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INVARIANT PSEUDO-DIFFERENTIAL OPERATORS ON A LIE GROUP

by ROBERT S. STRICHARTZ

ABSTRACT. - The invariant pseudo-differential operators on a Lie group G with Lie algebra $\mathfrak{G}$ are characterized in terms of a function on $\mathfrak{G}^*$ called the Lie symbol. A calculus of Lie symbols is developed in terms of the algebraic structure of $\mathfrak{G}$. We also define global Sobolev spaces for non-compact groups.

§ 1. Introduction.

Let G be a Lie group and $\mathfrak{G}$ its Lie algebra. The universal enveloping algebra $\mathcal{U}$ of $\mathfrak{G}$, which is the free algebra over the complexification of $\mathfrak{G}$ modulo the ideal generated by elements of the form $XY - YX - [XY]$, is naturally isomorphic to the algebra of left invariant partial differential operators on G. In this paper we define an analogous algebraic structure corresponding to classes of left invariant pseudo differential operators. Thus to every left invariant operator on G we associate a function on $\mathfrak{G}^*$ (the dual space of $\mathfrak{G}$ considered as a vector space) called its Lie symbol. The correspondence is bi-unique if we consider operators modulo smooth operators and Lie symbols modulo functions in $\mathcal{S}(\mathfrak{G}^*)$. Composition of operators corresponds to a rather complicated product operation on the Lie symbols. This operation is described in § 3, and it is effectively computable from the Lie algebra structure of $\mathfrak{G}$. In particular we may identity $\mathcal{U}$ with the algebra of polynomials on $\mathfrak{G}^*$ under this composition law.

The usual symbolic calculus for pseudo differential operators is of course still valid in this context. It has the disadvantage, however, that left invariance of the operator is not conveniently expressible in properties of the symbol, except for the top order symbol, and even then it is not clear that to every top order symbol, which is invariant under the induced action of G on its cotangent bundle, there corresponds an invariant operator.

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We refer the reader to Stetkaer-Hansen [20] and Rothschild [14] who prove the non-existence of bi-invariant pseudo-differential operators which are not differential operators for most non-compact non-abelian groups.

In Section 2 we review the facts about pseudo-differential operators we will need. In Section 3 we construct the algebra of Lie symbols and prove the main theorems of the paper. In Section 4 we extend the calculus of Section 3 to operators with mixed homogeneity. In Section 5 we prove that a compactly supported pseudo-differential operator of order zero is bounded in $L^p(G)$ and use this fact to define global Sobolev spaces. We show that the usual properties of Euclidean Sobolev spaces are true also in this context. The results of Section 5 actually depend only on the top order calculus. An appendix is devoted to sketching some aspects of the local L^p theory of pseudo-differential operators that do not seem to be adequately dealt with in the literature.

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§ 2. Pseudo-Differential Operators.

A C^∞ function $p(x, \xi)$ on $R^n \times R^n$ with compact support in the x variable belongs to S_ϱ^m for m real and $\frac{1}{2} < \varrho \leq 1$ if

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq c_{\alpha,\beta} (1 + |\xi|)^{m - \varrho|\beta| + (1-\varrho)|\alpha|} \tag{2.1}$$

for all multi-indices α, β. We denote $S_\varrho^\infty = \bigcup_m S_\varrho^m$ and $S_\varrho^{-\infty} = \bigcap_m S_\varrho^m$. Two functions in S_ϱ^m are *equivalent* if they differ by a function in $S_\varrho^{-\infty}$. Associated with $p(x, \xi)$ is the operator $p(x, D)$ given by

$$\begin{aligned} p(x, D)\, u(x) &= (2\pi)^{-n} \int p(x, \xi)\, \hat{u}(\xi)\, e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \iint u(y)\, e^{i(x-y)\cdot\xi}\, p(x, \xi)\, dy\, d\xi \end{aligned} \tag{2.2}$$

where the iterated integral is taken first with respect to y and then ξ.

The classes S_ϱ^m and associated operators were introduced by Hörmander in [6] and the reader is referred to that paper for the proofs of the properties we state below:

1. $p(x, D)$ is a continuous operator on $C_{\text{com}}^{\infty}(R^n)$

2. If $p \in S_{\varrho}^{-\infty}$ then $p(x, D)\, u(x) = \int k(x, y)\, u(y)\, dy$ for some C^{∞} function $k(x, y)$ with compact support in x, and conversely.

3. The operator $p(x, D)$ is quasi-local, i. e. if u is C^{∞} on any open set Ω then $p(x, D)\, u$ is C^{∞} on Ω.

4. If $p \in S_{\varrho}^{0}$ then $\| p(x, D)\, u \|_2 \leq C_k \| u \|_2$ for $u \in C_{\text{com}}^{\infty}(|x| < k)$ for each k. The same result holds for L^q norms for $1 < q < \infty$ provided $\varrho = 1$, as will be shown in the appendix.

5. Given any sequence $m_j \searrow -\infty$ and $p_j \in S_{\varrho}^{m_j}$ there exists $p \in S_{\varrho}^{m_1}$, unique up to equivalence, such that $p - \sum_{j=1}^{k-1} p_j \in S_{\varrho}^{m_k}$. We write $p \backsim \Sigma p_j$. In fact we can find q_j equivalent to p_j such that $p = \Sigma q_j$ pointwise.

6. If $p \in S_{\varrho}^{m_1}$ and $q \in S_{\varrho}^{m_2}$ there exists $r \in S_{\varrho}^{m_1 + m_2}$ such that $p(x, D) \cdot$ $\cdot q(x, D) = r(x, D)$. In fact r is given up to equivalence by

$$r \backsim \sum_{\alpha} \frac{1}{\alpha\,!} \left(\frac{\partial}{i\,\partial \xi}\right)^{\alpha} p \left(\frac{\partial}{\partial x}\right)^{\alpha} q.$$

7. The class of operators $p(x, D)$ for $p \in S_{\varrho}^{m}$ is invariant under diffeomorphism.

We also consider a subspace of S_1^m, denoted S_h^m of symbols which are homogeneous in the following sense: $p \backsim \Sigma p_j$ where $p_j(x, t\xi) = t^{m_j} p_j(x, \xi)$ for $t \geq 1$ and $|\xi| \geq 1$ for $m_j \searrow -\infty$. These symbols and the corresponding operators were the first class (locally) of pseudo-differential operators considered in Kohn-Nirenberg [10]. They are preserved under the operations in 5, 6, 7 above. If $p \in S_h^m$ is a polynomial in ξ for each x, then $p(x, D)$ is a differential operator.

Now let M be a paracompact C^{∞} manifold with smooth measure dx equivalent to Lebesgue measure in every coordinate patch. We say an operator $T: C_{\text{com}}^{\infty}(M) \to C^{\infty}(M)$ is *smooth* if it has the form $Tu(x) = \int_M k(x, y)\, u(y)\, dy$ for some $k \in C^{\infty}(M \times M)$. An operator $T: C_{\text{com}}^{\infty}(M) \to C^{\infty}(M)$ is called a general pseudo-differential operator of class S_{ϱ}^{m} or S_h^m if for any coordinate patch U and $\varphi \in C_{\text{com}}^{\infty}(U)$ the operator $T(\varphi u)$ is the sum of a smooth operator and an operator $p(x, D)$ for $p \in S_\varrho^m$ or S_h^m in the local coordinates of U. Because of property 7 this definition is independent of the choice of local coordinates.

A pseudo-differential operator is said to be compactly supported if given any compact neighborhood Ω there exists a compact neighborhood Ω_1 such that Tu vanishes in Ω whenever u vanishes in Ω_1, and Tu has support in Ω_1 whenever u has support in Ω. It is easily seen that a compactly supported operator preserves $C_{\text{com}}^\infty(M)$ and by extension preserves $C^\infty(M)$. Thus two such operators may be composed, while two general pseudo-differential operators need not have a well-defined composition. Note, however, that a general operator may be composed with a compactly supported operator in either order.

While many pseudo-differential operators naturally arising on non-compact manifolds are not compactly supported, there is no real loss in generality in restricting consideration to compactly supported operators in view of the following observation: every pseudo-differential operator differs from a compactly supported one by a smooth operator. This may be seen by considering a partition of unity $\varphi_1, \ldots$ and observing that $T = \sum_j \sum_k M_{\varphi_j} T M_{\varphi_k}$ where M_{φ_j} denotes the operator $M_{\varphi_j} u(x) = \varphi_j(x)\, u(x)$. If we take the sum over those j and k such that $\varphi_j \varphi_k = 0$ we obtain a smooth operator, and the remaining summands give a compactly supported operator. Thus restricting operators to be compactly supported is merely a technical device to simplify the theory on non-compact manifolds.

§ 3. The calculus of Lie symbols.

