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Invariant States and Conditional Expectations of the Anticommutation Relations

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Abstract. The group G of unitary elements of a maximal abelian von Neumann algebra on a separable, complex Hilbert space H acts as a group of automorphisms on the CAR algebra $\mathscr{A}(H)$ over H. It is shown that the set of G-invariant states is a simplex, isomorphic to the set of regular probability measures on a w^*-compact set S of G-invariant generalized free states. The GNS Hilbert space induced by an arbitrary G-invariant state on $\mathscr{A}(H)$ supports a *-representation of $C(S)$; the canonical map of $\mathscr{A}(H)$ into $C(S)$ can then be locally implemented by a normal, G-invariant conditional expectation.

In this paper we shall define observable Fermion number densities on the spectra of complete one particle observables and study the classical fields which they generate.

Let $\mathscr{A}(H)$ denote the C^*-algebra of the Canonical Anticommutation Relations (CAR) over a complex, separable Hilbert space H. H will be fixed throughout and $\mathscr{A}(H)$ denoted by $\mathscr{A}$. $\mathscr{A}$ is generated algebraically by the range of an antilinear map $f \to a(f)$ of H into $\mathscr{A}$ obeying the CAR:

$$a(f)a(g)+a(g)a(f)=0 \qquad a^*(f)a(g)+a(g)a^*(f)=(g,f) \qquad \forall f,g\in H .$$

Let u be a unitary operator on H. Then the map $a(f)\to a(uf)$ extends uniquely to a *-automorphism α_u of $\mathscr{A}$. α_u is called the Bogoliubov automorphism induced by u.

Let $\mathcal{O}$ be a self-adjoint operator on $\mathscr{H}$. $\mathcal{O}$ shall be called complete if its spectral family generates a maximal abelian von Neumann algebra $\mathscr{Y}$ on H. Let (X, B, μ) denote the spectral measure space of $\mathcal{O}$. By the well known isomorphism theorem (I § 7 and III § 1, Corollary 3 of Ref. [3]), completeness of $\mathcal{O}$ leads to identification of $\mathscr{H}$ with $\mathscr{L}^2(X, B, \mu)$ and of $\mathscr{Y}$ with $\mathscr{L}^\infty(X, B, \mu)$.

When $\mathcal{O}$ has discrete spectrum, the number density N on X is defined for each $x\in X$ by $N_x = a^*(\delta_x)a(\delta_x)$ where δ_x is the Kroenecker δ-function at $x \in X$. The number density N generates a classical field which is isomorphic to the lattice gas. One can also isolate the field and density by symmetry considerations (as we have remarked before [16]).

An observable in $\mathscr{A}$ is called $\mathcal{O}$-diagonal if it is diagonal in the Fock representation with respect to the basis formed by anti-symmetric products of eigenvectors

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of $\mathcal{O}$. The algebra of $\mathcal{O}$-diagonal observables is, at once, the classical field generated by the number density and the fixed point algebra for the unitary group $G(\mathcal{Y})$ of $\mathcal{Y}$ (which acts as Bogoliubov automorphisms): The $\mathcal{O}$-diagonal part of a correlation function is simply its average over $G(\mathcal{Y})$. Now, the $\mathcal{O}$-diagonal part of the one point functions $a^*(f)a(g)$ defines an operator-valued measure on (X, B) which is continuous with respect to μ, and whose Radon-Nikodym derivative is N. That is, if $D(a^*(f)a(g))$ denotes the diagonal part of $a^*(f)a(g)$ then

$$D(a^*(f)a(g)) = \int f(x)\bar{g}(x)N_x d\mu(x) \quad \forall f, g \in \mathcal{L}^2(X, B, \mu).$$

Thus, by focusing in the mean-values of correlation functions over $G(\mathcal{Y})$ one is naturally lead to the number density and its field. This is also the case when $\mathcal{O}$ does not have discrete spectrum.

Since the unitary group $G(\mathcal{Y})$ of an arbitrary maximal abelian von Neumann algebra $\mathcal{Y}$ may have poor ergodic properties even locally on $\mathcal{A}$, we shall, in effect, approximate $G(\mathcal{Y})$ by a net of compact subgroups, each of which has a unique invariant mean or conditional expectation on $\mathcal{A}$. In Section 2, we associate to each partition of the spectrum of $\mathcal{Y}$ a conditional expectation on $\mathcal{A}$ which destroys the off-diagonal correlations between elements of the partition. Indexed by the partially ordered set of partitions, these conditional expectations form a net. In Section 3, we prove local convergence of the net in the representations induced by $G(\mathcal{Y})$-invariant states and isolate the local number densities and classical fields. In Section 4, we prove the set of $G(\mathcal{Y})$-invariant states to be a simplex and characterize its extreme points. Then, using the global consistency of the approximation scheme, we obtain the C^*-algebra of observables of the classical field over the spectrum of $\mathcal{Y}$.

The notation for von Neumann and C^*-algebras follows that of Refs. [4] and [5] with the following exceptions: If $\phi \in \mathcal{A}^*$ and $S \in \mathcal{A}$, $\langle \phi; S \rangle$ denotes the value of ϕ at S; if $f, g \in H$ (a Hilbert space), $\Lambda(f, g)$ denotes the vector form on $B(H)$ defined by $\langle \Lambda(f, g); S \rangle = (f, Sg)\ \forall S \in B(H)$; the σ-topology of a von Neumann algebra is defined in Ref. [16]. The closure of a set X in topology τ is denoted by $X^{-\tau}$, the closure of the linear span of X by $[X]^{-\tau}$. The symmetric group of degree N is denoted by $\mathcal{S}_N$ and sgn(p) denotes the signature of $p \in \mathcal{S}_N$.

0. Conditional Expectations

In this section we recall the definition of a conditional expectation on a C^*-algebra and reproduce, for completeness, two theorems giving sufficient conditions for its existence. These results are applied in § 2 to the definition of a net of partial diagonal part operators.

0.1. Definition [13]. Let $\mathcal{A}$ be a C^*-algebra with unit. A linear mapping $\mathcal{E}$ on $\mathcal{A}$ is said to be a conditional expectation if the following conditions are satisfied:

1. $\mathcal{E}(\mathbb{1}) = \mathbb{1}$.
2. $S \geq 0 \Rightarrow \mathcal{E}(S) \geq 0$.
3. $\mathcal{E}(S\mathcal{E}(T)) = \mathcal{E}(\mathcal{E}(S)T) = \mathcal{E}(S)\mathcal{E}(T) \quad \forall S, T \in \mathcal{A}$.

$\mathcal{E}$ is said to be faithful if $\mathcal{E}(S) = 0$, $S \geq 0 \Rightarrow S = 0$.

The following theorem is a slight variant of the principle result of [16.1]. The proof is in the Appendix.

0.2. Theorem. *Let $\mathscr{A}$ be a C^*-algebra with unit, and let G be an amenable discrete group of automorphisms of $\mathscr{A}$. Denote by $\mathscr{A}^G$ the C^*-algebra of all fixed points of $\mathscr{A}$ and by $\mathfrak{S}^G$ the set of all G-invariant states on $\mathscr{A}$. If there exists a C^*-subalgebra (with unit) $\mathscr{A}^M$ of $\mathscr{A}^G$ which separates $\mathfrak{S}^G$, then*

1. *If $\hat{\varrho}$ is a state on $\mathscr{A}^M$, there exists a unique G-invariant state ϱ on $\mathscr{A}$ extending $\hat{\varrho}$.*
2. *$\mathscr{A}^M = \mathscr{A}^G$.*
3. *If $S \in \mathscr{A}$, there exists $S^G \in \mathscr{A}$ such that $\{S^G\} = \mathscr{A}^M \cap \mathrm{Co}\{gS | g \in G\}^{-N}$.*
4. *The mapping $S \to S^G$ is a conditional expectation on $\mathscr{A}$ satisfying,*
   a) $S^G = S \quad \forall S \in \mathscr{A}^G$,
   b) $(gS)^G = S^G \quad \forall g \in G; \forall S \in \mathscr{A}$.

0.3. Definition [9]. Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space H and let G be a group of automorphisms of $\mathscr{A}$. Let $R^+(\mathscr{A}, G)$ denote the set of G-invariant, positive normal forms on $\mathscr{A}$. $\mathscr{A}$ is said to be G-finite if for every $S \in \mathscr{A}^+$, $S \neq 0$ there exists $\phi \in R^+(\mathscr{A}, G)$ such that $\langle \phi ; S \rangle \neq 0$.

Kovács and Szücs have obtained the following ergodic theorem.

0.4. Theorem. *Let $\mathscr{A}$ be a von Neumann algebra and let G be a group of automorphisms of $\mathscr{A}$. Let $\mathscr{A}^G$ denote the set of all fixed points of $\mathscr{A}$. Suppose that $\mathscr{A}$ is G-finite. Then, for every $T \in \mathscr{A}$, there exists $T^G \in \mathscr{A}$ such that $\{T^G\} = \mathscr{A}^G \cap \mathrm{Co}\{gT | g \in G\}^{-\sigma}$. The mapping $T \to T^G$ is a normal, faithful conditional expectation on $\mathscr{A}$ such that*

1. *$(gT)^G = T^G \quad \forall g \in G; \forall T \in \mathscr{A}$.*
2. *$T = T^G \quad \forall T \in \mathscr{A}^G$.*

The mapping $T \to T^G$ is called the G-canonical map.

