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Invariant Spherical Hyperfunctions on The Tangent Space of A Symmetric Space

Jiro Sekiguchi

Introduction

This paper deals with the study of invariant spherical hyperfunctions on the tangent space of a semisimple symmetric space.

First recall the results on invariant eigendistributions (=IED) on a real semisimple Lie algebra. Hence let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g}_{rs}$ be the totality of regular semisimple elements of $\mathfrak{g}$. Then Harish-Chandra showed the following famous theorem (cf. [HC1]):

Theorem I. *Every invariant eigendistribution on* $\mathfrak{g}$ *is a locally* L^1*-function.*

At one step of the proof of this theorem, he also showed the following theorem ([HC3]):

Theorem II. *Let* u *be an IED on* $\mathfrak{g}$. *Then the restriction of* u *to* $\mathfrak{g}_{rs}$ *determines* u *itself, that is, if* u *is zero on* $\mathfrak{g}_{rs}$, *then* u *is identically zero on the whole space* $\mathfrak{g}$.

In this paper, we attempt to obtain a generalization of the results mentioned above for invariant spherical hyperfunctions on the tangent. To explain our main result, we need some notation. Let σ be an involution of $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ be the direct sum decomposition for σ, where $\mathfrak{h}$ and $\mathfrak{q}$ are the 1 and -1 eigenspaces of σ, respectively. Then one can naturally define an invariant spherical hyperfunction (=ISH) on the vector space $\mathfrak{q}$ (§5). An ISH is, by definition, a hyperfunction solution of a holonomic system $\mathcal{M}_\lambda$ of differential equations on $\mathfrak{q}$. In the case where $\mathfrak{g}=\mathfrak{g}'\oplus\mathfrak{g}'$ ($\mathfrak{g}'$ is semisimple) and $\sigma(X, Y)=(Y, X)$ ($(X, Y)\in\mathfrak{g}$), every ISH on $\mathfrak{q}$ ($\simeq\mathfrak{g}'$) is nothing but an IED on $\mathfrak{g}'$. Hence the notion of an ISH is a natural generalization of that of an IED on a semisimple Lie algebra.

Since every ISH u on $\mathfrak{q}$ is real analytic on $\mathfrak{q}_{rs}$ (=the set of $\mathfrak{q}$-regular semisimple elements of $\mathfrak{q}$), it is rather easy to treat the restriction of u to

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$\mathfrak{q}_{rs}$. This and Theorem II above lead us to the following problem:

Problem. Does there exist a non-zero ISH u on $\mathfrak{q}$ such that Supp(u) is contained in $\mathfrak{q}-\mathfrak{q}_{rs}$?

An ISH u is called singular if Supp(u) is contained in $\mathfrak{q}-\mathfrak{q}_{rs}$. Then there exists a non-zero singular ISH on $\mathfrak{q}$ for some symmetric pair $(\mathfrak{g}, \mathfrak{h})$ (cf. § 6). But we can show the following theorem which is an analogue of Theorem II (cf. § 5, Theorem 5.2).

Theorem 1. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a symmetric pair. Assume that* $\delta_{\mathfrak{q}}(Z)>0$ *for every* $\mathfrak{q}$*-distinguished element* Z *of* $\mathfrak{q}$ *(for the definition of the number* $\delta_{\mathfrak{q}}(Z)$*, see* § 1*). Then there exists no non-zero singular invariant spherical hyperfunction on* $\mathfrak{q}$.

Our method of the proof of Theorem 1 is based on Atiyah's lecture note [A] which is an intelligent introduction to Theorem I of Harish-Chandra. We are going to explain the main part of the argument. Let $\square$ be the pseudo-Laplacian on $\mathfrak{q}$. Following Atiyah [A], for an arbitrary element Z of $\mathfrak{q}$, we calculate the radial component of $\square$ at Z by use of a specified local coordinate system near Z. The choice of this local coordinate system plays a fundamental role in the proof of Theorem 1. In the case where Z is nilpotent, the radial component of $\square$ for the local coordinate is already obtained by van Dijk [vD]. At any rate, if $\delta_{\mathfrak{q}}(Z)>0$ for an arbitrary $\mathfrak{q}$-distinguished element Z of $\mathfrak{q}$, the proof of Theorem 1 goes parallel to the arguments in [A]. So we obtain Theorem 1.

Needless to say, the assumption of Theorem 1 holds for a Riemannian symmetric pair or a pair of the form $(\mathfrak{g}'\oplus\mathfrak{g}', \mathfrak{g}')$ (which is regarded as the case of the Lie algebra $\mathfrak{g}'$). But for these cases, the conclusion of Theorem 1 is already known. In fact, in the former case, every ISH on $\mathfrak{q}$ is real analytic. On the other hand, Theorem 1 is reduced to Theorem II in the latter case.

The next problem is to determine whether for a given symmetric pair $(\mathfrak{g}, \mathfrak{h})$, the assumption of Theorem 1 holds or not. Hence it is important to classify all the symmetric pairs satisfying the assumption. Let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be the complexification of $(\mathfrak{g}, \mathfrak{h})$ and let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}_c$ such that $\mathfrak{k}_0=\mathfrak{h}_c\cap\mathfrak{g}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$. We shall show the next theorem in Section 6 (cf. Theorem 6.3).

Theorem 2. *Retain the notation above. Let* Σ *be the restricted root system of* $\mathfrak{g}_0$*. For every root* $\lambda\in\Sigma$*, let* m_λ *be the multiplicity of* λ*. Assume that* $m_\lambda+m_{2\lambda}\leq 2$ *for all* $\lambda\in\Sigma$*. (For example, this condition holds in the case where* $\mathfrak{g}_0$ *is a normal real form of* $\mathfrak{g}_c$*.) Then the assumption of Theorem*

1 *holds for the pair* $(\mathfrak{g}, \mathfrak{h})$.

Our proof of this theorem is based on the classification of all the nilpotent orbits of complex simple Lie algebras.

At an early stage of the preparation of this paper, the author proved Theorem 1 only in the case of invariant spherical distribution (=ISD). T. Oshima showed him Proposition 2.3 in Section 2 as well as its proof. Once the proposition is established, so is Theorem 1 not only for the case of distributions but also for the case of hyperfunctions. It is worthwhile to state a conjecture concerning the difference between ISD and ISH. Let $\mathcal{M}_\Lambda$ be the holonomic system introduced in Section 5. Note that an ISD is, by definition, a distribution solution of $\mathcal{M}_\Lambda$.

Conjecture. $\mathcal{M}_\Lambda$ is a regular holonomic system in the sense of Kashiwara-Kawai (cf. [HK]).

If this is true, one would show, as a corollary, that any ISH turns out to be an ISD. Hence, there exists no difference between ISH and ISD.

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§ 1. Semisimple and nilpotent elements associated with symmetric spaces

In this section, we discuss the structure of the tangent space of a symmetric space. This will be needed in the subsequent discussions.

(1.1) First introduce some standard notation. In this paper, $\mathbf{N}$ always means the set $\{0, 1, 2, \cdots\}$. If V is a real vector space, we denote by V_c its complexification. Also we denote by V^* and V_c^* the duals of V and V_c, respectively. Moreover $S(V_c)$ denotes the symmetric algebra over V_c. For a real Lie algebra $\mathfrak{g}$, $U(\mathfrak{g}_c)$ denotes the enveloping algebra of $\mathfrak{g}_c$ and for an element X of $\mathfrak{g}$ and a linear subspace V of $\mathfrak{g}$, we put $V_X = \{Z \in V; [Z, X]=0\}$.

(1.2) Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involution. As usual, $\mathfrak{h}$ and $\mathfrak{q}$ denote the 1 and -1 eigenspaces of σ, respectively. Then $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ is a direct sum decomposition. In this paper, the pair $(\mathfrak{g}, \mathfrak{h})$ obtained in this way is called a symmetric pair. If G is the adjoint group of $\mathfrak{g}$, then σ is lifted to G. For the sake of simplicity, the lifting of σ is denoted by the same letter. Define $G^\sigma = \{g \in G; \sigma g = g\}$ and let H be the identity component of G^σ. Then $\mathfrak{h}$ is the Lie algebra of H. Since $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, H acts on $\mathfrak{q}$. The coset space G/H is a semisimple symmetric space and the tangent space of G/H at eH is identified with $\mathfrak{q}$.

Let $\mathscr{S}$ and $\mathscr{N}$ be the totality of semisimple elements and that of nilpotent elements of $\mathfrak{g}$, respectively. Put $\mathscr{S}(\mathfrak{q})=\mathscr{S}\cap\mathfrak{q}$ and its elements are called semisimple elements of $\mathfrak{q}$. Similarly, put $\mathscr{N}(\mathfrak{q})=\mathscr{N}\cap\mathfrak{q}$ and its elements are nilpotent elements of $\mathfrak{q}$.

(1.3) First recall the Jordan decomposition of elements of $\mathfrak{q}$.

Lemma 1.1. *Take* $Z_0\in\mathfrak{q}$.

(i) *There exist unique elements* $A_0\in\mathscr{S}(\mathfrak{q})$ *and* $X_0\in\mathscr{N}(\mathfrak{q})$ *such that* $Z_0=A_0+X_0$ *and that* $[A_0, X_0]=0$.

(ii) $\mathfrak{g}_{Z_0}=\mathfrak{g}_{A_0}\cap\mathfrak{g}_{X_0}$.

Definition 1.2. (i) The number $\min\{\operatorname{codim}_{\mathfrak{q}} H\cdot Z;\ Z\in\mathfrak{q}\}$ is called the rank of $(\mathfrak{g}, \mathfrak{h})$ or that of $\mathfrak{q}$ and is denoted by rank $\mathfrak{q}$.

(ii) As element Z of $\mathfrak{q}$ is called $\mathfrak{q}$-regular if $\operatorname{codim}_{\mathfrak{q}} H\cdot Z$ conicides with rank $\mathfrak{q}$.

Lemma 1.3. *Let* $S(\mathfrak{q}_c)^H$ *be the totality of H-invariant elements of* $S(\mathfrak{q}_c)$. *Then there exist homogeneous elements* $P_1, \cdots, P_l$ ($l=$*rank* $\mathfrak{q}$) *such that* $S(\mathfrak{q}_c)=\mathbf{C}[P_1, \cdots, P_l]$.

This is due to Chevalley (cf. [KR]).

(1.4) We review semisimple elements of $\mathfrak{q}$. A linear subspace $\mathfrak{a}$ of $\mathfrak{q}$ is called a Cartan subspace of $\mathfrak{q}$ if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements. By definition, the dimension of $\mathfrak{a}$ coincides with rank $\mathfrak{q}$. It is known (cf. [OM]) that there are only a finite number of H-conjugate classes of Cartan subspaces of $\mathfrak{q}$. Now take a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ and fix it once for all. For a linear form λ on $\mathfrak{a}_c$, we define

$$\mathfrak{g}_c^\lambda=\{X\in\mathfrak{g}_c;\ [A, X]=\lambda(A)X\quad\text{for all } A\in\mathfrak{a}_c\}$$

and

$$\Sigma(\mathfrak{a})=\{\lambda\in\mathfrak{a}_c^*;\ \mathfrak{g}_c^\lambda\neq\{0\}\}-\{0\}.$$

Then $\Sigma(\mathfrak{a})$ becomes a root system. In fact, $\Sigma(\mathfrak{a})$ coincides with the restricted root system of an appropriate real form of $\mathfrak{g}_c$ (cf. [OS]). Define $\mathfrak{a}'=\{A\in\mathfrak{a};\ \lambda(A)\neq 0$ for all $A\in\mathfrak{a}\}$. Then it is clear that every element of $\mathfrak{a}'$ is $\mathfrak{q}$-regular. For this reason, this is called a $\mathfrak{q}$-regular semisimple element of $\mathfrak{a}$.

Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ be the totality of representatives of mutually not H-conjugate Cartan subspaces of $\mathfrak{q}$. Put $\mathfrak{q}'=\bigcup_{i=1}^k\bigcup_{h\in H} h\cdot\mathfrak{a}'$. Then $\mathfrak{q}'$

consists of $\mathfrak{q}$-regular elements and is Zariski dense in $\mathfrak{q}$. Note that not every $\mathfrak{q}$-regular element of $\mathfrak{q}$ is contained in $\mathfrak{q}'$. For example, as will be stated later, in some cases, there exist $\mathfrak{q}$-regular nilpotent elements in $\mathfrak{q}$ (cf. (1.9)).

(1.5) Take a semisimple element A_0 of $\mathfrak{q}$. Then its centralizer $\mathfrak{z}=\mathfrak{g}_{A_0}$ is reductive and is σ-invariant. Let $\mathfrak{c}$ be its center and let $\mathfrak{z}_s$ be its semisimple part. Now put $\mathfrak{c}^-=\mathfrak{c}\cap\mathfrak{q}$, $\mathfrak{z}^+=\mathfrak{z}\cap\mathfrak{h}$, $\mathfrak{z}_s^+=\mathfrak{z}_s\cap\mathfrak{h}$, $\mathfrak{z}^-=\mathfrak{z}\cap\mathfrak{q}$ and $\mathfrak{z}_s^-=\mathfrak{z}_s\cap\mathfrak{q}$. Then $\mathfrak{z}_s=\mathfrak{z}_s^++\mathfrak{z}_s^-$ is a direct sum and $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ is a symmetric pair.

Definition 1.4. The pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ thus obtained is called a sub-symmetric pair of $(\mathfrak{g}, \mathfrak{h})$.

(1.6) Next we review nilpotent elements of $\mathfrak{q}$. The following lemmas are fundamental in the subsequent discussion (cf. [KR], [vD]).

Lemma 1.5. (i) *H leaves $\mathcal{N}(\mathfrak{q})$ invariant and there are only a finite number of H-orbits of $\mathcal{N}(\mathfrak{q})$.*

(ii) $\mathcal{N}(\mathfrak{q})=\{X\in\mathfrak{q};\ P(X)=P(0)$ *for all* $P\in S(\mathfrak{q}_c)^H\}$.

Remark 1.6. It is known (cf. [KR]) that $\operatorname{codim}_{\mathfrak{q}}\mathcal{N}(\mathfrak{q})\leq l=\operatorname{rank}\mathfrak{q}$. But the equality does not hold in general. In fact, if $(\mathfrak{g}, \mathfrak{h})$ is Riemannian, then $\mathcal{N}(\mathfrak{q})$ consists of only one element 0 and therefore $\operatorname{codim}_{\mathfrak{q}}\mathcal{N}(\mathfrak{q})=\dim\mathfrak{q}<l$.

Lemma 1.7. *Let $X_0\in\mathcal{N}(\mathfrak{q})$. Then there exist $A_0\in\mathfrak{h}$ and $Y_0\in\mathfrak{q}$ such that*

$$[A_0, X_0]=2X_0,\quad [A_0, Y_0]=-2Y_0,\quad [X_0, Y_0]=A_0. \tag{1.6.1}$$

Definition 1.8. Retain the notation in Lemma 1.7. Then (A_0, X_0, Y_0) is called a normal S-triple and the Lie algebra $\mathfrak{l}=\mathbf{R}A_0+\mathbf{R}X_0+\mathbf{R}Y_0$ is a TDS.

By definition, $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Let θ_0 be a Cartan involution on $\mathfrak{l}$ defined by θ_0: $(A_0, X_0, Y_0)\rightarrow(-A_0, -Y_0, -X_0)$. It is clear that θ_0 commutes with $\sigma|\mathfrak{l}$.

Lemma 1.9 ([vD]). *θ_0 can be extended to a Cartan involution θ on $\mathfrak{g}$ which commutes with σ.*

(1.7) Let (A_0, X_0, Y_0) be a normal S-triple. The centralizer $\mathfrak{g}_{X_0}$ is left invariant by $\operatorname{ad}A_0$ and also by σ. This implies that $\mathfrak{g}_{X_0}=\mathfrak{h}_{X_0}+\mathfrak{q}_{X_0}$ is a direct sum and $\operatorname{ad}A_0$ leaves $\mathfrak{h}_{X_0}$ and $\mathfrak{q}_{X_0}$ invariant. Noting this, we can take a basis $w_1, \cdots, w_r$ of $\mathfrak{q}_{X_0}$ which are eigenvectors of ad A_0 ($r=\dim \mathfrak{q}_{X_0}$). Hence $[A_0, w_i]=n_i w_i$ for a number n_i ($1\leq i\leq r$). Then each n_i is a non-negative integer (cf. [vD]).

Definition 1.10. Define $\delta_{\mathfrak{q}}(X_0)=\sum_{i=1}^{r}(n_i+2)-\dim \mathfrak{q}$.

It is clear from the definition that $\delta_{\mathfrak{q}}(h\cdot X_0)=\delta_{\mathfrak{q}}(X_0)$ for all $h \in H$.

(1.8) Let $B(\ ,\ )$ be the Killing form on $\mathfrak{g}$. Then we write $\omega(X)=B(X, X)$ ($X \in \mathfrak{q}$) and call ω the Casimir polynomial on $\mathfrak{q}$. It is clear from the definition that ω is an H-invariant non-degenerate quadratic form on $\mathfrak{q}$.

Definition 1.11. X_0 is $\mathfrak{q}$-distinguished nilpotent if X_0 does not commute with any non-zero semisimple element of $\mathfrak{q}$.

