Annali di Matematica pura ed applicata
(IV), Vol. CLXXV (1998), pp. 93-118

# Invariant Regions for the Nernst-Planck Equations (*).

GIOVANNI CIMATTI - ILARIA FRAGALÀ

**Abstract.** – *We consider a coupling between the Nernst-Planck equations and the Navier-Stokes system; we study the stationary and the evolution problems. The crucial property turns out to be the existence of an invariant region. An asymptotic result in the case of Neumann boundary conditions is also given.*

## 1. – Introduction.

In this paper we study a model describing ionic concentrations, electric potential, and velocity field in an electrolitic solution.

The model is a coupling between the Nernst-Planck equations and the Navier-Stokes system. We refer to the books [1], [12], [13] and [14] for a detailed description of the physical background, and simply summarize here how the basic equations are derived.

If $N$ ionic species, with concentrations $P_i(x, t)$ and $N_i(x, t)$ of positive and negative charges respectively, $i = 1, \ldots, N$, are assumed to exist in the solution, the corresponding current densities $\boldsymbol{J}_i$ are given by:

$$\boldsymbol{J}_i^+ = -D_i^+ \nabla P_i + k_i^+ P_i \boldsymbol{E} + P_i \boldsymbol{v} \,, \tag{1.1}$$

$$\boldsymbol{J}_i^- = -D_i^- \nabla N_i - k_i^- N_i \boldsymbol{E} + N_i \boldsymbol{v} \,, \tag{1.2}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{v}$ is the velocity of the fluid, $D_i^\pm > 0$ are the diffusion coefficients, and $k_i^\pm > 0$ are the ionic mobilities. Since a quasi-stationary situation is assumed with all magnetic effects considered as negligible, the electric field is derivable from an electric potential $\Phi$:

$$\boldsymbol{E} = -\nabla \Phi \,. \tag{1.3}$$

(*) Entrata in Redazione l'11 novembre 1996.

Indirizzo degli AA.: Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, 56127 Pisa, Italy.

E-mail: cimatti@dm.unipi.it; fragala@dm.unipi.it

The balance equations are

$$(P_i)_t + \nabla \cdot \boldsymbol{J}_i^+ = R(P_i, N_i), \tag{1.4}$$

$$(N_i)_t + \nabla \cdot \boldsymbol{J}_i^- = R(P_i, N_i), \tag{1.5}$$

where

$$R(P_i, N_i) = \alpha_i P_{i_0} N_{i_0} - \alpha_i P_i N_i \tag{1.6}$$

is the law of mass describing the process of dissociation and recombination which occurs in the solution.

In (1.6) $P_{i_0} = N_{i_0} > 0$ is the value of the concentrations at equilibrium, while $\alpha_i$ is a recombination constant, typical of each ionic family, which cannot exceed the Langevin limit value, (see [11])

$$(\alpha_i)_{\mathrm{LIM}} = \frac{k_i^+ + k_i^-}{\varepsilon} . \tag{1.7}$$

If we define

$$A_i = \frac{\alpha_i}{(\alpha_i)_{\mathrm{LIM}}} , \tag{1.8}$$

then

$$A_i \leqslant 1 . \tag{1.9}$$

Experimental studies, see [3] and [1], show that in liquid dielectrics $A_i$ is approximately equal to one.

Substituting (1.1) and (1.2) into (1.4) and (1.5) and taking into account (1.3) and (1.6), we have:

$$(P_i)_t = D_i^+ \Delta P_i + k_i^+ \nabla \cdot (P_i \nabla \Phi) - \nabla \cdot (P_i \boldsymbol{v}) + \alpha_i P_{i_0} N_{i_0} - \alpha_i P_i N_i , \tag{1.10}$$

$$(N_i)_t = D_i^- \Delta N_i - k_i^- \nabla \cdot (N_i \nabla \Phi) - \nabla \cdot (N_i \boldsymbol{v}) + \alpha_i P_{i_0} N_{i_0} - \alpha_i P_i N_i . \tag{1.11}$$

This system is coupled with the Poisson equation

$$-\varepsilon \Delta \Phi = \sum_{j=1}^{N} (P_j - N_j), \tag{1.12}$$

and the Navier-Stokes equations

$$\varrho(\boldsymbol{v}_t + \boldsymbol{v}_{x_k} v_k) = -\nabla \pi + \eta \Delta \boldsymbol{v} - \sum_{j=1}^{N} (P_j - N_j) \nabla \Phi , \tag{1.13}$$

$$\nabla \cdot \boldsymbol{v} = 0 . \tag{1.14}$$

We remark that a solution of (1.10)-(1.14) is physically meaningful only if it satisfies the condition $P_i(x, t) \geqslant 0$, $N_i(x, t) \geqslant 0$. We take $k_0 \Phi_0 / d_0$ as a characteristic velocity, and define the following non dimensional starred quantities:

$$v = \frac{k_0 \Phi_0}{d_0} v^*, \qquad x_i = d_0 x_i^*, \qquad t = \frac{d_0^2}{k_0 \Phi_0} t^*,$$

$$\Phi = \Phi_0 \Phi^*, \qquad P_i = P_{i_0} P_i^*, \qquad N_i = N_{i_0} N_i^*,$$

$$c_i^* = \frac{P_{i_0} d_0^2}{\varepsilon \Phi_0}, \qquad D_i^{\pm} = D_0 (D_i^{\pm})^*, \qquad k_i^{\pm} = k_0 (k_i^{\pm})^*,$$

$$\pi = \varrho \left( \frac{k_0 \Phi_0}{d_0} \right)^2 \pi^*, \qquad R^* = \frac{\varrho k_0 \Phi_0}{\eta}, \qquad T^* = \frac{\varepsilon \Phi_0}{k \eta}.$$

If we let $h_i = c_i A_i (k_i^+ + k_i^-)$ and suppress the star in the nondimensional quantities, we arrive at the system

$$(P_i)_t = D_i^+ \Delta P_i + k_i^+ \nabla \cdot (P_i \nabla \Phi) - \nabla \cdot (P_i v) + h_i (1 - P_i N_i), \tag{1.15}$$

$$(N_i)_t = D_i^- \Delta N_i - k_i^- \nabla \cdot (N_i \nabla \Phi) - \nabla \cdot (N_i v) + h_i (1 - P_i N_i), \tag{1.16}$$

$$-\Delta \Phi = \sum_{j=1}^{N} c_j (P_j - N_j), \tag{1.17}$$

$$\nabla \cdot v = 0, \tag{1.18}$$

$$R(v_t + v_{x_k} v_k) = -R \nabla \pi + \Delta v - T \sum_{j=1}^{N} c_j (P_j - N_j) \nabla \Phi. \tag{1.19}$$

This paper is organized as follows. Section 2 deals with the stationary problem. The evolution problem is studied in parts 3 and 4, where the existence of an invariant region for the concentrations is discussed in detail, and a theorem of existence and uniqueness for the initial-boundary value problem is presented. Finally, in section 5 the decay of the concentrations to a simple equilibrium solution is proved under the assumption of Neumann boundary conditions for both concentrations and potential.

We end this introduction with a list of preliminary results and notations that we use throughout the paper.

Let $\Omega$ be an open, bounded and connected subset of $\mathbb{R}^n$, with a regular boundary $\partial\Omega$. We use the customary spaces $C^{m,\alpha}(\overline{\Omega})$, $L^p(\Omega)$, $H^{mp}(\Omega)$, $L^q((0, T); H^{mp}(\Omega))$, referring to the book [2] for definitions and properties. We use the notations

$$\boldsymbol{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega), \qquad \boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega).$$

Scalar product and norm are denoted by $( , )$ and $|\cdot|$ in both $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$, while for $H_0^1(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ we write $((u, v)) = (\nabla u, \nabla v)$ and $\|u\| = (\nabla u, \nabla u)^{1/2}$.

Let $A_1$: $H_0^1(\Omega) \to H^{-1}(\Omega)$ be the canonical isomorphism given by

$$\langle A_1 \varphi, \xi \rangle = ((\varphi, \xi)), \qquad \forall \xi \in H_0^1(\Omega). \tag{1.20}$$

We still denote by $A_1$ the corresponding operator in $D(A_1) = H_0^1(\Omega) \cap H^2(\Omega)$ with range in $L^2(\Omega)$. We recall the Poincaré inequality

$$\lambda_1 |\xi|^2 \leq |\nabla \xi|^2, \qquad \forall \xi \in H_0^1(\Omega) \tag{1.21}$$

where $\lambda_1$ is the first eigenvalue to the operator $A_1$. Let

$$\mathcal{V} = \{ \boldsymbol{v} \in (C_0^\infty(\Omega) \times C_0^\infty(\Omega)); \nabla \cdot \boldsymbol{v} = 0 \}$$

and denote by $\boldsymbol{H}$ and $\boldsymbol{V}$ the closures of $\mathcal{V}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$. Let $A_0$: $\boldsymbol{V} \to \boldsymbol{V}'$ be the canonical isomorphism given by

$$\langle A_0 \boldsymbol{v}, \boldsymbol{w} \rangle = ((\boldsymbol{v}, \boldsymbol{w})), \qquad \forall \boldsymbol{w} \in \boldsymbol{V}. \tag{1.22}$$

As above we denote by $A_0$ the corresponding operator in $D(A_0) = \boldsymbol{V} \cap H^2(\Omega)$ with range in $\boldsymbol{H}$. The counterpart of (1.21) is now

$$\mu_1 |\boldsymbol{w}|^2 \leq |\nabla \boldsymbol{w}|^2, \qquad \forall \boldsymbol{w} \in \mathcal{V}, \tag{1.23}$$

where $\mu_1$ is the first eigenvalue of the operator $A_0$.
As usual with problems involving the Navier-Stokes equations, we set

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega u_i D_i v_j w_j \, dx,$$

whenever the integral exists. The identity

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \tag{1.24}$$

follows immediately from the definition, while the proof of the following lemmas can be found in [16].