1. The CAR

Let H be a separable, complex Hilbert space, and let $\mathscr{A}(H)$ be the C^*-algebra of the Canonical Anticommutation Relations (CAR) over H [7]. As stated in the introduction, $\mathscr{A}(H)$ is generated by the range of an anti-linear mapping $a: H \to \mathscr{A}(H)$ satisfying the CAR: a is continuous, with $\|a(f)\| = \|f\| \ \forall f \in H$; and $\mathscr{A}(H)$ is the closed linear span of the n-point correlation operators defined, for $N, M \in \mathbb{Z}^+$, $\{f_n\}_{n=1}^N, \{g_m\}_{m=1}^M \subset H$ by

$$A_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M) \equiv a^*(f_N) \ldots a^*(f_1) a(g_1) \ldots a(g_M).$$

By convention, $A_{00} \equiv 1$, $A_{10}(f) \equiv a^*(f)$, $A_{01}(g) \equiv a(g)$.

Let $\mathfrak{S}$ be the set of all states on $\mathscr{A}(H)$. A state $\omega \in \mathfrak{S}$ is determined by linearity and by continuity from its n-point correlation functions defined, for $N, M \in \mathbb{Z}^+$, $\{f_n\}_{n=1}^N, \{g_m\}_{m=1}^M \subset H$, by

$$W_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M) \equiv \langle \omega ; A_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M) \rangle .$$

A state ω is said to be a gauge invariant generalized free state if its n-point correlation functions have the form;

$$W_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M) = \delta_{M,N} \det\{W_{11}(f_n, g_m)\} \ \forall N, M \in \mathbb{Z}^+ ; \quad \{f_n\}_{n=1}^N, \{g_m\}_{m=1}^M \subset H .$$

$F(H)$ denotes the set of all gauge invariant generalized free states on $\mathscr{A}$. If ω is a gauge invariant generalized free state there exists a unique operator A on H such $W_{11}(f;g)=(g,Af)\,\forall f,g\in H$. It follows that $0\leqq A\leqq 1$. Conversely, if A is an operator on H such that $0\leqq A\leqq 1$, then there exists [2] a unique gauge invariant generalized free state ω_A such that

$$\langle\omega_A;a^*(f)a(g)\rangle=(g,Af)\quad\forall f,g\in H.$$

Let u be a unitary operator on H. Then, there exists a unique *-automorphism α_u of $\mathscr{A}(H)$ such that $\alpha_u(a(f))=a(uf)\,\forall f\in H$. The mapping $\alpha:U(H)\to\mathrm{Aut}\mathscr{A}(H)$ of the unitary group $U(H)$ of H into the automorphisms group of $\mathscr{A}(H)$ is a strongly continuous homomorphism, when the former is equipped with its strong operator topology.

If H' is a closed subspace of H, we shall denote by $\mathscr{A}(H')$ the C^*-subalgebra of $\mathscr{A}(H)$ generated by

$$\{A_{NM}(f_1,\dots,f_N;g_1,\dots,g_M)|N,M\in\mathbb{Z}^+;\{f_n\}_{n=1}^N,\{g_m\}_{m=1}^M\subset H'\}.$$

2. A Net of Conditional Expectations

Let $\mathscr{Y}$ be a maximal abelian von Neumann algebra on H, and let $G(\mathscr{Y})$ denote the unitary group of $\mathscr{Y}$.

2.0. Definition. Let $\tilde{E}=\{E_\alpha\}_{\alpha\in I}$ be a family of non-null, mutually orthogonal projectors in $\mathscr{Y}$ such that $\Sigma E_\alpha=\mathbb{1}_H$. $\tilde{E}$ is called a $\mathscr{Y}$-partition of H. Let $\Gamma(\mathscr{Y})$ denote the set of all $\mathscr{Y}$-partitions of H.

The set $\Gamma(\mathscr{Y})$ is partially ordered by refinement. That is, $\tilde{E}$ is said to refine $\tilde{F}$ if, for each $E\in\tilde{E}$ and $F\in\tilde{F}$, we have $EF=0$ or E. We write $\tilde{E}\geqq\tilde{F}$. One readily verifies that $(\Gamma(\mathscr{Y}),\geqq)$ is a partially ordered and directed set with $\tilde{E}\vee\tilde{F}\equiv\{E_kF_j\}$.

The directed set $(\Gamma(\mathscr{Y}),\geqq)$ will serve to index *a* net of conditional expectations. The remainder of this section is aimed at defining a conditional expectation for each partition $\tilde{E}\in\Gamma(\mathscr{Y})$.

Let $\tilde{E}$ be a $\mathscr{Y}$-partition of H. We shall denote by $G(\tilde{E})$ the unitary group of the abelian von Neumann algebra $\mathscr{Y}(\tilde{E})$ generated by $\tilde{E}$. $G(\tilde{E})$ is represented in $\mathrm{Aut}\mathscr{A}$ by α (see above). We denote the set of $G(\tilde{E})$-invariant elements of $\mathscr{A}$ (resp. $\mathfrak{S}$) by $\mathscr{A}^{G(\tilde{E})}$ (resp. $\mathfrak{S}^{G(\tilde{E})}$). If $\tilde{E}=\{\mathbb{1}\}$, then $G(\tilde{E})$ is the gauge group θ.

2.1. Lemma. *Let $\tilde{E}=\{E_j\}_{j\in J}$ be a $\mathscr{Y}$-partition of H; let, for each $j\in J$, $\mathscr{A}(E_jH)^\theta$ be the C^*-subalgebra of $\mathscr{A}$ generated by*

$$\{A_{NM}(f_1,\dots,f_N;g_1,\dots,g_M)|N=M\in\mathbb{Z}^+;\{f_n\}_{n=1}^N,\{g_m\}_{m=1}^M\subset E_jH\};$$

and, let $\bigotimes_j\mathscr{A}(E_jH)^\theta$ denote the C^*-subalgebra of $\mathscr{A}$ generated by $\bigcup_j\mathscr{A}(E_jH)^\theta$. Then, $\bigotimes_j\mathscr{A}(E_jH)^\theta$ separates $\mathfrak{S}^{G(\tilde{E})}$.

Proof. Let ϕ and $\phi^1\in\mathfrak{S}^{G(E)}$ with $\phi\neq\phi^1$. There exists $N,M\in\mathbb{Z}^+$, $\{f_n\}_{n=1}^N$, $\{g_m\}_{m=1}^M\subset H$ such that $\langle\phi-\phi^1;A_{NM}(f_1,\dots,f_N;g_1,\dots,g_M)\rangle\neq 0$. By linearity, and by continuity, there exists $\{j_n\}_{n=1}^N,\{i_m\}_{m=1}^M\subset J$ such that

$$\langle\phi-\phi^1;A_{NM}(E_{j_1}f_1,\dots,E_{j_N}f_N;E_{i_1}g_1,\dots,E_{i_M}g_M)\rangle\neq 0.$$

Let, for each $j \in J$, $N_j \equiv \mathrm{Card}\{n | j = j_n\}$ and $M_j \equiv \mathrm{Card}\{m | j = i_m\}$. Now, if $N_{j_0} \neq M_{j_0}$ for some $j_0 \in J$, $G(\tilde{E})$-invariance of $\phi - \phi^1$ implies, with the choice $\mathbb{1} + [\exp(i\Pi/|N_{j_0} - M_{j_0}|) - 1]\, E_{j_0} \in G(\tilde{E})$, that the LHS of preceding expression vanishes. Thus, $N_j = M_j \; \forall j \in J$. But this implies

$$A_{NM}(E_{j_1} f_1, \ldots, E_{j_N} f_N; E_{i_1} g_1, \ldots, E_{i_M} g_M) \subset \bigotimes_j \mathscr{A}(E_j H)^\theta . \quad \square$$

Since $G(\tilde{E})$ is amenable, and since $\bigotimes_j \mathscr{A}(E_j H)^\theta \subseteq \mathscr{A}^{G(\tilde{E})}$, Lemma 2.1 fulfills the remaining hypothesis of Theorem 0.2. We have therefore the following:

2.2. Theorem. *Let $\tilde{E}$ be a $\mathscr{Y}$-partition of H. There exists a unique $G(\tilde{E})$-invariant conditional expectation $\mathscr{E}(\ |\tilde{E})$ on $\mathscr{A}$ whose range is $\bigotimes \mathscr{A}(E_j H)^\theta = \mathscr{A}^{G(\tilde{E})}$.*

2.3. Corollary. $\mathscr{E}(a^*(f) a(g) | \tilde{E}) = \Sigma_j a^*(E_j f)\, a(E_j g) \quad \forall f, g \in H.$

2.4. Corollary. *Let $\tilde{E} \geqq \tilde{F}$ be two $\mathscr{Y}$-partitions of H. Then,*

$\mathscr{E}(\mathscr{E}(S|\tilde{F})|\tilde{E}) = \mathscr{E}(S|\tilde{E}) \quad \forall S \in \mathscr{A}.$

Proof. Since $G(\tilde{E}) \supseteqq G(\tilde{F})$ the result is immediate from Theorem 0.2 (3).