Lemma 1.12 ([vD]). *The following conditions on X_0 are mutually equivalent.*

(i) *X_0 is $\mathfrak{q}$-distinguished nilpotent.*
(ii) *$\omega(X)=0$ for all $X \in \mathfrak{q}_{X_0}$.*
(iii) *$\omega(X)=0$ for all $X \in \mathfrak{q}_{Y_0}$.*
(iv) *$n_i>0$ ($1\leq i\leq r$).*
(v) *$\mathfrak{q}_{X_0}\cap \mathfrak{q}_{Y_0}=0$.*

(1.9) A nilpotent element $X_0 \in \mathfrak{q}$ is called $\mathfrak{q}$-regular if $\dim \mathfrak{q}_{X_0}=\operatorname{rank} \mathfrak{q}$ $(=l)$, or equivalently, $\operatorname{codim}_{\mathfrak{q}} H\cdot X_0=l$ (cf. Definition 1.2).

If each of $\mathfrak{g}$ and $\mathfrak{h}$ is a normal real form of a complex semisimple Lie algebra, then $\mathcal{N}(\mathfrak{q})$ actually contains $\mathfrak{q}$-regular nilpotent elements. But, in some cases, this does not occur. A typical example is the case of Riemannian symmetric pairs (cf. Remark 1.6).

Let $X_0 \in \mathfrak{q}$ be $\mathfrak{q}$-regular nilpotent. Then X_0 has the following properties (cf. [KR], [vD]):

(i) $\dim \mathfrak{q}_{X_0}=l$.

(ii) If $n_1, \cdots, n_l$ are the integers defined as in (1.7), one may take the homogeneous generators $P_1, \cdots, P_l$ of $S(\mathfrak{q}_c)^H$ in such a way that $n_i+2=2\deg P_i$ ($i=1, \cdots, l$).

(iii) X_0 is $\mathfrak{q}$-distinguished.

(iv) $\delta_{\mathfrak{q}}(X_0)=l+\sum_{\lambda\in\Sigma^+,(1/2)\lambda\notin\Sigma}(2-m_\lambda-m_{2\lambda})$.

(Here Σ is the root system defined as in (1.4) and Σ^+ is a positive system of Σ.)

(1.10) Let $Z_0 \in \mathfrak{q}$ and let $Z_0=A_0+X_0$ be as in Lemma 1.1. Since A_0 is semisimple, we can define the sub-symmetric pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ and the vector spaces $\mathfrak{z}_s^-$, $\mathfrak{c}^-$, as we did in (1.5). By definition, X_0 is a nilpotent element of $\mathfrak{z}_{\mathfrak{s}}^{-}$. Hence the number $\delta_{\mathfrak{z}_{\mathfrak{s}}^{-}}(X_0)$ is defined similarly. Then we put $\delta_{\mathfrak{q}}(Z_0)=\delta_{\mathfrak{z}_{\mathfrak{s}}^{-}}(X_0)$.

Definition 1.13. Z_0 is $\mathfrak{q}$-distinguished if X_0 is $\mathfrak{z}_{\mathfrak{s}}^{-}$-distinguished nilpotent.

If $Z_0 \in \mathfrak{q}$ is $\mathfrak{q}$-regular, then Z_0 is $\mathfrak{q}$-distinguished. In fact, if $Z_0=A_0+X_0$ is the Jordan decomposition of Z_0 as above, then X_0 is $\mathfrak{z}_{\mathfrak{s}}^{-}$-regular nilpotent.

(1.11) Let A be a $\mathfrak{q}$-regular semisimple element of $\mathfrak{q}$. Then $\mathfrak{g}_A$ is left invariant by σ. So $\mathfrak{g}_A=\mathfrak{h}_A+\mathfrak{q}_A$ is a direct sum. Note that $\mathfrak{a}=\mathfrak{q}_A$ is a Cartan subspace of $\mathfrak{q}$ and that $\mathfrak{h}_A$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Then it easily follows that the number $m=\dim\mathfrak{a}-\dim\mathfrak{h}_A$ does not depend on the choice of A.

Lemma 1.14 (cf. [KR]). $\dim\mathfrak{q}_X-\dim\mathfrak{h}_X=m$ *for all* $X \in \mathfrak{q}$.

Corollary 1.15. $\dim\mathfrak{q}_X=\frac{1}{2}(\dim\mathfrak{g}_X+m)$ *for all* $X \in \mathfrak{q}$.

§ 2. Preliminaries from differential equations

(2.1) Let M be a connected open subset of $\mathbf{R}^m$ containing the origin and $x=(x_1, \cdots, x_m)$ its Cartesian coordinate system. First introduce some standard notation. As usual, put $D_{x_i}=\partial/\partial x_i$ $(i=1, \cdots, m)$. For any multi-index $\alpha=(\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$, $|\alpha|=\alpha_1+\cdots+\alpha_m$. Moreover, $x^\alpha=x_1^{\alpha_1}\cdots x_m^{\alpha_m}$, $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_m}^{\alpha_m}$. Let $\mathscr{A}(M)$ and $\mathscr{B}(M)$ denote the set of analytic functions on M and that of hyperfunctions on M, respectively. Moreover, $\mathscr{B}_M$ denotes the sheaf of hyperfunctions on M.

In this paper, differential operators always mean those having analytic functions as coefficients. Let $P(x, D_x)$ be a differential operator defined on M. Then P is expressed as follows:

$$P=\sum_{\alpha\in N^m, |\alpha|\leq d} a_\alpha(x)D_x^\alpha.$$

If $a_\alpha(x)$ is not identically zero for some α ($|\alpha|=d$), then d is called the order of P and is denoted by ord P. Moreover, $\sigma(P)(x, \xi)=\sum_{|\alpha|=d} a_\alpha(x)\xi^\alpha$ is called its principal symbol, where $\xi=(\xi_1, \cdots, \xi_m)$ is the conormal variable and $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$. For any $x_0 \in M$, $P(x_0, D_x)$ is called the local expression of P at x_0.

Define, for any $p \in S(\mathbf{C}^m)$, a differential operator $\partial(p)$ on $\mathbf{R}^m$ as follows. First, if $v=(v_1, \cdots, v_m) \in \mathbf{R}^m$, then define the vector field $\partial(v)$ on $\mathbf{R}^m$ by

$$(\partial(v)f)(x)=\frac{d}{dt}f(x+tv)|_{t=0},$$

where $f(x)$ is an arbitrary analytic function on $\mathbf{R}^m$. If $v_1, \cdots, v_k \in \mathbf{R}^m$, then $\partial(v_1 \cdots v_k) = \partial(v_1) \cdots \partial(v_k)$. Extending ∂ linearly to the symmetric algebra $S(\mathbf{C}^m)$, we have thus defined $\partial(p)$ for any $p \in S(\mathbf{C}^m)$. It is clear from the definition that for every $p \in S(\mathbf{C}^m)$, the local expression of $\partial(p)$ at every point of $\mathbf{R}^m$ coincides with itself.

(2.2) Let N be a connected open subset of $\mathbf{R}^n$ containing the origin and $y = (y_1, \cdots, y_n)$ as its Cartesian coordinate. We always identify N with a subset $\{0\} \times N$ of $M \times N$. The next lemma is fundamental (cf. [SKK]).

Lemma 2.1. *Let $u(x, y)$ be a hyperfunction on $M \times N$. If $u(x, y)$ satisfies the differential equations*

$$D_{x_i} u = 0 \quad (i = 1, \cdots, m),$$

then $u(x, y)$ is constant with respect to the variable x, that is $u(x, y)|_{x=0} = v(y)$ is a well-defined hyperfunction on N and $u(x, y) = v(x)$ holds on $M \times N$.

(2.3) In the rest of this section, we give propositions which play fundamental roles in the proof of Theorem 5.2 in Section 5.

Proposition 2.2. *Let $P(x, y, D_x, D_y)$ be a differential operator of order 2 defined on $M \times N$ and let $u(x, y)$ be a hyperfunction solution of the differential equation $Pu = 0$. Assume the following conditions:*

(i) *$\sigma(P)(0, y, \xi, 0)$ is a non-degenerate quadratic form of ξ for all $y \in N$.*

(ii) Supp $(u) \subset N$.

Then $u = 0$.

This follows from Chap. III, Prop. 2.1.3 in [SKK].

Proposition 2.3. *Let P be a differential operator defined on $M \times N$ of order 2 and of the following form:*

$$P = \sum_{j=1}^{m} a_j(x, y) D_{x_1} D_{x_j} + b(x, y) D_{x_1} + R_1(x, y, D'_x, D_y) x_1 D_{x_1} + R_0(x, y, D'_x, D_y),$$

where $D'_x = (D_{x_2}, \cdots, D_{x_m})$. Assume that

$a_1(x, y) = \lambda_1 x_1,$

$a_j(x, y) = \lambda_j x_j +$ *(terms of order* ≥ 2 *with respect to* x),

$b(x, y) = \mu +$ *(terms of order* ≥ 1 *with respect to* x),

$\sigma(R_0)(0, y, \xi_2, \cdots, \xi_m, 0) = 0,$

where each λ_j *is a constant such that* $\mathrm{Re}\,\lambda_j>0$ and μ is also a constant.

If $u(x, y)$ *is a hyperfunction on* $M\times N$ *satisfying that*

$$Pu=0, \qquad \mathrm{Supp}\,(u)\subset N,$$

then $u(x, y)$ *is expressed of the form*

$$u(x, y)=\sum f_\alpha(y)\delta^{(\alpha)}(x) \qquad (\textit{finite sum}),$$

where each $f_\alpha(x)$ *is a hyperfunction on* N.

Remark. T. Oshima has shown the author the proof of Proposition 2.3. So he expresses his hearty thanks to T. Oshima.

Proof. Let Z be an open subset of $\mathbf{C}^m$ such that $Z\cap\mathbf{R}^m=M$. Similarly, let $W\subset\mathbf{C}^n$ be an open set such that $W\cap\mathbf{R}^n=N$. Identify W with the subset $\{0\}\times W$ of $Z\times W$. Let $\mathcal{O}_{Z\times W}$ be the sheaf of holomorphic functions on $Z\times W$ and put $\mathcal{J}=\sum_{j=1}^m \mathcal{O}_{Z\times W}x_j$. It follows from the definition that $(Px_1)\mathcal{J}^d\subseteq\mathcal{J}^d$ for any integer $d>0$.

To prove the proposition, we need a lemma.

Lemma 2.4. *Choose a positive integer h such that*

$$\mathrm{Re}\left(\frac{h}{m}\lambda_j+\mu\right)>0 \qquad (j=1,\cdots,m).$$

Then for any open neighbourhood U *of the origin* $0\in Z\times W$, *there exists a small open neighbourhood* $V\subset U$ *of* 0 *satisfying the conditions* (i), (ii):

(i) Px_1: $\mathcal{J}^h(V)\to\mathcal{J}^h(V)$ *is injective.*

(ii) *For any* $f\in\mathcal{J}^h(U)$, *there exists a* $u\in\mathcal{J}^h(V)$ *such that* $Px_1u=f$ *holds on* V.

Assuming this lemma for a moment, we continue the proof of Proposition 2.3. Note that the adjoint P^* of P also satisfies the assumption of the proposition. Let T be a sufficiently small complex neighbourhood of the origin and let K be a compact subset of $T\cap N$. Replacing each point of K with the origin and applying Lemma 2.4, we find that

$$P^*x_1:\ \mathcal{J}^h(K)\to\mathcal{J}^h(K) \tag{2.3.1}$$

is a topological automorphism as a DFS-space. Here h is a sufficiently large integer satisfying the conditions (i), (ii) of Lemma 2.4 and

$$\mathcal{J}^h(K)=\varinjlim_U \mathcal{J}^h(U) \quad (U \text{ runs through complex neighbourhoods of } K).$$

Note that h is independent of the choice of K. Now put

$$\mathscr{S}_h=\{u\in\mathscr{B}_{M\times N};\ fu=0\ \text{for all}\ f\in\mathscr{J}^h\}.$$

Since $\mathscr{S}_h\simeq\sum_{|\alpha|<h}\mathscr{B}_N\delta^{(\alpha)}(x)$, $\mathscr{S}_h$ is regarded as not only a sheaf on $M\times N$ but also that on N. Taking the dual of the isomorphism (2.3.1), we obtain the isomorphism

$$H_K^0(M\times N,\ \mathscr{B}_{M\times N})\ \mathrm{mod}\ \Gamma(K,\mathscr{S}_h)\xleftarrow{x_1P}H_K^0(M\times N,\ \mathscr{B}_{M\times N})\ \mathrm{mod}\ \Gamma(K,\mathscr{S}_h).$$

Define the presheaf $\mathscr{F}$ on $M\times N$ as follows. For every open subset D of $M\times N$, $\Gamma(D,\mathscr{F})=H^0_{D\cap N}(D,\mathscr{B}_{M\times N})\ \mathrm{mod}\ \Gamma(D,\mathscr{S}_h)$. Since both $\mathscr{B}_{M\times N}$ and $\mathscr{S}_h$ are flabby sheaves, we find that $\mathscr{F}$ becomes a flabby sheaf. Hence, if D is contained in T, it follows that

$$\mathscr{F}(D)\xleftarrow{x_1P}\mathscr{F}(D)$$

is an isomorphism. This implies the proposition.

Proof of Lemma 2.4. First introduce a partial order on $\mathbf{N}^m$. Let $\alpha=(\alpha_1,\cdots,\alpha_m)$, $\alpha'=(\alpha_1',\cdots,\alpha_m')\in\mathbf{N}^m$. We define that $\alpha<\alpha'$ if and only if one of the conditions (a), (b) holds:

(a) $\alpha_1<\alpha_1'$.

(b) $\alpha_1=\alpha_1'$ and $|\alpha|<|\alpha'|$.

Consider the differential equation

$$(Px_1)u=f. \tag{2.3.2}$$

Assume that $f(x,y)$ and $u(x,y)$ are formal power series of the forms

$$f(x,y)=\sum_{\alpha\in\mathbf{N}^m,\beta\in N^n,|\alpha|\geq h}f_{\alpha\beta}x^\alpha y^\beta\qquad(f_{\alpha\beta}\in\mathbf{C}) \tag{2.3.3}$$

$$u(x,y)=\sum_{\alpha\in\mathbf{N}^m,\beta\in\mathbf{N}^n,|\alpha|\geq h}u_{\alpha\beta}x^\alpha y^\beta\qquad(u_{\alpha\beta}\in\mathbf{C}). \tag{2.3.4}$$

Note that if $u(x,y)$ is of the form (2.3.4), then $f=Px_1u$ is expressed of the form (2.3.3). In the sequel, we always assume that f and u are the ones as in (2.3.3) and (2.3.4). Put

$$\begin{aligned}P_0&=\Big(\sum_{j=1}^m\lambda_jx_jD_{x_1}D_{x_j}+\mu D_{x_1}\Big)x_1\\&=(x_1D_{x_1}+1)\Big(\sum_{j=1}^m\lambda_jx_jD_{x_j}+\mu\Big).\end{aligned}$$

Then (2.3.2) is rewritten of the form

$$P_0 u = (P_0 - P x_1) u + f. \tag{2.3.5}$$

Comparing the coefficients of $x^\alpha y^\beta$ of both sides in (2.3.5), we find that

$$\begin{aligned} &(\text{left hand side}) = (\alpha_1 + 1)\Big(\sum_{j=1}^{m} \lambda_j \alpha_j + \mu\Big) u_{\alpha\beta} \\ &(\text{right hand side}) = f_{\alpha\beta} + \sum_{\substack{\alpha' < \alpha \\ |\alpha'| + |\beta'| \le |\alpha| + |\beta| + 2}} c^{\alpha'\beta'}_{\alpha'\beta'} u_{\alpha'\beta'}, \end{aligned} \tag{2.3.6}$$

where $c^{\alpha\beta}_{\alpha'\beta'}$ are constants independent of the choices of f and u. From the choice of h, we find that $\mathrm{Re}\,(\sum_{j=1}^{m} \lambda_j \alpha_j + \mu) > 0$ for all $\alpha \in \mathbf{N}^m$, $|\alpha| \ge h$. It follows from (2.3.5) and (2.3.6) that $u(x, y)$ is uniquely determined by $f(x, y)$.