Let G be a Lie group and $\mathfrak{G}$ its Lie algebra. We use lower case roman letters to denote elements of G, upper case for elements of $\mathfrak{G}$, and greek letters for elements of $\mathfrak{G}^*$. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{G}$, and $\xi_1, \ldots, \xi_n$ the dual basis for $\mathfrak{G}^*$. The pairing between $\mathfrak{G}$ and $\mathfrak{G}^*$ is denoted by the dot product $X \cdot \xi$.

We shall deal only with Lie algebra products which are associated $[\,..[XY]Z]\ldots]W]$ and which we denote bracket $(XYZ\ldots W)$. For $X \in \mathfrak{G}$ we let $ad_k X(Y) = \text{bracket}\,(XY^k)$; $ad_k X$ should be regarded as a polynomial map of $\mathfrak{G}$ to itself.

There exists $\varepsilon > 0$ such that the exponential map $\exp: \mathfrak{G} \to G$ is a diffeomorphism of $|X| < \varepsilon$ onto a neighborhood of the identity of G. For $X \in \mathfrak{G}$ we associate the left invariant first order partial differential operator $Xu(x) = \dfrac{d}{dt} u(x \exp tX)\,|_{t=0}$.

We shall use the Campbell-Hausdorff formula (see [7]) which states that for sufficiently small X and Y we have $\exp X \exp Y = \exp Z$ where

$$Z = \sum_m \sum_{p_i q_i} \frac{(-1)^{m-1}}{m\Sigma\,(p_i + q_i)} \frac{\operatorname{bracket}(X^{p_1} Y^{q_1} X^{p_2} Y^{q_2} \ldots Y^{q_m})}{p_1!\, q_1!\, p_2!\, q_2! \ldots p_m!\, q_m!} \tag{3.1}$$

where m varies over the positive integers and, for each m, p_i and q_i for $i = 1, \ldots, m$ vary over the positive integers subject to $p_i + q_i > 0$. As a consequence we have $\exp tX \exp Y = \exp Z_t$ where

$$Z_t = Y + tX + t \sum_{k=1}^{\infty} c_k \operatorname{bracket}(XY^k) + 0\,(t^2) \tag{3.2}$$

as $t \to 0$. The actual determination of the constants c_k from (3.1) is a combinatorial problem.

Let us now consider a left invariant compactly supported pseudo-differential operator T on G of class S_ϱ^m. Now G has a distinguished set of coordinate patches, namely for each $x \in G$ we may coordinatize a neighborhood of x by $x \exp X$ for $|X| < \varepsilon$. Thus for each $x \in G$ there exists $p_x(X, \xi) \in S_\varrho^m$ such that

$$Tu\,(x \exp X) = (2\pi)^{-n} \iint u\,(x \exp Y)\, e^{i(X-Y)\cdot\xi}\, p_x(X, \xi)\, \psi(Y)\, dY\, d\xi \tag{3.3}$$

$$+ \text{ a smooth operator}$$

where $\psi \in C_{\text{com}}^\infty(|X| < \varepsilon)$ and $\psi \equiv 1$ in a neighborhood of the origin.

Now the left invariance of T means we can choose p_x to be independent of x by modifying the smooth operator in (3.3). The requirement that (3.3) define the same operator in every coordinate neighborhood places additional restrictions on $p_x(X, \xi)$ which we shall not study directly. Instead we set $X = 0$ in (3.3) and obtain

$$Tu\,(x) = (2\pi)^{-n} \iint u\,(x \exp Y)\, e^{-iY\cdot\xi}\, p\,(\xi)\, \psi\,(Y)\, dY\, d\xi + u * \varphi \tag{3.4}$$

where $p(\xi) = p(0, \xi)$, $\varphi \in C_{\text{com}}^\infty(G)$ and $u * \varphi = \int_G u\,(xy^{-1})\, \varphi\,(y)\, dy$. Here we have deduced the form of the smooth operator from the fact that T is invariant and compactly supported. We note that $p(\xi) \in S_\varrho^m$ or S_h^m if $p(X, \xi)$ is. By $p(\xi) \in S_\varrho^m$ we mean (2.1) holds without X derivatives.

Thus we see that every left invariant pseudo-differential operator must have the form (3.4). The converse is also true:

THEOREM 1: T is a left invariant compactly supported pseudo-differential operator of class S_ϱ^m or S_h^m if and only if it has the form (3.4). The correspondence $T <\!\!\longrightarrow p(\xi)$ is bi-unique between classes of operators modulo convolutions with C_{com}^∞ functions and classes of functions in S_ϱ^m modulo functions in $\mathcal{S}(\mathfrak{G}^*)$. T is a differential operator if and only if $p(\xi)$ may be chosen to be a polynomial.

PROOF: We must show how to go from (3.4) back to (3.3), deducing the form of $p(X, \xi)$ from $p(\xi)$. To do this we replace x by $x \exp X$ in (3.4) and set $\exp X \exp Y = \exp Z$. To do this we must take X sufficiently small, and Y may be forced to be small by shrinking the support of ψ. We note that changing ψ away from the origin only results in a change in the smooth operator $u * \varphi$. Thus we have

$$Tu(x \exp X) = (2\pi)^{-n} \iint u(x \exp Z)\, e^{-iY\cdot\xi}\, p(\xi)\, \psi(Y)\, J(X, Z)\, dZ\, d\xi \tag{3.5}$$

$$+ a \quad \text{smooth operator}$$

where Y may be determined from $\exp Y = \exp(-X) \exp Z$ by the Campbell-Hausdorff formula (3.1). Now we claim that for small X and Z that $-Y = W(X, Z)(X - Z)$ where $W(X, Z)$ is an invertible linear transformation on $\mathfrak{G}$ which depends on X and Z in a C^∞ manner. Indeed (3.1) gives $-Y = X - Z + \frac{1}{2}[X, Z] + \dots$ where each subsequent term is obtained by multiplying by a constant and applying adX and adY in various orders to $[XZ] = [(X - Z)Z] + [ZZ] = -adZ(X - Z)$. Thus $-Y = X - - Z + Q(adX, adZ)(X - Z)$ where Q is a non-commutative power series with values in the space of linear transformations on $\mathfrak{G}$. If we take X and Z small enough then $\| adX \|$ and $\| adZ \|$ will be small and the power series will converge because of the decrease of the coefficients in (3.1). Since Q has no constant terms we may make $\| Q(adX, adZ) \| < 1$ and obtain the invertibility of $W(X, Z) = I + Q(adX, adZ)$. Thus

$$\psi(X)\, Tu(x \exp X) = (2\pi)^{-n} \iint u(x \exp Z)\, e^{i(X-Z)\cdot W(X, Z)^* \xi}\, p(\xi).$$

$$J(X, Z)\, \psi(X)\, \psi(-W(X, Z)(X - Z))\, dZ\, d\xi$$

$$+ a \quad \text{smooth operator.}$$

Changing variables in the ξ integration we obtain

$$(3.6)\qquad \psi(X)\,Tu(x\exp X) = (2\pi)^{-n}\iint u(x\exp Z)\,e^{i(X-Z)\cdot\xi}\,p\,(W(X,Z)^{*-1}\xi)\cdot R(X,Z)\,dZ\,d\xi + a \text{ smooth operator}$$

where $R(X,Z)\in C^{\infty}_{\text{com}}(\mathfrak{G}\times\mathfrak{G})$. We note that

$$(3.7)\qquad \left|\left(\frac{\partial}{\partial X}\right)^{\alpha}\left(\frac{\partial}{\partial Z}\right)^{\beta}\left(\frac{\partial}{\partial \xi}\right)^{\gamma}(p\,(W(X,Z)^{*-1}\xi)\,R(X,Z))\right| \leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{m-\varrho|\gamma|+(1-\varrho)(|\alpha|+|\beta|)}.$$

Thus we may apply a lemma of Kuranishi (see [13], p. 155) which allows us to conclude from (3.6) and (3.7) that T is a pseudo-differential operator of class S^m_ϱ, and

$$(3.8)\qquad p\,(X,\xi) \backsim \sum_{\alpha}\frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial Z}\right)^{\alpha}(p\,(W(X,Z)^{*-1}\xi)\,R(X,Z))\big|_{Z=X}.$$

We note that $p(X,\xi)\in S^m_h$ if $p(\xi)$ does, and $p(X,\xi)$ may be taken to be a polynomial in ξ if $p(\xi)$ may. Also $p(\xi)\in S^{-\infty}_\varrho$ if and only if $p(\xi)\in\mathcal{S}$. The theorem follows immediately from the above and the properties of pseudo-differential operators outlined in § 2. Q. E. D..

We call $\mathcal{E}_p = S^{\infty}_{\varrho}/\mathcal{S}$ the space of *Lie symbols* of class ϱ, and loosely speaking we call $p(\xi)$ the Lie symbol of T. We call $p(X,\xi)$ the full symbol of T.