2.5. Corollary. *Let $\tilde{E} \geqq \tilde{F}$ be two $\mathscr{Y}$-partitions of H. Then, $\mathscr{E}(\mathscr{A}|\tilde{E}) \subseteqq \mathscr{E}(\mathscr{A}|\tilde{F})$.*

We have therefore defined a net $\{\mathscr{E}(\ |\tilde{E})\}_{\tilde{E} \in \Gamma(\mathscr{Y})}$ of conditional expectations on $\mathscr{A}$ with decreasing range (2.5) and the lattice property 2.4.

We now turn to the implementation of this net on some representations of $\mathscr{A}$.

Let $\tilde{E}$ be a $\mathscr{Y}$-partition of H; let ϕ be a $G(\tilde{E})$-invariant state on $\mathscr{A}$; let $(\pi_\phi, U_\phi, H_\phi, \Phi)$ be the cyclic, covariant representation of $(\mathscr{A}, G(\tilde{E}))$ associated to $\mathscr{A}$ via the Gelfand-Naimark-Segal construction [12, 6.22]. Since $\mathscr{A}$ is simple [15], it follows that π_ϕ is invertible. We can therefore define $\mathscr{E}_\phi(\ |\tilde{E}) \equiv \pi_\phi \mathscr{E}(\pi_\phi^{-1}\ |\tilde{E}) : \pi_\phi(\mathscr{A}) \to \pi_\phi(\mathscr{A}^{G(\tilde{E})})$. For each $g \in G(\tilde{E})$ define the automorphism $\hat{\alpha}_g : \pi_\phi(\mathscr{A})'' \to \pi_\phi(\mathscr{A})''$ by $\hat{\alpha}_g S = U_\phi(g) S U_\phi(g)^* \; \forall S \in \pi_\phi(\mathscr{A})''$. The mapping $\hat{\alpha} : G(\tilde{E}) \to \mathrm{Aut}\, \pi_\phi(\mathscr{A})''$ is a homomorphism such that $\hat{\alpha}_g \pi_\phi(S) = \pi_\phi(\alpha_g S) \; \forall S \in \mathscr{A}$. Accordingly, if we can prove that $\pi_\phi(\mathscr{A})''$ is $G(\tilde{E})$-finite (0.3), the $G(\tilde{E})$-canonical map (0.4) will be a normal, $G(\tilde{E})$-invariant conditional expectation on $\pi_\phi(\mathscr{A})''$ whose restriction to $\pi_\phi(\mathscr{A})$ is $\mathscr{E}_\phi(\ |\tilde{E})$.

By continuity, it will be the unique normal extension of $\mathscr{E}_\phi(\ |\tilde{E})$ to $\pi_\phi(\mathscr{A})''$. In that event we shall denote the $G(\tilde{E})$-canonical map by $\hat{\mathscr{E}}_\phi(\ |\tilde{E})$.

2.6. Lemma. *Let $\tilde{E} = \{E_j\}_{j \in J}$ be a $\mathscr{Y}$-partition of H, and let ϕ be a $G(\tilde{E})$-invariant state. Then with the above notation, $\pi_\phi(\mathscr{A})''$ is $G(\tilde{E})$-finite.*

Proof. It suffices to exhibit a complete orthonormal set of simultaneous eigenfunctions for $U_\phi(G(\tilde{E}))$. To this end, let $\mathscr{M} = \bigcup_j \{E_j f | f \in H\}$, and let

$$\mathscr{M}_\phi \equiv \bigcup_{N,M} \{\pi_\phi(A_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M))\Phi | \{f_n\}_{n=1}^N, \{g_m\}_{m=1}^M \subset \mathscr{M}\} .$$

Clearly, the linear span of $\mathscr{M}_\phi$ is dense in H_ϕ. Since $U_\phi(g)\Phi = \Phi \; \forall g \in G(\tilde{E})$, there exists, for $g \in G(\tilde{E})$ and $\Psi \in \mathscr{M}_\phi$, a unique $\theta(g, \Psi) \in [0, 2\Pi)$ such that $U_\phi(g)\Psi = \exp[i\theta(g, \Psi)]\, \Psi$. Two vectors Ψ and $\Psi^1 \in \mathscr{M}_\phi$ are said to be equivalent ($\approx$) if $\theta(g, \Psi) = \theta(g, \Psi^1) \; \forall g \in G(\tilde{E})$. $\mathscr{M}_\phi$ is thus divided into disjoint equivalence classes $\mathscr{M}_\phi^\alpha$.

Let $\theta(g, \alpha) \equiv \theta(g, \Psi)$ for $\Psi \in \mathscr{M}_\phi^\alpha$, $g \in G(\tilde{E})$, and let P_ϕ^α denote the projector upon the closed linear span H_ϕ^α of $\mathcal{M}_\phi^\alpha$. One readily sees that $P_\phi^\alpha P_\phi^{\alpha'}=0\ \forall\ \alpha\neq\alpha'$, $\Sigma P_\phi^\alpha=\mathbb{1}$, and

$$U_\phi(g)=\Sigma_\alpha \exp(i\theta(g,\alpha))P_\phi^\alpha\ \forall\ g\in G(\tilde{E}).\quad\square$$

From Lemma 2.6 and the preceding discussion we have the following.

2.7. Theorem. *Let ϕ be a $G(\mathcal{Y})$-invariant state on $\mathcal{A}$. Then, with the above notation and definitions, there exists a net $\{\hat{\mathcal{E}}_\phi(\ |\tilde{E})\}_{\tilde{E}\in\Gamma(\mathcal{Y})}$ of normal conditional expectations on $\pi_\phi(\mathcal{A})''$ such that*

1. For each $\tilde{E}\in\Gamma(\mathcal{Y})$, $\hat{\mathcal{E}}_\phi(\ |\tilde{E})$ is a normal, $G(\tilde{E})$-invariant conditional expectation on $\pi_\phi(\mathcal{A})''$ whose range is $(\pi_\phi(\mathcal{A})'')^{G(\tilde{E})}$.

2. $\hat{\mathcal{E}}_\phi(\ |\tilde{E})$ is the unique normal extension of $\mathcal{E}_\phi(\ |\tilde{E})$.

3. If $\tilde{E}\geqq\tilde{F}$, then

$$\hat{\mathcal{E}}_\phi(\hat{\mathcal{E}}_\phi(S|\tilde{F})|\tilde{E})=\hat{\mathcal{E}}_\phi(S|\tilde{E})\ \forall\ S\in\pi_\phi(\mathcal{A})''.$$

4. If $\tilde{E}\geqq\tilde{F}$, then

$$\hat{\mathcal{E}}_\phi(\pi_\phi(\mathcal{A})''|\tilde{E})\subseteqq\hat{\mathcal{E}}_\phi(\pi_\phi(\mathcal{A})''|\tilde{F}).$$

Remark. In the remainder of the paper we suppress the "$\hat{\ }$".

3. A Local Theory

In this section we investigate the convergence of the net $\{\mathcal{E}_\phi(\ |\tilde{E})\}_{\tilde{E}\in\Gamma(\mathcal{Y})}$ of normal conditional expectations on $\pi_\phi(\mathcal{A})''$ defined in Theorem 2.7 for a maximal abelian von Neumann algebra $\mathcal{Y}$ on H and $G(\mathcal{Y})$-invariant state on $\mathcal{A}$.

Due to the totally dissimilar behavior of the net for the atomic and non-atomic parts of $\mathcal{Y}$ we make the following:

3.0. Definition. Let $\mathcal{Y}$ be an abelian von Neumann algebra on H. A non-null projector $A\in\mathcal{Y}$ is said to be an atom of $\mathcal{Y}$ if for every projector P of $\mathcal{Y}$ we have $AP=0$ or A. Denote the set of all atoms of $\mathcal{Y}$ by $A(\mathcal{Y})=\{A_i\}_{i\in I}$; let $Y_A=\vee A(\mathcal{Y})$ and $Y_N=1-Y_A$.

We remark that (1) since distinct atoms are mutually orthogonal, $Y_A=\Sigma A_i$, and (2) since H is separable, $\mathrm{Card}\, I\leqq\aleph_0$.

Let $\Omega\in H$ be a separating vector for the maximal abelian von Neumann algebra $\mathcal{Y}$ on H [4, 1 § 7]. For each element F of $\mathcal{Y}$ we define the vector $f=F\Omega\in H$. The mapping $F\to f$ is an injection of $\mathcal{Y}$ onto a dense subset of H. Throughout the remainder of this section Ω will denote a fixed separating vector for $\mathcal{Y}$.