We are going to prove by the method of majorants that if f is convergent, then the formal power series solution u of (2.3.2) is actually convergent. First we replace the coordinate (x, y) with $(x_1^3, x_2, \cdots, x_m, y)$ and rewrite this by (x, y) for the sake of simplicity. Then $(R_1 x_1 D_{x_1} + R_0) x_1$ is changed into the form

$$\begin{aligned} & x_1\{\tilde{R}_1(x, y, x_1 D'_x, x_1 D_y) x_1 D_{x_1} + \tilde{R}_0\,(x, y, x_1 D'_x, x_1 D_y)\} \\ & = x_1\{\sum_{\alpha \in N^{m-1}, \beta \in N^n, |\alpha| + |\beta| \le 2} e_{\alpha\beta}(x, y)(x_1 D'_x)^\alpha (x_1 D_y)^\beta\}. \end{aligned} \tag{2.3.7}$$

Here $(x_1 D'_x)^\alpha = x_1^{|\alpha|} D_{x_2}^{\alpha_2} \cdots D_{x_m}^{\alpha_m}$, etc, and each $e_{\alpha\beta}(x, y)$ is an analytic function. Moreover λ_1 is changed into $3\lambda_1$. For the sake of simplicity, we rewrite this by λ_1. Put $s = x_2 + \cdots + x_m$ and $t = y_1 + \cdots + y_n$, and choose positive constants ε, C, $C' > 0$ satisfying the conditions

$$\varepsilon |\alpha| \le \Big|\sum_{j=1}^{m} \lambda_j \alpha_j + \mu\Big| \qquad (\forall \alpha \in \mathbf{N}^m,\ |\alpha| \ge h) \tag{2.3.8.1}$$

$$a_j(x, y) - \lambda_j x_j \ll \frac{\varepsilon C'(x_1 + s)^2}{C - x_1 - s - t} \qquad (j = 2, \cdots, m) \tag{2.3.8.2}$$

$$b(x, y) - \mu \ll \frac{C'(x_1 + s)}{C - x_1 - s - t} \tag{2.3.8.3}$$

$$f(x, y) \ll \frac{C'(x_1 + s)^h}{C - x_1 - s - t} \tag{2.3.8.4}$$

$$e_{\alpha\beta}(x, y) \ll \frac{C'}{C - x_1 - s - t} \qquad \text{for all } (\alpha, \beta). \tag{2.3.8.5}$$

Put $\hat{f}(x, y)=C'(x_1+s)^h/(C-x_1-s-t)(=\sum\hat{f}_{\alpha\beta}x^\alpha y^\beta)$ and let $x_1\hat{R}$ be the differential operator which is obtained from the one in (2.3.7) by changing each coefficient $e_{\alpha\beta}(x, y)$ with $C'/(C-x_1-s-t)$, that is, $\hat{R}$ is defined by

$$\hat{R}=\frac{C'}{C-x_1-s-t}\sum_{\alpha,\beta}(x_1D'_x)^\alpha(x_1D_y)^\beta. \tag{2.3.9}$$

Moreover define

$$\hat{P}_0=\varepsilon(x_1D_{x_1}+1)\Big(\sum_{j=1}^m x_jD_{x_j}\Big)$$

$$\hat{S}=\frac{C'(x_1+s)}{C-x_1-s-t}\Big\{\varepsilon(x_1+s)\Big(\sum_{j=2}^m x_jD_{x_j}\Big)+1\Big\}(x_1D_{x_1}+1)+x_1\hat{R}$$

and consider the differential equation

$$(\hat{P}_0-\hat{S})\hat{u}=\hat{f}. \tag{2.3.10}$$

Then, by means of an argument similar to the above one, one finds that for the given $\hat{f}(x, y)$, there exists a unique formal power series $\hat{u}(x, y)=\sum_{|\alpha|\geq h}\hat{u}_{\alpha\beta}x^\alpha y^\beta$ satisfying (2.3.10). In particular, as in (2.3.6), we obtain the relations

$$\varepsilon|\alpha|(\alpha_1+1)\hat{u}_{\alpha\beta}=\hat{f}_{\alpha\beta}+\sum\hat{c}^{\alpha\beta}_{\alpha'\beta'}\hat{u}_{\alpha'\beta'}. \tag{2.3.11}$$

Comparing (2.3.6) and (2.3.11), we find that $\hat{u}(x, y)$ is a majorant of $u(x, y)$. On the other hand, one easily shows that $\hat{u}(x, y)$ is expressed as a power series of x_1, s, t. Noting this, we may assume that $\hat{u}(x, y)=v(x_1, s, t)$ for a power series $v(x_1, s, t)$ of x_1, s, t. Then, from the definition, we find that

$$\hat{P}_0\hat{u}=\varepsilon(x_1D_{x_1}+1)(x_1D_{x_1}+sD_s)v,$$

$$\hat{S}\hat{u}=\Big[\frac{C'(x_1+s)}{C-x_1-s-t}\{\varepsilon(x_1+s)sD_s+1\}(x_1D_{x_1}+1)+x_1R'\Big]v,$$

where $R'=(C'/(C-x_1-s-t))\sum_{\alpha,\beta}(x_1D_s)^{|\alpha|}(x_1D_t)^{|\beta|}$ (cf. (2.3.9)). Then (2.3.10) turns out to be

$$(P'_0-S')v=\hat{f}, \tag{2.3.12}$$

where P'_0 and S' are defined by

$$P'_0=\varepsilon(x_1D_{x_1}+1)(x_1D_{x_1}+sD_s)$$

$$S'=\frac{C's}{C-x_1-s-t}\{(\varepsilon s^2D_s+1)(x_1D_{x_1}+1)\}$$

$$+x_1\left\{R'+\frac{\varepsilon C'(x_1+2s)}{C-x_1-s-t}sD_s(x_1D_{x_1}+1)+\frac{C'}{C-x_1-s-t}(x_1D_{x_1}+1)\right\}$$

At this stage, we consider the differential equation

$$(P'_0-S')v=g. \tag{2.3.13}$$

As in the previous case, assume that $g(x_1, s, t)$ and $v(x_1, s, t)$ are power series of the forms

$$g(x_1, s, t)=\sum_{i+j\geq h} g_{ijk}x_1^i s^j t^k \tag{2.3.14}$$

$$v(x_1, s, t)=\sum_{i+j\geq h} v_{ijk}x_1^i s^j t^k. \tag{2.3.15}$$

If $g=\hat{f}$, then (2.3.13) is reduced to (2.3.12). For this reason, from now on, we treat (2.3.13) instead of (2.3.12). Then by an argument similar to the above one, it suffices to show that if $g(x_1, s, t)$ is convergent, so is the solution $v(x_1, s, t)$ of (2.3.13). To accomplish this, introduce the function $\eta(x_1, s, t)$ defined by

$$\left(1-\frac{C's^2}{C-x_1-s-t}\right)\frac{\partial\eta}{\partial s}=\frac{C's}{C-x_1-s-t}\eta, \qquad \eta(x_1, 0, t)=1. \tag{2.3.16}$$

Using η, we change the coordinates by $(x_1', s', t')=(x_1, \eta s, t)$ and rewrite the differential equation (2.3.13) into that of x_1', s', t'. For the sake of simplicity, we exchange x_1', s', t' with x_1, s, t. Then (2.3.13) turns out to be

$$[\{\varepsilon(3x_1D_{x_1}+sD_s)+\phi(x_1, s, t)\}(x_1D_{x_1}+1)+x_1R'']v=g. \tag{2.3.17}$$

where $\phi(x_1, s, t)$ is an analytic function defined in a neighbourhood of the origin, such that $\phi(0)=0$. In virtue of the discussion above, it suffices to show that if $g(x_1, s, t)$ is a convergent power series of the form (2.3.14), then the power series solution $v(x_1, s, t)$ of (2.3.17) is also convergent. We may assume that

$$\phi\ll\hat{\phi}=\frac{x_1+s+t}{C-x_1-s-t}, \qquad g\ll\hat{g}=\frac{C'(x_1+s)^h}{C-x_1-s-t} \tag{2.3.18}$$

holds for the constants $C, C'>0$ and consider the differential equation

$$\{\varepsilon(3x_1D_{x_1}+1)(3x_1D_{x_1}+sD_s)-\hat{\phi}(3x_1D_{x_1}+1)-x_1\hat{R}''\}\hat{v}=\hat{g}, \tag{2.3.19}$$

where $\hat{R}''$ is a differential operator constructed from R'' in the same way obtaining $\hat{R}$ from $(R_1x_1D_{x_1}+R_0)x_1$. Then we find that $\hat{v}(x_1, s, t)$ is a majorant of $v(x_1^3, s, t)$. On the other hand, the differential equation (2.3.19) has regular singularities along the hypersurface $\{x_1=0\}$ (cf. [O]). Then in virtue of [O], we conclude that $\hat{v}(x_1, s, t)$ is convergent.

We have thus proved that $\hat{v}(x, y)=v(x_1, s, t)$ is convergent and therefore that the solution $u(x, y)$ of (2.3.2) is convergent. This implies (i) and (ii). In the above discussions, we assumed that $m>1$. But the case $m=1$ is easier to prove than the case $m>1$. In fact, in this case, the original equation (2.3.2) has regular singularities. q.e.d.

As a corollary of Proposition 2.3, we obtain the next one.

Proposition 2.5. *Retain the notation and the assumption in Proposition 2.3. Moreover assume that*

$$\sum_{j=1}^{m} \lambda_j(\alpha_j+1)\neq b(0) \qquad \text{for all } \alpha\in \mathbf{N}^m. \tag{2.3.20}$$

Then $u=0$.

Proof. Let $u(x, y)$ be a non-zero hyperfunction on $M\times N$ such that $Pu=0$ and that $\mathrm{Supp}(u)\subset N$. Then, by means of Proposition 2.3, we may assume from the first that $u(x, y)=\sum_{|\alpha|\leq d} f_\alpha(y)\delta^{(\alpha)}(x)$, $f_\alpha(y)\neq 0$ for some α ($|\alpha|=d$).

Let $f(y)$ be an arbitrary hyperfunction on N and take $\alpha\in\mathbf{N}^m$. Then it follows that

$$\begin{aligned} P(f(y)\delta^{(\alpha)}(x)) &= \{-\sum_{j=1}^{m}\lambda_j(\alpha_j+1)+\mu\} f(y)\delta^{(\alpha')}(x) \\ &\quad + \sum_{\beta\in\mathbf{N}^m, \beta<\alpha'} g_\beta(y)\delta^{(\beta)}(x), \end{aligned} \tag{2.3.21}$$

where $\alpha'=(\alpha_1+1, \alpha_2, \cdots, \alpha_m)$ and each $g_\beta(y)$ is a hyperfunction on N. Noting (2.3.21), we conclude from the assumption (2.3.20) that $u=0$. This contradicts the assumption. q.e.d.

§ 3. Radial components of differential operators

(3.1) Following [HC2] and [vD], we introduce some notation. For any $X\in\mathfrak{q}_c$, L_X is the linear endomorphism of $S(\mathfrak{q}_c)$ defined by $L_X(p)=Xp$. On the other hand, for any $X\in\mathfrak{h}_c$, d_X is the unique derivation of $S(\mathfrak{q}_c)$ which coincides on $\mathfrak{q}_c$ with $\mathrm{ad}_{\mathfrak{g}_c}(X)$. Furthermore, for any $Y\in\mathfrak{q}_c$, define $\sigma_Y(X)=L_{[X,Y]}+d_X (X\in\mathfrak{h}_c)$. Then σ_Y is a representation of $\mathfrak{h}_c$ on $S(\mathfrak{q}_c)$. We can extend σ_Y to the representation of $U(\mathfrak{h}_c)$ on $S(\mathfrak{q}_c)$ and denote it by the same letter. For $Y\in\mathfrak{q}$, let Γ_Y be the linear mapping of $U(\mathfrak{h}_c)\otimes S(\mathfrak{q}_c)$

to $S(\mathfrak{q}_c)$ defined by $\Gamma_Y(g\otimes p)=\sigma_Y(g)p$ ($g\in U(\mathfrak{h}_c)$, $p\in S(\mathfrak{q}_c)$).

(3.2) Now take an element X_0 of $\mathfrak{q}$. Let U be a linear subspace of $\mathfrak{q}$ such that $\mathfrak{q}=U+[X_0,\mathfrak{h}]$ is a direct sum. On the other hand, let V be a linear subspace of $\mathfrak{h}$ such that $\mathfrak{h}=V+\mathfrak{h}_{X_0}$ is a direct sum. For any $u\in U$, define the linear mapping ϕ_u of $V\times U$ to $\mathfrak{q}$ by $\phi_u(v,w)=w+[X_0+u,v]$. Then ϕ_0 is bijective. Noting this, we define $\zeta(u)=\det(\phi_u\circ\phi_0^{-1})$ $(u\in U)$ which is a polynomial on U with respect to an arbitrary linear coordinate. For the linear subspace V of $\mathfrak{h}$, $S(V_c)$ is naturally identified with the subspace of $S(\mathfrak{h}_c)$. Let $S_d(\mathfrak{h}_c)$ be the totality of homogeneous elements of degree d in $S(\mathfrak{h}_c)$ for every $d\in\mathbf{N}$. Put $S_d(V_c)=S(V_c)\cap S_d(\mathfrak{h}_c)$. Let λ be the symmetrization of $S(\mathfrak{h}_c)$ onto $U(\mathfrak{h}_c)$. Using λ, we define $\mathfrak{S}_d(V_c)=\lambda(S_d(V_c))$, $\mathfrak{S}(V_c)=\lambda(S(V_c))$ and $\mathfrak{S}_+(V_c)=\sum_{d\geq 1}\mathfrak{S}_d(V_c)$.

Lemmas 3.1–3.3 below are shown by arguments similar to Lemmas 8 and 9 in [HC2].

Lemma 3.1. *Let $u\in U$. If $\zeta(u)\neq 0$, then Γ_{X_0+u} is a bijective mapping of $\mathfrak{S}(V_c)\otimes S(U_c)$ onto $S(\mathfrak{q}_c)$.*

Lemma 3.2. *Take $p\in S(\mathfrak{q}_c)$. Then there exist a non-negative integer r and a polynomial mapping γ_p of U to $\mathfrak{S}(V_c)\otimes S(U_c)$ such that $\Gamma_{X_0+u}(\gamma_p(u))=\zeta(u)^r p$ for any $u\in U$.*

In the above lemma, a polynomial mapping is used in the following sense. Let A and B be vector spaces. Assume that $\dim A<\infty$. Then $p\colon A\to B$ is called a polynomial mapping if $p(a)=\sum_i f_i(a)v_i$ (finite sum), where $v_i\in B$ and each $f_i(a)$ is a polynomial on A with respect to a linear coordinate system.

Now put $U'=\{u\in U;\ \zeta(u)\neq 0\}$.

Lemma 3.3. *Fix $p\in S(\mathfrak{q}_c)$. Then for any $u\in U'$, there exist unique elements $\alpha_u(p)\in S(U_c)$ and $\beta_u(p)\in\mathfrak{S}_+(V_c)\otimes S(U_c)$ such that $p-\alpha_u(p)=$ $=\Gamma_{X_0+u}(\beta_u(p))$. Moreover $d^0\alpha_u(p)\leq d^0p$. (Here d^0q denotes the degree of q.)*

Definition 3.4. $\alpha_u(p)$ is called the radial component of p at u (with respect to X_0, U, V).

(3.3) For any $p\in S(\mathfrak{q}_c)$, we define the differential operator $\partial(p)$ as in Section 2. On the other hand, for any $X\in\mathfrak{h}$, define a vector field $\tau(X)$ on $\mathfrak{q}$ as follows:

$$(\tau(X)f)(Y)=\frac{d}{dt}f(Y+t[Y,X])|_{t=0}\qquad (Y\in\mathfrak{q}).$$

(Here $f(Y)$ is an arbitrary analytic function on $\mathfrak{q}$.)

Let D be a differential operator on an open subset Ω of $\mathfrak{q}$. Then we define a differential operator $\Delta(D)$ on the open subset $U'\cap(\Omega-X_0)$ by the same way as in [HC2, p. 283] (see also [vD]).

Let $\mathscr{D}(\Omega)$ denote the ring of differential operators on Ω and let $\mathscr{T}(\Omega)$ be the left ideal of $\mathscr{D}(\Omega)$ generated by the vector fields $\tau(X)$ ($X\in\mathfrak{h}$).

Definition 3.5. (i) A hyperfunction u on Ω is locally invariant if $\tau(X)u=0$ for all $X\in\mathfrak{h}$.

(ii) A differential operator D on Ω is locally invariant if $[\tau(X), D]=0$ for all $X\in\mathfrak{h}$.

(3.4) Return to the situation in (3.2). Take a basis $u_1, \cdots, u_m$ of U and let $x_1, \cdots, x_m$ be its coordinates. Similarly, take a basis $b_1, \cdots, b_n$ of V and let $y_1, \cdots, y_n$ be the coordinate with respect to this V-basis.

Lemma 3.6. *Define the analytic mapping F of $\mathbf{R}^m\times\mathbf{R}^n$ to $\mathfrak{q}$ by*

(3.4.1) $$F(x, y)=e^{y_1b_1}\cdots e^{y_nb_n}(X_0+\sum_{i=1}^{m}x_iu_i).$$

Then $F(0, 0)=X_0$ and dF is non-singular at $(0, 0)$.

This lemma follows from the definition.

Let U_0 and V_0 be open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$ containing the origins, such that $U_0\subset U'$ and that $F|U_0\times V_0$ is an isomorphism. Put $\Omega_0=F(U_0\times V_0)$. Then Ω_0 is an open neighbourhood of X_0 and is in a one to one correspondence with $U_0\times V_0$. For simplicity, put $f^\sim(x, y)=f(F(x, y))$ for any $f\in\mathscr{A}(\Omega_0)$. Similarly, for any $D\in\mathscr{D}(\Omega_0)$, define a differential operator $D^\sim$ on $U_0\times V_0$ by $(D^\sim h)(x, y)=(D(h\circ F^{-1})(F(x, y)))$. Since F is bijective, $D^\sim$ is well-defined. Let $\tilde{\mathscr{T}}(U_0\times V_0)$ be the left ideal of $\mathscr{D}(U_0\times V_0)$ generated by the vector fields $\tau(X)^\sim$ ($\forall X\in\mathfrak{h}$).