LEMMA 1.1. – *If $n = 2$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$, we have*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq 2^{1/2} |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{1/2} \|\boldsymbol{v}\| |\boldsymbol{w}|^{1/2} \|\boldsymbol{w}\|^{1/2}. \tag{1.25}$$

LEMMA 1.2. – *If $n = 2$ and $u \in L^2((0, T); H^1(\Omega)) \cap L^\infty((0, T); L^2(\Omega))$, then*

$$u \in L^4((0, T); L^4(\Omega)).$$

Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$, let $\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v})$ be the distribution defined by

$$\langle \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \qquad \forall \boldsymbol{w} \in \mathcal{V}.$$

For $\boldsymbol{u} = \boldsymbol{v}$, we write

$$\boldsymbol{B}(\boldsymbol{u}) = b(\boldsymbol{u}, \boldsymbol{u}).$$

LEMMA 1.3. – *If $n = 2$, $\boldsymbol{B}(\boldsymbol{u})$ belongs to $\mathcal{V}'$ and satisfies the estimate*

$$\|\boldsymbol{B}(\boldsymbol{u})\|_{\mathcal{V}'} \leq c \|\boldsymbol{u}\|^2_{L^4(\Omega)}, \tag{1.26}$$

*where $c$ is a positive constant.*

## 2. – The stationary problem.

The steady-state system corresponding to (1.15)-(1.19), i.e.

$$D_i^+ \Delta P_i + k_i^+ \nabla\cdot(P_i \nabla\Phi) - \nabla\cdot(P_i \boldsymbol{v}) + h_i(1 - P_i N_i) = 0 , \tag{2.1}$$

$$D_i^- \Delta N_i - k_i^- \nabla\cdot(N_i \nabla\Phi) - \nabla\cdot(N_i \boldsymbol{v}) + h_i(1 - P_i N_i) = 0 , \tag{2.2}$$

$$-\Delta\Phi = \sum_{j=1}^{N} c_j(P_j - N_j), \tag{2.3}$$

$$\nabla\cdot\boldsymbol{v} = 0 , \tag{2.4}$$

$$R\boldsymbol{v}_{xk} v_k = -R\nabla\pi + \Delta\boldsymbol{v} - T\sum_{j=1}^{N} c_j(P_j - N_j)\,\nabla\Phi , \tag{2.5}$$

has the trivial equilibrium solution

$$(P_1, \dots, P_N, N_1, \dots, N_N, \Phi, \boldsymbol{v}) = (1, \dots, 1, 1, \dots, 1, \Phi_A, 0), \tag{2.6}$$

where $\Phi_A$ is an harmonic function defined in $\overline{\Omega}$. Let us consider (2.1)-(2.5) with boundary conditions of the form

$$P_i = P_{ib}(x, \lambda), \qquad N_i = N_{ib}(x, \lambda) \text{ on } \partial\Omega , \tag{2.7}$$

$$\Phi = \Phi_b \qquad \text{on } \Omega , \tag{2.8}$$

$$\boldsymbol{v} = 0 \qquad \text{on } \partial\Omega , \tag{2.9}$$

where $\lambda \in (-\bar{\lambda}, \bar{\lambda})$ is a parameter. Assume

$$P_{ib}, N_{ib} \in C^{2,\alpha}(\partial\Omega), \qquad \forall\lambda \in (-\bar{\lambda}, \bar{\lambda}), \tag{2.10}$$

$$P_{ib}, N_{ib} > 0 \quad \text{on } \partial\Omega, \qquad \forall\lambda \in (-\bar{\lambda}, \bar{\lambda}), \tag{2.11}$$

$$P_{ib}(x, 0) = N_{ib}(x, 0) = 1 \qquad \text{on } \partial\Omega . \tag{2.12}$$

Our goal is to prove the existence of an unique continuous branch of physically acceptable solutions starting from the trivial one (2.6). We shall treat the problem in Hölder spaces, assuming $\partial\Omega$ so regular that $C^{2,\alpha}(\partial\Omega)$ is meaningful. Let

$$X = (C^{2,\alpha}(\overline{\Omega}))^{2N} \times C^{2,\alpha}(\overline{\Omega}) \times \mathcal{W} ,$$

$$Y = (C^{0,\alpha}(\overline{\Omega}))^{2N} \times C^{0,\alpha}(\overline{\Omega}) \times (C^{0,\alpha}(\overline{\Omega}))^3 \times (C^{2,\alpha}(\partial\Omega))^{2N} \times C^{2,\alpha}(\partial\Omega) \times (C^{2,\alpha}(\partial\Omega))^3 ,$$

where

$$\mathcal{W} = \{\boldsymbol{v} \in (C^{2,\alpha}(\overline{\Omega}))^3;\ \nabla\cdot\boldsymbol{v} \text{ in } \Omega\} .$$

Define the map $F: X \times (-\bar{\lambda}, \bar{\lambda}) \to Y$ as follows: if $\boldsymbol{u} = (P_1, \ldots, P_N, N_1, \ldots, N_N, \Phi, \boldsymbol{v})$, then

$$F(\boldsymbol{u}, \lambda) = \begin{pmatrix} -D_i^+ \Delta P_i - k_i^+ \nabla \cdot (P_i \nabla \Phi) + \nabla \cdot (P_i \boldsymbol{v}) - h_i(1 - P_i N_i) \\ -D_i^- \Delta N_i + k_i^- \nabla \cdot (N_i \nabla \Phi) + \nabla \cdot (N_i \boldsymbol{v}) - h_i(1 - P_i N_i) \\ -\Delta \Phi - \sum_{j=1}^{N} c_j (P_j - N_j) \\ R \boldsymbol{v}_{xk} v_k - R \nabla \pi + \Delta \boldsymbol{v} - T \sum_{j=1}^{N} c_j (P_j - N_j) \nabla \Phi \\ \nabla \cdot \boldsymbol{v} \\ P_i \,|_{\partial\Omega} - P_{ib}(x, \lambda) \\ N_i \,|_{\partial\Omega} - N_{ib}(x, \lambda) \\ \Phi \,|_{\partial\Omega} - \Phi_b \\ \boldsymbol{v} \,|_{\partial\Omega} \end{pmatrix}.$$

The operators $F(\boldsymbol{u}, \lambda)$ and $F_{\boldsymbol{u}}(\boldsymbol{u}, \lambda)$ are both continuous, and we have $F(\boldsymbol{u}_0, 0) = 0$, with $\boldsymbol{u}_0 = (1, \ldots, 1, 1, \ldots, 1, \Phi_A, \boldsymbol{0})$. To prove that $F_u(\boldsymbol{u}_0, 0)$ is continuously invertible, we need the following algebraic

LEMMA 2.1. – *Let us consider the* $2 \times 2$ *matrices:*

$$D_i = \begin{bmatrix} -c_i k_i^+ - h_i & c_i k_i^+ - h_i \\ c_i k_i^- - h_i & -c_i k_i^- - h_i \end{bmatrix}, \qquad i = 1, \ldots, N, \tag{2.13}$$

$$L_{ij} = \begin{bmatrix} -c_j k_i^+ & c_j k_i^+ \\ c_j k_i^- & -c_j k_i^- \end{bmatrix}, \qquad i, j = 1, \ldots, N. \tag{2.14}$$

*Let* $M$ *be the* $2N \times 2N$ *matrix defined by the blocks* $D_i$ *and* $L_{ij}$ *as*

$$M = \begin{bmatrix} D_1 & L_{12} & L_{13} & \ldots & L_{1N} \\ L_{21} & D_2 & L_{23} & \ldots & L_{2N} \\ L_{31} & L_{32} & D_3 & \ldots & L_{3N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{N1} & L_{N2} & L_{N3} & \ldots & D_N \end{bmatrix}. \tag{2.15}$$

*Then* $M$ *has the following eigenvalues*

$$\left\{ -2h_1, \ldots, -2h_N, -\sum_{j=1}^{N} c_j (k_j^+ + k_j^-), 0 \right\}, \tag{2.16}$$

*and the corresponding multiplicities are*

$$\{1, \ldots, 1, 1, N-1\}. \tag{2.17}$$

PROOF. – We first prove by elementary operations the equality

$$\det(M-\lambda I)=\left[\prod_{i=1}^{N}(-2h_i-\lambda)\right]\cdot\det(M^*), \tag{2.18}$$

where $M^*=[m_{ij}^*]_{i,j=1,\ldots,N}$ is the $N\times N$ matrix defined by

$$m_{ij}^*=\begin{cases}1 & \text{if } i\neq j\,,\\ 1+\dfrac{\lambda}{c_i(k_i^+ + k_i^-)} & \text{if } i=j\,.\end{cases}$$

To this aim, take the matrix $(M-\lambda I)$ and replace the $(2k-1)$-th column by the sum of the $(2k-1)$-th column plus the $(2k)$-th column, for $k=1,\ldots,N$; then replace the $(2k-1)$-th row by the difference between the $(2k-1)$-th row and the $(2k)$-th row, for $k=1,\ldots,N$. We get

$$\det(M-\lambda I)=\left[\prod_{i=1}^{N}(-2h_i-\lambda)\right]\cdot\det(M^{**}),$$

where the matrix $M^{**}=[m_{ij}^{**}]_{i,j=1,\ldots,N}$ is given by

$$m_{ij}^{**}=\begin{cases}c_j(k_i^+ + k_i^-) & \text{if } i\neq j\,,\\ c_i(k_i^+ + k_i^-)+\lambda & \text{if } i=j\,.\end{cases}$$

We obtain (2.18) if we divide the $(i)$-th row of $M^{**}$ by $(k_i^+ + k_i^-)$, for $i=1,\ldots,N$, and the $(j)$-th column by $c_j$, for $j=1,\ldots,N$. Let now

$$\gamma_i=c_i(k_i^+ + k_i^-),\qquad i=1,\ldots,N\,.$$

We prove, by induction on $N$, that

$$\det(M^*)=\left(\prod_{i=1}^{N}\gamma_i\right)\cdot\lambda^{N-1}\cdot\left(-\lambda-\sum_{i=1}^{N}\gamma_i\right).$$