Let P_N denote the set of all projectors P of $\mathcal{Y}$ such that $PY_N=P$. Clearly P_N is a Boolean σ-algebra; the mapping $P\to\omega(P)\equiv(\Omega, P\Omega)$ is a finite positive measure on P_N; and (P_N,ω) is a finite, separable, non-atomic measure algebra. Therefore, (P_N,ω) is isomorphic to $([0,\omega(Y_N)], dx)$ [8, 41 C].

3.1. Definition. Let $\tilde{E}^0\equiv\{Y_N\}\cup A(\mathcal{Y})$.

An increasing sequence $\{\tilde{E}^n\}_{n=1}^\infty$ of $\mathcal{Y}$-partitions of H is said to be dense if 1) $\tilde{E}^1\geqq\tilde{E}^0$, and if 2) to every projector $F\in P_N$, and to every $\varepsilon>0$, there corresponds a positive integer n_0 and a projector E which is a union of projectors of $\tilde{E}^{n_0}$ and is such that

$$\omega(F(1-E)+E(1-F))<\varepsilon.$$

Theorems 41 B and 41 C of Ref. [8] assure the existence of an abundance of dense, increasing $\mathscr{Y}$-partitions of H.

3.a A Local Convergence Theorem

3.2. Theorem. *Let $\mathscr{Y}$ be a maximal abelian von Neumann algebra on H; let ϕ be a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$; let $\{\mathscr{E}_\phi(\ |\tilde{E})\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$ be the net of conditional expectations on $\pi_\phi(\mathscr{A})''$ defined in Theorem 2.7. Then, there exists a unique normal conditional expectation $\mathscr{E}_\phi(\ |\mathscr{Y})$ on $\pi_\phi(\mathscr{A})''$ such that*

$$\sigma-\lim_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})=\mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y}) \qquad \forall\, S\in\mathscr{A}.$$

Further, 0) $\mathscr{E}_\phi(\ |\mathscr{Y})$ has abelian range.

1) $\mathscr{E}_\phi(U_\phi(g)SU_\phi(g)^*|\mathscr{Y})=\mathscr{E}_\phi(S|\mathscr{Y}) \quad \forall\, g\in G(\mathscr{Y})$.

2) $U_\phi(g)\mathscr{E}_\phi(S|\mathscr{Y})\,U_\phi(g)^*=\mathscr{E}_\phi(S|\mathscr{Y}) \quad \forall\, S\in\pi_\phi(\mathscr{A})''$.

3)
$$\begin{aligned}&\mathscr{E}_\phi(\pi_\phi(A_{NM}(f_1,\ldots,f_N;g_1,\ldots,g_M))|\mathscr{Y})\\ &=\delta_{M,N}\textstyle\sum_{p\in\mathscr{P}_N}\operatorname{sgn}(p)\prod_{n=1}^N\mathscr{E}_\phi(\pi_\phi(A_{11}(f_n;g_{p(n)}))|\mathscr{Y})\\ &\equiv\delta_{M,N}\operatorname{Det}\{\mathscr{E}_\phi(\pi_\phi(A_{11}(f_n;g_m))|\mathscr{Y})\}\ \forall\, N,M\in\mathbb{Z}^+;\\ &\quad\forall\,\{f_n\}_{n=1}^N,\{g_m\}_{m=1}^M\subset H.\end{aligned}$$

4) $\mathscr{E}_\phi(\pi_\phi(\mathscr{A})''|\mathscr{Y})=\{\mathscr{E}_\phi(\pi_\phi(A_{11}(f;g))|\mathscr{Y})|f,g\in H\}''$.

The proof of this result is punctuated with several lemmas and propositions.

3.3. Remark. The proof of σ-convergence of the nets of operators $\{\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$, which results in Proposition 3.11, employs the following artifice. Since, for each $S\in\mathscr{A}$ and $\mathscr{Y}$-partition $\tilde{E}$, $\|\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})\|\leqq\|S\|$ there exists, by σ-compactness of bounded spheres of $\pi_\phi(\mathscr{A})''$, a σ-convergent subnet: We chose one and denote its limit point by $\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})$, finally proving that

$$\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})=\sigma\text{-}\lim_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}).$$

This method of proof has the advantage that if P is a projector on H_ϕ, and if the net $\{P\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})P\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$ can be shown to be ultra-weakly convergent, then its limit point is $P\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})P$. We remark further that since the weak-operator and ultra-weak operator topologies on $B(H_\phi)$ coincide on bounded spheres, weak operator convergence of the required nets is sufficient to ensure σ-convergence.

The following argument appears often enough to warrant abbreviation: If two vectors Ψ, $\Psi^1\in H_\phi$ are eigenvectors of a unitary operator g with different eigenvalues, then they are orthogonal. We shall say "$(\Psi,\Psi^1)=0$ due to g-invariance", leaving the task of verifying that Ψ and Ψ^1 belong to different eigensubspaces of g to the reader.

Preparatory to proving Proposition 3.11, we make the following definitions and remarks. Let $D\in\mathscr{Y}$ be a projector; let $G(D)=\{\mathbb{1}+(g-1)D|g\in G(\mathscr{Y})\}\subseteqq G(\mathscr{Y})$; and if ϕ is a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$, let $H_\phi(D)=\{\Psi\in H_\phi|U_\phi(g)\Psi=\Psi\ \forall\, g\in G(D)\}$.

Define the family of projectors $\{Q_\phi(D)_i\}_{i=1}^4$ as follows (recall that $d \equiv D\Omega$ where Ω is a fixed cyclic vector for $\mathscr{Y}$ in H).

$$Q_\phi(D)_1 H_\phi \equiv [\pi_\phi(a(d))H_\phi(D)]^-$$

$$Q_\phi(D)_2 H_\phi \equiv [\pi_\phi(a^*(d))H_\phi(D)]^-$$

$$Q_\phi(D)_3 H_\phi \equiv [\pi_\phi(a(d)a^*(d))H_\phi(D)]^-$$

$$Q_\phi(D)_4 H_\phi \equiv [\pi_\phi(a^*(d)a(d))H_\phi(D)]^- .$$

From the CAR and $U_\phi(1-2D)$-invariance it follows that these projectors are pairwise orthogonal. Let $Q_\phi(D) \equiv \Sigma Q_\phi(D)_i$.

If D and D' be two orthogonal projectors of $\mathscr{Y}$, then $[Q_\phi(D), Q_\phi(D')]=0$. In fact, let $O_\phi(D)$ [resp. $O_\phi(D')$] denote the projector upon $H_\phi(D)$ [resp. $H_\phi(D')$]. Since $O_\phi(D) \in U_\phi(G(D))''$ and $O_\phi(D') \in U_\phi(G(D'))''$ it follows that $[O_\phi(D), O_\phi(D')]=0$. From this, and the fact that $O_\phi(D') \in \pi_\phi(\mathscr{A}(\mathbb{C}d))'$, $O_\phi(D')Q_\phi(D)H_\phi \subseteqq Q_\phi(D)H_\phi$. Hence $[O_\phi(D'), Q_\phi(D)]=0$ and $Q_\phi(D)O_\phi(D')H_\phi \subseteqq O_\phi(D')H_\phi$. Consequently,

$$Q_\phi(D)Q_\phi(D')H_\phi \subseteqq Q_\phi(D')H_\phi ,$$

proving the assertion.

3.4. Lemma. *Let D be a projector of $\mathscr{Y}$. Then, with the above definitions and notation,*

$$\text{strong}\lim_{\Gamma(\mathscr{Y})} Q_\phi(D)\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D) = Q_\phi(D)\mathscr{E}_\phi^0(\pi_\phi(N(d))|\mathscr{Y})Q_\phi(D)$$

$$(N(d) \equiv a^*(d)a(d)).$$

Proof. We prove, for $i, i'=1, \ldots, 4$, that each of the nets $(*)_{ii'} \equiv \{Q_\phi(D)_i\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_{i'}\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$ is strong-operator convergent. However, by $U_\phi((\mathbb{1}-D)+iD)$-invariance, the (i, i')-net vanishes for

$$i=1;\, i'=2,3,4 \qquad i=3;\, i'=1,2$$

$$i=2;\, i'=1,3,4 \qquad i=4;\, i'=1,2 .$$

Thus, the proof reduces to six cases:

Case 1. $i=i'=1$. We prove the bounded net of positive operators to be increasing and therefore [4, Appendix 1] strong operator convergent. Let $\tilde{E} \geqq \tilde{F}$, and let $F_k = \Sigma_{j\in J(k)}E_j$, with the obvious notation. Let $\Psi = \pi_\phi(a(d))\chi$ with $\chi \in H_\phi(D)$. Then

$$(\Psi, \{\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{F}) - \mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})\}\Psi)$$

$$= \Sigma_k(\chi, \pi_\phi[a^*(d)\{a^*(F_kd)a(F_kd) - \Sigma_{j\in J(k)}a^*(E_jd)a(E_jd)\}a(d)]\chi)$$

$$= \Sigma_k(\chi, \pi_\phi[a^*((\mathbb{1}-F_k)d)\{-\}a((\mathbb{1}-F_k)d)]\chi)$$

$$+ \Sigma_k(\chi, \pi_\phi[a^*(F_kd)\{-\}a(F_kd)]\chi)$$

$$+ \Sigma_k(\chi, \pi_\phi[a^*((\mathbb{1}-F_k)d)\{-\}a(F_kd)]\chi) + \text{c.c.}$$