Lemma 3.7. *D_{y_i} $(1\leq i\leq n)$ are contained in $\tilde{\mathscr{T}}(U_0\times V_0)$.*

Proof. Define analytic functions $f_{ij}(y)$ by $\mathrm{Ad}(e^{y_1b_1}\cdots e^{y_{i-1}b_{i-1}})b_i=\sum_{j=1}^{k}f_{ij}(y)X_j$, where $\{X_1, \cdots, X_k\}$ is a basis of $\mathfrak{h}$ ($k=\dim\mathfrak{h}$). Then it follows that

$$D_{y_i}=\sum_{j=1}^{k}f_{ij}(y)\tau(X_j)^\sim\in\tilde{\mathscr{T}}(U_0\times V_0).\qquad \text{q.e.d.}$$

Lemma 3.8. *For each i $(1\leq i\leq n)$, the local expression of $\tau(b_i)^\sim$ at $(x, 0)$ coincides with $-D_{y_i}$.*

Proof. Take $h \in \mathscr{A}(U_0 \times V_0)$ and calculate $(\tau(b_i)^{\sim} h)(x, 0)$. For simplicity, put $f = h \circ F^{-1} \in \mathscr{A}(\Omega_0)$. Then

$$\begin{aligned}(\tau(b_i)^{\sim} h)(x, 0) &= (\tau(b_i) f)(F(x, 0)) \\ &= (\tau(b_i) f)(X_0 + \sum_j x_j u_j) \\ &= \frac{d}{dt} f(e^{-tb_i}(X_0 + \sum_j x_j u_j))|_{t=0}.\end{aligned}$$

Since $e^{-tb_i}(X_0 + \sum_j x_j u_j) = F(x, y)$ with $y_k = -\delta_{ki} t$ $(1 \leq k \leq n)$, we find that

$$\frac{d}{dt} f(e^{-tb_i}(X_0 + \sum_j x_j u_j))|_{t=0} = -(D_{y_i} h)(x, 0).$$

Hence it follows that $(\tau(b_i)^{\sim} h)(x, 0) = -(D_{y_i} h)(x, 0)$. This holds for any $h \in \mathscr{A}(U_0 \times V_0)$ and the lemma follows. q.e.d.

Lemma 3.9. *Take $X_1, \cdots, X_k \in \mathfrak{h}$ and $Y_1, \cdots, Y_l \in \mathfrak{q}$ and define the differential operator P on $\mathfrak{q}$ by*

$$(Pf)(Y) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \frac{\partial^l}{\partial s_1 \cdots \partial s_l} f(e^{t_1 X_1} \cdots e^{t_k X_k}(Y + \sum_{j=1}^{l} s_j Y_j))|_{t=s=0}$$

for any $f \in \mathscr{A}(\mathfrak{q})$. On the other hand, define $g = X_1 \cdots X_k \in U(\mathfrak{h}_c)$ and $p = Y_1 \cdots Y_l \in S(\mathfrak{q}_c)$. Then the local expression of P at $Y \in \mathfrak{q}$ coincides with $\partial(\Gamma_Y(g \otimes p))$.

This follows from the definitions of $\Gamma_Y(g \otimes p)$ and P.

(3.5) If $p \in S(\mathfrak{q}_c)$, then $\partial(p)$ is a differential operator on $\mathfrak{q}$ and also $\partial(p)^{\sim}$ is a differential operator on $U_0 \times V_0$. Then it follows that

$$\partial(p)^{\sim} = \sum_{\alpha \in \mathbf{N}^m} \sum_{\beta \in \mathbf{N}^n} a_{\alpha\beta}(x, y) D_x^{\alpha} D_y^{\beta}, \tag{3.5.1}$$

where each $a_{\alpha\beta}(x, y)$ is an analytic function on $U_0 \times V_0$. On the other hand, let $\alpha_u(p)$ be the radial component of p at $u \in U'$. For simplicity, put $u^{\alpha} = u_1^{\alpha_1} \cdots u_m^{\alpha_m} \in S(U_c)$ $(\alpha \in \mathbf{N}^m)$. Then, from Lemma 3.3, we can define analytic functions $f_{\alpha}(x)$ $(\alpha \in \mathbf{N}^m)$ such that $\alpha_u(p) = \sum_{\alpha \in \mathbf{N}^m} f_{\alpha}(x) u^{\alpha}$ for any $u = \sum_{j=1}^{m} x_j u_j \in U'$. Corresponding to this expression, we define the differential operator $Q(x, D_x)$ on $U' (\subset \mathbf{R}^m)$ by

$$Q(x, D_x) = \sum_{\alpha \in \mathbf{N}^m} f_{\alpha}(x) D_x^{\alpha}. \tag{3.5.2}$$

Lemma 3.10. *Q coincides with $\sum_{\alpha\in\mathbf{N}^m} a_{\alpha 0}(x, 0)D_x^\alpha$.*

Proof. It follows from Lemma 3.8 and 3.9 that for any $\alpha\in\mathbf{N}^m$ and $\beta\in\mathbf{N}^n$, $D_x^\alpha D_y^\beta$ coincides with the local expression of $\partial(\Gamma_{X_0+\sum x_j u_j}(b^\beta\otimes u^\alpha))^\sim$ at $(x, 0)$, where b^β denotes $b_1^{\beta_1}\cdots b_n^{\beta_n}$ for all $\beta\in\mathbf{N}^n$. Hence the local expression of $\partial(p)^\sim$ at $(x, 0)\in U_0\times V_0$ coincides with that of

$$(3.5.3)\qquad \sum_{\alpha\in\mathbf{N}^m}\sum_{\beta\in\mathbf{N}^n} a_{\alpha\beta}(x, 0)\partial(\Gamma_{X_0+\sum x_j u_j}(b^\beta\otimes u^\alpha))^\sim.$$

On the other hand, from Lemma 3.3, we can choose $g_j\in\mathfrak{S}_+(V_c)$, $p_j\in S(U_c)$, polynomials $q_j(u)$ on U $(1\leq i\leq N)$ and an integer $r\geq 0$ such that

$$(3.5.4)\qquad p=\alpha_u(p)+\zeta(u)^{-r}\sum_{j=1}^N q_j(u)\Gamma_{X_0+u}(g_j\otimes p_j)$$

for $u\in U'$. Put $\zeta^\sim(x)=\zeta(u)$, $q_j^\sim(x)=q_j(u)$ for $u=\sum_i x_i u_i$. Then we find that the local expression of $\partial(p)^\sim$ at $(x, 0)$ coincides with that of

$$(3.5.5)\qquad Q+\zeta^\sim(x)^{-r}\sum_{j=1}^N q_j^\sim(x)\partial(\Gamma_{X_0+\sum x_j u_j}(g_j\otimes p_j))^\sim.$$

Comparing (3.5.3) and (3.5.5) and noting (3.5.2), we obtain the lemma. q.e.d.

For any $p\in S(\mathfrak{q}_c)$, we denote by $\Delta(\partial(p)^\sim)$ the differential operator $Q(x, D_x)$ defined in (3.5.2) and call it the radial component of $\partial(p)^\sim$ on U_0. By definition, $\Delta(\partial(p)^\sim)$ coincides with the differential operator $\partial(\alpha_u(p))$ on U_0 by the correspondence $(x_1, \cdots, x_m)\to\sum_j x_j u_j$. In this sense, $\Delta(\partial(p)^\sim)$ is nothing but $\Delta(\partial(p))$ defined in (3.3). Hence we confuse these ones in the sequel.

(3.6) Let u be a locally invariant hyperfunction defined on the open subset Ω_0 of $\mathfrak{q}$ and let p be an element of $S(\mathfrak{q}_c)^H$. The next lemma is a direct consequence of Lemmas 3.7 and 2.1.

Lemma 3.11. *The hyperfunction $u^\sim(x, y)$ on $U_0\times V_0$ is constant with respect to the variable y.*

This combined with the assumption on p implies that $u^\sim(x, y)$ and $(\partial(p)u)^\sim(x, y)$ can be restricted on the closed subset $U_0\simeq U_0\times\{0\}\subset U_0\times V_0$. Then the radial component $\Delta(\partial(p)^\sim)$ has the following meaning.

Theorem 3.12. *Let u and p be as above. Put $v(x)=u^\sim(x, 0)$, the restriction of u to U_0. Then*

$$(\partial(p)^\sim u)(x, 0)=\Delta(\partial(p)^\sim)v(x).$$

This follows from the definition of $\Delta(\partial(p)^{\sim})$ and Lemmas 3.7–3.10 (cf. [HC2, Lemma 10]).

§ 4. The radial component of the pseudo-Laplacian on $\mathfrak{q}$

(4.1) Retain the notation in the previous sections. Let ω be the Casimir polynomial on $\mathfrak{q}$.

Definition 4.1. The differential operator $\partial(\omega)$ is called the pseudo-Laplacian on $\mathfrak{q}$.

The purpose of this section is to calculate the radial components of $\partial(\omega)$ for certain elements of $\mathfrak{q}$.

(4.2) Let A_0 be a semisimple element of $\mathfrak{q}$. Similarly as in (1.5), define $\mathfrak{z}$, $\mathfrak{z}^{\pm}$, $\mathfrak{z}_s$, $\mathfrak{z}_s^{\pm}$, $\mathfrak{c}^-$. Take a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ containing both A_0 and $\mathfrak{c}^-$ and let $\Sigma=\Sigma(\mathfrak{a})$ be the root system of $(\mathfrak{g}_c, \mathfrak{a}_c)$ (cf. (1.4)). For any root $\lambda \in \Sigma$, choose a **C**-basis $X_{\lambda,1}, \cdots, X_{\lambda,m_\lambda}$ of $\mathfrak{g}_c^\lambda$ ($m_\lambda=\dim_{\mathbf{C}}\mathfrak{g}_c^\lambda$) such that $B(X_{\lambda,i}, \sigma X_{\lambda,j})=-\delta_{ij}$ $(i,j=1, \cdots, m_\lambda)$. Put $\mathfrak{a}^-=\mathfrak{a}\cap\mathfrak{z}_s^-$. Then $\mathfrak{a}=\mathfrak{a}^-+\mathfrak{c}^-$, and $\mathfrak{a}^-$ and $\mathfrak{c}^-$ are orthogonal with respect to the Killing form. Let $A_1, \cdots, A_p$ be a **C**-basis of $\mathfrak{a}_c^-$ such that $B(A_i, A_j)=\delta_{ij}$ and let $C_1, \cdots, C_q$ be a **C**-basis of $\mathfrak{c}_c^-$ such that $B(C_i, C_j)=\delta_{ij}$. As usual, identify $\mathfrak{q}_c$ with its dual by the Killing form. Then it follows from the definition that

$$\omega=\sum_{i=1}^{p}A_i^2+\sum_{i=1}^{q}C_i^2+\frac{1}{2}\sum_{\lambda\in\Sigma^+}\sum_{j=1}^{m_\lambda}(X_{\lambda,i}-\sigma X_{\lambda,i})^2. \tag{4.2.1}$$

(Here Σ^+ is the positive system of Σ for an order fixed hereafter.) Define

$$\Sigma_0=\{\lambda\in\Sigma;\ \lambda(A_0)=0\}$$
$$\Sigma_1=\{\lambda\in\Sigma;\ \lambda(A_0)\neq 0\}$$
$$\Sigma_j^+=\Sigma^+\cap\Sigma_j \quad (j=0, 1).$$

Using these, we also define

$$V_c^{\pm}=\sum_{\lambda\in\Sigma_1^+}\sum_{j=1}^{m_\lambda}\mathbf{C}(X_{\lambda,i}\pm\sigma X_{\lambda,i}) \tag{4.2.2}$$
$$V^+=V_c^+\cap\mathfrak{h},\quad V^-=V_c^-\cap\mathfrak{q}.$$

Then it is clear that $\mathfrak{h}=\mathfrak{z}^++V^+$ and $\mathfrak{q}=\mathfrak{z}^-+V^-$ are direct sums. Moreover, we have $[A_0, \mathfrak{h}]=[A_0, V^+]=V^-$ and $\dim V^+=\dim V^-$. Defining the linear mapping η_u of $\mathfrak{z}^-\times V^+$ to $\mathfrak{q}$ by $\eta_u(w, v)=w+[u, v]$, we find that η_{A_0} is bijective. Hence $\xi(u)=\det(\eta_u\circ\eta_{A_0}^{-1})$ $(u\in\mathfrak{z}^-)$ is well-defined. The next lemma follows from the definition.

Lemma 4.2. $\xi(A)=\prod_{\lambda\in\Sigma_1^+}[\lambda(A)/\lambda(A_0)]^{m_\lambda}$ *for all* $A\in\mathfrak{a}$.

Put $(\mathfrak{z}^-)'=\{u\in\mathfrak{z}^-;\ \xi(u)\neq 0\}$. Then the following statements are consequences of the results in (3.2).

Lemma 4.3. *For any* $Z\in(\mathfrak{z}^-)'$, Γ_Z *defines a bijective mapping of* $\mathfrak{S}(V_c^+)\otimes S(\mathfrak{z}_c^-)$ *to* $S(\mathfrak{q}_c)$.

Corollary 1. *Fix* $p\in S(\mathfrak{q}_c)$. *Then for any* $Z\in(\mathfrak{z}^-)'$, *there exist unique elements* $\alpha_Z(p)\in S(\mathfrak{z}_c^-)$ *and* $\beta_Z(p)\in\mathfrak{S}_+(V_c^+)\otimes S(\mathfrak{z}_c^-)$ *such that* $p-\alpha_Z(p)=\Gamma_Z(\beta_Z(p))$. *Moreover* $d^0\alpha_Z(p)\leq d^0p$.

Corollary 2. *For any* $p\in S(\mathfrak{q}_c)$, *we can choose an integer* $r\geq 0$ *such that the mappings* $Z\to\xi(Z)^r\alpha_Z(p)$ *and* $Z\to\xi(Z)^r\beta_Z(p)$ $(Z\in(\mathfrak{z}_c^-)')$ *can be extended to polynomial mappings of* $\mathfrak{z}_c^-$ *into* $S(\mathfrak{z}_c^-)$ *and* $\mathfrak{S}(V_c^+)\otimes S(\mathfrak{z}_c^-)$, *respectively.*

Let ω_s (resp. ω_c) be the restriction of ω to $\mathfrak{z}_s^-$(resp. $\mathfrak{c}^-$). Then ω_s (resp. ω_c) is a non-degenerate quadratic form on $\mathfrak{z}_s^-$ (resp. $\mathfrak{c}^-$). Using ω_s and ω_c, identify $\mathfrak{z}_s^-$ with its dual and $\mathfrak{c}^-$ with its dual. Then it follows that

$$\omega_s=\sum_{i=1}^{p}A_i^2+\frac{1}{2}\sum_{\lambda\in\Sigma_0^+}\sum_{j=1}^{m_\lambda}(X_{\lambda,i}-\sigma X_{\lambda,i})^2 \tag{4.2.3}$$
$$\omega_c=\sum_{i=1}^{q}C_i^2.$$

Lemma 4.4. *Let* $\Delta(\partial(\omega))$ *be the differential operator on* $\mathfrak{z}^-$ *obtained from* $\partial(\omega)$ *as we did in* (3.5). *Then*

$$\Delta(\partial(\omega))=\xi(u)^{-1/2}(\partial(\omega_c)+\partial(\omega_s))\circ\xi(u)^{1/2}-\mu(u),$$

where $\mu(u)=\xi(u)^{-1/2}(\partial(\omega_c)+\partial(\omega_s))\xi(u)^{1/2}$ *is an analytic function on* $(\mathfrak{z}^-)'$.

This lemma is shown by an argument similar to Lemma 18 in [HC3]. In the proof, we use the expression of ω in (4.2.1) and those of ω_s, ω_c in (4.2.3). Note that in our case, the function $\mu(u)$ does not vanish in general.

Let H^+ and H_s^+ be the analytic subgroups of H corresponding to $\mathfrak{z}^+$ and $\mathfrak{z}_s^+$, respectively. Then it follows from the definition that both $\xi(u)$ and $\mu(u)$ are H^+-invariant.

(4.3) Take a nilpotent element X_0 of $\mathfrak{z}_s^-$ and let (B_0, X_0, Y_0) be a normal S-triple for the pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$. In particular, $B_0\in\mathfrak{z}_s^+$ and $Y_0\in\mathfrak{z}_s^-$. Then we have the next lemma which is shown in the way similar to the proof of Lemma 1.12. Though ω_s is not the Casimir polynomial on $\mathfrak{z}_s^-$, we only use in the proof that ω_s is non-degenerate.

Lemma 4.5. *The following conditions are equivalent.*
(i) *X_0 is $\mathfrak{z}_s^-$-distinguished.*
(ii) *$\omega_s(X)=0$ for all $X \in (\mathfrak{z}_s^-)_{X_0}$.*
(iii) *$\omega_s(X)=0$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$.*
(iv) *$(\mathfrak{z}_s^-)_{X_0} \cap (\mathfrak{z}_s^-)_{Y_0}=0$.*

It follows from Lemma 1.9 that there exists a Cartan involution θ on $\mathfrak{z}_s$ commuting with σ such that θ: $(B_0, X_0, Y_0) \to (-B_0, -Y_0, -X_0)$. Then $-B(X, \theta(Y))$ $(X, Y \in \mathfrak{z}_s)$ is a positive definite bilinear form on $\mathfrak{z}_s$. This defines a Euclidean structure on $\mathfrak{z}_s$. The adjoint of $\mathrm{ad}_{\mathfrak{z}_s} X (X \in \mathfrak{z}_s)$ with respect to this structure is given by $-(\mathrm{ad}_{\mathfrak{g}} \theta X)|\mathfrak{z}_s$. Put $U=(\mathfrak{z}_s^-)_{Y_0}$ and choose an orthonormal basis $u_1, \cdots, u_m$ of U such that $[B_0, u_i]=-\lambda_i u_i$ $(1 \le i \le m)$ $(m=\dim U)$. Then each λ_i is a non-negative integer. These follow from the arguments in [vD, §1] (cf. §1). Since Y_0 is contained in $(\mathfrak{z}_s^-)_{Y_0}=U$, we may assume that $u_1=cY_0$ $(c=1/\|Y_0\|)$. So $\lambda_1=2$. Let $x_1, \cdots, x_m$ be the coordinates with respect to this basis of U.