When $N=1$ the assertion is obviously true. Suppose that the formula holds whenever the size of $M^*$ is less or equal to $N-1$. Replace the first row of $M^*$ by the difference between the first and the second row, and the first column by the difference between the first and second column. It follows

$$\det(M^*)=\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\cdot\lambda\cdot\left[\left(\prod_{i=2}^{N}\gamma_i\right)\cdot\lambda^{N-2}\cdot\left(-\lambda-\sum_{i=2}^{N}\gamma_i\right)\right]+$$

$$-\left(-\frac{1}{\gamma_2}\right)^2\cdot\lambda^2\cdot\left[\left(\prod_{i=3}^{N}\gamma_i\right)\cdot\lambda^{N-3}\cdot\left(-\lambda-\sum_{i=3}^{N}\gamma_i\right)\right]=\left(\prod_{i=1}^{N}\gamma_i\right)\cdot\lambda^{N-1}\cdot\left(-\lambda-\sum_{i=1}^{N}\gamma_i\right)$$

and the lemma is proved. ∎

COROLLARY 2.2. – *The quadratic form*

$$(2.19)\qquad Q(P_1, N_1, \ldots, P_N, N_N) = -\left[\sum_{i=1}^{N} k_i^+ P_i\right]\left[\sum_{j=1}^{N} c_j(P_j - N_j)\right] + \\ -\sum_{i=1}^{N} h_i P_i(P_i + N_i) + \left[\sum_{i=1}^{N} k_i^- N_i\right]\left[\sum_{j=1}^{N} c_j(P_j - N_j)\right] - \sum_{i=1}^{N} h_i N_i(P_i + N_i)$$

*is negative semidefinite.*

PROOF. – This is an immediate consequence of Lemma 2.1 since

$Q(P_1, N_1, \ldots, P_N, N_N) = M(P_1, N_1, \ldots, P_N, N_N)\cdot(P_1, N_1, \ldots, P_N, N_N)$. ∎

LEMMA 2.3. – *The operator $F_u(\boldsymbol{u}_0, 0)$ is continuously invertible.*

PROOF. – By the closed graph theorem, we only need to prove that the linear boundary value problem:

$$(2.20)\qquad D_i^+ \Delta P_i + k_i^+ \nabla P_i \cdot \nabla \Phi_A - k_i^+ \sum_{j=1}^{N} c_j(P_j - N_j) - h_i(P_i + N_i) = 0\,,$$

$$(2.21)\qquad D_i^- \Delta N_i - k_i^- \nabla N_i \cdot \nabla \Phi_A + k_i^- \sum_{j=1}^{N} c_j(P_j - N_j) - h_i(P_i + N_i) = 0\,,$$

$$(2.22)\qquad -\Delta \Phi = \sum_{j=1}^{N} c_j(P_j - N_j)\,,$$

$$(2.23)\qquad A_0 \boldsymbol{v} - T \sum_{j=1}^{N} c_j(P_j - N_j)\,\nabla \Phi_A = 0\,,$$

$$(2.24)\qquad P_i = 0 \quad \text{on } \partial\Omega\,,$$

$$(2.25)\qquad N_i = 0 \quad \text{on } \partial\Omega\,,$$

$$(2.26)\qquad \Phi = 0 \quad \text{on } \partial\Omega\,,$$

$$(2.27)\qquad \boldsymbol{v} = 0 \quad \text{on } \partial\Omega\,,$$

has only the trivial solution $(P_1, N_1, \ldots, P_N, N_N) = (0, 0, \ldots, 0, 0)$. We observe that equations (2.20) and (2.21) are uncoupled with (2.22) and (2.23). Therefore it is sufficient to show that the unique solution of (2.20)-(2.21), which satisfies also the boundary conditions (2.24)-(2.25), is the trivial one. Let us multiply (2.20) by $P_i$ and (2.21) by $N_i$. Since $\Phi_A$ is harmonic we have, after integration by parts,

$$-D_i^+ \int_\Omega |\nabla P_i|^2 - k_i^+ \int_\Omega P_i \sum_{j=1}^{N} c_j(P_j - N_j) - h_i \int_\Omega P_i(P_i + N_i) = 0\,,$$

$$-D_i^- \int_\Omega |\nabla N_i|^2 + k_i^- \int_\Omega N_i \sum_{j=1}^{N} c_j(P_j - N_j) - h_i \int_\Omega N_i(P_i + N_i) = 0\,.$$

The sum over $i$ gives us

$$-\sum_{i=1}^{N} D_1^+ \int_\Omega |\nabla P_i|^2 - \sum_{i=1}^{N} D_i^- \int_\Omega |\nabla N_i|^2 + \int_\Omega Q(P_1, N_1, \ldots, P_N, N_N) = 0 .$$

By Corollary 2.2 we get $P_i = N_i = 0$. ■

The implicit function theorem in Banach spaces can now be applied to the equation $F(\boldsymbol{u}, \lambda) = 0$. We conclude that there exists a local branch of solutions to the corresponding boundary value problem (2.1)-(2.9), i.e.

$$(P_1(x, \lambda), \ldots, P_N(x, \lambda), N_1(x, \lambda), \ldots, N_N(x, \lambda), \Phi(x, \lambda), \boldsymbol{v}(x, \lambda)) , \tag{2.28}$$

which is defined for $|\lambda| < \bar{\lambda}$, with $\bar{\lambda}$ small enough. The solutions given by (2.28) are physically realistic because we have the following

LEMMA 2.4. – *The concentrations $P_i(x, \lambda)$, $N_i(x, \lambda)$ are positive in $\overline{\Omega}$ for $|\lambda| < \bar{\lambda}$.*

PROOF. – Define

$$m(\lambda) = \min \{\inf_{\overline{\Omega}} P_i(x, \lambda), \inf_{\overline{\Omega}} N_i(x, \lambda) \colon i = 1, \ldots, N\} .$$

Then $m(0) > 0$, and we can consider

$$\lambda_0 = \sup \{\lambda \in [0, \bar{\lambda}); m(\lambda) > 0\} .$$

If, by contradiction, $\lambda_0 < \bar{\lambda}$, there exists a point $x_0 \in \overline{\Omega}$ and a concentration, e.g. $P_1$, such that

$$P_1(x, \lambda_0) \geqslant P_1(x_0, \lambda_0) = 0 , \qquad \forall x \in \overline{\Omega} .$$

Since $P_{1b}(x, \lambda) > 0$, we cannot have $x_0 \in \partial\Omega$ and therefore $x_0$ must be an interior point of $\overline{\Omega}$. In this case, putting $x = x_0$ into the equation for the concentration $P_1$, we obtain

$$-D_i^+ \Delta P_1 = h_1 \leqslant 0 .$$

This proves the required positivity. ■

It would clearly be interesting to prove a result of existence valid for arbitrary boundary conditions. For the case of one ionic family, i.e. for the system

$$-D^+ \Delta P + (\boldsymbol{v} - k^+ \nabla\Phi)\cdot\nabla P + ck^+ P(P - N) - h(1 - PN) = 0 , \tag{2.29}$$

$$-D^- \Delta N + (\boldsymbol{v} + k^- \nabla\Phi)\cdot\nabla N - ck^- N(P - N) - h(1 - PN) = 0 , \tag{2.30}$$

$$-\Delta\Phi = c(P - N) , \tag{2.31}$$

$$R(\boldsymbol{v}_t - \boldsymbol{v}_{x_k} v_k) = -R\nabla\pi + \Delta\boldsymbol{v} - Tc(P - N)\,\nabla\Phi , \tag{2.32}$$

$$\nabla\cdot\boldsymbol{v} = \boldsymbol{0} , \tag{2.33}$$

$$P = N = 0 \quad \text{on } \partial\Omega, \tag{2.34}$$

$$\Phi = \Phi_b \quad \text{on } \partial\Omega, \tag{2.35}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{2.36}$$

we have

THEOREM 2.5. – *Let*

$$h \geq \max\left\{\frac{ck^+}{2}, \frac{ck^-}{2}\right\}. \tag{2.37}$$

*Then there exists at least one solution of problem* (2.29)-(2.36) *which satisfies*

$$P(x) \geq 0, \quad N(x) \geq 0 \quad \text{in } \overline{\Omega}. \tag{2.38}$$

PROOF. – We apply the Schauder fixed point theorem. Let

$$\Sigma = \{(P, N) \in \boldsymbol{L}^2(\Omega);\ P(x) \geq 0, N(x) \geq 0 \text{ a.e}, \ |P|^2 \leq \eta^+, \ |N|^2 \leq \eta^-\},$$

where

$$\eta^{\pm} = \frac{h^4 |\Omega|}{(D^{\pm})^2 \lambda_1^2}$$

($|\Omega|$ denotes the Lebesgue measure of $\Omega$).
$\Sigma$ is a closed and convex subset of $\boldsymbol{L}^2(\Omega)$. Define the map

$$T: \Sigma \ni (\tilde{P}, \tilde{N}) \mapsto (P, N) \in \boldsymbol{L}^2(\Omega),$$

via the following linear problem:

$$\tilde{\Phi} - \Phi_b \in H_0^1(\Omega), \quad \int_\Omega \nabla\tilde{\Phi}\cdot\nabla\xi = c\int_\Omega (\tilde{P} - \tilde{N})\,\xi, \quad \forall\xi \in H_0^1(\Omega), \tag{2.39}$$

$$\tilde{\boldsymbol{v}} \in \boldsymbol{V}, \quad Rb(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}, \boldsymbol{u}) + ((\tilde{\boldsymbol{v}}, \boldsymbol{u})) = -Tc\int_\Omega (\tilde{P} - \tilde{N})\,\nabla\tilde{\Phi}\cdot\boldsymbol{u}, \quad \forall \boldsymbol{u} \in \boldsymbol{V}, \tag{2.40}$$

$$P \in H_0^1(\Omega), \quad D^+\int_\Omega \nabla P\cdot\nabla\xi + k^+\int_\Omega P\nabla\tilde{\Phi}\cdot\nabla\xi - \int_\Omega P\tilde{\boldsymbol{v}}\cdot\nabla\xi + \tag{2.41}$$

$$+ h\int_\Omega P\tilde{N}\xi = h\int_\Omega \xi, \quad \forall\xi \in H_0^1(\Omega),$$

$$N \in H_0^1(\Omega), \quad D^- \int_\Omega \nabla N \cdot \nabla \xi - k^- \int_\Omega N \nabla \widetilde{\Phi} \cdot \nabla \xi - \int_\Omega N \tilde{v} \cdot \nabla \xi + \\ + h \int_\Omega \widetilde{P} N \xi = h \int_\Omega \xi, \qquad \forall \xi \in H_0^1(\Omega). \tag{2.42}$$

$T$ is well defined and continuous by standard arguments from the linear theory. We claim that $T$ is compact and $T(\Sigma) \subseteq \Sigma$. Setting $\xi = P$ in (2.39) and using (2.41), we find:

$$D^+ \int_\Omega |\nabla P|^2 + \int_\Omega \left[ \frac{ck^+}{2} \widetilde{P} + \left( h - \frac{ck^+}{2} \right) \widetilde{N} \right] P^2 = h \int_\Omega P.$$

