follows at once that the δ^* corresponding to the invariant point a in n-dimensional space approaches 0 as n increases, inasmuch as the norm of $Sf \geqq c/2$. This completes the proof of the theorem.

A simple illustration of the Theorem VII is given by the non-linear integral equation

$$\phi(s) = \lambda \int_0^1 K(s, t)\phi^2(t)dt,$$

where $K(s, t)$ is continuous and has a positive lower bound. The right hand side of this equation defines a transformation of functions $\phi(s)$, an invariant direction of which yields a solution of the integral equation. The hypotheses of the theorem are fulfilled, as may be verified by use of the law of the mean, so that a function ϕ exists for which the equation has a continuous normalized solution. A continuous solution, not normalized, evidently exists, then, for every real $\lambda \neq 0$.

9. The existence of inverse directions in n-space

In this, and in the following paragraph, we shall consider the existence of points inverse to a given point for a given transformation. We begin with a transformation in n-space.

Theorem VIII. *Let T_λ denote a transformation $x' = x + \lambda\varphi(x)$, in which the functions $\varphi(x)$ are bounded, B, and continuous, on the hypersphere H_n, $\sum x_i^2 = n-1$. Let λ_0 be a number such that the transformation T_λ transforms no point of H_n into the origin for any value of λ in the closed interval* $(0, \lambda_0)$. *Let b be any point on H_n.*

Then there exists an inverse point a of H_n, i. e., such that $\rho b = T_{\lambda_0}(a)$, $\rho > 0$.

In other words, any direction has an inverse by T_{λ_0} provided $Q_\lambda : x' = -\lambda\varphi(x)$ has no invariant direction for $0 \leqq \lambda \leqq \lambda_0$.

The proof of the theorem resembles that of Theorem VI and an outline will be sufficient. First assume that the functions $f(x)$ are polynomials, and consider the equations

$$x + \lambda\varphi(x) - \rho b = 0, \qquad \sum x_i^2 = n - 1$$

in the region R_{n+2} : $|x_i| \leqq 2\sqrt{n}$, $\lambda^2 \leqq \lambda_0^2$, $0 \leqq \rho \leqq 1 + B$, B being an upper bound for $\lambda_0\Phi(x)$ on H_n where $\Phi^2(x) = \sum \phi_i^2(x)/(n-1)$. From the first equations we obtain by squaring and adding

$$\rho^2(n-1) = (n-1) + 2\lambda\sum x_i\varphi_i(x) + \lambda^2\sum \varphi_i^2$$

so that $|\rho| < 1+B$ for any solution $|\lambda| < \lambda_0$. For $\lambda=0$ the equations have a unique solution, for which the functional determinant with respect to the set x and ρ is not 0. The second solution on the boundary of R_{n+2} which by the lemma of §1 exists, necessarily corresponds to the value λ_0 of λ, and a positive value of ρ. If the x have here the values a, then a is the required inverse, and $\rho b = T_{\lambda_0}a$.

If the functions $f(x)$ are merely continuous, the conclusion remains valid, and the proof by a limiting process is immediate.

10. The existence of inverse directions in function space

A theorem in function space which corresponds to Theorem VIII may be formulated as follows.

THEOREM IX. *Let $f' = Sf$ denote a one-valued transformation applicable to continuous functions on H_f and of the form $Sf = f + \lambda Qf$, where $\lambda_0\, Qf$ is bounded, B, and equicontinuous, $\eta(\epsilon)$, with $\eta(\epsilon)$ convex. If, in addition, S is continuous, and transforms no function f on H_f into 0 for $0 \leqq \lambda \leqq \lambda_0$ then, for any continuous function g of H_f there exists a direction inverse to g under S_{λ_0} i. e., such that $\rho g = S_{\lambda_0}f$, $\rho > 0$.*

Take any set of variables x_i such that $\sum x_i^2 = n-1$ and form the corresponding polygonal function $\pi(s, x)$. It is not difficult to establish the inequality

$$\frac{1}{3} \leqq \int_0^1 \pi^2(s, x)dx \leqq \frac{4}{3}.$$

In fact this flows at once from the evident equality

$$\int_{s_i}^{s_i+1} \pi^2(s, x)dx = \frac{1}{3(n-1)}\,(x_i^2 + x_ix_{i+1} + x_{i+1}^2)$$

by summation, since

$$\frac{x_i^2 + x_{i+1}^2}{2} \leqq x_i^2 + x_ix_{i+1} + x_{i+1}^2 \leqq 2(x_i^2 + x_{i+1}^2).$$

Hence, if we normalize $\pi(s, x)$ to $f(s, x)$, the multiplicative factor required lies between $\sqrt{3}$ and $\sqrt{3/4}$, and it evidently approaches 1 as n becomes infinite, provided $\pi(s, x)$, approaches a limit uniformly.

Write now $f' = \pi + \lambda Q f$ and

$$x_i' = f'(s_i, x), \ (s_i = \frac{i-1}{n-1}; i = 1, 2, \ldots, n),$$

thus defining a transformation T_λ of R_n of the form in theorem VIII, namely

$$x_i' = x_i + \lambda Q(f(s_i, x))$$

which is clearly one-valued and continuous. Furthermore, by hypothesis $\lambda Q f$ is bounded, B, and continuous, $\eta(\epsilon)$, for the function $f(s, x)$ on H_f.

Hence by the theorem of §9 we can assert the existence of an inverse point under T unless for $0 \leqq \lambda \leqq \lambda_0$ and some x on H_n,

$$0 = f(s_i, x) + \lambda Q f$$

for $s = s_i$ $(i = 1, 2, \ldots, n)$. Here $\lambda Q f$ is bounded, B, and continuous, $\eta(\epsilon)$, so that $f(s, x)$ is bounded, $B + \eta(1/(n-1))$, and, on account of its polygonal character, continuous, $\eta(\epsilon)$. If such a function $f(s, x)$ exists for indefinitely large values of n, we infer the existence of a limit function f_0, bounded, B, and equicontinuous $\eta(\epsilon)$, such that $S_\lambda(f_0) = 0$ for a value of λ between 0 and λ_0. Furthermore since the approximating functions are bounded and equicontinuous, it follows that $\int_0^1 f_0^2 ds = 1$, on account of the property $\sum x_i^2 = n-1$, which holds at every stage. This is in contradiction with the hypotheses of the theorem.

Hence, by Theorem VIII, we have for some set x_i

$$g(s) = \pi(s, x) + \lambda_0 Qf$$

for $s = s_i$ $(i = 1, 2, \ldots, n)$. The form of this equation shows that $\pi(s, x)$ is bounded, $2B$, and continuous, $2\eta(\epsilon)$. Consequently, as n increases without limit, there exists a set of polygonal functions $\pi(s, x)$ approaching a limit function f_0 uniformly, where f_0 is bounded, $2B$, and continuous, $2\eta(\epsilon)$. The same limit is approached by $f(s, x)$ of course, and f_0 will lie on H_f. We have then $g = S_\lambda f_0$, and the theorem is proved.

It should be remarked that λ need not be positive in the above reasoning, and that an inverse direction exists for any value of λ between the largest negative and smallest positive values of λ for which $Sf = 0$ for some function of R_f'.

In the linear case the field of functions on which the transformation operates extends at once to all continuous functions. Also the factor ρ can be suppressed since for invariant directions we can replace f/ρ by f. The above result evidently includes what is essential for the solution of integral equations of the Fredholm type, at least in the case of symmetric kernels.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.