For any $u \in U$, similarly as in (3.2), we define the linear mapping ψ_u of $U \times [\mathfrak{z}_s^-, Y_0]$ to $\mathfrak{q}$ by $\psi_u(w, v)=w+[X_0+u, v]$. Since $\mathfrak{z}_s^+=[\mathfrak{z}_s^-, Y_0]+(\mathfrak{z}_s^+)_{X_0}$ is a direct sum, it follows that ψ_0 is bijective. Noting this, we put $\kappa(u)=\det(\psi_u \circ \psi_0^{-1})$ $(u \in U)$ and $U'=\{u \in U;\ \kappa(u) \neq 0\}$ as in (3.2). Let $b_1, \cdots, b_n$ be an arbitrary basis of $[\mathfrak{z}_s^-, Y_0]$ and let $y_1, \cdots, y_n$ be the coordinates with respect to this basis on $[\mathfrak{z}_s^-, Y_0]$. Using the coordinates y on $[\mathfrak{z}_s^-, Y_0]$ and x on U, define the mapping Ψ of $\mathbf{R}^m \times \mathbf{R}^n$ to $\mathfrak{z}_s^-$ by the same way as (3.4.1):

$$\Psi(x, y)=e^{y_1 b_1} \cdots e^{y_n b_n}\Big(X_0+\sum_{j=1}^{m} x_j u_j\Big). \tag{4.3.1}$$

Then it follow from Lemma 3.6 that $\Psi(0, 0)=X_0$ and $d\Psi$ is non-singular at $(0, 0)$. Hence we define U_0, V_0 and Ω_0 as in (3.2), in particular, $0 \in U_0 \subset \mathbf{R}^m$, $0 \in V_0 \subset \mathbf{R}^n$ and $\Psi | U_0 \times V_0$ is a diffeomorphism and $\Omega_0=F(U_0 \times V_0)$. Then Ω_0 is an open neighbourhood of X_0 in $\mathfrak{z}_s^-$ and (x, y) is regarded as a coordinate system on Ω_0.

Since ω_s is contained in $S(\mathfrak{z}_s^-)$, we can define the radial component $\alpha_u(\omega_s)$ of ω_s at $u \in U'$ with respect to X_0, U, $[\mathfrak{z}_s^-, Y_0]$ (cf. Definition 3.4). Let $\Delta(\partial(\omega_s))$ be the differential operator on U' such that $\Delta_u(\partial(\omega_s))=\partial(\alpha_u(\omega_s))$ for all $u \in U'$. ($\Delta_u(\partial(\omega_s))$ denotes the local expression of $\partial(\omega_s)$ at u.)

Lemma 4.6. *The homogeneous part of degree 2 of $\Delta_0(\partial(\omega_s))$ is zero if and only if X_0 is $\mathfrak{z}_s^-$-distinguished.*

This is shown by an argument similar to Lemma 33 in [HC2] (cf. [vD, Lemma 13]).

Theorem 4.7. *Assume that X_0 is $\mathfrak{z}_s^-$-distinguished. Then there exist analytic functions $a_{ij}(x)$ and $a_i(x)$ on U_0 satisfying $a_{ij}(0)=0$ $(1<i, j\leq m)$, such that*

$$(4.3.2) \qquad ||X_0|| \Delta(\partial(\omega_s)) = 2x_1 \frac{\partial^2}{\partial x_1^2} + (\dim \mathfrak{z}_s^-) \frac{\partial}{\partial x_1} + \sum_{i=2}^{m} (\lambda_i+2) x_i \frac{\partial^2}{\partial x_1 \partial x_i} + \sum_{1<i\leq j\leq m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=2}^{m} a_i(x) \frac{\partial}{\partial x_i}.$$

This is shown by an argument similar to Theorem 14 in [vD]. It should be noted that though ω_s is not the Casimir polynomial on $\mathfrak{z}_s^-$, we only used in the proof that ω_s is non-degenerate.

Remark 4.8. A prototype of the formula in Theorem 4.7 seems to have first appeared in Atiyah's lecture note [A] as a literature. T. Kawazoe and the author obtained the formula (4.3.2) a few years ago being suggested by the note [A]. Afterwards, T. Oshima informed the author of van Dijk's article [vD] in which the formula was also shown. Our method was based on direct calculations by using the expression in (4.2.1). On the other hand, van Dijk's method is on the Euler operator and is therefore very simple and elegant compared with ours.

Last we comment on the differential operator $\partial(\omega_c)$. Since ω_c is a quadratic form on the vector space $\mathfrak{c}^-$, with respect to an appropriate basis $c_1, \cdots, c_q$ of $\mathfrak{c}^-$ $(q=\dim \mathfrak{c}^-)$, we have

$$(4.3.3) \qquad \partial(\omega_c) = D_{t_1}^2 + \cdots + D_{t_{q_1}}^2 - D_{t_{q_1+1}}^2 - \cdots - D_{t_q}^2,$$

where $t_1, \cdots, t_q$ are the coordinates with respect to the basis and q_1 is a certain integer such that $q_1 \leq q$.

§ 5. Invariant spherical hyperfunctions on $\mathfrak{q}$

(5.1) For an arbitrary linear form Λ on $\mathfrak{a}_c$, we define a system of differential equations $\mathcal{M}_\Lambda$ on $\mathfrak{q}$ defined by

$$(5.1.1) \qquad \mathcal{M}_\Lambda \quad \begin{cases} \partial(P)u = P(\Lambda)u & \text{for all } P \in S(\mathfrak{q}_c)^H \\ \tau(Y)u = 0 & \text{for all } Y \in \mathfrak{h}. \end{cases}$$

It is provable that $\mathcal{M}_\Lambda$ is holonomic in the sense of Sato-Kashiwara.

Definition 5.1. A hyperfunction solution u to the system $\mathcal{M}_\Lambda$ is called an invariant spherical hyperfunction (=ISH) on $\mathfrak{q}$ with the infinitesimal character Λ. Moreover, an ISH u is called singular if Supp (u) is contained in the set $\mathfrak{q}-\mathfrak{q}'$.

First we give a remark on the relation between ISH's and invariant eigendistributions (=IED) on a semisimiple Lie algebra. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and put $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_0$. Define the involution σ on $\mathfrak{g}$ by $\sigma(X, Y)=(Y, X)$ $(X, Y\in\mathfrak{g}_0)$. Then we obtain the direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ for σ. In this case, $\mathfrak{h}=\{(X, X); X\in\mathfrak{g}_0\}$ and $\mathfrak{q}=\{(X, -X); X\in\mathfrak{g}_0\}$. Hence $\mathfrak{h}\simeq\mathfrak{q}\simeq\mathfrak{g}_0$. Put $G_0=\mathrm{Int}(\mathfrak{g}_0)$ and $G=\mathrm{Int}(\mathfrak{g})$. Then $G=G_0\times G_0$ and H is equal to the diagonal subgroup $\{(g, g); g\in G_0\}$ of G. Moreover the action of H on $\mathfrak{q}$ is identified with the adjoint action of G_0 on $\mathfrak{g}_0$. Under the identification, an ISH on $\mathfrak{q}$ is nothing but an IED on $\mathfrak{g}_0$.

As to a singular IED on a semisimple Lie algebra, Harish-Chandra obtained the following result which plays a fundamental role in the proof of his famous theorem that any IED on a real semisimple Lie algebra is L^1-local (cf. [HC1]).

Theorem HC ([HC3]). *There exists no non-zero singular invariant eigendistribution on a real semisimple Lie algebra.*

As will be seen in Section 6, there exists a non-zero singular ISH on $\mathfrak{q}$ for some symmetric pair $(\mathfrak{g}, \mathfrak{h})$ and therefore an analogue of Theorem HC to the case of the tangent space of a symmetric space does not hold in general. But we can prove an analogue for symmetric pairs which satisfy some additional condition.

(5.2) The main result of this paper is the following theorem which is an analogue of Theorem HC.

Theorem 5.2. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a symmetric pair. If* $\delta_{\mathfrak{q}}(Z)>0$ *holds for any* $\mathfrak{q}$*-distinguished element* Z *of* $\mathfrak{q}$*, then there exists no non-zero singular invariant spherical hyperfunction on* $\mathfrak{q}$.

Remark. We shall discuss the condition of the theorem in Section 6.

Proof. Let u be a singular ISH on $\mathfrak{q}$. Then, by definition, $\mathrm{Supp}\,(u)$ is contained in $\mathfrak{q}-\mathfrak{q}'$. Assuming that $u\neq 0$, we lead a contradiction. Hence assume that $u\neq 0$ and therefore that $\mathrm{Supp}\,(u)\neq\emptyset$. Take $Z_0\in\mathrm{Supp}\,(u)$. Let $Z_0=A_0+X_0$ be its Jordan decomposition, where $A_0\in\mathscr{S}(\mathfrak{q})$ and $X_0\in\mathscr{N}(\mathfrak{q})$. As in (1.5), define $\mathfrak{z}^{\pm}$, $\mathfrak{z}_s^{\pm}$, $\mathfrak{c}^-$ for the semisimple element A_0. Now S_k is the set of those Z_0 such that rank $\mathfrak{z}_s^-=k$. Then $\mathrm{Supp}\,(u)=\bigcup_{k=0}^{l} S_k$, where $l=\mathrm{rank}\,\mathfrak{q}$. Since $\mathrm{Supp}\,(u)\subset\mathfrak{q}-\mathfrak{q}'$, it follows that A_0 is not $\mathfrak{q}$-regular. This implies that $S_0=\emptyset$.

Now assume that $S_1=\cdots=S_{k-1}=\emptyset$ and $S_k\neq\emptyset$. Then we may assume from the first that Z_0 is contained in S_k. Retain the notation above. Let H_s^+ be the analytic subgroup of H corresponding to $\mathfrak{z}_s^+$. Then H_s^+ acts on $\mathfrak{z}_s^-$. Since $\mathcal{N}(\mathfrak{z}_s^-)$ is decomposed into finitely many H_s^+-orbits (cf. Lemma 1.5), we can write $\mathcal{N}(\mathfrak{z}_s^-)=O_1\cup\cdots\cup O_\nu$, where the O_j are disjoint H_s^+-orbits and for $1\leq j\leq\nu$, $\mathcal{N}_j=O_j\cup\cdots\cup O_\nu$ is a closed set containing O_j as an open subset. Since $Z_0=A_0+X_0$ is contained in $\mathrm{Supp}(u)$ and also in $A_0+\mathcal{N}(\mathfrak{z}_s^-)$, it follows that $\mathrm{Supp}(u)\cap(A_0+\mathcal{N}(\mathfrak{z}_s^-))\neq\emptyset$. Noting this, we may assume that $\mathrm{Supp}(u)\cap(A_0+O_i)=\emptyset$ for $i=1,\cdots,j-1$ and $\mathrm{Supp}(u)\cap(A_0+O_j)\neq\emptyset$ for some j. It is clear that $\mathrm{Supp}(u)\cap(A_0+O_j)$ is contained in S_k. Hence we may assume from the first that X_0 is contained in O_j.

For the element A_0, we define V^+ as in (4.2.2). Let $f_1,\cdots,f_d$ be a basis of V^+ and let $z_1,\cdots,z_d$ be the coordinates with respect to this basis of V^+. Similarly, we retain the notation in (4.3). So let (B_0, X_0, Y_0) be a normal S-triple. Then we define $U=(\mathfrak{z}_s^-)_{Y_0}$. Take the basis $u_1,\cdots,u_m$ of U and its coordinates $x_1,\cdots,x_m$ as in (4.3). Moreover let $b_1,\cdots,b_n$ be the basis of $[\mathfrak{z}_s^-, Y_0]$ and $y_1,\cdots,y_n$ its coordinates. Last let $c_1,\cdots,c_q$ be the basis of $\mathfrak{c}^-$ and $t_1,\cdots,t_q$ its coordinates. Using these coordinates, we define the map Φ of $\mathbf{R}^q\times\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R}^d$ to $\mathfrak{q}$ by

$$\begin{aligned}&\Phi(t_1,\cdots,t_q,x_1,\cdots,x_m,y_1,\cdots,y_n,z_1,\cdots,z_d)\\&=e^{z_1f_1}\cdots e^{z_df_d}\Big(A_0+\sum_{i=1}^q t_ic_i+e^{y_1b_1}\cdots e^{y_nb_n}\Big(X_0+\sum_{\alpha=1}^m x_\alpha u_\alpha\Big)\Big)\\&=e^{z_1f_1}\cdots e^{z_df_d}\Big(A_0+\sum t_ic_i+\Psi(x,y)\Big),\end{aligned}$$

where $\Psi(x,y)$ is the mapping defined in (4.3.1). Then it follows that $\Phi(0)=A_0+X_0$ and Φ induces a diffeomorphism between an open neighbourhood of the origin in $\mathbf{R}^{q+m+n+d}$ and that of Z_0 in $\mathfrak{q}$. Hence take open subsets T_0, U_0, V_0 and W_0 of $\mathbf{R}^q$, $\mathbf{R}^m$, $\mathbf{R}^n$ and $\mathbf{R}^d$ containing the origins, respectively, such that $\Phi|T_0\times U_0\times V_0\times W_0$ is bijective. Put $\Xi=\Phi(T_0\times U_0\times V_0\times W_0)$ and the following identifications are used in the subsequent discussion:

$$\begin{aligned}&T_0\simeq T_0\times\{0\}\subset T_0\times U_0\simeq T_0\times U_0\times\{0\}\subset T_0\times U_0\times V_0\\&\simeq T_0\times U_0\times V_0\times\{0\}\subset T_0\times U_0\times V_0\times W_0.\end{aligned}$$

Replacing U_0 and V_0 with small ones if necessary, we can show by an argument similar to Lemma 22 in [V] (cf. [vD, Lemma 17]) that $\mathcal{N}_j\cap\{\Phi(t,x,y,z)\in\Xi;\, t=z=0\}\subseteq O_j$.

Since u is locally invariant, it follows from Lemma 3.11 that $u^{\sim}(t,x,y,z)=u(\Phi(t,x,y,z))$ which is a hyperfunction on $T_0\times U_0\times V_0\times W_0$ is constant with respect to the variables $z_1,\cdots,z_d$. So put $u_1(t,x,y)=u^{\sim}(t,x,y,z)|_{z=0}$. This is a well-defined hyperfunction on the open subset $T_0\times U_0\times V_0$ of $\mathbf{R}^{q+m+n}$ containing the origin $0=\Phi^{-1}(Z_0)$. Moreover

$u^{\sim}(t, x, y, z)=u_1(t, x, y)$ holds on $T_0\times U_0\times V_0\times W_0$. On the other hand, since u is an ISH, we have

$$(\partial(\omega)-\omega(\Lambda))u=0$$

for a linear form Λ on $\mathfrak{a}_c$ which is the complexification of a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$. Let ξ be the function on $\mathfrak{z}^-$ defined for A_0 as in (4.2). Replacing T_0, U_0, V_0 with small ones if necessary, we may assume from the first that $\xi\neq 0$ on $\Phi(T_0\times U_0\times V_0)$. So there exists an analytic function $\rho(t, x, y)$ on $T_0\times U_0\times V_0$ such that $\xi(\Phi(t, x, y, 0))=\rho(t, x, y)^2$ holds on $T_0\times U_0\times V_0$. Put $u_2(t, x, y)=\rho(t, x, y)^{-1}u_1(t, x, y)$. Then it follows from Lemma 4.4 that

$$\{\partial(\omega_c)^{\sim}+\partial(\omega_s)^{\sim}-\mu^{\sim}(t, x, y)-\omega(\Lambda)\}u_2=0. \tag{5.2.1}$$

where $\mu^{\sim}=\mu\circ\Phi$ (cf. Lemma 4.4). Since $\rho\circ\Phi^{-1}$ is H_s^+-invariant, it follows that

$$\tau(Y)^{\sim}u_2=0 \qquad \text{for all } Y\in\mathfrak{z}^+, \tag{5.2.2}$$

where the vector field $\tau(Y)^{\sim}$ is defined similarly as we did in (3.4). Since $\mathfrak{z}^+$ commutes with $\mathfrak{c}^-$, we find from Lemma 3.11 and (5.2.2) that

$$\frac{\partial}{\partial y_i}u_2(t, x, y)=0 \quad (i=1, \cdots, n).$$

Hence $u_2(t, x, y)$ is constant with respect to y. Noting this, we put $u_3(t, x)=u_2(t, x, y)|_{y=0}$. Then $u_3(t, x)$ is a well-defined hyperfunction on the open subset $T_0\times U_0$ of $\mathbf{R}^{q+m}$ containing $0=\Phi^{-1}(Z_0)$. The assumption combined with Lemma 17 (iii) in [vD] implies that Supp (u_3) is contained in the set T_0. On the other hand, u_3 satisfies the differential equation

$$(\Delta(\partial(\omega_s))+\partial(\omega_c)-\mu^{\sim}(t, x, 0)-\omega(\Lambda))u_3=0. \tag{5.2.3}$$

We can show under the assumption of the theorem that there exists no hyperfunction $u_3(y, t)$ satisfying the differential equation (5.2.3) and that Supp $(u_3)\subset T_0$. In fact, if Z_0 is not $\mathfrak{q}$-distinguished, that is, if X_0 is not $\mathfrak{z}_s^-$-distinguished nilpotent, it follows from (5.2.3), Lemma 4.6 and Proposition 2.2 that $u_3=0$. On the other hand, if X is $\mathfrak{z}_s^-$-distinguished nilpotent, by means of the formula (4.3.2) for $\Delta(\partial(\omega_s))$ and the assumption $\delta_{\mathfrak{z}_s^-}(X_0)>0$, we can conclude from Proposition 2.5 that $u_3=0$. Note that the condition $\delta_{\mathfrak{z}_s^-}(X_0)>0$ implies the condition (2.3.20) of Proposition 2.5. So we find that $u_3=0$ on $T_0\times U_0$, in other words, $u=0$ in the neighbourhood Ξ of Z_0 in $\mathfrak{q}$. This contradicts the assumption that $Z_0\in$ Supp (u). We have thus shown the theorem completely.