We get the following estimate:

$$D^+ \int_\Omega |\nabla P|^2 \leq h \int_\Omega |P| \leq \frac{D^+ \lambda_1}{2} \int_\Omega |P|^2 + \frac{h^4 |\Omega|}{2D^+ \lambda_1} \leq \frac{D_1}{2} \int_\Omega |\nabla P|^2 + \frac{h^4 |\Omega|}{2D^+ \lambda_1},$$

where we have used hypothesis (2.37), the Young inequality, and inequality (1.21). Therefore we have:

$$\int_\Omega |\nabla P|^2 \leq \frac{h^4 |\Omega|}{(D^+)^2 \lambda_1}, \qquad \int_\Omega |P|^2 \leq \frac{h^4 |\Omega|}{(D^+)^2 \lambda_1^2} = \eta^+ .$$

In a similar way, we find the analogous estimates for the concentration $N$. Thus $T(\Sigma)$ is compact by Rellich's Theorem. Setting $\xi = P^- = \min\{0, P\}$ in (2.41) we obtain

$$D^+ \int_{\{x \in \Omega :\, P(x) \leq 0\}} |\nabla P|^2 + \int_{\{x \in \Omega :\, P(x) \leq 0\}} \left[ \frac{ck^+}{2} \widetilde{P} + \left( h - \frac{ck^+}{2} \right) \widetilde{N} \right] P^2 = h \int_{\{x \in \Omega :\, P(x) \leq 0\}} P.$$

It follows $P^- = 0$ a.e. on $\Omega$. After repeating the same argument for $N$, we conclude that $T(\Sigma) \subseteq \Sigma$. This completes the proof. ■

REMARK 2.6. – In liquid dielectrics, $h$ is approximately equal to $c(k^+ + k^-)$ by Langevin's theory. Therefore (2.37) is quite accettable from the physical point of view.

## 3. – Invariant regions for the concentrations.

We recall the definition of invariant region for weakly coupled parabolic systems, together with related results. We refer for the proof to the excellent book by Smoller [15]; we also quote, among the numerous papers related to invariant regions, [4], [5],

[6], [7], [8] and [17]. Attention will be focused on the solutions of

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i + \sum_{h=1}^{n} b_{ih}(x, t) \frac{\partial u_i}{\partial x_h} = f_i(\boldsymbol{u}), \tag{3.1}$$

where $\boldsymbol{u} = (u_1, \ldots, u_m)$, $d_i > 0$ for $i = 1, \ldots, m$, and $\boldsymbol{F}(\boldsymbol{u}) = (f_1(\boldsymbol{u}), \ldots, f_m(\boldsymbol{u}))$ is a smooth vector field defined on $\mathbb{R}^m$. Let $G_k$, $k = 1, \ldots, s$, be real functions of class $C^1$ on $\mathbb{R}^m$ such that $\nabla G_k \neq \boldsymbol{0}$. The set

$$\Sigma = \bigcap_{k=1}^{s} \{\boldsymbol{u} \in \mathbb{R}^m;\ G_k(\boldsymbol{u}) \leq 0\} \tag{3.2}$$

is called an *invariant region* for (3.1) if any regular solution $\boldsymbol{u}(x, t)$ of (3.1), such that

$$\boldsymbol{u}(x, 0) \in \Sigma, \qquad \forall x \in \Omega, \tag{3.3}$$

$$\boldsymbol{u}(x, t) \in \Sigma, \qquad \forall (x, t) \in \partial\Omega \times (0, T), \tag{3.4}$$

satisfies

$$\boldsymbol{u}(x, t) \in \Sigma, \qquad \forall (x, t) \in \Omega \times (0, T). \tag{3.5}$$

THEOREM 3.1. – *Let $\Sigma$, defined by* (3.2), *be an invariant region for* (3.1). *Then at every point $\boldsymbol{u} \in \partial\Sigma$ where $G_k(\boldsymbol{u}) = 0$, we have*

$$\nabla G_k(\boldsymbol{u}) \cdot \boldsymbol{F}(\boldsymbol{u}) \leq 0. \tag{3.6}$$

THEOREM 3.2. – *Let $d_i \neq d_j$ when $i \neq j$. If $\Sigma$ is an invariant region for* (3.1), *then it has the special form*

$$\Sigma = \bigcap_{i=1}^{m} \{\boldsymbol{u} \in \mathbb{R}^m;\ a_i \leq u_i \leq b_i\}. \tag{3.7}$$

We also quote the following sufficient conditions.

THEOREM 3.3. – *Assume $d_i = d > 0$ and let $\Sigma$, given by* (3.2), *be convex. Moreover, suppose $\boldsymbol{F}(\boldsymbol{u})$ never points out of $\Sigma$ for $\boldsymbol{u}$ on the boundary of $\Sigma$. Then $\Sigma$ is an invariant region for* (3.1).

THEOREM 3.4. – *Let $\Sigma$ be given by* (3.7) *and assume $\boldsymbol{F}(\boldsymbol{u})$ never points out of $\Sigma$ whenever $\boldsymbol{u}$ belongs to $\partial\Sigma$. Then $\Sigma$ is an invariant region for* (3.1).

Equations (1.12) and (1.13) are not of the type (3.1); however, they can be written in a form suitable for applying Theorems 3.1 and 3.4. More precisely, if we substitute (1.14) in (1.12) and (1.13), and take into account (1.15), we obtain

$$(P_i)_t = D_i^+ \Delta P_i + (k_i^+ \nabla\Phi - \boldsymbol{v}) \cdot \nabla P_i + f_i(\boldsymbol{P}, \boldsymbol{N}), \tag{3.8}$$

$$(N_i)_t = D_i^- \Delta N_i - (k_i^- \nabla\Phi + \boldsymbol{v}) \cdot \nabla N_i + g_i(\boldsymbol{P}, \boldsymbol{N}), \tag{3.9}$$

where

$$f_i(\boldsymbol{P}, \boldsymbol{N}) = -k_i^+ P_i \sum_{j=1}^{N} c_j(P_j - N_j) + h_i(1 - P_i N_i), \tag{3.10}$$

$$g_i(\boldsymbol{P}, \boldsymbol{N}) = k_i^- N_i \sum_{j=1}^{N} c_j(P_j - N_j) + h_i(1 - P_i N_i). \tag{3.11}$$

The cases $N = 1$ and $N > 1$ are considerably different. In fact, we have

THEOREM 3.5. – *Let $N = 1$ and define*

$$Q = \{(P, N) \in \mathbb{R}^2 : 0 \leq P \leq 1 + r, \, 0 \leq N \leq 1 + r\},$$

*where $r$ is a positive constant. Then, there exists $\hat{r} > 0$ such that $Q$ is an invariant region for the concentrations for all $r \geq \hat{r}$.*

PROOF. – In the present case of one ionic family, (3.10)-(3.11) become

$$f(P, N) = -kcP(P - N) + h(1 - PN),$$

$$g(P, N) = kcN(P - N) + h(1 - PN).$$

To apply Theorem 3.4, we put

$$\partial Q = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where

$$\Gamma_1 = \{(P, N); \, N = 0, \, 0 \leq P \leq 1 + r\},$$

$$\Gamma_2 = \{(P, N); \, P = 1 + r, \, 0 \leq N \leq 1 + r\},$$

$$\Gamma_3 = \{(P, N); \, N = 1 + r, \, 0 \leq P \leq 1 + r\},$$

$$\Gamma_4 = \{(P, N); \, P = 0, \, 0 \leq N \leq 1 + r\}.$$

If $\boldsymbol{\nu}$ is the exterior unit normal vector to $\partial Q$ and $\boldsymbol{F}(P, N) = (f(P, N), g(P, N))$, we have

$$\boldsymbol{F} \cdot \boldsymbol{\nu}|_{\Gamma_1} = -h < 0, \qquad P \in [0, 1 + r]$$

and, in the same way,

$$\boldsymbol{F} \cdot \boldsymbol{\nu}|_{\Gamma_4} < 0, \qquad N \in [0, 1 + r].$$

Moreover, $\boldsymbol{F} \cdot \boldsymbol{\nu}|_{\Gamma_2}$ is a linear function of $N$, which is negative for $N = 0$ and $N = 1 + r$ if $r \geq \hat{r}$, with $\hat{r}$ sufficiently large. Similarly we have $\boldsymbol{F} \cdot \boldsymbol{\nu}|_{\Gamma_3} < 0$. Hence, by Theorem 3.4, the result follows. ■

THEOREM 3.6. – *Assume* $N \geq 1$ *and*

$$D_i^+ = D_i^- = D\,, \qquad i = 1, \ldots, N\,. \tag{3.12}$$

*Let* $r, \alpha_1, \ldots, \alpha_N$ *be positive constants and define*

$$\varrho_i = c_i(P_i - N_i)\,, \qquad \varrho = \sum_{j=1}^{N} \varrho_j\,,$$

$$Q = \{(\boldsymbol{P}, \boldsymbol{N}) \in \mathbb{R}^{2n};\ 0 \leq P_i,\ N_i \leq 1 + r,\ i = 1, \ldots, N\}\,,$$

$$S^+ = \{(\boldsymbol{P}, \boldsymbol{N}) \in \mathbb{R}^{2n};\ \varrho \geq 0,\ \varrho_i \leq \alpha_i,\ i = 1, \ldots, N\}\,,$$

$$S^- = \{(\boldsymbol{P}, \boldsymbol{N}) \in \mathbb{R}^{2n};\ \varrho \leq 0,\ \varrho_i \geq \alpha_i,\ i = 1, \ldots, N\}\,.$$

*Then, there exists a positive constant* $\hat{r}$ *such that the convex sets*

$$\Sigma^+ = Q \cap S^+\,, \qquad \Sigma^- = Q \cap S^-$$

*are invariant for the concentrations when* $r \geq \hat{r}$.