Now, (1) the first term on the RHS vanishes by $U_\phi(\mathbb{1}-2DE_j)$-invariance for $j\in J(k)$ and all k; (2) the third and fourth terms vanish by $U_\phi(\mathbb{1}-2D(\mathbb{1}-F_k))$-invariance for all k; (3) since $a(F_kd)^2=0$, we are left with

$$-\Sigma_k(\chi, \pi_\phi[a^*(F_kd)\{\Sigma_{j\in J(k)}a^*(E_jd)a(E_jd)\}a(F_kd)]\chi)\leqq 0\,.$$

Since $\Psi\in\pi_\phi(a(d))H_\phi(D)$ was arbitrary, we have by continuity $\tilde{E}\geqq\tilde{F}\Rightarrow Q_\phi(D)_1\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_1\geqq Q_\phi(D)_1\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{F})Q_\phi(D)_1$.

Case 2. $i=i'=2$. We prove the net $(*)_{22}$ of positive operators to be decreasing and therefore strong-operator convergent. Let $\tilde{E}\geqq\tilde{F}$ and let $F_k=\Sigma_{j\in J(k)}E_j$. Let $\Psi=\pi_\phi(a^*(d))\chi$ with $\chi\in H_\phi(D)$. Then, by an argument similar to that of Case 1, we have;

$$(\Psi, [\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{F})-\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})]\Psi)$$

$$=\Sigma_k(\chi, \pi_\phi[a(F_kd)\{\|F_kd\|^2-\Sigma_{j\in J(k)}a^*(E_jd)a(E_jd)\}a^*(F_kd)]\chi)\geqq 0$$

since

$$\Sigma_{j\in J(k)}\|a^*(E_jd)a(E_jd)\|\leqq\Sigma_{j\in J(k)}\|E_jd\|^2=\|F_kd\|^2\,.$$

Since $\Psi\in\pi_\phi(a^*(d))H_\phi(D)$ was arbitrary, we have, by continuity,

$$\tilde{E}\geqq\tilde{F}\Rightarrow Q_\phi(D)_2\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_2\leqq Q_\phi(D)_2\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{F})Q_\phi(D)_2\,.$$

Case 3. $i=i'=3$. We have (CAR),

$$a(d)a^*(d)\mathscr{E}(N(d)|\tilde{E})a(d)a^*(d)=\|d\|^2a(d)\mathscr{E}(N(d)|\tilde{E})a^*(d)$$

$$-a(d)a^*(d)\Sigma_j\|E_jd\|^4\,.$$

Therefore,

$$Q_\phi(D)_3\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_3$$

$$=\|d\|^{-2}Q_\phi(D)_3\pi_\phi(a(d))\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})\pi_\phi(a^*(d))Q_\phi(D)_3$$

$$-\|d\|^{-4}Q_\phi(D)_3\pi_\phi(a(d)a^*(d))Q_\phi(D)_3[\Sigma_j\|E_jd\|^4]\,.$$

Now, (1) since $\pi_\phi(a^*(d))Q_\phi(D)_3=Q_\phi(D)_2\pi_\phi(a^*(d))$, strong-operator convergence of the net of first term on the RHS follows from Case 2; and since the net in $\mathbb{C}$, $\{\Sigma_j\|E_jd\|^4\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$ is positive and decreasing, the net of second terms is norm and, a fortiori, strong-operator convergent.

Case 4. $i=i'=4$. By an argument similar to that of Case 3, we have

$$Q_\phi(D)_4\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_4$$

$$=\|d\|^{-4}Q_\phi(D)_4\pi_\phi(N(d))Q_\phi(D)_4(\Sigma_j\|E_jd\|^4)$$

$$+\|d\|^{-2}Q_\phi(D)_4\pi_\phi(a^*(d))\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})\pi_\phi(a(d))Q_\phi(D)_4\,.$$

Now, since $\pi_\phi(a(d))Q_\phi(D)_4=Q_\phi(D)_1\pi_\phi(a(d))$, strong-operator convergence of the net of second terms on the RHS follows from Case 1; (2) Convergence of the net of first terms follows as in Case 3.

Case 5. $i=3$, $i'=4$. We have (CAR)

$$Q_\phi(D)_3\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_4$$
$$=\|d\|^{-2}Q_\phi(D)_3\pi_\phi[a^*(\Sigma\|E_jd\|^2E_jd)a(d)]Q_\phi(D)_4\,.$$

Now let, for each $\mathscr{Y}$-partition $\tilde{E}$ of H,

$$f(d,\tilde{E})=\Sigma_j\|E_jd\|^2E_jd.\ \text{Let}\ f(d,\mathscr{Y})=\Sigma\|A_id\|^2A_id\,.$$

Let, for each $M\in\mathbb{Z}^+$, $\tilde{E}^M$ be a $\mathscr{Y}$-partition of H such that 1) $\tilde{E}^M\geqq\tilde{E}^0\equiv A(\mathscr{Y})\cup\{Y_N\}$, and 2) $(\Omega,DY_NE_j^M\Omega)=0$ or $(\Omega,DY_N\Omega)/M$ $\forall j$. Let $\tilde{F}\geqq\tilde{E}^M$ for fixed, but arbitrary, $M:E_j^M=\Sigma_{k\in K(j)}F_k$.

We have, $\|f(d,\mathscr{Y})-f(d,\tilde{F})\|^2=\Sigma_k\|F_kY_Nd\|^4=\Sigma_j\Sigma_{K(j)}\|F_kY_Nd\|^4\leqq\Sigma_j\|E_j^MY_Nd\|^4=(\Omega,DY_N\Omega)^2/M$. Thus $\lim_{\Gamma(\mathscr{Y})}f(d,\tilde{E})=f(d,\mathscr{Y})$, and consequently

$$\lim_{\Gamma(\mathscr{Y})}Q_\phi(D)_3\mathscr{E}_\phi(\pi_\phi(N(d))|\tilde{E})Q_\phi(D)_4$$
$$=\|d\|^{-2}Q_\phi(D)_3\pi_\phi[a^*(f(d,\mathscr{Y}))a(d)]Q_\phi(D)_4\,.$$

Case 6. $i=4$, $i'=3$. Case 6 follows from Case 5 by virtue of *norm* convergence. □

3.5. Lemma. *Let $\tilde{D}$ be a $\mathscr{Y}$-partition of H; let $\{F_n\}_{n=1}^N$ be a finite set of mutually orthogonal projectors of $\mathscr{Y}(\tilde{D})$ (i.e. $F_n=\Sigma_{j\in J(n)}D_j$); and let $P_\phi(\tilde{D})H_\phi\equiv[\pi_\phi(\mathscr{A}([\tilde{D}\Omega]^-))\Phi]^-$.*
Then

$$\text{S-lim}_{\Gamma(\mathscr{Y})^N}P_\phi(\tilde{D})\prod_{n=1}^N\mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}^n)P_\phi(\tilde{D})$$
$$=P_\phi(\tilde{D})\prod_{n=1}^N\mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y})P_\phi(\tilde{D})\,.$$

Proof. Let $Q_\phi(n)\equiv\prod_{j\in J(n)}Q_\phi(D_j)$. It follows from Lemma 3.4 that, for each finite subset $J_0(n)\subseteqq J(n)$,

$$\text{S-lim}_{\Gamma(\mathscr{Y})}Q_\phi(n)\mathscr{E}_\phi(\pi_\phi(N(\Sigma_{j\in J_0(n)}D_jf_n))|\tilde{E})Q_\phi(n)$$
$$=Q_\phi(n)\mathscr{E}_\phi^0(\pi_\phi(N(\Sigma_{j\in J_0(n)}D_jf_n))|\mathscr{Y})Q_\phi(n)\,.$$

Since, for $\tilde{E}\geqq\tilde{D}$, we have

$$\|Q_\phi(n)\,\{\mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E})-\mathscr{E}_\phi(\pi_\phi(N(\Sigma_{j\in J_0(n)}D_jf_n))|\tilde{E})\}Q_\phi(n)\|$$
$$\leqq\|(\mathbb{1}-\Sigma_{j\in J_0(n)}D_j)f_n\|^2\,,$$

it follows that

$$\|Q_\phi(n)\,\{\mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y})-\mathscr{E}_\phi^0(\pi_\phi(N(\Sigma_{j\in J_0(n)}D_jf_n))|\mathscr{Y})\}Q_\phi(n)\|$$
$$\leqq\|(\mathbb{1}-\Sigma_{j\in J_0(n)}D_j)f_n\|^2\,.$$