We turn now to the question of what operation on the Lie symbols corresponds to composition of operators. From property 6 of § 2 it is clear that the key step in answering this question is to compute $\left(\frac{\partial}{\partial X}\right)^{\alpha} p\,(X,\xi)\,|_{X=0}$ in terms of $p(\xi)$. While it is in principle possible to deduce this from (3.8), we have found it easier to do this by computing $X_j Tu$ using (3.3) and (3.4) and comparing the two results.

LEMMA 1: If T has full symbol $p(X,\xi)$ then $X_j T$ has full symbol $q_j(X,\xi)$ where

$$(3.9)\qquad q_j(X,\xi) = i\left(X_j + \sum_{1}^{\infty}(-1)^k c_k\,ad_k\,X_j(X)\right)\cdot\xi\,p\,(X,\xi) + \left(X_j + \sum_{1}^{\infty}(-1)^k c_k\,ad_k\,X_j(X)\right)\cdot\nabla_X\,p\,(X,\xi)$$

and Lie symbol $q_j(\xi)$ given by

$$q_j(\xi) = iX_j \cdot \xi p(\xi) + \frac{\partial}{\partial X_j} p(X, \xi)\,|_{X=0}\,. \tag{3.10}$$

Here c_k are the constants appearing in (3.2).

PROOF: We have, using (3.3),

$$X_j\, Tu\,(x \exp X) = \frac{d}{dt}\, Tu\,(x \exp X \exp tX_j)\,|_{t=0} =$$

$$\frac{d}{dt}\,(2\pi)^{-n} \iint u\,(x \exp Y)\, e^{i(Z_t - Y)\cdot\xi}\, p\,(Z_t, \xi)\, \psi\,(Y)\, dY\, d\xi\,|_{t=0}$$

$$+\, a \quad \text{smooth operator}$$

where $Z_t = X + tX_j + t \sum_1^\infty (-1)^k c_k\, ad_k\, X_j(X) + 0\,(t^2)$ from (3.2). We may now differentiate, ignoring the $0\,(t^2)$ terms, and set $t = 0$ to obtain (3.9). Setting $X = 0$ we obtain (3.10). Q. E. D..

LEMMA 2: If T has Lie symbol $p\,(\xi)$ then $X_j T$ has Lie symbol $q\,(\xi)$ where

$$q\,(\xi) \backsim iX_j \cdot \xi p\,(\xi) + i\xi \cdot \sum_{k=1}^\infty c_k\, ad_k\, X_j \left(\frac{\partial}{i\,\partial\xi}\right) p\,(\xi). \tag{3.11}$$

PROOF: We have, using (3.4),

$$X_j\, Tu\,(x) = \frac{d}{dt}\,(2\pi)^{-n} \iint u\,(x \exp tX_j \exp Y)\, e^{-Y\cdot\xi}\, p\,(\xi)\, \psi\,(Y)\, dY\, d\xi\,|_{t=0}$$

$$+\, a \quad \text{smooth operator.}$$

For fixed t we set $\exp tX_j \exp Y = \exp Z$ or $\exp Y = \exp(-tX_j) \exp Z$ and change the Y integration to a Z integration, absorbing the cange in ψ into the smooth operator:

$$X_j\, Tu = \frac{d}{dt}\,(2\pi)^{-n} \iint u\,(x \exp Z)\, e^{i\left(-Z + tX_j + t \sum_1^\infty c_k ad_k X_j(Z) + 0(t^2)\right)\cdot\xi}$$

$$\cdot\, p\,(\xi)\, \det \left(\frac{\partial}{\partial Z} \left(Z - tX_j - t \sum_1^\infty c_k\, ad_k\, X_j\,(Z) + 0\,(t^2)\right)\right) \cdot$$

$$\cdot\, \psi\,(Z)\, dZ\, d\xi\,|_{t=0} + a \quad \text{smooth operator}$$

where we have used (3.2). Using the fact that $\frac{d}{dt}\det(I + tA + 0(t^2))|_{t=0} = tr\,A$ we have

$$(3.12)\qquad X_j\,Tu(x) = (2\pi)^{-n}\iint u(x\exp Z)\,e^{-iZ\cdot\xi}\,p(\xi)\Big(iX_j\cdot\xi + i\Big(\sum_{k=1}^{\infty} c_k\,ad_k\,X_j(Z)\Big)\cdot\xi - tr\,\frac{\partial}{\partial Z}\sum_{1}^{\infty} c_k\,ad_k\,X_j(Z)\Big)\,\psi(Z)\,dZ\,d\xi + a \text{ smooth operator.}$$

Since this is a Fourier integral we may integrate by parts and neglect boundary terms to replace the polynomials in Z by differentiations of $p(\xi)$. Indeed $\left(\frac{\partial}{i\,\partial\xi}\right)^{\alpha} e^{-iZ\cdot\xi} = (-1)^{|\alpha|} Z^{\alpha} e^{-iZ\cdot\xi}$ so $ad_k\,X_j\left(\frac{\partial}{i\,\partial\xi}\right)e^{-iZ\cdot\xi} = (-1)^{|\alpha|}\,ad_k\cdot X_j(Z)\,e^{-iZ\cdot\xi}$. By Leibnitz' rule we have

$$iad_k\,X_j\left(\frac{\partial}{i\,\partial\xi}\right)\cdot\xi\,e^{-iZ\cdot\xi} = (-1)^{|\alpha|}\Big(iad_k\,X_j(Z) - tr\,\frac{\partial}{\partial Z}\,ad_k\,X_j(Z)\Big)\cdot e^{-iZ\cdot\xi}$$

so when we integrate by parts we have

$$\iint u(x\exp Z)\,e^{-iZ\cdot\xi}\,p(\xi)\Big(iad_k\,X_j(Z) - tr\,\frac{\partial}{\partial Z}\,ad_k\,X_j(Z)\Big)\,\psi(z)\,dZ\,d\xi = \iint u(x\exp Z)\,e^{-iZ\cdot\xi}\,i\xi\cdot ad_k\,X_j\left(\frac{\partial}{i\,\partial\xi}\right)p(\xi)\,\psi(Z)\,dZ\,d\xi.$$

Substituting this in (3.12) for $k = 1, \dots, N$ and handling the remainder in the usual way we deduce (3.11). Q. E. D..

LEMMA 3: There exist infinite order partial differential operators $A(\alpha)$ on $\mathfrak{G}^*$ with polynomial coefficients such that

$$(3.13)\qquad \left(\frac{\partial}{\partial X}\right)^{\alpha} p(X,\xi)\,|_{X=0} \backsim A(\alpha)\,p(\xi).$$

If we write $A(\alpha) = \sum_{\beta,\gamma} c(\alpha,\beta,\gamma)\,\xi^{\beta}\left(\frac{\partial}{\partial\xi}\right)^{\gamma}$ then non-zero terms occur only when $|\beta| \leq |\gamma|$ and $|\beta| \leq |\alpha|$. Thus $A(\alpha)$ preserves the classes $S_{\varrho}^{m}/\mathcal{S}$ and $S_{h}^{m}/\mathcal{S}$. $A(\alpha)$ may be computed recursively from the following relation:

$$(3.14)\qquad \prod_{r=1}^{m}\left(i\xi_{j(r)} + \frac{\partial}{\partial X_{j(r)}}\right)p(X,\xi)\,|_{X=0} = \text{sym}\prod_{r=1}^{m}\left(i\xi_{j(r)} + B(X_{j(r)})\right)p(\xi)$$

where the symmetric product sym Π is the average of the product for all $m!$ orderings of the factors and

$$B(X_j) = i\xi \cdot \sum_{k=1}^{\infty} c_k \, ad_k \, X_j \left(\frac{\partial}{i\,\partial\xi}\right). \tag{3.15}$$

An explicit algorithm for computing $A(\alpha)$ is: 1) expand out the right side of (3.14); 2) commute all terms where $B(X_j)$ stands to the left of ξ_k by replacing $B(X_j)\,\xi_k$ by $\xi_k\,B(X_j) + [B(X_j), \xi_k]$; 3) similarly commute any multiple commutator of $B(X_j)$ which stands to the left of ξ_k; 4) delete all terms in which any ξ_k occurs outside a commutator.

PROOF: We deduce relation (3.14) by computing the Lie symbol of sym $\prod_{r=1}^{m} X_{j(r)}\,T$ by iterating Lemma 1 and 2. Indeed by iterating Lemma 2 we obtain easily the right side of (3.14). In iterating Lemma 1 we must be careful to apply all the differential operators before setting $X = 0$; thus we obtain that the Lie symbol is

$$\text{sym} \prod_{r=1}^{m} \left[i \left(X_{j(r)} + \sum_{1}^{\infty} (-1)^k c_k \, ad_k \, X_{j(r)}(X) \right) \cdot \xi + \right.$$
$$\left. \left(X_{j(r)} + \sum_{1}^{\infty} (-1)^k c_k \, ad_k \, X_{j(r)}(X) \right) \cdot \nabla_X \right] p(X, \xi)\,|_{X=0}\,.$$

Now it appears that in addition to the commuting terms on the left side of (3.14) there are many more terms arising from applying the X derivatives to the coefficients. We claim, however, that all those terms that survive setting $X = 0$ will cancel when the product is symmetrized. To see this we observe that $ad_k\,X_j(X)$ is a homogeneous polynomial of degree k in X and $\frac{\partial}{\partial X_{j(1)}} \cdots \frac{\partial}{\partial X_{j(k)}}\, ad_k\, X_j(X) =$ bracket $(X_j\, X_{j(1)} \ldots X_{j(k)}) +$ similar brackets. All terms that arise contain such expressions, and by interchanging the role of X_j and $X_{j(1)}$ we obtain exactly the negative since each index $j(r)$ occurs only once in each term. Thus they all cancel in the symmetric product.