PROOF. – In view of (3.12), we can apply Theorem 3.4 to the vector field $\boldsymbol{F}(\boldsymbol{P}, \boldsymbol{N})$ and the convex set $\Sigma^+$, where $\boldsymbol{F}(\boldsymbol{P}, \boldsymbol{N}) = (f_i(\boldsymbol{P}, \boldsymbol{N}), g_i(\boldsymbol{P}, \boldsymbol{N}))$, $i = 1, \ldots, N$. The boundary of $\Sigma^+$ can be decomposed as follows:

$$\partial\Sigma^+ = \{(\partial S^+) \cap Q\} \cup \{(\partial Q) \cap S^+\}\,. \tag{3.13}$$

Using the notation $\Gamma_y^a$ to denote the set where $y = a$, we can write

$$\begin{cases} \partial S^+ = \left\{\Gamma_\varrho^0 \cup \bigcup_{i=1}^{N} \Gamma_\varrho^{ai}\right\} \cap S^+\,, \\ \partial Q = \left\{\left(\bigcup_{i=1}^{N} \Gamma_{P_i}^0\right) \cup \left(\bigcup_{i=1}^{N} \Gamma_{N_i}^0\right) \cup \left(\bigcup_{i=1}^{N} \Gamma_{P_i}^{1+r}\right) \cup \left(\bigcup_{i=1}^{N} \Gamma_{N_i}^{1+r}\right)\right\} \cap Q\,. \end{cases} \tag{3.14}$$

Then, successively we have

I. $\quad \boldsymbol{F}\cdot\boldsymbol{\nu}|_{\Gamma_\varrho^0} = \sum_{i=1}^{N} f_i - g_i\,|_{\varrho = 0} = 0,$

II. $\quad \boldsymbol{F}\cdot\boldsymbol{\nu}|_{\Gamma_{\varrho i}^{ai}} = f_i - g_i\,|_{\varrho_i = \alpha_i} = -k(P_i + N_i)\,\varrho\,|_{\varrho_i = \alpha_i} \leq 0$ on $S^+$,

III.*a* $\quad \boldsymbol{F}\cdot\boldsymbol{\nu}|_{\Gamma_{P_i}^0} = -f_i\,|_{P_i = 0} = -h_i < 0,$

III.*b* $\quad \boldsymbol{F}\cdot\boldsymbol{\nu}|_{\Gamma_{N_i}^0} = -g_i\,|_{N_i = 0} = -h_i < 0,$

IV.*a* $\quad \boldsymbol{F}\cdot\boldsymbol{\nu}|_{\Gamma_{P_i}^{1+r}} = f_i\,|_{P_i = 1+r} = \varphi_i(P_1, \ldots, \widehat{P}\widehat{P}_i, \ldots, N_N)$, with

$$\varphi_i(P_1, \ldots, \widehat{P}_i, \ldots, N_N) = -k(1+r)\left\{c_i[(1+r) - N_i] + \sum_{\substack{j=1 \\ j \neq i}}^{N} \varrho_j\right\} + h_i[1 - (1+r)\,N_i]\,.$$

We claim that the function $\varphi_i$ is nonpositive on the polyhedron $\mathcal{P}$ of $\mathbb{R}^{2N-1}$ given by

$$\mathcal{P}=\left\{\begin{array}{ll} 0\leq P_j\leq 1+r\,, & \forall j=1,\dots,N\,,\ j\neq i\\ 0\leq N_j\leq 1+r\,, & \forall j=1,\dots,N\\ c_i[(1+r)-N_i]+\sum\limits_{\substack{j=1\\ j\neq i}}^{N}\varrho_j\geq 0 & \\ c_i[(1+r)-N_i]\leq\alpha_i & \\ \varrho_j\leq\alpha_j\,, \quad \forall j=1,\dots,N\,,\ j\neq i & \end{array}\right\}.$$

Since $\alpha_i\geq 0$, $\mathcal{P}$ is non empty. For $r$ large enough, we can write $\mathcal{P}$ as

$$\mathcal{P}=\tilde{\mathcal{P}}\cap\left\{c_i(1+r)-\alpha_i\leq c_iN_i\leq c_i(1+r),\ \sum_{\substack{j=1\\ j\neq i}}^{N}\varrho_j\geq c_i[N_i-(1+r)]\right\},$$

where

$$\tilde{\mathcal{P}}=\{0\leq P_j\leq 1+r,\ 0\leq N_j\leq 1+r,\ \varrho_j\leq\alpha_j\ \ \forall j=1,\dots,N,\ j\neq i\}.$$

We observe that $\varphi_i$ is a linear function of its variables, and therefore it will be non positive on $\mathcal{P}$, provided it is non positive on $\partial\mathcal{P}$. Iterating this argument $(2N-1)$ times, we conclude that we only need to check the sign of $\varphi_i$ on the vertices of the polyhedron $\mathcal{P}$. Let us consider the superior and inferior faces

$$\mathcal{F}^+=\tilde{\mathcal{P}}\cap\{c_i(1+r)\}\,,\qquad \mathcal{F}^-=\tilde{\mathcal{P}}\cap\{c_i(1+r)-\alpha_i\}$$

and the halfspace

$$\mathcal{H}=\left\{\sum_{\substack{j=1\\ j\neq i}}\varrho_j\geq c_i[N_i-(1+r)]\right\}.$$

The intersection between each face with the halfspace gives

$$\mathcal{F}^+\cap\mathcal{H}=\left\{\begin{array}{l} 0\leq P_j\leq 1+r,\ \ 0\leq N_j\leq 1+r,\ \ \varrho_j\leq\alpha_j,\ \ \forall j=1,\dots,N,\ \ j\neq i\\ \sum\limits_{\substack{j=1\\ j\neq i}}^{N}\varrho_j\geq 0\,,\qquad c_iN_i=c_i(1+r) \end{array}\right\},$$

$$\mathcal{F}^-\cap\mathcal{H}=\left\{\begin{array}{l} 0\leq P_j\leq 1+r,\ \ 0\leq N_j\leq 1+r,\ \ \varrho_j\leq\alpha_j,\ \ \forall j=1,\dots,N,\ \ j\neq i\\ \sum\limits_{\substack{j=1\\ j\neq i}}^{N}\varrho_j\geq -\alpha_i\,,\qquad c_iN_i=c_i(1+r)-\alpha_i \end{array}\right\}.$$

If we recall the expression of $\varphi_i$, we obtain

$$\varphi_i|_{\mathcal{F}^+ \cap \mathcal{K}} = -k(1+r)\cdot \sum_{\substack{j=1 \\ j\neq i}}^{N} \varrho_j + h_i[1-(1+r)^2] \leqslant h_i[1-(1+r)^2] \leqslant 0\,,$$

where we have chosen $r$ large enough in the last inequality. Similarly we have, again for $r$ large enough,

$$\varphi_i|_{\mathcal{F}^- \cap \mathcal{K}} = -k(1+r)\cdot \left[\alpha_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \varrho_j\right] + h_i\left\{1-(1+r)\left[(1+r) - \frac{\alpha_i}{c_i}\right]\right\} \leqslant$$

$$\leqslant h_i\left\{1-(1+r)\left[(1+r) - \frac{\alpha_i}{c_i}\right]\right\} \leqslant 0\,.$$

In particular $\varphi_i$ is nonnegative on all the vertices of $\mathcal{P}$. The last remaining condition

IV.*b* $\boldsymbol{F}\cdot\boldsymbol{v}|_{\Gamma_{N_i}^{+r}} \leqslant 0$ can be proved in the same way as before, and the conclusion follows. ■

Theorem 3.1 implies a non-existence result:

THEOREM 3.7. – *If* $N>1$, *then hyperrectangular sets of the form*

$$\Sigma = \bigcap_{i=1}^{N} \{(\boldsymbol{P},\boldsymbol{N}) \in \mathbb{R}^{2N};\ 0 \leqslant P_i \leqslant \alpha_i,\ 0 \leqslant N_i \leqslant \beta_i,\ i=1,\ldots,N\}$$

*cannot be invariant for the concentrations.*

PROOF. – Let us suppose, by contradiction, $\Sigma$ to be invariant. Consider the two parts of $\partial\Sigma$ given by

$$\Gamma_1 = \{P_1 = \alpha_1\}\,, \qquad \Gamma_2 = \{N_N = \beta_N\}\,.$$

We have

$$\boldsymbol{F}\cdot\boldsymbol{v}|_{\Gamma_1} = f_1(\alpha_1, P_2, \ldots, N_1, \ldots, N_N) =$$

$$= -k_1\alpha_1\left[c_1(\alpha_1 - N_1) + \sum_{j=2}^{N} c_j(P_j - N_j)\right] h_1(1-N_1)$$

and, in particular,

$$f_1(\alpha_1, 0, \ldots, 0, \beta_N) = -k_1\alpha_1[c_1\alpha_1 - c_N\beta_N] + h_1 \leqslant 0\,;$$

on the other hand

$$\boldsymbol{F}\cdot\boldsymbol{v}|_{\Gamma_2} = g_N(P_1, P_2, \ldots, N_{N-1}, \beta_N) =$$

$$= k_N\beta_N\left[\sum_{j=1}^{N-1} c_j(P_j - N_j) + c_N(P_N - \beta_N)\right] + h_N(1 - P_N\beta_N)$$

and thus

$$g_N(\alpha_1, 0, \ldots, 0, \beta_N) = k_N \beta_N [c_1 \alpha_1 - c_N \beta_N] + h_N \leqslant 0 .$$

Therefore we get the coupled inequalities

$$\begin{cases} -k_1 \alpha_1 [c_1 \alpha_1 - c_N \beta_N] + h_1 \leqslant 0 , \\ k_N \beta_N [c_1 \alpha_1 - c_N \beta_N] + h_N \leqslant 0 . \end{cases}$$

Put $\omega = c_1 \alpha_1 - c_N \beta_N$, then

$$\frac{h_1}{k_1 \alpha_1} \leqslant \omega \leqslant - \frac{h_N}{k_N \beta_N} ,$$

which is impossible. Hence, by Theorem 3.4, $\Sigma$ is not invariant. ■

As and immediate consequence of Theorems 3.7 and 3.2 we have

COROLLARY 3.8. – *If $N > 1$ and the diffusion coefficients $D_i^+$, $D_i^-$ are different, no invariant region can exist for the concentrations.*

## 4. – The evolution problem.

In this section we prove existence and uniqueness for the initial-boundary value problem given by equations (1.15)-(1.19), coupled with

(4.1) $$\boldsymbol{P} = \boldsymbol{P}_b , \qquad \boldsymbol{N} = \boldsymbol{N}_b \quad \text{on } \partial\Omega ,$$