It then follows, by the triangle and Schwartz inequalities, from the above remarks that

$$\text{S-lim}_{\Gamma(\mathscr{Y})}Q_\phi(n)\mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E})Q_\phi(n)=Q_\phi(n)\mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y})Q_\phi(n)\,.$$

By joint strong-operator continuity of products on bounded sets, and by virtue of the fact that, for $n \neq n'$,

$$0=[Q_\phi(n), Q_\phi(n')]=[\mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}), Q_\phi(n')]$$
$$=[\mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}), \mathscr{E}_\phi(\pi_\phi(N(f_{n'}))|\tilde{F})],$$

we have

$$\text{S-lim}_{\Gamma(\mathscr{Y})^N}\prod_{n=1}^N Q_\phi(n)\prod_{n=1}^N \mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}^n)\prod_{n=1}^N Q_\phi(n)$$
$$=\prod_{n=1}^N Q_\phi(n)\prod_{n=1}^N \mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y})\prod_{n=1}^N Q_\phi(n).$$

However, since $P_\phi(\tilde{D})\leqq Q_\phi(D_j)\ \forall j$; it follows that $\prod_{n=1}^N Q_\phi(n)\geqq P_\phi(\tilde{D})$: This, and the preceding statement, proves the lemma. □

3.6. Lemma. *Let $\{F_n\}_{n=1}^N$ be a finite family of mutually orthogonal projectors of $\mathscr{Y}$. Then,*

$$\text{S-lim}_{\Gamma(\mathscr{Y})^N}\prod_{n=1}^N \mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}^n)=\prod_{n=1}^N \mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y}).$$

Proof. Let $\Gamma(\mathscr{Y}|\{F_n\})\equiv\{\tilde{D}\in\Gamma(\mathscr{Y})|F_n\in\tilde{D}''\ \forall n\}$. Clearly, $\Gamma(\mathscr{Y}|\{F_n\})$ is a directed set and cofinal with $\Gamma(\mathscr{Y})$: thus, since [13, 1.3.1 and Remark, p. 46] the linear span of the projectors of $\mathscr{Y}$ is norm-dense in $\mathscr{Y}$, since Ω is cyclic for $\mathscr{Y}$ in H, since Φ is cyclic for $\pi_\phi(\mathscr{A})$ in H_ϕ and since the n-point correlation operators $\{A_{NM}\}$ are continuous and multilinear, it follows that

$$\text{S-lim}_{\Gamma(\mathscr{Y}|\{F_n\})}P_\phi(\tilde{D})=\mathbb{1}.$$

The result then follows from Lemma 3.5 via the inequality, valid for

$$\tilde{D}\in\Gamma(\mathscr{Y}|\{F_n\}),$$
$$|(\Psi,[\prod_{n=1}^N \mathscr{E}_\phi(\pi_\phi(N(f_n))|\tilde{E}^n)-\prod_{n=1}^N \mathscr{E}_\phi^0(\pi_\phi(N(f_n))|\mathscr{Y})]\Psi)|$$
$$\leqq|(P_\phi(\tilde{D})\Psi,[-]P_\phi(\tilde{D})\Psi)|+2\prod_{n=1}^N\|f_n\|^2\ \|(\mathbb{1}-P_\phi(\tilde{D}))\Psi\|$$
$$\cdot(2\|\Psi\|+\|(\mathbb{1}-P_\phi(\tilde{D}))\Psi\|).\quad □$$

3.7. Lemma. *Let $\{S_n\}$ be a Cauchy Sequence in $\mathscr{A}$ such that*

$$\text{w-lim}_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(S_n)|\tilde{E})=\mathscr{E}_\phi^0(\pi_\phi(S_n)|\mathscr{Y})$$

for each $n\in\mathbb{Z}^+$. Then, if $S=\text{n-lim}_{n\to\infty}S_n$,

$$\text{w-lim}_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})=\mathscr{E}_\phi^0(\pi_\phi(S)|\mathscr{Y})=\text{n-lim}_{n\to\infty}\mathscr{E}_\phi^0(\pi_\phi(S_n)|\mathscr{Y}).$$

The proof is immediate from linearity and uniform boundedness (i.e. $\|\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})\|\leqq\|S\|\ \forall\ \tilde{E}\in\Gamma(\mathscr{Y})$) of the maps $\{\mathscr{E}_\phi(\ |\tilde{E})\}_{\tilde{E}\in\Gamma(\mathscr{Y})}$.

3.8. Lemma. *Let $N, M\in\mathbb{Z}^+$, and let $\{f_n\}_{n=1}^N, \{g_m\}_{m=1}^M\subset H$. Then,*

$$\text{w-lim}_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(A_{NM}(f_1,\ldots,f_N;g_1,\ldots,g_M))|\tilde{E})$$
$$=\mathscr{E}_\phi^0(\pi_\phi(A_{NM}(f_1,\ldots,f_N;g_1,\ldots,g_M))|\mathscr{Y})$$
$$=\delta_{MN}\sum_{p\in\mathscr{P}_N}\text{sgn}(p)\prod_{n=1}^N \mathscr{E}_\phi^0(\pi_\phi(A_{11}(f_n;g_{p(n)}))|\mathscr{Y}).$$

As implied by the notation, the order of the product is unimportant.

Proof. Let $\tilde{D}=\{D_k\}$ be a $\mathscr{Y}$-partition of H. Let, for each $1\leqq n\leqq N$ (resp. $1\leqq m\leqq M$), D_{k_n} (resp. D_{κ_m}) be chosen from $\tilde{D}$; let $f_n=d_{k_n}\ \forall\, 1\leqq n\leqq N$ (resp. $g_m=d_{\kappa_m}\ \forall\, 1\leqq m\leqq M$). By the familiar invariance argument (e.g. Proof of 2.1) we have $\mathscr{E}_\phi(\pi_\phi(A_{NM}(f_1,\ldots,f_N;g_1,\ldots,g_M))|\tilde{D})$

$$=\begin{cases} 0 & \text{if (1) } k_n=k_{n'} \text{ for } n\neq n' \\ & \text{or (2) } \kappa_m=\kappa_{m'} \text{ for } m\neq m' \\ & \text{or (3) } \{k_n\}_{n=1}^N\neq\{\kappa_m\}_{m=1}^M \\ \operatorname{sgn}(p)\prod_{n=1}^N E_\phi(\pi_\phi(A_{11}(d_{k_n};d_{k_n}))|\tilde{D}), \text{ where} & \\ & \text{— otherwise} \\ p \text{ is the unique permutation such that } k_n=\kappa_{p(n)}\ \forall\, n & \end{cases}$$

$$=\delta_{M,N}\sum_{p\in\mathscr{S}_N}\operatorname{sgn}(p)\prod_{n=1}^N\mathscr{E}_\phi(\pi_\phi(A_{11}(f_n;g_{p(n)}))|\tilde{D}).$$

Taking into account the orthogality of the $\{D_k\}$, one has, by operating on both sides by $\mathscr{E}_\phi(\ |\tilde{E})$, the above equality for $\tilde{D}$ replaced by $\tilde{E}\geqq\tilde{D}$. The result then follows for this special case from Lemma 3.6. Clearly, the order of the product is irrelevant. Since the linear span of the set of all projectors of $\mathscr{Y}$ is norm dense in $\mathscr{Y}$ [13, 1.3.1], and since Ω is cyclic for $\mathscr{Y}$ in H, the general case follows from linearity and continuity (3.7). □

3.9. Definition. Let $D_\phi(\mathscr{Y})$ [resp. $D_\phi(\mathscr{Y}_A)$, $D_\phi(\mathscr{Y}_N)$] denote the C^*-algebra on H_ϕ generated by

$$\{\mathscr{E}_\phi^0(\pi_\phi(a^*(f)a(g))|\mathscr{Y})|f,g\in H \text{ (resp. } Y_AH,\ Y_NH)\}\cup\{\mathbb{1}\}.$$

We remark that (2.3) $D_\phi(\mathscr{Y})$ is generated by $D_\phi(\mathscr{Y}_A)\cup D_\phi(\mathscr{Y}_N)$.