Once (3.14) has been established the rest of the Lemma follows easily. We observe that $B(X_j)$ contains terms of the form $\xi^\beta \left(\frac{\partial}{\partial\xi}\right)^\gamma$ for $|\beta| = 1$ and $|\gamma| \geq 1$ so the corresponding observations about the terms in $A(\alpha)$ follow. The algorithm for computing $A(\alpha)$ is easily established by induction.

Q. E. D.

THEOREM 2: Let S and T be left invariant compactly supported pseudo-differential operators of class S_ϱ^m and $S_\varrho^{m'}$ respectively. Let S have Lie symbol $p(\xi)$ and T have Lie symbol $q(\xi)$. Then ST is a left invariant compactly supported pseudo differential operators of class $S_\varrho^{m+m'}$ with Lie symbol

$$r(\xi) \backsim \sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\, A(\alpha)\, q(\xi). \tag{3.16}$$

PROOF: It is obvious that ST has all the properties except perhaps (3.16). Now the terms $\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\,\xi^\beta \left(\frac{\partial}{i\,\partial\xi}\right)^\gamma q(\xi)$ appearing in (3.16) are in $S_\varrho^{m+m'+|\beta|-\varrho(|\alpha|+|\gamma|)}$ so there are only a finite number of order $> M$ for any real M $\left(\text{here we use } |\beta| \leq |\alpha|,\ |\beta| \leq |\gamma| \text{ and } \varrho > \frac{1}{2}\right)$. If we denote the sum of these terms by $r_M(\xi)$ and the Lie symbol of ST by $r(\xi)$ it suffices to show $r(\xi) - r_M(\xi) \in S_\varrho^M$ to establish (3.16). But we know from Lemma 3 and property 6 of § 2 that there exists a pseudo differential operator R_M (which need not be left invariant) with symbol $r'_M(X, \xi)$ in the coordinate neighborhood about the identity with $r(X,\xi) - r'_M(X,\xi) \in S_\varrho^M$ and $r_M(\xi) - r'_M(0,\xi) \in S_\varrho^M$. Indeed we take $r'_M(X,\xi) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(X,\xi)\cdot$ $\cdot\left(\frac{\partial}{\partial X}\right)^\alpha q(X,\xi)$ for large enough k (so that $m + m' - k\varrho \leq M$). The desired result follows by setting $X = 0$. Q. E. D.

REMARK: It is clear that the product (3.16) preserves the class S_h^m and also the subclass of homogenebus polynomials. Thus we have obtained a faithful representation of $\mathcal{U}$ as the polynomials on $\mathfrak{G}^*$ with the product (3.16), in this case $\backsim$ may be replaced by $=$.

We next consider adjoints. We do not use the standard adjoint formula for pseudo-differential operators because it gives the adjoint with respect to the wrong measure.

THEOREM 3: Let T be a left invariant compactly supported pseudo-differential operators of class S_ϱ^m with Lie symbol $p(\xi)$. Then the adjoint operator T^* with respect to left Haar measure is also a left invariant compactly supported pseudo-differential operators of class S_ϱ^m and T^* has symbol $q(\xi) = \overline{p(\xi)}$ if G is unimodular or

$$q(\xi) \backsim \sum_\alpha \frac{1}{\alpha!}(\xi^0)^\alpha \left(\frac{\partial}{i\,\partial\xi}\right)^\alpha \overline{p(\xi)} \tag{3.17}$$

in general where

(3.18) $$\Delta(\exp X) = e^{X\cdot\xi^0},$$

Δ the modular function.

PROOF: By definition $\int_G Tu(x)\overline{v(x)}\,dx = \int_G u(x)\overline{T^*v(x)}\,dx$ so we compute, neglecting smooth operators,

$$\int_G Tu(x)\overline{v(x)}\,dx = (2\pi)^{-n}\int_G\iint u(x\exp Y)\,e^{-iY\cdot\xi}\,p(\xi)\,\overline{v(x)}\,\psi(Y)\,dY\,d\xi\,dx$$

$$= (2\pi)^{-n}\iiint u(x)\,e^{-iY\cdot\xi}\,p(\xi)\,\overline{v(x\exp(-Y))}\,\psi(Y)\,\Delta(\exp Y)\,dY\,d\xi\,dx$$

$$= (2\pi)^{-n}\int_G u(x)\overline{\iint v(x\exp Y)\,e^{-iY\cdot\xi}\,\Delta(\exp(-Y))\,\psi(-Y)\,\overline{p(\xi)}\,dY\,d\xi}\,dx.$$

Now $\Delta(\exp X) = e^{X\cdot\xi^0}$ for some $\xi^0\in\mathfrak{G}^*$; in fact $X\cdot\xi^0 = tr\ adX$ (see Helgason [5] pp. 366-367). Thus we conclude that

$$T^*v(x) = (2\pi)^{-n}\iint v(x\exp Y)\,e^{-iY\cdot\xi}\,e^{Y\cdot\xi^0}\,\overline{p(\xi)}\,\psi(Y)\,dY\,d\xi.$$

Now this appears to require an analytic continuation of $\overline{p(\xi)}$ to $\overline{p(\xi+i\xi^0)}$ to obtain the form (3.4). Since the Lie symbol is only defined modulo functions in $\mathcal{S}$, however, we have available a substitute for analytic continuation even when $\bar{p}$ isn't analytic. Indeed we let $\overline{p_k(\xi)} = \sum_{|\alpha|\le k}\frac{1}{\alpha!}(\xi^0)^\alpha\cdot\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha\overline{p(\xi)}$; note this is just the partial power series expansion of $\overline{p(\xi+i\xi^0)}$ if it existed. In order to establish (3.17) we murt show

(3.19) $$(2\pi)^{-n}\iint v(x\exp Y)\,e^{-iY\cdot\xi}\left(e^{Y\cdot\xi^0}\overline{p(\xi)} - \bar{p}_k(\xi)\right)\psi(Y)\,dY\,d\xi$$

is a pseudo-differential operator of class S_ϱ^M for any real M provided k is large enough. But by integration by parts we easily transform (3.19) into

(3.20) $$(2\pi)^{-n}\iint v(x\exp Y)\,e^{-iY\cdot\xi}\,\overline{p(\xi)}\left(e^{Y\cdot\xi^0} - \sum_{j=0}^{k}\frac{1}{j!}(Y\cdot\xi^0)^j\right)\psi(Y)\,dY\,d\xi.$$

We now repeat the argument given in the proof of Theorem 1, replacing (3.4) by (3.20). We may thus conclude that (3.20) defines a pseudo-differential operator whose symbol is given by a variant of (3.8). Note that since $e^{Y\cdot\xi^0} - \sum_{j=0}^{k} \frac{1}{j!}(Y\cdot\xi^0)^j$ vanishes to order k, the function $R(X, Z)$ in (3.8) will contain a factor that vanishes to order k in $X - Z$. Thus we must take $\alpha \geq k$ in (3.8) in order to obtain a non-zero contribution, so in fact the symbol is of class $S_\varrho^{m-k(2\varrho-1)}$. Since $\varrho > \frac{1}{2}$ we may make $m - k(2\varrho - 1)$ arbitrarily small by taking k large. Q. E. D.

All the above results have obvious extensions to systems of operators. It is not necessary to formulate results for vector bundles with group action over G because all such bundles are trivial. Indeed if $\pi: E \longrightarrow G$ is a vector bundle on which G acts by L_x covering left multiplication $(\pi(L_x e) = x\pi(e))$ then we map E to $G \times \pi^{-1}(1)$ by $e \longrightarrow (\pi(e), L_{\pi(e)^{-1}}\, e)$. It is clear this a bundle isomorphism and $L_x e \longrightarrow (x\pi(e), L_{(x\pi(e))^{-1}} L_x e) = (x\pi(e), L_{\pi(e)^{-1}}\, e)$ so the induced action on $G \times \pi^{-1}(1)$ is left multiplication cross the identity. Note, however, if we are also given an action R_x of G on E covering right multiplication it need not be trivialized by this map.

For our next result we do not assume T compactly supported, which means the convolution hernel φ in (3.4) need not have compact support.