(4.2) $$\boldsymbol{P}(x, 0) = \boldsymbol{P}_0(x), \qquad \boldsymbol{N}(x, 0) = \boldsymbol{N}_0(x) \quad \text{in } \Omega ,$$

(4.3) $$\Phi = \Phi_b \quad \text{on } \partial\Omega ,$$

(4.4) $$\boldsymbol{v} = 0 \quad \text{on } \partial\Omega ,$$

(4.5) $$\boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x) \quad \text{in } \Omega .$$

Since uniqueness for the Navier-Stokes equations in dimension 3 is an open problem, we limit our treatment to the case $\Omega \subseteq \mathbb{R}^2$. However, the result of existence remains true in higher dimensions. Our proof is based on the existence of the invariant region given by Theorems 3.5 and 3.6. No special hypotheses on the data $\boldsymbol{P}_0$, $\boldsymbol{N}_0$, $\boldsymbol{P}_b$, $\boldsymbol{N}_b$, apart from regularity, are needed if $N = 1$. When $N > 1$, we assume

(4.6) $$D_i^+ = D_i^- = D > 0 , \qquad i = 1, \ldots, N$$

and the existence of positive constants $\hat{r}$ and $\alpha_i$ such that, with the notations of Theorem 3.6,

(4.7)$_1$ $$(\boldsymbol{P}_b, \boldsymbol{N}_b) \in \Sigma^+ , \qquad \forall x \in \partial\Omega , \qquad (\boldsymbol{P}_0, \boldsymbol{N}_0) \in \Sigma^+ , \qquad \forall x \in \Omega ,$$

or

$$(4.7)_2 \qquad (\boldsymbol{P}_b, \boldsymbol{N}_b) \in \Sigma^-, \quad \forall x \in \partial\Omega, \quad (\boldsymbol{P}_0, \boldsymbol{N}_0) \in \Sigma^-, \quad \forall x \in \Omega .$$

We recall below a lemma of functional analysis [9], and a result of regularity [10].

LEMMA 4.1. – *Let $B$ be a Banach space and $T(w, \lambda)$ a continuous and compact mapping of $B \times [0, 1]$ into $B$, such that $T(w, 0) = \overline{u} \in B$ for every $w \in B$. Suppose there exists a constant $M$ such that*

$$\|w\|_B \leq M ,$$

*for all $(w, \lambda) \in B \times [0, 1]$ satisfying*

$$w = T(w, \lambda). \tag{4.8}$$

*Then the map $T(w, 1)$ of $B$ into itself has a fixed point.*

THEOREM 4.2. – *Let $q > 2$, $s > 2$, $\partial\Omega \in C^2$ and define $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times \times (0, T)$. Consider the problem*

$$u_t = d\Delta u + \sum_{i=1}^{n} a_i(x, t)\, u_{x_i} + a(x, t)\, u , \tag{4.9}$$

$$u = u_b \quad on\ S_T , \qquad u(x, 0) = u_0(x) \quad \text{in } \Omega , \tag{4.10}$$

*and suppose*

$$\|a_i\|_{L^s(Q_T)} < \infty , \qquad \|a\|_{L^s(Q_T)} < \infty , \tag{4.11}$$

$$u_0 \in W^{2-2/q,\, q}(\Omega), \qquad u_b \in W^{1-1/2q,\, q}((0, T); W^{2-1/q,\, q}(\Omega)) , \tag{4.12}$$

$$u_0 \,|_S = u_b(x, 0). \tag{4.13}$$

*Then problem (4.9)-(4.10) has a unique solution $u \in W^{1,q}((0, T); W^{2,q}(\Omega))$ which satisfies*

$$\|u\|_{W^{1,q}((0, T), W^{2,q}(\Omega))} \leq c\left\{\|u_0\|_{W^{2-2/q,\, q}(\Omega)} + \|u_b\|_{W^{1-1/2q,\, q}((0, T);\, W^{2-1/q,\, q}(\Omega))}\right\}. \tag{4.14}$$

*Furthermore $u$ satisfies an Hölder condition in $\overline{Q}_T$.*

The main result of this section is the following

THEOREM 4.3. – *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^3$ such that $\partial\Omega \in C^2$ and let $Q_T = \Omega \times (0, T)$. Assume $q > 2$ and*

$$P_{i0}, N_{i0} \in W^{2-2/q,\, q}(\Omega), \qquad P_{i0}, N_{i0} \geq 0 \quad in\ \Omega , \tag{4.15}$$

$$P_{ib}, N_{ib} \in W^{2-1/q,\, q}(\Omega), \qquad P_{ib}, N_{ib} \geq 0 \quad in\ \Omega . \tag{4.16}$$

*Let (4.6) and $(4.7)_1$ or $(4.7)_2$ hold. Then there exists a unique solution $(\boldsymbol{P}, \boldsymbol{N}, \Phi, \boldsymbol{v})$ to* *problem* (1.15)-(1.19), (4.1)-(4.5) *such that*

$$P_i,\, N_i \in W^{1,q}((0,\,T);\, W^{2,q}(\Omega))\,, \tag{4.17}$$

$$\Phi \in W^{1,q}((0,\,T);\, W^{4,q}(\Omega))\,, \tag{4.18}$$

$$\boldsymbol{v} \in L^2((0,\,T);\, V) \cap L^\infty((0,\,T);\, \boldsymbol{H})\,. \tag{4.19}$$

*Moreover,* $P_i,\, N_i$ *are nonnegative functions which satisfy an Hölder condition in* $\overline{Q}_T$.

PROOF. To prove existence, we apply Lemma 4.1. Let $B = \boldsymbol{L}^2((0,\,T);\, \boldsymbol{H}^1(\Omega)) \cap \boldsymbol{L}^\infty((0,\,T);\, \boldsymbol{L}^2(\Omega))$ and $\tilde{\boldsymbol{w}} = (\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{N}}) \in B^{2N}$. Define $T\colon B^{2N} \times [0,\,1] \ni \tilde{w} \mapsto w \in B^{2N}$ via the following linear problem:

$$P_i - P_{ib} \in B\,, \qquad (P_i)_t = D_i^+\, \Delta P_i + \tag{4.20}$$

$$+ \lambda[(k_i^+\, \nabla\tilde{\Phi} - \tilde{\boldsymbol{v}})\cdot\nabla P_i - c_i k_i^+\, P_i(\tilde{P}_i - \tilde{N}_i) + h(1 - P_i\tilde{N}_i)]\,,$$

$$N_i - N_{ib} \in B\,, \qquad (N_i)_t = D_i^-\, \Delta N_i + \tag{4.21}$$

$$+ \lambda[-(k_i^-\, \nabla\tilde{\Phi} + \tilde{\boldsymbol{v}})\cdot\nabla N + c_i k_i^-\, N_i(\tilde{P}_i - \tilde{N}_i) + h(1 - \tilde{P}_i N_i)]\,,$$

$$P_i(x,\,0) = P_{i0}(x)\,, \qquad N_i(x,\,0) = N_{i0}(x)\,, \tag{4.22}$$

$$\tilde{\Phi} - \Phi_b \in L^\infty((0,\,T);\, H_0^1(\Omega))\,, \qquad -\Delta\tilde{\Phi} = \sum_{j=1}^N c_j(P_j - N_j)\,, \tag{4.23}$$

$$\tilde{\boldsymbol{v}} \in L^2((0,\,T);\, \boldsymbol{v}) \cap L^\infty((0,\,T);\, \boldsymbol{H})\,, \qquad R(\tilde{\boldsymbol{v}}_t + \tilde{\boldsymbol{v}}_{xk}\tilde{v}_k) = \tag{4.24}$$

$$= -R\nabla\pi + \Delta\tilde{\boldsymbol{v}} - \lambda\left[T \sum_{j=1}^N c_j(P_j - N_j)\, \nabla\Phi\right],$$

$$\nabla\cdot\tilde{\boldsymbol{v}} = 0\,, \tag{4.25}$$

$$\tilde{\boldsymbol{v}} = 0 \quad \text{on } \partial\Omega\,, \qquad \tilde{\boldsymbol{v}}(x,\,0) = \boldsymbol{v}_0(x)\,. \tag{4.26}$$

As by Lemma 1.2 we have $(\tilde{P}_i, \tilde{N}_i) \in L^4(Q_T)$, by elliptic regularity (4.23) implies

$$\nabla\tilde{\Phi} \in L^2((0,\,T);\, H^2(\Omega)) \cap L^\infty((0,\,T);\, H^1(\Omega))$$

and

$$\nabla\tilde{\Phi} \in L^4((0,\,T);\, L^r(\Omega))\,, \qquad r < \infty\,.$$

Considering again Lemma 1.2, we infer

$$(\tilde{P} - \tilde{N})\, \nabla\tilde{\Phi} \in L^s((0,\,T),\, L^4(\Omega))\,, \qquad s < 4\,,$$

$$(\nabla\tilde{\Phi} - \tilde{\boldsymbol{v}}) \in L^4(Q_T)\,.$$

Applying Theorem 4.2, we conclude that

$$P_i,\, N_i \in W^{1,q}((0,\, T);\, W^{2,q}(\Omega))\,, \qquad q > 2\,, \tag{4.27}$$

$$P_i,\, N_i \in C^{0,\alpha}(\overline{Q}_T)\,. \tag{4.28}$$

Because of (4.14) $T$ is compact; the proof of the continuity of $T$ is quite elementary and is omitted. For every $w \in B$, we have $T(w,\, 0) = \overline{u}$, where $\overline{u}_i = (\overline{P}_i, \overline{N}_i)$ is the solution of

$$\begin{cases} (\overline{P}_i)_t = D_i^+ \Delta \overline{P}_i\,, \quad \overline{P}_i = \overline{P}_{ib} \ \text{ on } \partial\Omega\,, \quad \overline{P}_i(x,\, 0) = P_{i0}(x) & \text{in } \Omega\,, \\ (\overline{N}_i)_t = D_i^- \Delta \overline{N}_i\,, \quad \overline{N}_i = \overline{N}_{ib} \ \text{ on } \partial\Omega\,, \quad \overline{N}_i(x,\, 0) = N_{i0}(x) & \text{in } \Omega\,. \end{cases} \tag{4.29}$$