3.10. Lemma. *Let $\mathscr{Z}_\phi=\pi_\phi(\mathscr{A})''\cap\pi_\phi(\mathscr{A})'$. Then $D_\phi(\mathscr{Y}_N)\subseteqq\mathscr{Z}_\phi$ and $D_\phi(\mathscr{Y})$ is abelian.*

Proof. Since, by Lemma 3.8, $D_\phi(\mathscr{Y}_N)\subseteqq\pi_\phi(\mathscr{A})''$, it remains to prove that $D_\phi(\mathscr{Y}_N)\subseteqq\pi_\phi(\mathscr{A})'$. It suffices to prove that $[\pi_\phi(a(f)),\mathscr{E}_\phi^0(\pi_\phi(a^*(g)a(l))|\mathscr{Y})]=0\ \forall f\in H;\ \forall\, g,l\in Y_NH$. Since Ω is clearly cyclic for $Y_N\mathscr{Y}$ in Y_NH it suffices by linearity and continuity to prove that, if F,G,K be projectors of $\mathscr{Y}$, then $[\pi_\phi(a(f)),\mathscr{E}_\phi^0(\pi_\phi(A_{11}(Y_Ng;Y_Nk))|\mathscr{Y})=0$; or,

$$\text{w-lim}_{\Gamma(\mathscr{Y})}[\pi_\phi(a(f)),\mathscr{E}_\phi(\pi_\phi(A_{11}(Y_Ng;Y_Nk))|\tilde{E})]=0.$$

In fact, let $\tilde{F}\geqq\tilde{E}^N$ with the notation of Lemma 3.4, Case 5 but with $D=Y_N$. It then follows, by an elementary calculation, that $\|[\pi_\phi(a(f)),\mathscr{E}_\phi(\pi_\phi(A_{11}(Y_Ng;Y_Nk))|\tilde{F})]\|^2\leqq(\Omega,Y_N\Omega)^3/N$. This proves the first assertion. To prove that $D_\phi(\mathscr{Y})$ is abelian, it remains only to show that $D_\phi(\mathscr{Y}_A)$ is abelian. But, since $\mathscr{E}_\phi^0(\pi_\phi(A_{11}(Y_Af;Y_Ag))|\mathscr{Y})=\Sigma\pi_\phi(a^*(A_if)a(A_ig))$, it follows that $D_\phi(\mathscr{Y}_A)$ is generated by $\{\pi_\phi(N(a_i))\}\cup\{\mathbb{1}\}$ and these clearly commute. □

We now put together the results obtained so far.

3.11. Proposition. *There exists a unique linear, continuous mapping $\mathscr{E}_\phi^0(\ |\mathscr{Y})$: $\pi_\phi(\mathscr{A})\to\pi_\phi(\mathscr{A})''$ such that*

$$\mathscr{E}_\phi^0(\pi_\phi(S)|\mathscr{Y})=\sigma\text{-lim}_{\Gamma(\mathscr{Y})}\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})\quad\forall\, S\in\mathscr{A}.$$

Further,

0) $\mathscr{E}^0_\phi(\mathbb{1}|\mathscr{Y})=\mathbb{1}$.

1) $\|\mathscr{E}^0_\phi(\ |\mathscr{Y})\|=1$.

2) $S\geqq 0 \Rightarrow \mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})\geqq 0$.

3) $\mathscr{E}^0_\phi(\pi_\phi(S^*)|\mathscr{Y})=\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})^* \quad \forall S\in\mathscr{A}$.

4) $\mathscr{E}^0_\phi(\mathscr{U}_\phi(g)\pi_\phi(S)\mathscr{U}_\phi(g)^*|\mathscr{Y})=\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})$
$=\mathscr{U}_\phi(g)\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})\mathscr{U}_\phi(g)^* \quad \forall S\in\mathscr{A};\forall g\in G(\mathscr{Y})$.

5) *For* $N, M\in\mathbb{Z}^+$, $\{f_n\}_{n=1}^N$, $\{g_m\}_{m=1}^M\subset H$, *we have*

$$\mathscr{E}^0_\phi(\pi_\phi(A_{NM}(f_1,\ldots,f_N;g_1,\ldots,g_M))|\mathscr{Y})=\delta_{MN}\det\{\mathscr{E}^0_\phi(\pi_\phi(A_{11}(f_n;g_m))|\mathscr{Y})\}.$$

6) $E^0_\phi(\pi_\phi(\mathscr{A})|\mathscr{Y})\subseteqq D_\phi(\mathscr{Y})$.

Proof. Since $\mathscr{A}$ is the closed linear span of the n-point operators $\{A_{NM}\}$, existence of $\mathscr{E}^0_\phi(\ |\mathscr{Y})$ follows by linearity and by continuity (3.7) from Lemma 3.8: uniqueness is immediate. 0), 1), 2), and 3) follow from the properties of the conditional expectations $\{\mathscr{E}_\phi(\ |\tilde{E})\}$. It follows from the $G(\tilde{E})$-invariance of $\mathscr{E}_\phi(\ |\tilde{E})$ for $\tilde{E}\in\Gamma(\mathscr{Y})$ that 4), holds for $g\in G^0(\mathscr{Y})\equiv\bigcup_{\Gamma(\mathscr{Y})}G(\tilde{E})$. Since $G^0(\mathscr{Y})$ is norm-dense in $G(\mathscr{Y})$ (by trivial modification of the proof of [13, 1.3.1]), and since α and $\mathscr{U}_\phi$ are strongly continuous, 4) holds for all $g\in G(\mathscr{Y})$. 5) follows from 3.8, and 6) from 5). □

3.12. Proposition. *There exists a unique normal conditional expectation* $\mathscr{E}_\phi(\ |\mathscr{Y})$ *on* $\pi_\phi(\mathscr{A})''$ *which extends* $E^0_\phi(\ |\mathscr{Y})$. *Further,* $\mathscr{E}_\phi(\pi_\phi(\mathscr{A})''|\mathscr{Y})=D_\phi(\mathscr{Y})''$.

Proof. We first prove existence of a, necessarily unique, normal linear map extending $\mathscr{E}^0_\phi(\ |\mathscr{Y})$. To this end, let $H^A_\phi\equiv\{\pi_\phi(\mathscr{A}(Y_AH))\Phi\}^-$, and let $M_\phi H_\phi\equiv[D_\phi(\mathscr{Y}_N)H^A_\phi]^-$. Since $M_\phi\in D_\phi(\mathscr{Y})'$, and since $\pi_\phi(\mathscr{A}(Y_NH))\subset D_\phi(\mathscr{Y})'$ [recalling that $G_\phi(\mathscr{Y}_N)\subseteqq\mathscr{Z}_\phi$], the central support of M_ϕ in $G_\phi(\mathscr{Y})''$ is $\mathbb{1}$ (i.e. $[D_\phi(\mathscr{Y})'M_\phi H_\phi]^-\supseteqq[\pi_\phi(\mathscr{A}(Y_NH))D_\phi(\mathscr{Y}_N)\pi_\phi(\mathscr{A}(Y_AH))\Phi]^-\supseteqq[\pi_\phi(\mathscr{A}(H)\Phi]^-=H_\phi$). Therefore, the restriction map

$R:D_\phi(\mathscr{Y})''\to D_\phi(\mathscr{Y})''M_\phi$ is an isomorphism [4; 1, § 2, Proposition 2]. On the other hand, if $S\in\mathscr{A}$, we have

$$\mathscr{U}_\phi(g)\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)\mathscr{U}_\phi(g)^*=\mathscr{U}_\phi(1+Y_N(g-1))\mathscr{E}_\phi(\pi_\phi(S))|\tilde{E}^0)\mathscr{U}_\phi(1+Y_N(g-1))^* \quad \forall g\in G.$$

Further, since $\mathscr{U}_\phi(G(Y_N))\subseteqq\{D_\phi(\mathscr{Y}_N)\pi_\phi(\mathscr{A}(Y_AH))\}'$, it follows that $\mathscr{U}_\phi(g)M_\phi=M_\phi\ \forall g\in G(Y_N)$. Combining these two observations, we have

$$M_\phi\mathscr{U}_\phi(g)\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)\mathscr{U}_\phi(g)^*M_\phi=M_\phi\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)M_\phi\ \forall g\in G(\mathscr{Y}).$$

Recalling 0.2(3), it follows from linearity and continuity that

$$M_\phi\mathscr{E}_\phi(\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)|\tilde{C})M_\phi=M_\phi\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)M_\phi\ \forall\tilde{C}\in\Gamma(\mathscr{Y}).$$

Therefore (2.7(4)),

$$M_\phi\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})M_\phi=M_\phi\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)M_\phi\ \forall S\in\mathscr{A}.$$

This proves that $\mathscr{E}^0_\phi(\pi_\phi(S)|\mathscr{Y})=R^{-1}(M_\phi\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^0)M_\phi)\ \forall S\in\mathscr{A}$. Now, since the mapping $S(\in\pi_\phi(\mathscr{A})'')\to M_\phi\mathscr{E}_\phi(S|\tilde{E}^0)M_\phi$ is ultra-weakly continuous it follows that $M_\phi\mathscr{E}_\phi(S|\tilde{E}^0)M_\phi\in D_\phi(\mathscr{Y})''M_\phi\ \forall S\in\pi_\phi(\mathscr{A})''$. We define $\mathscr{E}_\phi(\ |\mathscr{Y})\equiv R^{-1}(M_\phi\mathscr{E}_\phi(\ |\tilde{E}^0)M_\phi)$

on $\pi_\phi(\mathscr{A})''$. $\mathscr{E}_\phi(\ |\mathscr{Y})$ is clearly a normal linear map extending $\mathscr{E}^0_\phi(\ |\mathscr{Y})$. To prove $\mathscr{E}_\phi(\ |\mathscr{Y})$ is a conditional expectation only (3) of Definition 0.1 remains.