THEOREM 4: Let T be a left invariant pseudo-differential operator of class S_ϱ^m with Lie symbol $p(\xi)$. If we have $\varrho = 1$ and

$$|p(\xi)| \geq c\,|\xi|^m \text{ for } |\xi| \geq c' \text{ then } T \text{ is elliptic.} \tag{3.21}$$

If we assume merely that

$$|p(\xi)| \geq c\,|\xi|^{m'} \text{ for } |\xi| \geq c' \text{ and some } m', \text{ and} \tag{3.22}$$

$$\left|\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\right| \leq c_\alpha\,|p(\xi)|\,(1 + |\xi|)^{-\varrho|\alpha|} \tag{3.23}$$

then T is hypoelliptic.

PROOF: This is a simple variant of Theorem 4.2 of Hörmander [6]. The point is that we are assuming only estimates on the Lie symbol and not on the full symbol.

We construct a parametrix S of class S_0^{-m} for the elliptic case and class $S_\varrho^{-m'}$ for the hypoelliptic case. The theorem then follows by standard arguments. We construct only a right parametrix because the hypotheses of the theorem are easily seen to hold for T^*.

Thus we need to construct a Lie symbol $q(\xi)$ with $\sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\cdot$ $\cdot A(\alpha)\,q(\xi) \backsim 1$. We do this by setting $q_0(\xi) = p(\xi)^{-1}$ for large ξ and C^∞ elsewhere and then solving $p(\xi)\,q_{j+1}(\xi) \backsim -\sum_{\alpha\neq 0} \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\,A(\alpha)\,q_j(\xi)$. We have $q_0 \in S_\varrho^{-m'}$ and by induction $q_j \in S_\varrho^{-m'-j(2\varrho-1)}$. Thus we may set

$$q(\xi) \backsim \sum_{j=0}^{\infty} q_j(\xi) \in S_\varrho^{-m'}$$

and obtain

$$\sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\,A(\alpha)\,q(\xi) \backsim \sum_\alpha \sum_j \frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\,A(\alpha)\,q_j(\xi) \backsim p(\xi)\,q_0(\xi)$$

$$+\sum_{j=1}^{\infty}\left(p(\xi)\,q_j(\xi) + \sum_{\alpha\neq 0}\frac{1}{\alpha!}\left(\frac{\partial}{i\,\partial\xi}\right)^\alpha p(\xi)\,A(\alpha)\,q_{j-1}(\xi)\right).$$

The last terms cancel and $p(\xi)\,q_0(\xi) \backsim 1$. Q. E. D.

REMARK: While the theorem applies to differential operators given by non commutative polynomials in $X_1, \dots, X_n$, except in the elliptic case it may be difficult to verify the hypotheses since computing the Lie symbol of such an operator requires using (3.4).

§ 4. Pseudo-differential operators with mixed homogeneity.

In this section we indicate how to extend the results of Section 3 to operators with mixed homogeneity. The Euclidean theory of singular integral operators with mixed homogeneity may be found in Fabes and Riviere [4]. The compatibility requirements we place on the structure constant of the Lie algebra may be verified for many examples of nilpotent groups. However they do not apply to the singular integral operators studied by Knapp and Stein [9].

Let $a = (a_1, \dots, a_n)$ be a fixed n-tuple of positive reals. Let $\delta(r)\,x = (r^{a_1} x_1, \dots, r^{a_n} x_n)$ and let $w(x)$ be a C^∞ positive valued function on $R^n \setminus \{0\}$ which satisfies $w(\delta(r)\,x) = r\,w(x)$. It is easy to see that such functions exist and and any two give equivalent theories. We then define the class S_a^m, in analogy with S_1^m, to be the set of C^∞ functions on $R^n \times R^n$ with compact support in the x variable, and satisfyng

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1 + w(\xi))^{m-a\cdot\beta} \tag{4.1}$$

for all α, β. We may then define an operator $p(x, D)$ by (2.2) and verify properties 1-6 of Section 2. Property 7 does not hold in general since $w(\xi)$ is not invariant under rotations. Nevertheless if we are given a fixed set of coordinate neighborhoods and local coordinates for a manifold M we may define pseudodifferential operators of class S_a^m as in Section 2; the definition now will depend on the choice ef local coordinates.

We will also need a version of Kuranishi's lemma: if $p(x, \xi, y)$ is a C^∞ function on $R^n \times R^n \times R^n$ with compact support in the x and y variables and satisfyng

$$|D_x^\alpha D_\xi^\beta D_y^\gamma p(x, \xi, y)| \leq C_{\alpha, \beta, \gamma}(1 + w(\xi))^{m - a \cdot \beta} \tag{4.2}$$

then

$$(2\pi)^{-n} \iint u(y)\, e^{i(x-y)\cdot\xi}\, p(x, \xi, y)\, dy\, d\xi \tag{4.3}$$

defines a pseudo-differential operator of clas S_a^m with symbol $q(x, \xi)$ satisfing

$$q(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \left(\frac{\partial}{i\partial\xi}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\alpha p(x, \xi\; y)\,|_{y=x}\,. \tag{4.4}$$

The proof is a trivial modification of the proof given in Nirenberg [13].

Now let G ve a Lie group with Lie algebra $\mathfrak{G}$. We say that a basis $X_1, \dots, X_n$ of $\mathfrak{G}$ with dual basis $\xi_1, \dots \xi_n$ of $\mathfrak{G}^*$ is *compatible* with a if the structure constans $c_{ijk} = [X_i, X_j]\cdot\xi_k$ vanish whenever $a_k > a_j$.

THEOREM 5: Assume there exists a basis $X_1, \dots, X_n$ for $\mathfrak{G}$ compatible with a. Then the class of left-invariant compactly supported pseudo-differential operators of class S_a^m with respect to the canonical coordinate systems may be described in analogy with the class S_1^m as follows: they are of the form

$$Tu(x) = (2\pi)^{-n} \iint u(x \exp Y)\, e^{-iY\cdot\xi}\, p(\xi)\, \psi(Y)\, dY\, d\xi + u * \varphi \tag{4.5}$$

where φ, ψ are as in theorem 1 and $p(\xi) \in S_a^m(\mathfrak{G}^*)$, that is $p \in C^\infty(\mathfrak{G}^*)$ and

$$|D_\xi^\beta p(\xi)| \leq c_\beta (1 + w(\xi))^{m - a\cdot\beta} \tag{4.6}$$

for all β.

The correspondence $T <\!\!\longrightarrow p$ is bi-unique between operators mod smooth operators and Lie symbols mod simbols in $\mathcal{S}(\mathfrak{G}^*)$.

PROOF: We may use the same argument as in the proof of theorem 1 with the appropriate modificdtions except for one point: to establish the analogue of (3.7) we must use the fact that $X_1, \dots, X_n$ is compatile with a. Indeed the compatibility implies $(W(X, Z)\, X_j)\cdot \xi_k = 0$ if $a_k > a_j$, hence $b_{jk} = X_j \,.\, W(X, Z)^{*-1} \xi_k = 0$ if $a_k > a_j$ because matrices of block triangular form are preserved under inversion. Thus we have

$$
\begin{aligned}
|(p(W(X, Z)^{*-1}\xi))| &\leq c(1 + w(\Sigma\, b_{jk}\, \xi_k))^m \\
&\leq c'(1 + \sum_j |\sum_k b_{jk}\, \xi_k|^{1/a_j})^m \\
&\leq c''(1 + \sum_k |\xi_k|^{1/a_k})^m \leq c'''(1 + w(\xi))^m
\end{aligned}
$$

since $b_{jk} \neq 0$ only if $\dfrac{1}{a_j} \leq \dfrac{1}{a_k}$. Reasoning similarly we obtain

$$
\left| \left(\frac{\partial}{\partial X}\right)^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} \left(\frac{\partial}{\partial Z}\right)^{\gamma} (P(W(X, Z)^{*-1}\xi)\, R(X, Z) \right| \leq c_{\alpha, \beta, \gamma}(1 + w(\xi))^{m - a \cdot \beta}
$$

which is the desired analogue of (3.7). Q. E. D.

REMARK: A similar argument may be used to show that the class of S_a^m operators is invariant under those diffeomorphisms whose Jacobian matrices at each point are block triangular in the sense that $\mathrm{J}(x)_{jk} = 0$ if $a_j > a_k$.

Theorems 2 and 3 are easily seen to hold in this context also.

§ 5. Global Sobolev Spaces.

In this section we construct Banach space $L_\alpha^p(G)$ for real α and $1 < p < \infty$ whith the property that $L_0^p(G) = L^p(G, dx)$ for left Haar measure dx, and if T is a left invariant compactly supported pseudo-differential operator of class S_ϱ^m then T is continuous from $L_{\alpha+m}^p$ to L_α^p. Here and throughout we assume that $p = 2$ unless $\varrho = 1$. The requirement that T be compactly supported is important, since the theory of L^p estimates for convolutions with kernels which are not compactly supported is quite involved (see Stein [17]).