It remains to prove that all solutions of (4.8) are a priori bounded. If $w = (P_i,\, N_i)$ satisfies (4.8) we have (4.28) and this in turn implies, by the regularity of the solutions of the Navier-Stokes system

$$\nabla \Phi - \boldsymbol{v} \in C^{0,\alpha}(Q_T)\,. \tag{4.30}$$

If $N = 1$, (4.30) allows us to apply Theorem 3.5 and to conclude that there exists $r > 0$, independent of $\lambda \in [0,\, 1]$, such that

$$0 \leq P(x,\, t) \leq 1 + r\,, \qquad (x,\, t) \in \overline{Q}_T\,, \tag{4.31}$$

$$0 \leq N(x,\, t) \leq 1 + r\,, \qquad (x,\, t) \in \overline{Q}_T\,, \tag{4.32}$$

where the index $i$ is omitted as we are dealing with only one ionic family. When $N > 1$ we use Theorem 3.6, recalling hypotheses (4.6) and (4.7), and we conclude:

$$(\boldsymbol{P}(x,\, t),\, \boldsymbol{N}(x,\, t)) \in \Sigma^+\,, \qquad (x,\, t) \in \overline{Q}_T\,, \tag{4.33}$$

or

$$(\boldsymbol{P}(x,\, t),\, \boldsymbol{N}(x,\, t)) \in \Sigma^-\,, \qquad (x,\, t) \in \overline{Q}_T\,. \tag{4.34}$$

By the usual bootstrap argument, $w = (P_i,\, N_i)$ is bounded in the $B$-norm. Hence, by Lemma 4.1, there exists at least one solution $(P_i,\, N_i,\, \Phi,\, \boldsymbol{v})$ to the initial-boundary value problem (1.15)-(1.19), (4.1)-(4.5). Moreover, the crucial positivity condition for $P_i$ and $N_i$ is satisfied by (4.31)-(4.34).

Uniqueness is proved as an elementary application of the Gronwall inequality. Let $(\boldsymbol{P},\, \boldsymbol{N},\, \Phi,\, \boldsymbol{v}), (\boldsymbol{P}',\, \boldsymbol{N}',\, \Phi',\, \boldsymbol{v}')$ be two solutions and define $p_i = P_i - P_i'$, $n_i = N_i - N_i'$, $\varphi = \Phi - \Phi'$, $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}'$. Then $(\boldsymbol{p},\, \boldsymbol{n},\, \varphi,\, \boldsymbol{w})$ solves

$$(p_i)_t = D_i^+ \Delta p_i + k_i^+ \nabla\cdot(p_i \nabla \Phi) + k_i^+ \nabla\cdot(P_i' \nabla\varphi) - \nabla\cdot(P_i' \boldsymbol{w}) + \\ - \nabla\cdot(p_i \boldsymbol{v}) - h_i(p_i N_i + P_i' n_i)\,, \tag{4.35}$$

$$(n_i)_t = D_i^- \Delta n_i - k_i^- \nabla\cdot(n_i \nabla \Phi) - k_i^- \nabla\cdot(N_i' \nabla\varphi) - \nabla\cdot(N_i' \boldsymbol{w}) + \\ - \nabla\cdot(n_i \boldsymbol{v}) - h_i(p_i N_i + P_i' n_i)\,, \tag{4.36}$$

(4.37) $$p_i = 0 \quad \text{on } \partial\Omega\,, \qquad p_i(x,0) = 0 \quad \text{in } \Omega\,,$$

(4.38) $$n_i = 0 \quad \text{on } \partial\Omega\,, \qquad n_i(x,0) = 0 \quad \text{in } \Omega\,,$$

(4.39) $$-\Delta\varphi = \sum_{j=1}^{N} c_j(p_j - n_j)\,,$$

(4.40) $$\varphi = 0 \quad \text{on } \partial\Omega\,,$$

(4.41) $$R(\boldsymbol{w}_t + B\boldsymbol{v} - B\boldsymbol{v}') + A_1\boldsymbol{w} = -T\sum_{j=1}^{N} c_j(P_j\nabla\varphi + p_j\nabla\Phi' - N_j\nabla\varphi - n_j\nabla\Phi')\,,$$

(4.42) $$\nabla\cdot\boldsymbol{w} = 0\,,$$

(4.43) $$\boldsymbol{w} = \boldsymbol{0} \quad \text{on } \partial\Omega\,, \qquad \boldsymbol{w}(x,0) = \boldsymbol{0} \quad \text{on } \Omega\,.$$

Define $D = \min\{D_i^+, D_i^-, i = 1, \dots, N\}$. From (4.39) we have

(4.44) $$|\nabla\varphi(t)|^2 \leq C_1(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2)\,.$$

Let us multiply (4.35) by $p_i$ and (4.36) by $n_i$ and integrate by parts over $\Omega$. Using (4.44) and the elementary inequality

$$|ab| \leq \frac{a^2}{2\eta} + \frac{\eta}{2}b^2\,,$$

we find, for $\eta > 0$ and for suitable constants $C_i = C_i(\eta)$,

$$\frac{1}{2}\frac{d}{dt}(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2) + D(|\nabla\boldsymbol{p}(t)|^2 + |\nabla\boldsymbol{n}(t)|^2) \leq$$
$$\leq C_2(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2) + \eta C_3(|\nabla\boldsymbol{p}(t)|^2 + |\nabla\boldsymbol{n}(t)|^2) + C_4|\boldsymbol{w}(t)|^2\,.$$

Choosing $\eta = D/C_3$, we have

(4.45) $$\frac{d}{dt}(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2) \leq C_5(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2) + C_6|\boldsymbol{w}(t)|^2\,.$$

We take, a.e. in $t$, the scalar product of (4.41) with $\boldsymbol{w}(t)$ in the duality between $\mathcal{V}$ and $\mathcal{V}'$, and use (4.44). We get

(4.46) $$R\left[\frac{d}{dt}|\boldsymbol{w}(t)|^2 + 2b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{w}(t)) - 2b(\boldsymbol{v}'(t), \boldsymbol{v}'(t), \boldsymbol{w}(t))\right] +$$
$$+ 2\|\boldsymbol{w}(t)\|^2 \leq C_7(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2)\,.$$

Proceeding as in the proof of uniqueness for the 2-dimensional Navier-Stokes system, we have, using Lemma 2.1 and Young's inequality,

(4.47) $$\frac{d}{dt}|\boldsymbol{w}(t)|^2 \leq C_8(|\boldsymbol{p}(t)|^2 + |\boldsymbol{n}(t)|^2) + C_9\|\boldsymbol{v}(t)\|^2|\boldsymbol{w}(t)|^2\,.$$

Adding (4.45) and (4.47) we obtain

$$\frac{d}{dt}(|\boldsymbol{p}(t)|^2+|\boldsymbol{n}(t)|^2+|\boldsymbol{w}(t)|^2)\leqslant (C_7+C_8)(|\boldsymbol{p}(t)|^2+|\boldsymbol{n}(t)|^2)+ \\ +(C_6+C_9\|\boldsymbol{v}(t)\|^2)\,|\boldsymbol{w}(t)|^2\,. \tag{4.48}$$

The function $t\mapsto\|v(t)\|^2$ is integrable, therefore we conclude, by Gronwall's inequality,

$$|\boldsymbol{p}(t)|^2+|\boldsymbol{n}(t)|^2+|\boldsymbol{w}(t)|^2=0 \tag{4.49}$$

and, by (4.39), $|\varphi(t)|=0$. Thus, the solution is unique. ■

## 5. – Decay to equilibrium with no flux at the boundary.

It is reasonable to expect that the asymptotic behaviour of our initial-boundary value problem could be quite complex; here we limit ourselves to treat a particular case, in which we have convergence to the trivial solution of the stationary problem, i.e.

$$(P_1,\dots,P_N,N_1,\dots,N_N,\Phi,\boldsymbol{v})=(1,\dots,1,1,\dots,1,\Phi_A,0)\,. \tag{5.1}$$

We suppose the medium electrically insulated. This implies

$$\frac{\partial\Phi}{\partial\boldsymbol{\nu}}=0 \quad \text{on } \partial\Omega\,. \tag{5.2}$$

Moreover, the concentrations $(P_i, N_i)$ are subjected to homogeneous Neumann boundary data

$$\frac{\partial P_i}{\partial\boldsymbol{\nu}}=0\,,\qquad \frac{\partial N_i}{\partial\boldsymbol{\nu}}=0\quad \text{on } \partial\Omega\,. \tag{5.3}$$

The other conditions are unchanged:

$$P_i(x,0)=P_{i0}(x)\,,\qquad N_i(x,0)=N_{i0}(x)\quad \text{in } \Omega\,, \tag{5.4}$$

$$\boldsymbol{v}=0\quad \text{on } \partial\Omega\,, \tag{5.5}$$

$$\boldsymbol{v}(x,0)=\boldsymbol{v}_0(x)\quad \text{in } \Omega\,. \tag{5.6}$$

If $(\boldsymbol{P},\boldsymbol{N},\Phi,\boldsymbol{v})$ is a solution corresponding to the new boundary conditions, we have, integrating (1.12),

$$\sum_{j=1}^{N}\int_\Omega [P_j(x,t)-N_j(x,t)]\,dx=0\,. \tag{5.7}$$

As a consequence, the total electric charge is zero and the present problem makes sense only if the compatibility condition

$$\sum_{j=1}^{N} \int_{\Omega} [P_{0j}(x) - N_{0j}(x)]\, dx = 0\,, \tag{5.8}$$

is satisfied. With minor changes, all considerations regarding the existence of invariant regions can be repeated even with Neumann boundary conditions. For the present problem it is also possible to give a result of existence and uniqueness similar to Theorem 4.3.