This theorem is restated in a slightly familiar form.

Theorem 5.3. *Retain the assumption in Theorem* 5.2 *for a symmetric pair* $(\mathfrak{g}, \mathfrak{h})$. *Let* $u(X)$ *be an invariant spherical hyperfunction on* $\mathfrak{q}$. *Then one has*:

(i) *The restriction* $u|\mathfrak{q}'$ *to* $\mathfrak{q}'$ *is real analytic.*

(ii) *If* $u|\mathfrak{q}'=0$, *then* $u=0$ *on the whole space* $\mathfrak{q}$.

Proof. (ii) is a direct consequence of Theorem 5.2. On the other hand, (i) is proved if one can show the existence of an elliptic operator P defined in a neighbourhood of an arbitrary element of $\mathfrak{q}'$ such that $Pu=0$. We are going to show this along the line of [A]. Hence let X_0 be an arbitrary element of $\mathfrak{q}'$. Then there exists a Cartan subspace $\mathfrak{a}$ containing X_0. Clearly $\mathfrak{a}$ is uniquely determined by X_0 and X_0 is contained in $\mathfrak{a}'$ (cf. (1.4)). Take a basis $\{h_1, \cdots, h_l\}$ of $\mathfrak{a}$. Then $Z=\sum_{i=1}^{l} h_i^2$ is contained in $S(\mathfrak{a}_c)$. If W is the Weyl group of $\sum(\mathfrak{a})$ which is the root system of $(\mathfrak{g}_c, \mathfrak{a}_c)$ (cf. (1.4)), the product $\hat{Z}=\prod_{w\in W} wZ$ is clearly W-invariant. Since it follows from Chevalley's theorem that $S(\mathfrak{q}_c)^H$ is isomorphic to the subring of $S(\mathfrak{a}_c)$ consisting of W-invariant elements, we denote the element of $S(\mathfrak{q}_c)^H$ corresponding to $\hat{Z}$ by the same letter. On the other hand, let $e_1, \cdots, e_p$ be an arbitrary basis of $\mathfrak{a}^\perp \cap \mathfrak{q}$. Using the basis, we define $P=\partial(\hat{Z})+\sum_{j=1}^{p} \tau(e_j)^2$. Let $u(X)$ be an ISH. Then, by definition, u satisfies the differential equation $(P-c)u=0$ for some complex number c. Since $\sigma(P-c)=\sigma(P)$ does not vanish in a neighbourhood U of X_0, we conclude that u is real analytic on U. Hence the result follows. q.e.d.

§ 6. A condition on $\mathfrak{q}$-distinguished elements

(6.1) Theorem 5.2 leads us to the following problem:

Problem A. Classify the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ satisfying the condition (C):

(C) $\delta_{\mathfrak{q}}(Z)>0$ for any $\mathfrak{q}$-distinguished element Z of $\mathfrak{q}$.

If the classification of H-orbits of $\mathcal{N}(\mathfrak{q})$ is accomplished, one would easily check the condition (C) for each $\mathfrak{q}$-distinguished nilpotent element. But the classification seems not to be done at present (cf. [S]). For this reason, we restrict our attention to look for examples of symmetric pairs which satisfy (C).

(6.2) Before entering into treating the subject, we show that there exists a non-zero singular ISH for some symmetric pair. Hence consider the pair $(\mathfrak{g}, \mathfrak{h})=(\mathfrak{so}(1, n+1), \mathfrak{so}(1, n))$. In this case, $\mathfrak{q}$ is identified with the real vector space $V=\mathbf{R}^{n+1}$ with the canonical coordinate $x=(x_0, x_1, \cdots, x_n)$ and $H\simeq SO_0(1, n)$. Then the action of H on $\mathfrak{q}$ is identified with the natural action of $SO_0(1, n)$ on V. In this case, the pseudo-Laplacian on $\mathfrak{q}$ coincides with

$$P=-D_{x_0}^2+D_{x_1}^2+\cdots+D_{x_n}^2$$

up to a constant factor. Then the system of differential equations introduced in (5.1) is rewritten in the following form for a certain complex number λ:

$$\mathcal{M}_\lambda \quad \begin{cases} (P-\lambda)u=0 & \\ (x_0 D_{x_i}+x_i D_{x_0})u=0 & (i=1, \cdots, n) \\ (x_i D_{x_j}-x_j D_{x_i})u=0 & (1\leq i, j\leq n), \end{cases}$$

Define $f(x)=-x_0^2+x_1^2+\cdots+x_n^2$. Then the nilpotent subvariety $\mathcal{N}(\mathfrak{q})$ of $\mathfrak{q}$ is identified with the set $\mathcal{N}(V)=\{x\in V; f(x)=0\}$. It is easy to see that $V-\mathcal{N}(V)$ consists of $\mathfrak{q}$-regular semisimple elements and that $\mathcal{N}(V)-\{0\}$ consists of $\mathfrak{q}$-distinguished nilpotent elements. Take $X\in\mathcal{N}(V)-\{0\}$. Then it follows from direct calculation that $\mathfrak{q}_X=\mathbf{R}X$. This implies that $\delta_{\mathfrak{q}}(X)=4-\dim V=3-n$. Hence if $n\geq 3$, the condition (C) does not hold. Moreover, if n is odd and $n\geq 3$, there actually exists a singular ISH on V. In fact, consider the distribution $v_s(x)=|f(x)|_+^s$ on $V(s\in\mathbf{C})$. By definition, $v_s(x)$ coincides with $|f(x)|^s$ if $f(x)\geq 0$ and coincides with 0 if $f(x)<0$. Then it follows that $Pv_s=4s(s+(n-1)/2)v_{s-1}$. Since v_s is meromorphically dependent on s and has a simple pole at $s=-(n-1)/2$, one can put $u(x)=((s+(n-1)/2)v_s(x))|_{s=(1-n)/2}$. Then it is clear that $u(x)$ satisfies the differential equations $\mathcal{M}_\lambda$ for $\lambda=0$. Moreover Supp(u) coincides with $\mathcal{N}(V)$. Hence $u(x)$ is a singular ISH. (Cf. [A])

In the sequel, we restrict our attention to look for examples of symmetric pairs satisfying the condition (C).

(6.3) In this paragraph, we give some known examples satisfying the condition (C).

(6.3.1) Let $(\mathfrak{g}, \mathfrak{h})$ be a Riemannian symmetric pair. Then the assumption holds in this case. In fact, $\mathcal{N}(\mathfrak{q})=\{0\}$ holds in this case and moreover, any sub-symmetric pair of $(\mathfrak{g}, \mathfrak{h})$ is also Riemannian.

(6.3.2) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair such that $\mathfrak{g}_c$ is identified with $\mathfrak{h}_c\oplus\mathfrak{h}_c$. In this case, $(\mathfrak{g}_c, \mathfrak{h}_c)$ is identified with $(\mathfrak{h}_c\oplus\mathfrak{h}_c, \mathfrak{h}_c)$. So one can check easily that the condition (C) holds for the pair $(\mathfrak{g}, \mathfrak{h})$ (cf. [A], [vD]).

(6.4) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be its complexification. Let X_0 be an element of $\mathcal{N}(\mathfrak{q}_c)$, the nilpotent subvariety of $\mathfrak{q}_c$. Then one can define a number $\tilde{\delta}_{\mathfrak{q}_c}(X_0)$ quite similarly as the number introduced in Definition 1.11. The precise one is given as follows. Let (A_0, X_0, Y_0) be a normal S-triple (cf. [KR]). Then ad A_0 leaves $(\mathfrak{q}_c)_{X_0}$ invariant. So we can choose a basis $w_1, \cdots, w_r$ of $(\mathfrak{q}_c)_{X_0}$ $(r=\dim_{\mathbf{C}} \mathfrak{q}_c)$ such that each w_i is an eigenvector of ad A_0 (cf. (1.7)). Define the number n_i by $[A_0, X_0]=n_i w_i$ for each i.

Definition 6.1. $\tilde{\delta}_{\mathfrak{q}_c}(X_0)=\sum_{i=1}^{r}(n_i+2)-\dim_{\mathbf{C}} \mathfrak{q}_c$.

It is clear from the definition that $\tilde{\delta}_{\mathfrak{q}_c}(X_0)=\delta_{\mathfrak{q}}(X_0)$ if X_0 is contained in $\mathcal{N}(\mathfrak{q})$. Note that $\mathcal{N}(\mathfrak{q}_c)$ always contains a $\mathfrak{q}_c$-regular element of $\mathfrak{q}_c$, that is, there exists a nilpotent element $X_0 \in \mathfrak{q}_c$ such that $\dim_{\mathbf{C}}(\mathfrak{q}_c)_{X_0}=\dim_{\mathbf{C}} \mathfrak{q}_c-\operatorname{rank} \mathfrak{q}$. For any $Z \in \mathfrak{q}_c$, we also define the number $\tilde{\delta}_{\mathfrak{q}_c}(Z)$ by the same way as we did in (1.10).

Lemma 6.2. *Retain the notation above. Assume that $\mathfrak{g}_c$ is simple. If $\tilde{\delta}_{\mathfrak{q}_c}(X_0)>0$ holds for any $\mathfrak{q}_c$-regular nilpotent element X_0 of $\mathfrak{q}_c$, then $(\mathfrak{g}_c, \mathfrak{h}_c)$ is isomorphic to one of the following pairs*:

(I) $(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$
(II) $(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C})+\mathfrak{sl}(n, \mathbf{C})+\mathbf{C})$
(III) $(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C})+\mathbf{C})$
(IV) $(\mathfrak{so}(2n+k, \mathbf{C}), \mathfrak{so}(n+k, \mathbf{C})+\mathfrak{so}(n, \mathbf{C}))$ $(k=0, 1, 2)$
(V) $(\mathfrak{e}_6, \mathfrak{sp}(4, \mathbf{C}))$
(VI) $(\mathfrak{e}_6, \mathfrak{sl}(6, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$
(VII) $(\mathfrak{e}_7, \mathfrak{sl}(8, \mathbf{C}))$
(VIII) $(\mathfrak{e}_8, \mathfrak{so}(16, \mathbf{C}))$
(IX) $(\mathfrak{f}_4, \mathfrak{sp}(3, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$
(X) $(\mathfrak{g}_2, \mathfrak{sl}(2, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$
(XI) $(\mathfrak{so}(p+q, \mathbf{C}), \mathfrak{so}(p, \mathbf{C})+\mathfrak{so}(q, \mathbf{C}))$ $(q+2<p<2q+2)$.

This lemma is shown by using the classification of symmetric pairs and the results in (1.9) (cf. [He]).

Theorem 6.3. *Let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be a complex symmetric pair. Assume that $(\mathfrak{g}_c, \mathfrak{h}_c)$ is isomorphic to one of the pairs* (I)-(X) *in Lemma* 6.2. *Then $\tilde{\delta}_{\mathfrak{q}_c}(Z)>0$ for any $\mathfrak{q}_c$-distinguished element Z of $\mathfrak{q}_c$.*

Remark. If $(\mathfrak{g}_c, \mathfrak{h}_c)$ is the pair in (XI), there exists a $\mathfrak{q}_c$-distinguished element $Z \in \mathfrak{q}_c$ such that $\tilde{\delta}_{\mathfrak{q}_c}(Z)\leq 0$.

To prove this theorem, we need a simple lemma.

Lemma 6.4. *Let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be a complex symmetric pair and let Σ be its* *root system defined as in* (1.4), *that is, if* $(\mathfrak{g}_0, \mathfrak{k}_0)$ *is a Riemannian symmetric pair such that* $\mathfrak{g}_0$ *and* $\mathfrak{k}_0$ *are real forms of* $\mathfrak{g}_c$ *and* $\mathfrak{h}_c$, *respectively, then* Σ *is the restricted root system of* $\mathfrak{g}_0$. *Assume that* $\dim_{\mathbf{C}} \mathfrak{g}_c^{\alpha} + \dim_{\mathbf{C}} \mathfrak{g}_c^{2\alpha} \leq 2$ *for all* $\alpha \in \Sigma$. *Then each irreducible factor of* $(\mathfrak{g}_c, \mathfrak{h}_c)$ *is isomorphic to one of the pairs in* (I)-(X) *or the pair* $(\mathfrak{g}_c' \oplus \mathfrak{g}_c', \mathfrak{g}_c')$ *for some complex simple Lie algebra* $\mathfrak{g}_c'$.

Proof. It is clear that there exists a Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_c$ and $\mathfrak{k}$ is that of $\mathfrak{h}_c$. Then Σ is nothing but the restricted root system of $\mathfrak{g}$. Then the lemma follows from Table in [W, p. 33].

Corollary 6.5. *Let* $(\mathfrak{g}_c, \mathfrak{h}_c)$ *be a complex symmetric pair and let* $(\mathfrak{g}_c', \mathfrak{h}_c')$ *be its sub-symmetric pair. If* $(\mathfrak{g}_c, \mathfrak{h}_c)$ *satisfies the assumption of Lemma* 6.4, *so does* $(\mathfrak{g}_c', \mathfrak{h}_c')$.

By means of Corollary 6.5, to prove Theorem 6.3, it suffices to show the following.

Lemma 6.6. *Retain the notation and the assumption in Theorem* 6.3. *Then* $\tilde{\delta}_{\mathfrak{q}_c}(X) > 0$ *for all* $\mathfrak{q}_c$*-distinguished nilpotent element* $X \in \mathfrak{q}_c$.

(6.5) We are going to prove Lemma 6.6 for the cases (I)-(X), separately.

(6.5.1) *Proof for the case* (I).

Let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be a symmetric pair isomorphic to $(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$ for some integer n. It is easy to see that, in this case, every $\mathfrak{q}_c$-distinguished nilpotent element is $\mathfrak{g}_c$-regular. Hence the claim for this case follows from Lemma 6.2.

(6.5.2) *Proof for the case* (II).

(a) Let $\mathfrak{g}_c = \mathfrak{sl}(2n, \mathbf{C})$ and let σ be the involution of $\mathfrak{g}_c$ defined by

$$\sigma X = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} X \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \quad (X \in \mathfrak{g}_c).$$

Then

$$\mathfrak{h}_c = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}; A, B \in \mathfrak{gl}(n, \mathbf{C}), \operatorname{tr}(A+B) = 0 \right\}$$

$$\mathfrak{q}_c = \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}; A, B \in \mathfrak{gl}(n, \mathbf{C}) \right\}.$$

Moreover, define

$$H_c=\left\{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix};\ A, B \in \mathrm{GL}(n, \mathbf{C}),\ \det(AB)=1\right\}.$$

(b) For each positive integer d, define the matrices I_d, J_d, K_d, L_d in the following way:

$$I_d=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad d\times d \text{ matrix},$$

$$J_d=\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad d\times d \text{ matrix},$$

$$K_d=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{bmatrix} \quad (d+1)\times d \text{ matrix},$$

$$L_d=\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \ddots \\ 0 & & & 1 \end{bmatrix} \quad d\times (d+1) \text{ matrix}.$$

(c) Let $\eta=(p_1, \cdots, p_k)$ be a partition of $2n$, that is, each p_i is an integer, $p_1+\cdots+p_k=2n$ and $p_1\geq\cdots\geq p_k>0$.

Moreover, define subsets I_1, I_2, I_3, I_4 of $\{1, \cdots, k\}$ associated with the partition η as follows:

(i) $\{1, \cdots, k\}=I_1\cup I_2\cup I_3\cup I_4$ (disjoint union)

(ii) p_i is even for all $i \in I_1\cup I_2$.

(iii) p_i is odd for all $i \in I_3\cup I_4$.

(iv) $|I_3|=|I_4|$.

Let $\mathscr{Q}_n$ be the totality of $(\eta, \{I_1, I_2, I_3, I_4\})$ defined as above. Note that there are several possibilities of the choices of $\{I_1, I_2, I_3, I_4\}$ for a given partition η.