If G is compact we do not need the group structure and the theory is well-known [16]. If G is not compact the best we can do with these local methods is to define Frechet spaces $L^p_{\alpha,\,\mathrm{com}}$ and $L^p_{\alpha,\,\mathrm{loc}}$ of compactly supported and locally L^p_α functions. We will of course want $L^p_{\alpha,\,\mathrm{com}} \subseteq L^p_\alpha \subset \subset L^p_{\alpha,\,\mathrm{loc}}$.

Using the results of Section 3 it will be easy to construct the spaces L^p_α using the local theory (property 4 of Section 2) together with the following simple principle of uniform localization:

LEMMA 5: Let φ be any non-zero continuous function with compact support on G. Then $f \in L^p$ if and only if $\left(\int\limits_G |f(xy)\,\varphi(y)|^p\,dy\right)^{1/p} \in L^p$ with equivalence of norms.

PROOF:

$$\left\| \left(\int\limits_G |f(xy)\,\varphi(y)|^p\,dy\right)^{1/p} \right\|_p^p = \iint |f(xy)\,\varphi(y)|^p\,dy\,dx$$

$$= \iint |f(x)\,\varphi(y)|^p\,\Delta(y^{-1})\,dx\,dy = c\,\|f\|_p^p \qquad \text{Q. E. D.}$$

THEOREM 6: Let T be a left invariant compact supported pseudo-differential operator of class S^0_ϱ. Then T is bounded in L^p for $p = 2$ and $1 < p < \infty$ if $\varrho = 1$. The L^p boundedness for $1 < p < \infty$ remains true for operators of class S^0_a.

PROOF: Let $\varphi \in C^\infty_{\mathrm{com}}(G)$. Since T is compactly supported there exists $\psi \in C^\infty_{\mathrm{com}}(G)$ with $\psi \equiv 1$ on a neighborhood of the support of φ such that $\varphi\,Tu = \varphi T(\psi u)$. From the local theory we have $\|\varphi Tu\|_p \leq c\,\|\psi u\|_p$. Applying this to $L_x\,u$ and using the invariance of T we obtain $\int |Tu(xy)\,\varphi(y)|^p\,dy \leq \leq c^p \int |u(xy)\,\psi(y)|^p\,dy$. The theorem follows by integrating with respect to x and applying the lemma Q. E. D.

DEFINITION:

Let $P_\alpha\,u(x) = (2\pi)^{-n} \iint u(x \exp Y)\,e^{-iY\cdot\xi}\,(1 + |\xi|^2)^{\alpha/2}\,\psi(Y)\,dY\,d\xi.$

This will play the role of the Riesz potentisl of order $-\alpha$ in the Euclidean case [1]. We note that P_α is of class S_1^α and is elliptic. These are the only properties of P_α we will use. If we could show that P_α (or some substitute) were invertible then it could play the role of the Euclidean Bessel potential and much of what follows could be simplified.

For $\alpha \geq 0$ we define $L_\alpha^p \subseteq L^p$ to be completion of $C_{\mathrm{com}}^\infty(G)$ in the norm $\| u : L_\alpha^p \| = (\| u \|_p^p + \| P_\alpha u \|_p^p)^{1/p}$. For $\alpha < 0$ we define $L_\alpha^p \subseteq \mathcal{D}'$ to be the completion of $C_{\mathrm{com}}^\infty(G)$ in the norm $\| u : L_\alpha^p \| = \inf \{ (\| f \|_p^p + \| g \|_p^p)^{1/p} : u = f + P_{-\alpha} g\}$. Note that the L_α^p norm is invariant under left translations.

THEOREM 7: Let T be a left invariant compactly supported pseudo-differential operator of class S_ϱ^m. Then

$$\| Tu : L_\alpha^p \| \leq c \| u : L_{\alpha+m}^p \| \tag{5.1}$$

for any α and $p = 2$ or $1 < p < \infty$ if $\varrho = 1$. Thus T extends to a bounded operator from $L_{\alpha+m}^p$ to L_α^p.

PROOF: Since P_α is elliptic for all α we may write $T = P_{-\alpha} Q_0 P_{\alpha+m} + R$ where Q_0 is of class S_ϱ^0 and R is convolution with a C_{com}^∞ function. It is easy to show that R is bounded from L_α^p to L_β^p for any α and β, so it sufficies to show

$$\| P_{-\alpha} u : L_\alpha^p \| \leq \| u : L_0^p \| \text{ and} \tag{5.2}$$

$$\| P_\alpha u : L_0^p \| \leq \| u : L_\alpha^p \| \text{ for all } \alpha. \tag{5.3}$$

First let $\alpha \geq 0$. Then (5.3) is obvious and if $u \in L^p$ then $P_{-\alpha} u \in L^p$ and $P_\alpha P_{-\alpha} u \in L^p$ since both $P_{-\alpha}$ and $P_\alpha P_{-\alpha}$ are of class S_ϱ^0. Thus (5.2) holds.

Next let $\alpha < 0$. Then (5.2) is obvious and if $u = f + P_{-\alpha} g$ with f, $g \in L^p$ then $P_\alpha u = P_\alpha f + P_\alpha P_{-\alpha} g \in L^p$ since both P_α and $P_\alpha P_{-\alpha}$ are of class S_ϱ^0. Thus (5.3) holds. Q. E. D.

COROLLARY:

(a) $L_\alpha^p \subseteq L_\beta^p$ if $\beta < \alpha$.

(b) $\| u : L_\alpha^p \|$ and $\| u : L_{\alpha-1}^p \| + \sum_{j=1}^n \| X_j u : L_{\alpha-1}^p \|$ are equivalent norms.

(c) There exists a smooth convolution operrator $R_\alpha u = \varphi_\alpha * u$, $\varphi_\alpha \in C_{\mathrm{com}}^\infty$, such that $\| u : L_\alpha^p u \|$ and $\| P_\alpha u \|_p + \| R_\alpha u \|_p$ are equivalent norms.

(*d*) $\left| \int u(x)\, \overline{v(x)}\, dx \right| \leq c \, \| u : L^p_\alpha \| \, \| v : L^{p'}_{-\alpha} \|$ for $\frac{1}{p} + \frac{1}{p'} = 1$. In this way L^p_α and $L^{p'}_{-\alpha}$ are dual spaces.

(*e*) If k is a positive integer then $u \in L^p_k$ if and only if $u \in L^p$ and $X_{j(1)} \dots X_{j(k)} u \in L^p$ in the distribution sense for all sequences $j(1), \dots, j(k)$.

PROOF: (*a*) The identity operator is of class S^0_1 hence of class $S^{\alpha-\beta}_1$.

(*b*) We have $\| X_j u : L^p_{\alpha-1} \| \leq c \, \| u : L^p_\alpha \|$ since X_j is of class S^1_1, and $\| u : L^p_{\alpha-1} \| \leq c \, \| u : L^p_\alpha \|$ by (*a*). Conversely, since $\sum_{j=1}^{n} X_j^2$ is elliptic by theorem 4, we may write $I = \sum_{j=1}^{n} T_j X_j + R$ where T_j is of class S^{-1}_1 and R is smooth, and deduce

$$\| u : L^p_\alpha \| \leq c \left(\sum_{j=1}^{n} \| X_j u : L^p_{\alpha-1} \| + \| u : L^p_{\alpha-1} \| \right).$$

(*c*) We have $\| P_\alpha u \|_p + \| R_\alpha u \|_p \leq c \, \| u : L^p_\alpha \|$ immediately from the theorem. For the converse we write $I = P_{-\alpha} Q_0 P_\alpha + R_\alpha$. If $\alpha < 0$ we have $u = R_\alpha u + P_{-\alpha} (Q_0 P_\alpha u)$ so $\| u : L^p_\alpha \| \leq (\| R_\alpha u \|^p_p + \| Q_0 P_\alpha u \|^p_p)^{1/p}$. If $\alpha > 0$ $\| u \|_p \leq \| P_{-\alpha} Q_0 (P_\alpha u) \|_p + \| R_\alpha u \|_p \leq c \, \| P_\alpha u \|_p + \| R_\alpha u \|_p$ since $P_{-\alpha} Q_0$ is of order ≤ 0 so $\| u : L^p_\alpha \| \leq c' \, \| P_\alpha u \|_p + \| R_\alpha u \|_p$.

(*d*) We may as well assume $\alpha > 0$. Write $v = f + P_{-\alpha} g$ with $f, g \in L^{p'}$. Then

$$\begin{aligned} \left| \int u(x)\, \overline{v(x)}\, dx \right| &= \left| \int u(x)\, \overline{f(x)}\, dx + \int P^*_{-\alpha} u(x)\, \overline{g(x)}\, dx \right| \\ &\leq \| u \|_p \, \| f \|_{p'} + \| P^*_{-\alpha} u \|_p \, \| g \|_{p'} \\ &\leq c \, \| u : L^p_\alpha \| \, \| v : L^{p'}_{-\alpha} \| . \end{aligned}$$

Thus $L^p_\alpha \subseteq (L^{p'}_{-\alpha})^*$ and $L^{p'}_{-\alpha} \subseteq (L^p_\alpha)^*$. It remains to establish the reverse inclusions.