Let

$$\overline{P}_i(t) = \frac{1}{|\Omega|} \int_{\Omega} P_i(x, t)\, dx\,, \qquad \overline{N}_i(t) = \frac{1}{|\Omega|} \int_{\Omega} N_i(x, t)\, dx\,. \tag{5.9}$$

Our result on the large time behaviour of solutions holds if the diffusion is sufficiently intense, and is based on the following

LEMMA 5.1. – *Define*

$$D = \min\{D_i^+, D_i^-, i = 1, \ldots, N\}\,.$$

*Let* $(\boldsymbol{P}, \boldsymbol{N}, \Phi, \boldsymbol{v})$ *be any solution which admits a bounded invariant region in* $\mathbb{R}^{2N}$ *for the concentrations* $(\boldsymbol{P}, \boldsymbol{N})$. *Then there exist positive constants* $\overline{D}$, $\tau$, $\gamma$, *and* $\gamma'$, *such that, when*

$$D \geq \overline{D}\,,$$

*we have*

$$|\nabla \boldsymbol{P}(t)|^2 + |\nabla \boldsymbol{N}(t)|^2 \leq \gamma e^{-\tau t}\,, \tag{5.10}$$

$$|\boldsymbol{P}(\cdot, t) - \overline{\boldsymbol{P}}(t)| \leq \gamma' e^{-\tau t}\,, \tag{5.11}$$

$$|\boldsymbol{N}(\cdot, t) - \overline{\boldsymbol{N}}(t)| \leq \gamma' e^{-\tau t}\,. \tag{5.12}$$

PROOF. – Let $\Sigma$ be an invariant region, i.e. if $(\boldsymbol{P}(x, 0), \boldsymbol{N}(x, 0)) \in \Sigma$ for every $x \in \Omega$, then $(\boldsymbol{P}(x, t), \boldsymbol{N}(x, t)) \in \Sigma$ for every $x \in \Omega$ and $t \in (0, T)$. Set

$$\mu_1 = \sup_{(\boldsymbol{P}, \boldsymbol{N}) \in \Sigma} |\nabla \Phi(x, t)|\,, \qquad \mu_2 = \sup_{(\boldsymbol{P}, \boldsymbol{N}) \in \Sigma} |\boldsymbol{v}(x, t)|\,,$$

$$\mu_3 = \sup_{(\boldsymbol{P}, \boldsymbol{N}) \in \Sigma} |\boldsymbol{P}|\,, \qquad \mu_4 = \sup_{(\boldsymbol{P}, \boldsymbol{N}) \in \Sigma} |\boldsymbol{N}|\,.$$

By the definition of invariant region and the regularity of $\Phi$ and $\boldsymbol{v}$, the constants $\mu_k$, $k = 1, \ldots, 4$, are well defined. Put

$$\Psi(t) = \frac{1}{2} (|\nabla \boldsymbol{P}(t)|^2 + |\nabla \boldsymbol{N}(t)|^2)\,. \tag{5.13}$$

Calculating $\nabla P_{jt}$ and $\nabla N_{jt}$ from (1.15) and (1.16), we estimate

$$\dot{\Psi}(t) = \sum_{j=1}^{N} [(\nabla P_j, \nabla P_{jt}) + (\nabla N_j, \nabla N_{jt})]. \tag{5.14}$$

Integrating by parts, in view of (5.2)-(5.6) and using the Cauchy-Schwartz inequality, we get

$$\dot{\Psi}(t) \leqslant -D(|\Delta \boldsymbol{P}(t)|^2 + |\Delta \boldsymbol{N}(t)|^2) + C_1(|\Delta \boldsymbol{P}(t)|^2 + |\Delta \boldsymbol{N}(t)|^2) + \\ + C_2(|\nabla \boldsymbol{P}(t)|^2 + |\nabla \boldsymbol{N}(t)|^2), \tag{5.15}$$

where $C_1$ and $C_2$ are constants which depend only on the $\mu_k$'s. The Neumann's conditions (5.3) imply the existence of a positive constant $m$, depending only on $\Omega$, such that

$$|\Delta \boldsymbol{P}(t)|^2 \geqslant m|\nabla \boldsymbol{P}(t)|^2, \qquad |\Delta \boldsymbol{N}(t)|^2 \geqslant m|\nabla \boldsymbol{N}(t)|^2. \tag{5.16}$$

Therefore, if $\tau = Dm - C_2 - C_1 m$ and

$$D > \frac{C_2 + C_1 m}{m}, \tag{5.17}$$

we infer

$$\dot{\Psi}(t) \leqslant [C_2 + (C_1 - D)\, m](|\nabla \boldsymbol{P}(t)|^2 + |\nabla \boldsymbol{N}(t)|^2), \tag{5.18}$$

i.e.

$$\dot{\Psi}(t) \leqslant -\tau \Psi(t). \tag{5.19}$$

This implies (5.10) and, by the Poincaré inequality, (5.11) and (5.12). ∎

The spatial averages of $P_i$ and $N_i$ satisfy the following coupled equations, obtained integrating (1.15) and (1.16) over $\Omega$:

$$\begin{cases} \dfrac{d\overline{P}_i}{dt} = h_i\left(1 - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega P_i N_i\, dx\right), \\ \dfrac{d\overline{N}_i}{dt} = h_i\left(1 - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega P_i N_i\, dx\right). \end{cases} \tag{5.20}$$

Defining

$$g_i(t) = \frac{h_i}{|\Omega|}\int_\Omega [\overline{P}_i(t)\overline{N}_i(t) - P_i(x, t)\, N_i(x, t)]\, dx, \tag{5.21}$$

we can write (5.20) as a system of ordinary differential equations; more precisely we have

$$\text{(5.22)}\qquad \begin{cases} \dfrac{d\overline{P}_i}{dt}(t) = h_i(1 - \overline{P}_i(t)\,\overline{N}_i(t)) + g_i(t), \\ \dfrac{d\overline{N}_i}{dt}(t) = h_i(1 - \overline{P}_i(t)\,\overline{N}_i(t)) + g_i(t). \end{cases}$$

The term $g_i(t)$ can be estimated using (5.11) and (5.12). We obtain

$$\text{(5.23)}\qquad |g_i(t)| \leq C_i e^{-\tau t},$$

where the constants $C_i$ depend only on $\mu_3$ and $\mu_4$. Let $\sigma > 0$. By (5.23) there exists a time $\hat{t}(\sigma)$ such that

$$\text{(5.24)}\qquad |g_i(t)| \leq \sigma, \qquad \forall t > \hat{t}(\sigma).$$

THEOREM 5.2. – *Suppose*

$$\text{(5.25)}\qquad \overline{P}_i(0) = \overline{P}_{i0} = \overline{N}_{i0} = \overline{N}_i(0), \qquad i = 1, \ldots, N.$$

*Then we have*

$$\text{(5.26)}\qquad \lim_{t\to\infty} P_i(x, t) = 1, \qquad \lim_{t\to\infty} N_i(x, t) = 1 \quad \textit{in } L^2(\Omega), \qquad i = 1, \ldots, N.$$

PROOF. – In view of (5.25) and (5.22), we obtain $\overline{P}_i(t) = \overline{N}_i(t)$. Therefore (5.22) reduces to a single equation. Let $\widehat{P}_i = \overline{P}_i(\hat{t})$ and consider the following Cauchy problems

$$\text{(5.27)}\qquad \begin{cases} \dfrac{d\overline{P}_i}{dt} = h_i(1 - \overline{P}_i^2) + g_i(t), \\ \overline{P}_i(\hat{t}) = \widehat{P}_i, \end{cases}$$

$$\text{(5.28)}\qquad \begin{cases} \dfrac{d\varphi_i^{\pm}}{dt} = h_i[1 - (\varphi_i^{\pm})^2] \pm \sigma, \\ \varphi^{\pm}(\hat{t}) = \widehat{P}_i. \end{cases}$$

Assume $1 - \sigma/h_i > 0$. Comparing the solutions of (5.27) and (5.28), (see [18]), we obtain

$$\text{(5.29)}\qquad \sqrt{1 + \frac{\sigma}{h_i}} \geq \limsup_{t\to\infty} \overline{P}_i(t) \geq \liminf_{t\to\infty} \overline{P}_i(t) \geq \sqrt{1 - \frac{\sigma}{h_i}}.$$

Since $\sigma > 0$ is arbitrary, it follows

$$\lim_{t \to \infty} \overline{P}_i(t) = 1 .$$

Taking into account (5.11) and (5.12), we have (5.26). ∎

## REFERENCES


[1] I. ADAMCZEWSKI, *Les phénomènes d'ionisation et de conduction dans les diélectriques liquides*, Masson, Paris (1968).

[2] R. A. ADAMS, *Sobolev Spaces*, Academic Press, London (1975).

[3] P. ATTEN, *Stabilité electrodynamique des liquides de faible conductivité*, J. Mécanique (3), 14 (1975), pp. 465-495.

[4] J. F. G. AUCHMUTY, *Positivity for elliptic and parabolic systems*, Proc. Roy. Soc. Edinburgh, Set A, **79** (1977-78), pp. 183-191.

[5] C. BARDOS, *Comportement asymptotique de solutions d'equations de réaction-diffusion*, J. Phys., Colloque C5 (Supplement au n. 8), **39** (1975), C5, pp. 61-67.

[6] K. N. CHUEN - C. CONLEY - J. SMOLLER, *Positively invariant regions for systems of nonlinear diffusion equations*, Ind. Univ. Math. J., (2), **26** (1977), pp. 373-391.

[7] E. CONWAY - D. HOFF - J. SMOLLER, *Large time behaviour of solutions of systems of nonlinear reaction-diffusion equations*, Siam J. Appl. Math., (1), **35** (1978), pp. 1-16.

[8] E. FEIREISL, *A stabilizing effect of vanishing diffusion in electrodynamics*, to appear.

[9] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York (1969).

[10] O. A. LADYZHENSKAYA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, Transl. of Math. Monographs, American Mathematical Society, Providence (1968).

[11] P. LANGEVIN, *Recherches sur les gaz ionisés*, Ann. Chimie Physique, **28** (1903), pp. 223-235.

[12] A. B. LIDIARD, *Ionic Conductivity*, Electrical Conductivity, Springer-Verlag, Berlin (1957).

[13] A. OCCHIALINI, *I gas compressi come dielettrici e come conduttori*, Tipografia Editrice F. Mariotti, Pisa (1906).

[14] I. RUBINSTEIN, *Electrodiffusion of Ions*, SIAM, Philadelphia (1990).

[15] J. SMOLLER, *Shock Waves and Reaction Diffusion Equations*, Springer-Verlag, Berlin (1983).

[16] R. TEMAM, *Navier-Stokes Equations*, North-Holland, Amsterdam (1984).

[17] H. F. WEINBERGER, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat. Univ. Roma (6), 8 (1975), pp. 295-310.

[18] Z. ZHANG, *Qualitative Theory of Differential Equations*, Transl. of Math. Monographs, American Mathematical Society, Providence (1968).