(d) For every $\zeta=(\eta, \{I_1, I_2, I_3, I_4\}) \in \mathscr{Q}_n$ ($\eta=(p_1, \cdots, p_k)$), define the matrix X_ζ as follows:

$$X_\zeta=\left[\begin{array}{c|c} 0 & \begin{matrix} X_1 & & \\ & \ddots & \\ & & X_k \end{matrix} \\ \hline \begin{matrix} Y_1 & & \\ & \ddots & \\ & & Y_k \end{matrix} & 0 \end{array}\right]$$

Here X_i, Y_j are the matrices defined by

$$X_i=\begin{cases} I_{q_i} & (i\in I_1)\\ J_{q_i} & (i\in I_2)\\ K_{q_i} & (i\in I_3)\\ L_{q_i} & (i\in I_4)\end{cases}\qquad Y_i=\begin{cases} J_{q_i} & (i\in I_1)\\ I_{q_i} & (i\in I_2)\\ L_{q_i} & (i\in I_3)\\ K_{q_i} & (i\in I_4).\end{cases}$$

(Here $q_i=[p_i/2]$.)

(e) It is clear from the definition that for each $\zeta\in\mathcal{Q}_n$, X_ζ is nilpotent and is contained in $\mathfrak{q}_c$. Moreover, the following lemma is a direct consequence of [S, § 3].

Lemma 6.7. *For any nilpotent element X of $\mathfrak{q}_c$, there exist $h\in H_c$ and $\zeta\in\mathcal{Q}_n$ such that $hXh^{-1}=X_\zeta$.*

As to $\mathfrak{q}_c$-distinguished nilpotent elements of $\mathfrak{q}_c$, one can easily find the following lemma.

Lemma 6.8. *Let $\zeta=(\eta, \{I_1, I_2, I_3, I_4\})\in\mathcal{Q}_n$ and retain the notation in (d). Then X_ζ is $\mathfrak{q}_c$-distinguished nilpotent if and only if $X_i=X_j$ and $Y_i=Y_j$ for every pair of indices (i,j) such that $p_i=p_j$.*

(f) Take $\zeta\in\mathcal{Q}_n$ and retain the notation in (d). Let $(A_\zeta, X_\zeta, Y_\zeta)$ be a normal S-triple. It is always possible to take A_ζ as a diagonal matrix and moreover such an A_ζ is uniquely determined. So we assume such an A_ζ is taken.

Let $X=\begin{bmatrix}0 & Z\\ W & 0\end{bmatrix}\in\mathfrak{q}_c$. We use the notation:

$Z=(Z_{ij})_{1\le i,j\le k}$, where Z_{ij} is an $r_i\times s_j$ matrix

$W=(W_{ij})_{1\le i,j\le k}$, where W_{ij} is an $s_i\times r_j$ matrix.

$\left(\text{Here } r_i=\begin{cases} q_i & \text{if } i\in I_1\cup I_2\cup I_4,\\ q_i+1 & \text{if } i\in I_3\end{cases}\quad s_i=\begin{cases} q_i & \text{if } i\in I_1\cup I_2\cup I_3\\ q_i+1 & \text{if } i\in I_4\end{cases}\right).$

Then define

$$U_{ij}=\left\{X=\begin{bmatrix}0 & Z\\ W & 0\end{bmatrix}\in\mathfrak{q}_c;\ Z_{\alpha\beta}=0,\ W_{\alpha\beta}=0 \text{ if } (\alpha,\beta)\neq(i,j)\right\}$$

$$V_{ij}=\{X\in U_{ij};\ [X, X_\zeta]=0\}.$$

It follows from the definition that

$$(\mathfrak{q}_c)_{X_\zeta}=\bigoplus_{i,j=1}^{k} V_{ij}$$

$$\dim U_{ij}=r_i s_j+r_j s_i.$$

Now take a basis $w_{ij}^1, \cdots, w_{ij}^{n_{ij}}$ of V_{ij} ($n_{ij}=\dim V_{ij}$) in such a way that $[A_\zeta, w_{ij}^\alpha]=f_{ij}^\alpha w_{ij}^\alpha$ ($\alpha=i, \cdots, n_{ij}$), where each $f_{ij}^\alpha \geq 0$ is an integer (cf. (1.7)). Using the notation introduced above, we define

$$\chi_{ij}=\sum_{\alpha=1}^{n_{ij}}(f_{ij}^\alpha+2)-\dim V_{ij}.$$

Then it is clear from the definition that

$$\tilde{\delta}_{\mathfrak{q}_c}(X_\zeta)=\sum_{i,j=1}^{k}\chi_{ij} \tag{6.5.2.1}$$

Lemma 6.9.

(i) $\chi_{ii}=\begin{cases} p_i/2 & (i \in I_1 \cup I_2) \\ 0 & (i \in I_3 \cup I_4). \end{cases}$

(ii) *If* $i, j \in I_1$ $(i\neq j)$ *or if* $i, j \in I_2$ $(i\neq j)$, *then* $\chi_{ij}=\chi_{ji}=\min(p_i, p_j)$.

(iii) *If* $i \in I_1, j \in I_2$ *or if* $i \in I_2, j \in I_1$, *then* $\chi_{ij}=\chi_{ji}=0$.

(iv) *If* $i, j \in I_3$ $(i<j)$ *or if* $i, j \in I_4$ $(i<j)$, *then* $\chi_{ij}=\chi_{ji}=(p_j-p_i)/2$.

(v) *If* $i \in I_3, j \in I_4$ *or if* $i \in I_4, j \in I_3$, *then* $\chi_{ij}=\chi_{ji}=(p_i+p_j+2)/2$.

(vi) *If* $i \in I_1 \cup I_2, j \in I_3 \cup I_4$ *or if* $i \in I_3 \cup I_4$, $j \in I_1 \cup I_2$, *then* $\chi_{ij}+\chi_{ji}=\min(p_i, p_j)$.

Since the proof of this lemma is elementary but rather lengthy, we omit it.

It follows from Lemma 6.9 that if (i, j) satisfies one of the conditions in (i)–(iii), (v), (vi) of the lemma, then $\chi_{ij}+\chi_{ji}\geq 0$. On the other hand, if (i, j) satisfies the condition in Lemma 6.9, (iv), then $\chi_{ij}+\chi_{ji}\leq 0$.

We now calculate the sum S:

$$S=\Big(\sum_{i,j\in I_3, i<j}+\sum_{i,j\in I_4, i<j}\Big)(\chi_{ij}+\chi_{ji}) + \sum_{i\in I_3, j\in I_4}(\chi_{ij}+\chi_{ji}).$$

Put $d=|I_3|=|I_4|$ and assume that $I_3=\{i_1, \cdots, i_d\}$ $(i_1<\cdots<i_d)$ and $I_4=\{j_1, \cdots, j_d\}$ $(i_1<\cdots<j_d)$. Then it follows from Lemma 6.9 that

$$\begin{aligned} S &=\sum_{1\leq\alpha<\beta\leq d}\{(\chi_{i_\alpha i_\beta}+\chi_{i_\beta i_\alpha})+(\chi_{j_\alpha j_\beta}+\chi_{j_\beta j_\alpha})\}+\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}(\chi_{i_\alpha j_\beta}+\chi_{j_\beta i_\alpha}) \\ &=\sum_{1\leq\alpha<\beta\leq d}\{(p_{i_\beta}-p_{i_\alpha})+(p_{j_\beta}-p_{j_\alpha})\}+\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}(p_{i_\alpha}+p_{j_\beta}+2) \\ &=\sum_{\alpha=1}^{d}(2\alpha-d-1)(p_{i_\alpha}+p_{j_\beta})+d\sum_{\alpha=1}^{d}(p_{i_\alpha}+p_{j_\beta})+2d^2 \end{aligned}$$

$$=\sum_{\alpha=1}^{d}(2\alpha-1)(p_{i_\alpha}+p_{j_\alpha})+2d^2.$$

Hence we find that $S\geq 0$. If $d>0$, this implies that $\tilde{\delta}_{\mathfrak{q}_c}(X)>0$. On the other hand, if $d=0$, then $S=0$. In this case, $I_1\cup I_2\neq\emptyset$ and $\chi_{ii}>0$ for all $i\in I_1\cup I_2$ (cf. Lemma 6.9 (i)). This combined with the formula (6.5.2.1) implies that $\tilde{\delta}_{\mathfrak{q}_c}(X)>0$.

We have thus shown the next lemma which implies the conclusion of Lemma 6.6 for the case (II).

Lemma 6.10. $\tilde{\delta}_{\mathfrak{q}_c}(X)>0$ *for all nilpotent element* $X\in\mathfrak{q}_c$.

(6.5.3) Let $(\mathfrak{g}_c, \mathfrak{h}_c)$ be a symmetric pair isomorphic to one of the pairs in (III) and (IV). Then one can prove the statement similar to Lemma 6.10 by an argument similar to that in (6.5.2). Hence Lemma 6.6 is true for the cases (III) and (IV).

(6.5.4) *Proof for the cases* (V)–(X).

To prove the conclusion of Lemma 6.6 for the cases (V)–(X), we use the classification of the nilpotent orbits of exceptional simple Lie algebras by Dynkin [D].

From now on, $\mathfrak{g}$ denotes a complex simple Lie algebra instead of real one for the sake of simplicity. Let X be a nilpotent element of $\mathfrak{g}$ and let (A, X, Y) be an S-triple. Then we can choose a basis $w_1, \cdots, w_p$ of $\mathfrak{g}_X$ ($p=\dim\mathfrak{g}_X$) such that each of $w_1, \cdots, w_p$ is an eigenvector of $\mathrm{ad}(A)$. So put $[A, w_i]=n_i w_i$ ($1\leq i\leq p$). Then $n_1, \cdots, n_p$ are non-negative integers. We may assume without loss of generality that $n_1\geq\cdots\geq n_k>n_{k+1}=\cdots=n_p=0$. Let σ be a complex linear involution of $\mathfrak{g}$ and let $\mathfrak{h}$ and $\mathfrak{q}$ be the 1-eigenspace and the (-1)-eigenspace of σ, respectively. Now assume that X is contained in $\mathfrak{q}$ and (A, X, Y) is a normal S-triple. Then we may assume from the first that each of $w_1, \cdots, w_p\in\mathfrak{g}_X$ is contained in $\mathfrak{h}$ or $\mathfrak{q}$. At this moment, there are two possibilities:

(a) $\dim\mathfrak{h}_X>p-k$.

(b) $\dim\mathfrak{h}_X\leq p-k$.

In the case (a), X is clearly not $\mathfrak{q}$-distinguished (cf. Lemma 1.12). On the other hand, in the case (b), we define

$$n(X)=\sum_{i=k-q+1}^{k}(n_i+2)-\dim\mathfrak{q}\qquad(q=\dim\mathfrak{q}_X).$$

If X is $\mathfrak{q}$-distinguished, it is clear from the definition that $\tilde{\delta}_{\mathfrak{q}}(X)\geq n(X)$. Then we obtain the next lemma from Tables I–V in an Appendix.

Lemma 6.11. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a complex symmetric pair.*

(i) *If* $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to one of the pairs* (V), (VII)–(X), *then* $n(X)>0$ *for all* $\mathfrak{q}$*-distinguished nilpotent element* X *of* $\mathfrak{q}$.

(ii) *Assume that* $(\mathfrak{g}, \mathfrak{h})$ *is the pair* (VI), *namely,* $(\mathfrak{g}, \mathfrak{h})\simeq(\mathfrak{e}_6, \mathfrak{sl}(6, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$. *If* X *is* $\mathfrak{q}$*-distinguished nilpotent and* $n(X)\leq 0$, *then the weighted Dynkin diagram* $\Delta(X)$ *of* X *is one of the following ones*:

$$\begin{matrix}22022, & 20202\\ 2 & 0\end{matrix}$$

We can also prove the next lemma. Since we need elementary but complicated arguments for the proof, we leave its proof in another paper.

Lemma 6.12. *Retain the notation in Lemma* 6.11 (ii). *Let* X *be a nilpotent element of* $\mathfrak{q}$. *Let* $u_1, \cdots, u_q$ *be a basis of* $\mathfrak{q}_X$. *Assume that* $[A, u_i]=m_i u_i$, *where* m_i *is a non-negative integer* $(m_1\geq\cdots\geq m_q)$.

(i) *Assume that* $\Delta(X)=\begin{matrix}22022\\2\end{matrix}$. *Then* $q=5$ *and* $(m_1, \cdots, m_5)=(14, 10, 10, 6, 2)$.

(ii) *Assume that* $\Delta(X)=\begin{matrix}20202\\0\end{matrix}$. *Then* $q=7$ *and* $(m_1, \cdots, m_7)$ *coincides with* (10, 10, 6, 6, 2, 2, 2) or (10, 8, 6, 4, 4, 2, 2).

The conclusion of Lemma 6.6 for the cases (V)–(X) follows from Lemma 6.11 and 6.12.

Corollary to Theorem 6.3. Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair such that $(\mathfrak{g}_c, \mathfrak{h}_c)$ is one of the pairs in (I)–(X). Then $(\mathfrak{g}, \mathfrak{h})$ satisfies the assumption of Theorem 5.2.

§ 7. Two conjectures

In this section, we present two conjectures on the system $\mathcal{M}_\Lambda$ defined in (5.1.1). It is provable that $\mathcal{M}_\Lambda$ is holonomic for any $\Lambda \in \mathfrak{a}_c^*$.

Conjecture 7.1. $\mathcal{M}_\Lambda$ is regular holonomic in the sense of Kashiwara-Kawai (cf. [HK]).

If this is true, one would conclude that any ISH on $\mathfrak{q}$ turns out to be a distribution. This follows from the general theory of regular holonomic systems.

Conjecture 7.1 is true in some cases. For example, if $\mathfrak{q}$ is of rank one, one can easily show that $\mathcal{M}_\Lambda$ is regular holonomic. On the other hand, if the pair $(\mathfrak{g}, \mathfrak{h})$ is the one in (6.3.2), then $\mathcal{M}_\Lambda$ is also regular holonomic. This is a deep result shown in [HK]. To prove this, Hotta and Kashiwara [HK] showed that $\mathcal{M}_\Lambda$ coincides with the minimal extension of the restriction $\mathcal{M}_\Lambda|(\mathfrak{g}_c)'$ to $(\mathfrak{g}_c)'$ which is the totality of regular semisimple elements of $\mathfrak{g}_c$. This leads us to the next conjecture.

Conjecture 7.2. Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair. Assume that $(\mathfrak{g}_c, \mathfrak{h}_c)$ satisfies the assumption of Lemma 6.4. Then the system $\mathcal{M}_\Lambda$ coincides with the minimal extension of the restriction $\mathcal{M}_\Lambda|(\mathfrak{q}_c)'$ to $(\mathfrak{q}_c)'$ which is the totality of the $\mathfrak{q}_c$-regular semisimple elements of $\mathfrak{q}_c$.

One easily finds that the assumption of Conjecture 7.2 is necessary. In fact, consider the symmetric pair $(\mathfrak{so}(n+1, 1), \mathfrak{so}(n, 1))$. In the case where n is odd and is greater than 2, the system $\mathcal{M}_\Lambda$ with $\Lambda=0$ has a non-trivial coherent quotient whose support is contained in the outside of the set of regular semisimple elements of $\mathfrak{q}_c$. This follows from the arguments in (6.2).

Appendix

We give a proof of Lemma 6.11 in this appendix.

Retain the notation in Lemma 6.11. Let X be a nilpotent element of $\mathfrak{g}$ and let $\Delta(X)$ be its weighted Dynkin diagram. We use the notation: $p=\dim_{\mathbf{C}} \mathfrak{g}_X$ and $q=\dim_{\mathbf{C}} \mathfrak{q}_X$. Note that p is determined in [E]. Also the number q is easily determined from p (cf. Corollary 1.15). Moreover one can decide the numbers $n_1, \cdots, n_p$ by using the weighted Dynkin diagram $\Delta(X)$ and the root system of $\mathfrak{g}$.

Noting these facts, we give the numbers $n(X)$ for all niloptent elements X of $\mathfrak{g}$ in Tables I–V. Explain the contents of the Tables. In each table, the weighted Dynkin diagram, the numbers $n_1, \cdots, n_p, p=\dim \mathfrak{g}_X, q=\dim \mathfrak{q}_X, n=n(X)$ are arranged in order. In the case where (*a*): $\dim \mathfrak{h}_X > p-k$ (cf. (6.3.4)) holds for the nilpotent element X, we write N instead of the number q. In each Table, the notation n^d means that the number n occurs with multiplicity d.

Only in the case of Table I, we must remark one comment more. In this case, we treat the case of the simple Lie algebra of type E_6. This coorresponds to the pairs in (V) and (VI). The numbers q and n are used in the former case. On the other hand, q' and n' are used instead of q and n in the latter case.