Suppose then $\lambda \in (L^p_\alpha)^*$. Let $B = \{ (u, P^*_\alpha u) : u \in L^p_\alpha \} \subseteq L^p \oplus L^p$. We define a linear functional λ_0 on B by $\lambda_0 (u, P^*_\alpha u) = \lambda(u)$. Since λ is bounded on L^p_α we have λ_0 bounded on B in the $L^p \oplus L^p$ norm, so by the Hahn-Banach theorem we may find an extension λ_1 bounded on $L^p \oplus L^p$. But $(L^p \oplus L^p)^* = L^{p'} \oplus L^{p'}$ so there exist $f, g \in L^{p'}$ such that $\lambda(u) = \int u(x) \overline{f(x)}\, dx + \int P^*_\alpha u(x)\, \overline{g(x)}\, dx$. Since C^∞_{com} is dense in $L^{p'}$ we may approximate f and g

in the $L^{p'}$ metric by $f_k, g_k \in C^\infty_{\text{com}}$ and so $\lambda(u) = \lim_{k\to\infty} \int u(x)\overline{(f_k(x) + P_\alpha g_k(x))}dx$. But $f_k + P_\alpha g_k \in C^\infty_{\text{com}}$ since P_α is compactly supported hence $\lambda \in L^{p'}_{-\alpha}$.

Next let $\lambda \in (L^{p'}_{-\alpha})^*$. Then $|\lambda(f)| \leq c\, \| f : L^{p'}_{-\alpha} \| \leq c\, \| f \|_{p'}$ so we may identify λ with $u \in L^p$, $\lambda(v) = \int v(x)\, \overline{u(x)}\, dx$ for every $v \in C^\infty_{\text{com}}$. It suffices to show $P_\alpha u \in L^p$. Note $P_\alpha u \in L^p_{-\alpha}$ is defined by taking $u_k \to u$ in L^p, $u_k \in C^\infty_{\text{com}}$, and $P_\alpha u = \lim_{k\to\infty} P_\alpha u_k$ in $L^p_{-\alpha}$. Thus for $\varphi \in C^\infty_{\text{com}}$ we have

$$\lim_{k\to\infty} \int \overline{P_\alpha u_k(x)}\, \varphi(x)\, dx = \lim_{k\to\infty} \int \overline{u_k(x)}\, P^*_\alpha u(x)\, dx = \int \overline{u(x)}\, P_\alpha^* \varphi(x)\, dx$$

$$= \lambda(P_\alpha^* \varphi)$$

and since $|\lambda(P^*_\alpha \varphi)| \leq c\, \| P^*_\alpha \varphi : L^{p'}_{-\alpha} \| \leq c\, \| \varphi \|_{p'}$ we conclude that $\lim_{k\to\infty} P_\alpha u_k$ may be identified with a function in L^p.

(*e*) This follows by induction from (*b*) and (*d*). Q.E.D..

REMARK: Part (*d*) above shows that by defining L^p_α to be the completion of C^∞_{com} we have not made the space too small; indeed $(L^{p'}_{-\alpha})^*$ is the largest space that might be considered to define L^p_α.

We next consider the extension of Lemma 5 to L^p_α.

THEOREM 8: Let $\varphi \in C^\infty_{\text{com}}$ be not identically zero. Then $\| u : L^p_\alpha \|$ and $\left(\int \| \varphi L_x u : L^p_\alpha \|^p dx \right)^{1/p}$ are equivalent norms for $\alpha \geq 0$.

PROOF: Let $\varphi_0 \in C^\infty_{\text{com}}$ be identically one on a large enough neighborhood of the support of φ so that $\varphi_0 P_\alpha(\varphi f) = P_\alpha(\varphi f)$. We may then write $P_\alpha(\varphi f) = QP_\alpha(f) + Rf$ where Q and R are compactly supported pseudodifferential operators of class S^0_1 and $S^{-\infty}_1$ respectively (not left invariant of course) such that $\varphi_0 Q = Q$ and $\varphi_0 R = R$. From the local theory we have

$$\| P_\alpha(\varphi f) \|_p + \| \varphi f \|_p \leq c\, (\| \varphi_0 P_\alpha(f) \|_p + \| \varphi_0 f \|_p). \tag{5.4}$$

If we substitute $L_x u$ for f in (5.4) and then take the L^p norm in x we obtain $\left(\int \| \varphi L_x u : L^p_\alpha \|^p dx \right)^{1/p} \leq c\, \| u : L^p_\alpha \|$ by Lemma 5.

For the converse we choose $\varphi_1, \varphi_2 \in C^\infty_{\text{com}}$ so that φ_1 is supported on a compact set on which φ is bounded away from zero, and $\varphi\varphi_2 = 1$ on a neighborhood of the support of φ_1. If necessary we modify the definition of P_α by changing the function $\psi(Y)$ to have so small a support that $\varphi_1 P_\alpha f = \varphi_1 P_\alpha(\varphi_2 \varphi f)$; this change does not affect the L^p_α spaces. Again from the local theory we obtain

$$(5.5) \qquad \| \varphi_1 P_\alpha f \|_p + \| \varphi_1 f \|_p \leq c\, (\| P_\alpha(\varphi f) \|_p + \| \varphi f \|_p$$

and the proof is completed as before. Q.E.D.

From this theorem we may pass from local properties of functions in L^p_α which do not depend on the group structure to global properties of the class L^p_α. In this way virtually all properties of the Euclidean L^p_α spaces proved in [3] and [19] generalize to $L^p_\alpha(G)$. We state a few of these, proving only the first since the other proof are analogous:

COROLLARY: (*a*) $L^p_\alpha \subseteq L^q_\beta$ continuously if $\alpha > \beta \geq 0$ and $\dfrac{1}{q} - \dfrac{1}{p} = \dfrac{\alpha - \beta}{n}$, $1 < p < q < \infty$.

(*b*) L^p_α forms an algebra under pointwise multiplication if $\alpha > n/p$.

(*c*) $f \in L^2_\alpha$ for $0 < \alpha < 1$ if and only if $f \in L^2$ and $\displaystyle\int_G \int_G | f(x \exp Y) - f(x) |^2 \frac{dY}{|Y|^{n+2\alpha}}\, dx < \infty$ with equivalence of norms.

(*d*) $f \in L^p_\alpha$ for $0 < \alpha < 1$ if and only if $f \in L^p$ and $S_\alpha f \in L^p$ with equivalence of norms, where

$$S_\alpha f = \left(\int_0^\infty \left(\int_{|Y| \leq 1} | f(x \exp tY) - f(x) |\, dY \right)^2 t^{-1-2\alpha}\, dt \right)^{1/2}.$$

PROOF: (*a*) Let $\varphi, \varphi_0 \in C^\infty_{\mathrm{com}}$ with $\varphi_0 \equiv 1$ on a neighborhood of the support of φ. It follows from the local theory (see Calderon [3] and Seely [16]) that $\| \varphi f : L^q_\beta \| \leq c \| \varphi_0 f : L^p_\alpha \|$. Thus $\| \varphi L_x u : L^q_\beta \| \leq c \| \varphi_0 L_x u : L^p_\alpha \| \leq c' \| u : L^p_\alpha \|$. If $u \in L^p_\alpha$ we have $\| \varphi L_x u : L^q_\beta \|$ in L^p and L^∞ hence L^q since $p < q < \infty$.

(*b*) (*c*) and (*d*): Use the local results in [19] and reason similarly.

Q. E. D.

A more general study of global Sobolev spaces on a Riemannian manifold is given in [2] with less precise results.

APPENDIX.

L^p THEORY OF PSEUDO-DIFFERENTIAL OPERATORS

Most works on pesudo-differential operators deal only only with the L^2 theory. The L^p theory for the class S_h^m is developed by Seeley in [15]; his method depends on spherical harmonic expansions and appears unlikely to generalize to the class S_1^m. Results for more general classes than S_1^m are given in Kagan [8] and extended in Kumao-go and Nagase [12]. However P. Szeptycki, in reviewing Kagan's paper (MR 37 # 4392), has cast doubt on its validity. In any case Kagan's paper is not available in translation so we present here a brief proof which covers the classes S_1^m and the more general S_a^m classes defined in Section 4. The reader will note that our method is really a straightforward generalization of the usual L^2 theory. In contrast with Seeley [15] we require more rapid decrease in the x variable for our symbols.