Table I: $(\mathfrak{e}_6, \mathfrak{sp}(4, \mathbf{C}))$ and $(\mathfrak{e}_6, \mathfrak{sl}(6, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$

$\Delta(X)$	$n_1, \cdots, n_p$	p	q	n	q'	n'
22222 2	22, 16, 14, 10, 8, 2	6	6	42	4	2
22022 2	16, 14, 10, 10, 8, 6, 4, 2	8	7	26	5	0

20202 2	$14, 10^3, 8, 6, 4, 4, 2, 0$	10	8	28	6	6
20202 0	$10, 10, 8, 8, 6, 6, 4^3, 2^3$	12	9	14	7	−2
21012 1	$10, 9^2, 8, 6, 5^2, 4, 3^2, 2, 0^3$	14	10	32	8	12
11011 2	$10, 8, 7^2, 6^2, 5^2, 4, 2^2, 1^2, 0$	14	10	17	8	2
11011 1	$8, 7^2, 6, 5^2, 4^3, 3^2, 2^2, 1^2, 0$	16	11	14	9	2
00200 2	$10, 6^8, 2, 0^8$	18	12	N	10	40
20002 2	$8, 6^5, 4^3, 2^5, 0^4$	18	12	28	10	14
00200 0	$6^2, 4^7, 2^9, 0^2$	20	13	18	11	8
01010 1	$6, 5^2, 4^3, 3^6, 2^4, 1^2, 0^4$	22	14	22	12	12
10101 0	$5^2, 4^4, 3^4, 2^5, 1^6, 0^3$	24	15	16	13	8
10001 2	$6, 4^5, 3^8, 2, 0^{11}$	26	16	N	14	34
01010 0	$4^3, 3^4, 2^9, 1^8, 0^4$	28	17	18	15	12
20002 0	$4^8, 2^8, 0^{14}$	30	18	N	16	40
10001 1	$4, 3^6, 2^8, 1^8, 0^9$	32	19	24	17	21
00000 2	$4, 2^{19}, 0^{16}$	36	21	N	19	36
00100 0	$3^2, 2^9, 1^{16}, 0^{11}$	38	22	30	20	24
10001 0	$2^8, 1^{16}, 0^{22}$	46	26	N	24	40
00000 1	$2, 1^{20}, 0^{35}$	56	31	N	29	N
00000 0	0^{78}	78	42	N	40	N

Table II: $(\mathfrak{e}_7, \mathfrak{sl}(8, \mathbf{C}))$

$\Delta(X)$	$n_1, \cdots, n_p$	p	q	n
222222 2	$34, 26, 22, 18, 14, 10, 2$	7	7	70
220222 2	$26, 22, 18, 16, 14, 10^2, 6, 2$	9	8	44
220202 2	$22, 18, 16, 14^2, 10^2, 8, 6, 2^2$	11	9	30
222020 0	$22, 16^3, 14, 10, 8^3, 2, 0^3$	13	10	70
202022 0	$18, 16, 14^2, 10^3, 8, 6^2, 4, 2^2$	13	10	22
210122 1	$18, 15^2, 14, 10^2, 9^2, 6, 5^2, 2, 0^3$	15	11	52
202020 0	$16, 14, 12^2, 10^2, 8^3, 6, 4^3, 2, 0$	15	11	28
202002 0	$14, 12, 10^4, 8^2, 6^3, 4^2, 2^4$	17	12	14
210102 1	$14, 11^2, 10^2, 9^2, 8, 6^2, 5^2, 3^2, 2^2, 0^3$	19	13	34
210110 1	$14, 11^2, 10, 9^2, 8^3, 6, 5^2, 3^2, 2^2, 0^3$	19	13	34
002020 0	$12^3, 10, 8^3, 6^5, 4^3, 2, 0^3$	19	13	34
220020 0	$14, 10^5, 8^3, 6, 4^4, 2, 0^6$	21	14	56
002002 0	$10^3, 8^3, 6^5, 4^4, 2^6$	21	14	10
010102 1	$10^2, 9, 8^2, 7^2, 6^3, 5^3, 3^4, 2^3, 0^3$	23	15	25
020020 0	$10^2, 8^4, 6^4, 4^6, 2^5, 0^2$	23	15	18
101012 0	$10, 9^2, 8^3, 6^2, 5^4, 4^2, 3^3, 2^2, 1^3, 0^3$	25	16	18
101020 0	$10, 9^2, 8^3, 6, 5^6, 4^3, 3^2, 2, 0^6$	25	16	42
200200 0	$10, 8^3, 6^8, 4^3, 2^7, 0^5$	25	16	24
201010 0	$10, 8, 7^4, 6^4, 5^4, 4, 2^4, 1^4, 0^4$	27	17	24

002000 0	$8^3, 6^5, 4^{10}, 2^6, 0^3$	27	17	22
101010 0	$8, 7^2, 6^3, 5^4, 4^5, 3^4, 2^4, 1^4, 0^2$	29	18	15
200022 0	$10, 8^7, 6, 4^7, 2, 0^{14}$	31	19	N
210001 1	$10, 7^4, 6^6, 5^4, 2^2, 1^4, 0^{10}$	31	19	53
200020 0	$8, 6^7, 4^9, 2^7, 0^9$	33	20	44
000200 0	$6^5, 4^{10}, 2^{15}, 0^3$	33	20	20
001010 0	$6^3, 5^4, 4^4, 3^8, 2^8, 1^4, 0^4$	35	21	20
010001 1	$6^2, 5^4, 4^5, 3^8, 2^8, 1^4, 0^6$	37	22	26
220000 0	$10, 6^{14}, 2, 0^{21}$	37	22	N
100101 0	$6, 5^4, 4^7, 3^6, 2^7, 1^8, 0^5$	39	23	24
020000 0	$6^2, 4^{13}, 2^{15}, 0^9$	39	23	38
101000 0	$6, 5^2, 4^7, 3^{10}, 2^6, 1^6, 0^9$	41	24	34
010010 0	$5^2, 4^6, 3^8, 2^{11}, 1^{10}, 0^6$	43	25	24
200002 0	$6, 4^{15}, 2^{10}, 0^{21}$	47	27	N
200010 0	$6, 4^7, 3^{16}, 2, 0^{24}$	49	28	N
000020 0	$4^{10}, 2^{22}, 0^{17}$	49	28	54
000000 2	$4^7, 2^{28}, 0^{14}$	49	28	42
001000 0	$4^3, 3^8, 2^{15}, 1^{16}, 0^9$	51	29	30
100010 0	$4, 3^8, 2^{16}, 1^{16}, 0^{16}$	57	32	42
000001 1	$3^6, 2^{16}, 1^{20}, 0^{21}$	63	35	50

200000 0	$4, 2^{31}, 0^{24}$	67	37	N
010000 0	$3^2, 2^{15}, 1^{28}, 0^{24}$	69	38	54
000002 0	$2^{27}, 0^{52}$	79	43	N
000010 0	$2^{10}, 1^{32}, 0^{39}$	81	44	N
100000 0	$2, 1^{32}, 0^{66}$	99	53	N
000000 0	0^{133}	133	70	N

Table III: $(\mathfrak{e}_8, \mathfrak{so}(16, \mathbf{C}))$

$\Delta(X)$	$n_1, \cdots, n_p$	p	q	n
2222222 2	$58, 46, 38, 34, 26, 22, 14, 2$	8	8	128
2202222 2	$46, 38, 34, 28, 26, 22, 18, 14, 10, 2$	10	9	82
2202022 2	$38, 34, 28, 26, 22^2, 18, 16, 14, 10, 6, 2$	12	10	56
2020222 0	$34, 28, 26^2, 22, 18^2, 16, 14, 10^2, 8, 2^2$	14	11	40
2101222 1	$34, 27^2, 26, 22, 18, 17^2, 14, 10, 9^2, 2, 0^3$	16	12	100
2020202 0	$28, 26, 22^2, 18^2, 16, 14^3, 10^2, 8, 6, 4, 2$	16	12	30
2020022 0	$26, 22^2, 20, 18, 16^2, 14^2, 12, 10^3, 6^2, 4, 2^2$	18	13	20
2101022 1	$26, 22, 21^2, 18, 16, 15^2, 14, 11^2, 10^2, 6, 5^2, 2, 0^3$	20	14	59
2020020 0	$22^2, 20, 18, 16, 14^3, 12^2, 10^4, 8, 6, 4, 2^3$	20	14	16
2101101 1	$22, 21^2, 18, 15^2, 14, 12^3, 11^2, 10, 9^2, 6, 3^2, 2, 0^3$	22	15	46
0020022 0	$22, 18^2, 16^3, 14^3, 10^3, 8^3, 6^2, 4, 2^4$	22	15	8
0101022 1	$22, 18, 17^2, 16, 15^2, 14^2, 10^2, 9^2, 8, 7^2, 6, 3^2, 2^2, 0^3$	24	16	38

0020020 0	$18^2, 16, 14^3, 12^3, 10^3, 8^3, 6^5, 4, 2^3$	24	16	10
1010122 0	$22, 17^2, 16^3, 15^2, 14, 10, 9^2, 8^3, 7^2, 2^2, 1^4, 0^3$	26	17	24
2002002 0	$18, 16^2, 14^3, 12, 10^6, 8^3, 6^3, 4^2, 2^4, 0$	26	17	14
2010102 0	$18, 16, 15^2, 14^2, 11^2, 10^3, 9^4, 8, 6^2, 5^2, 4, 2^2, 1^2, 0^3$	28	18	25
0020002 0	$16, 14^3, 12^2, 10^6, 8^3, 6^5, 4^4, 2^4$	28	18	6
1010102 0	$16, 14, 13^2, 12^2, 11^2, 10^2, 9^2, 8^3, 7^2, 6, 5^2, 4^3, 3^2, 2^2, 1^2, 0$	30	19	6
1010110 0	$15^2, 14, 12^3, 11^2, 10, 9^2, 8^3, 7^4, 6, 5^2, 4^3, 3^2, 2, 0^3$	30	19	26
2000222 0	$22, 16^7, 14, 10, 8^7, 2, 0^{14}$	32	20	N
2100012 1	$18, 15^4, 14, 10^6, 9^4, 6, 5^4, 2, 0^{10}$	32	20	95
1010101 0	$14, 13^2, 12, 11^2, 10^2, 9^2, 8^3, 7^4, 6^3, 5^2, 4^3, 3^2, 2^2, 1^2, 0$	32	20	8
2000202 0	$16, 14, 12^6, 10^2, 8^7, 6, 4^7, 2, 0^8$	34	21	62
0002002 0	$14, 12^2, 10^7, 8^5, 6^5, 4^5, 2^8, 0$	34	21	4
0010102 0	$14, 12, 11^2, 10^4, 9^4, 8^2, 7^2, 6^3, 5^2, 4^2, 3^4, 2^4, 1^2, 0^3$	36	22	13
1010100 0	$12^3, 11^2, 10, 9^2, 8^3, 7^4, 6^5, 5^4, 4^3, 3^2, 2^2, 1^2, 0^3$	36	22	18
0100012 1	$14, 11^4, 10^2, 9^4, 8^5, 6^2, 5^4, 3^4, 2^6, 0^6$	38	23	32
2000200 0	$12^3, 10^5, 8^3, 6^{13}, 4^3, 2^5, 0^6$	38	23	34
1001012 0	$14, 11^2, 10^5, 9^4, 8^3, 7^2, 6, 5^2, 4^4, 3^2, 2^2, 1^6, 0^6$	40	24	24
0002000 0	$10^4, 8^6, 6^{10}, 4^{10}, 2^{10}$	40	24	4
0010100 0	$10^3, 9^2, 8^3, 7^4, 6^5, 5^6, 4^4, 3^6, 2^6, 0^3$	42	25	16
0100010 1	$10^2, 9^4, 8, 7^4, 6^7, 5^4, 4^5, 3^8, 2^3, 0^6$	44	26	30

1001010 0	$10^2, 9^2, 8^4, 7^4, 6^4, 5^6, 4^7, 3^4, 2^4, 1^4, 0^3$	44	26	12
1010001 0	$10, 9^4, 8^3, 7^2, 6^5, 5^8, 4^7, 3^4, 2^2, 1^4, 0^6$	46	27	26
0100101 0	$10, 9^2, 8^3, 7^6, 6^4, 5^6, 4^6, 3^4, 2^7, 1^4, 0^3$	46	27	10
2000022 0	$14, 10^9, 8^7, 6, 4^8, 2, 0^{21}$	48	28	N
0010010 0	$9^2, 8^3, 7^4, 6^6, 5^6, 4^6, 3^8, 2^6, 1^4, 0^3$	48	28	12
2000020 0	$10^2, 8^8, 6^8, 4^{15}, 2^3, 0^{14}$	50	29	68
0000002 2	$10, 8^6, 6^{14}, 4^7, 2^{14}, 0^8$	50	29	34
2000101 0	$10, 9^2, 8^7, 6, 5^{14}, 4^7, 3^2, 2, 0^{17}$	52	30	84
0010002 0	$10, 8^3, 7^6, 6^8, 5^6, 4^3, 3^2, 2^7, 1^{10}, 0^6$	52	30	16
0100100 0	$8^3, 7^4, 6^5, 5^6, 4^{10}, 3^8, 2^7, 1^6, 0^3$	52	30	12
0000200 0	$8^3, 6^{12}, 4^{15}, 2^{18}, 0^6$	56	32	28
0010001 0	$8, 7^4, 6^5, 5^8, 4^9, 3^8, 2^9, 1^8, 0^4$	56	32	14
1000102 0	$10, 8, 7^8, 6^8, 5^8, 4, 2^8, 1^8, 0^{15}$	58	33	54
1000101 0	$8, 7^2, 6^7, 5^8, 4^9, 3^8, 2^8, 1^8, 0^9$	60	34	29
1001000 0	$7^4, 6^6, 5^4, 4^{10}, 3^{16}, 2^6, 1^4, 0^{10}$	60	34	36
0000012 1	$8^2, 7^6, 6^{13}, 5^6, 2^2, 1^{14}, 0^{21}$	64	36	77
0000000 2	$6^8, 4^{20}, 2^{28}, 0^8$	64	36	32
0010000 0	$6^5, 5^6, 4^{10}, 3^{14}, 2^{15}, 1^{10}, 0^6$	66	37	22
2000002 0	$8, 6^{11}, 4^{21}, 2^{11}, 0^{24}$	68	38	90
1000100 0	$6^3, 5^8, 4^8, 3^{16}, 2^{16}, 1^8, 0^{11}$	70	39	35

0000010 1	$6^2, 5^6, 4^{13}, 3^{12}, 2^{16}, 1^{14}, 0^9$	72	40	28
0100001 0	$6, 5^6, 4^{11}, 3^{16}, 2^{15}, 1^{14}, 0^{13}$	76	42	39
0000022 0	$10, 6^{26}, 2, 0^{52}$	80	44	N
0001000 0	$5^4, 4^{10}, 3^{16}, 2^{20}, 1^{20}, 0^{10}$	80	44	32
0000020 0	$6^2, 4^{25}, 2^{27}, 0^{28}$	82	45	88
0000101 0	$6, 5^2, 4^{15}, 3^{18}, 2^{10}, 1^{14}, 0^{24}$	84	46	68
1000010 0	$5^2, 4^{10}, 3^{16}, 2^{23}, 1^{18}, 0^{17}$	86	47	48
2000000 0	$4^{14}, 2^{50}, 0^{28}$	92	50	72
0100000 0	$4^7, 3^{14}, 2^{28}, 1^{28}, 0^{17}$	94	51	48
1000002 0	$5, 4^{12}, 3^{31}, 2, 0^{55}$	100	54	N
0000100 0	$4^3, 3^{16}, 2^{27}, 1^{32}, 0^{24}$	102	55	60
1000001 0	$4, 3^{12}, 2^{32}, 1^{32}, 0^{35}$	112	60	80
0000000 1	$3^8, 2^{28}, 1^{48}, 0^{36}$	120	64	80
0000002 0	$4, 2^{55}, 0^{78}$	134	71	N
0000010 0	$3^2, 2^{27}, 1^{52}, 0^{55}$	136	72	108
1000000 0	$2^{14}, 1^{64}, 0^{78}$	156	82	N
0000001 0	$2, 1^{56}, 0^{133}$	190	99	N
0000000 0	0^{248}	248	128	N

Table IV: $(\mathfrak{f}_4, \mathfrak{sp}(3, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$

$\Delta(X)$	$n_1, \cdots, n_p$	p	q	n

22⇒22	22, 14, 10, 2	4	4	28
22⇒02	14, 10, 10, 6, 4, 2	6	5	14
02⇒02	10, 10, 8, 6, 4, 2^3	8	6	10
10⇒12	10, 9, 9, 6, 3, 3, 2, 0^3	10	7	28
22⇒00	10, 6^5, 2, 0^3	10	7	28
02⇒00	6^2, 4^4, 2^6	12	8	8
10⇒10	6, 5^2, 4, 3^4, 2^3, 0^3	14	9	17
01⇒01	5^2, 4^3, 3^2, 2^2, 1^4, 0^3	16	10	14
20⇒01	6, 4^4, 3^4, 2, 0^6	16	10	28
00⇒10	4^3, 3^2, 2^6, 1^4, 0^3	18	11	13
00⇒02	4^7, 2, 0^{14}	22	13	N
20⇒00	4, 2^{13}, 0^8	22	13	24
01⇒00	3^2, 2^6, 1^{10}, 0^6	24	14	18
00⇒01	2^7, 1^8, 0^{15}	30	17	N
10⇒00	2, 1^{14}, 0^{21}	36	20	N
00⇒00	0^{52}	52	28	N

Table V: $(\mathfrak{g}_2, \mathfrak{sl}(2, \mathbf{C})+\mathfrak{sl}(2, \mathbf{C}))$

$\Delta(X)$	$n_1, \cdots, n_p$	p	q	n
2⇛2	10, 2	2	2	8
2⇛0	4, 2, 2, 2	4	3	4
0⇛1	3, 3, 2, 0, 0, 0	6	4	N
1⇛0	2, 1, 1, 1, 1, 0, 0, 0	8	5	8
0⇛0	0^{14}	14	8	N

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Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya
Tokyo, 158 Japan