Let $p(x, \xi)$ be a symbol of class S_1^r with rapid decrease in x. Thus we assume $p \in C^\infty(R^n \times R^n)$ and

$$\text{(A1)} \qquad |x|^N \left| \left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta p(x, \xi) \right| \leq c_{\alpha, \beta, N}(1 + |\xi|)^{r - |\beta|}$$

for all α, β and all non-negative N. For $f \in \mathcal{S}(R^n)$, $1 < p < \infty$, α real, define $\|f : L_\alpha^p\| = \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\alpha/2} \hat{f}(\xi))\|_p$ and let L_α^p denote the completion of $\mathcal{S}$ in the L_α^p norm. We wish to prove:

THEOREM 9: Under the above hypotheses the operator $p(x, D)$ is bounded from $L_{\alpha+r}^p$ to L_α^p, for $1 < p < \infty$ and all real α.

Our main tool, which replaces the Plancherel theorem in the L^2 theory, is the Marcinkiewicz multiplier theorem. There are many versions of this theorem; the following will suffice for our purposes (see [18]):

MARCINKIEWICZ MULTIPLIER THEOREM: Let M_p^p denote the space of bounded functions $m(\xi)$ on R^n such that $\mathcal{F}^{-1}(m(\xi)\hat{f}(\xi))$ is a bounded operator on L^p, with the operator norm. A sufficient condition for $m(\xi)$ to belong to M_p^p is that m be bounded and C^n on the complement of the set

$\{\xi : \xi_1 \, \xi_2 \ldots \xi_n = 0\}$ and $\xi^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha m(\xi)$ be bounded for all α such that each α_j is zero or one. We then have $\| m : M_p^p \| \leq c \left(\Sigma \left\| \xi^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha m(\xi) \right\|_\infty \right)$.

LEMMA 6. (Peetre's inequality): Far every real s there exist positive constants c and t such that

$$\| (1 + |\xi|^2)^{-s/2} (1 + |\xi + \eta|^2)^{s/2} : M_p^p(\xi) \| \leq c\,(1 + |\eta|)^t.$$

PROOF: In fact we may take $t = |s| + n$. We use the usual form of Peetre's inequality, $(1 + |\xi|^2)^{-s/2} (1 + |\xi + \eta|^2)^{s/2} \leq c\,(1 + |\eta|)^{|s|}$, and apply the Marcinkiewicz multiplier theorem; it is easy to see that the differentiation at most produces an additional factor of $(1 + |\eta|)^n$ Q. E. D.

LEMMA 7: Let $R(\xi, \eta) \in C^\infty(R^n \times R^n)$ satisfy

$$\| R(\xi, \eta + \xi)(1 + |\xi + \eta|^2)^{-r/2} : M_p^p(\xi) \| \leq c_N (1 + |\eta|)^{-N}$$

for every positive N. Then the operator T given by $(Tf)^\wedge(\xi) = \int R(\xi, \eta) \widehat{f(\eta)}\, dy$ is bounded from L_{s+r}^p to L_s^p for every real s.

PROOF; Let $f \in L_{s+r}^p$. Then $(1 + |\xi|^2)^{(s+r)/2} \widehat{f}(\xi) = \widehat{g}(\xi)$ for $g \in L^p$ with $\|f : L_{s+r}^p\| = \|g\|_p$. To show $Tf \in L_s^p$ we must show $\mathcal{F}^{-1}((1 + |\xi|^2)^{s/2}(Tf^\wedge \cdot \cdot(\xi)) \in L^p$. Thus we must show that the operator

$$Sg = \mathcal{F}^{-1}\left(\int R(\xi, \eta)\, \hat{g}(\eta)(1 + |\eta|^2)^{-(s+r)/2} (1 + |\xi|^2)^{s/2}\, d\eta \right)$$

is bounded in L^p. Replacing η by $\eta + \xi$ and applyng the hypotheses we obtain

$$\| Sg \|_p \leq C_N \int (1 + |\eta|)^{-N} \| \mathcal{F}^{-1} \,(\widehat{g}(\eta + \xi)(1 + |\eta + \xi|^2)^{-s/2} (1 + |\xi|^2)^{s/2} \|_p \, d\eta.$$

From Peetre's inequality we then obtain

$$\| Sg \|_p \leq c_N\, c \int (1 + |\eta|)^{-N} (1 + |\eta|)^t \, \| g \|_p \, d\eta \leq c' \| g \|_p$$

if we take N large enough. Q. E. D.

PROOF OF THEOREM: Since

$$(P(x, D)f)^{\wedge}(\xi) = \int \widehat{f}(\eta)\, \widehat{p}(\xi - \eta, \eta)\, d\eta$$

where the Fourier transform $\widehat{p}$ is taken in the x variables alone, it suffices to verify the conditions of lemma 7 for $R(\xi, \eta) = \widehat{p}(-\eta, \eta + \xi)$. After a change of variables this amounts to estimating $\| \widehat{p}(\eta, \xi)(1 + |\xi|^2)^{-r/2} : M_p^p(\xi) \|$. But from (A1) we obtain

$$\left| \left(\frac{\partial}{\partial \xi}\right)^{\beta} \widehat{p}(\eta, \xi) \right| \leq C_N (1 + |\eta|)^{-N} (1 + |\xi|)^{r - |\beta|}.$$

Thus an application of the Marcinkiewicz multiplier theorem gives

$$\| \widehat{p}(\eta, \xi)(1 + |\xi|^2)^{-r/2} : M_p^p(\xi) \| \leq C_N (1 + |\eta|)^{-N}$$

as desired. Q. E. D.

The above arguments may be modified to deal with the classes S_a^m defined in Section 4. We let $\lambda(\xi) \in C^\infty(R^n)$ be chosen equal to $w(\xi)$ in $|\xi| \geq 1$ and non-vanishing in $|\xi| \leq 1$. We define $L_{\lambda, a}^p$ to be the closure of $\mathcal{S}$ in the norm

$$\| f : L_{\lambda, a}^p \| = \| \mathcal{F}^{-1}(\lambda(\xi)^a \widehat{f}(\xi)) \|_p .$$

We may then show that an operator with symbol of class S_a^m satisfying

$$|x|^N \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} p(x, \xi) \right| \leq C_{\alpha, \beta, N}\, \lambda(\xi)^{m - a \cdot \beta}$$

is bounded from $L_{\lambda, a+m}^p$ to $L_{\lambda, a}^p$. The key point is that $\left| \xi^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\alpha} \lambda(\xi) \right| \leq c\, \lambda(\xi)$ so the Marcinkiewicz multiplier theorem may be applied as before.

REFERENCES

[1] N. ARONSZAJN & K. T. SMITH, *Theory of Bessel potentials* I, Ann. Inst. Fourier (Grenoble) 11 (1961), 385-475.

[2] R. D. ADAMS, N. ARONSZAJN, & M. S. HANNA, *Theory of Bessel potentials III*, Ann. Inst. Fourier (Grenoble) 19 (1969) 279-338.

[3] A. P. CALDERÓN, *Lebesgue spaces of differentiable functions and distributions*, Symp. on Pure Math 5 (1961), 33-49.

[4] E. B. FABES & N. M. RIVIÈRE, *Singular integrals with mixed homogeneity*, Studia Math. 27 (1966), 19-38.

[5] S. HELGASON, *Differential geometry and symmetric spaces*, Academic Press 1962.

[6] L. HÖRMANDER, *Pseudo-differential operators and hypoelliptic equations*, Symp. on Pure Math 10 (1968), 138-183.

[7] N. JACOBSON, *Lie Algebras*, Interscience 1962.

[8] V. M. KAGAN, *Boundedness of pseudo-differential operators in L_p*, Izv. Vysš. Učebn. Zaved Mathematica 1968, no. 6 (73) 35-44 (in Russian).

[9] A. W. KNAPP & E. M. STEIN, *Intertwining operators for semisimple groups*, to appear Annals of Math.

[10] J. J. KOHN & L. NIRENBERG, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math 18 (1965), 269-305.

[11] H. KUMANO-GO, *Pseudo-differential operators and the uniqueness of the Cauchy problem*, Comm. Pure Appl. Math. 22 (1969), 73-129.

[12] H. KUMANO-GO & M. NAGASE, *L^p-theory of pseudo-differential operators*, Proc. Jap. Acad. 46 (1970), 138-142.

[13] L. NIREMBERG, *Pseudo-differential operators*, Proc. Symp. Pure Math 16 (1970), 149-167.

[14] L. P. ROTHSCHILD, *Invariant pseudo-local operators on Lie groups* (preprint).

[15] R. T. SEELEY, *Refinement of the functional calculus of Calderon and Zygmund*, Proc. Kon. Ned. Akad. wan wet. 68 (1965), 521-531.

[16] R. T. SEELEY, *Singular integrals and boundary value problems*, Am. J. Math 88 (1966), 781-809.

[17] E. M. STEIN, *Some problems in harmonic analysis suggested by symmetric spaces and semisimple groups*, International Congress of Mathematicians, Nice 1970.

[18] E. M. STEIN, *Singular integrats and differentiability properties of functions, Princeton U.* Press 1970.

[19] R. STRICHARTZ, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. 16 (1967), 1031-1060.

[20] H. STETKAER-HANSEN, *Invariant pseudo-differential operators*, Math. Scand. (to appear).

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