We have,

$$\begin{aligned}\mathscr{E}_\phi(S\mathscr{E}_\phi(T|\mathscr{Y})|\mathscr{Y}) &= R^{-1}(M_\phi\mathscr{E}_\phi(S\mathscr{E}_\phi(T|\mathscr{Y})|\tilde{E}^0)M_\phi)\\ &= R^{-1}(M_\phi\mathscr{E}_\phi(S|\tilde{E}^0)\,M_\phi\mathscr{E}_\phi(T|\mathscr{Y})M_\phi\\ &= R^{-1}(M_\phi\mathscr{E}_\phi(S|\mathscr{Y})\mathscr{E}_\phi(T|\mathscr{Y})M_\phi)\\ &= \mathscr{E}_\phi(S|\mathscr{Y})\mathscr{E}_\phi(T|\mathscr{Y}) \quad \forall\, S, T\in\pi_\phi(\mathscr{A})'',\end{aligned}$$

where the second equality follows from $G(E^0)$-invariance $\mathscr{E}_\phi(T|\mathscr{Y})$. It is clear, from 3.11 and normality, that $E_\phi(\pi_\phi(\mathscr{A})''|\mathscr{Y})\subseteqq D_\phi(\mathscr{Y})''$. On the other hand, if $S\in D_\phi(\mathscr{Y})''$, we have

$$\mathscr{E}_\phi(S|\mathscr{Y}) = R^{-1}(M_\phi\mathscr{E}_\phi(S|\tilde{E}^0)M_\phi) = R^{-1}(SM_\phi) = S\,.$$

Hence

$$\mathscr{E}_\phi(\pi_\phi(\mathscr{A})''|\mathscr{Y}) = D_\phi(\mathscr{Y})''\,. \quad \square$$

The proof of Theorem 3.2 is immediate from Propositions 3.11 and 3.12. We shall need the following modification of 3.2.

3.13. Proposition. *Let $\{\tilde{E}^n\}_{n=1}^\infty$ be a dense, increasing sequence of $\mathscr{Y}$-partitions of H; let ϕ be a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$. Then,*

$$\sigma\text{-}\lim_{n\to\infty}\mathscr{E}_\phi(\pi_\phi(S)|\tilde{E}^n) = \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y}) \quad \forall\, S\in\mathscr{A}\,.$$

Proof. Let $\tilde{F}=\{F_j\}_{j=1}^\infty \geqq E^0$ be a $\mathscr{Y}$-partition of H ordered so that $\{(\Omega, Y_N F_j\Omega)\}$ is decreasing. For every $\varepsilon_1>0$ there exists $J(\varepsilon_1)$ (finite) such that $(\Omega, Y_N(1-\sum_{j=1}^J F_j)\Omega)<\varepsilon_1$. Furthermore, since $\{\tilde{E}^n\}_{n=1}^N$ is dense, we have for arbitrary ε_2, an n_0 such that, for each $1\leqq j\leqq J$, there exists a projector G'_j which is the sum of orthogonal projectors of $\tilde{E}^{n_0}$ and which satisfies $(\Omega, G'_j(\mathbb{1}-F_j)+F_j(\mathbb{1}-G'_j)\Omega)<\varepsilon_2$. Let, for each $1\leqq j\leqq J$, $G_j\equiv G'_j(1-\bigvee_{i<j}G'_i)$; then (1) $G_j\subseteqq G'_j$, (2) $G_jG_i=0\ \forall\, i\neq j$, (3) $\sum_j G_j=\bigvee_j G'_j$, and (4) G_j is the sum of orthogonal projectors of $\tilde{E}^{n_0}$. It follows directly that

$$(\Omega, F_j(\mathbb{1}-G_j)+G_j(\mathbb{1}-F_j)\Omega)<\varepsilon_2 J \qquad 1\leqq j\leqq J\,.$$

For arbitrary $n\geqq n_0$, and projector $D\in\mathscr{Y}$, we have

$$\begin{aligned}&\|\mathscr{E}(N(d)|\tilde{F}\vee\tilde{E}^n)-\mathscr{E}(N(d)|\tilde{E}^n)\| = \|\textstyle\sum_{j<J}\{\mathscr{E}(N(F_jd)|\tilde{E}^n)\\ &-\mathscr{E}(N(G_jd)|\tilde{E}^n)\}+\mathscr{E}(N((\mathbb{1}-\textstyle\sum_{j\leqq J}F_j)d)|\tilde{E}^n\vee\tilde{F})\\ &-\mathscr{E}(N((\mathbb{1}-\textstyle\sum_{j\leqq J}G_j)d)|\tilde{E}^n)\| < \textstyle\sum_{j\leqq J}\|N(F_jd)-N(G_jd)\|+\varepsilon_1\\ &+(\Omega,(\mathbb{1}-\textstyle\sum_{j\leqq J}G_j)D\Omega)\leqq 2\varepsilon_1+J^2\varepsilon_2+2J^{3/2}\|\Omega\|\varepsilon_2^{1/2}\,.\end{aligned}$$

To sum up, if $\tilde{F}$ is a $\mathscr{Y}$-partition of H, and if D is a projector of $\mathscr{Y}$, then for every $\varepsilon>0$ there exists n_0 such that $n\geqq n_0\Rightarrow$

$$\|\mathscr{E}(N(d)|\tilde{F}\vee\tilde{E}^n)-\mathscr{E}(N(d)|\tilde{E}^n)\|<\varepsilon\,.$$

The proof is then completed by simple modifications to 3.7, 3.8, and 3.11. $\square$

3.b Consequences of Local Convergence

The next lemma aims at establishing (Theorem 3.15) the map $\Omega_\phi : \mathscr{Y}_* \to D_\phi(\mathscr{Y})''$.

3.14. Lemma. *Let ϕ be a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$. Then, there exists a unique mapping $K_\phi : (D_\phi(\mathscr{Y})'')_* \to \mathscr{Y}$ such that*

$$(g, K_\phi(\psi)f) = \langle \psi ; \mathscr{E}_\phi(\pi_\phi(a^*(f)a(g))|\mathscr{Y})\rangle \quad \forall f, g \in H .$$

Further, K_ϕ is positive, linear and contracting.

Proof. Since, for each $\psi \in (D_\phi(\mathscr{Y})'')_*$, the mapping $(g, f) \to \langle \psi ; \mathscr{E}_\phi(\pi_\phi(a^*(f)a(g))|\mathscr{Y})\rangle$ is, by Theorem 3.2, a continuous sesquilinear form over H, there exists, by Riesz's theorem, a unique bounded linear operator, $K_\phi(\psi)$ on H such $(g, K_\phi(\psi)f) = \langle \psi ; \mathscr{E}_\phi(\pi_\phi(a^*(f)a(g))|\mathscr{Y})\rangle$. By (1) of Theorem 3.2, $hK_\phi(\psi)h^{-1} = K_\phi(\psi)\ \forall h \in G(\mathscr{Y})$. It follows that $K_\phi(\psi) \in \mathscr{Y}' = \mathscr{Y}$. This proves existence and uniqueness. Linearity and positivity are immediate. Finally,

$$\|K_\phi(\psi)\| = \sup_{\substack{f, g \in H: \\ \|f\| = \|g\| = 1}} |\langle \psi ; \mathscr{E}_\phi(\pi_\phi(a^*(f)a(g))|\mathscr{Y})\rangle| \leq \|\psi\| . \quad \square$$

To each $\Lambda \in \mathscr{Y}_*$, we can now assign a mean number operator.

3.15. Theorem. *There exists a unique map $\Omega_\phi : \mathscr{Y}_* \to D_\phi(\mathscr{Y})''$ such that*

$$\Omega_\phi(\Lambda(g, f)) = \mathscr{E}_\phi(\pi_\phi(a^*(f)a(g))|\mathscr{Y}) \quad \forall f, g \in H .$$

Further, Ω_ϕ is positive, linear and norm-reducing.

Proof. Uniqueness is immediate. Existence follows from Lemma 3.14 by defining $\Omega_\phi(\Lambda) = K_\phi^*(\Lambda)\ \forall \Lambda \in \mathscr{Y}_*$. $\square$

We can now make contact with the discrete case.

3.16. Corollary. *Let $A(\mathscr{Y}) = \{A_i\}_{i \in I}$ denote the atoms of $\mathscr{Y}$; define for each $i \in I$, $\Lambda_i \in \mathscr{Y}_*$ such that $\Lambda_i(A_j) = \delta_{ij}$. Then, $\Omega_\phi(\Lambda_i) = \pi_\phi(a^*(f_i)a(f_i))$ where f_i is any normalized vector in the range of A_i. Hence $\Omega_\phi(\Lambda_i)$ is a projector and $D_\phi(\mathscr{Y}_A)$ is generated by $\mathbb{1}_\phi$ and $\{\Omega_\phi(\Lambda_i) | i \in I\}$.*

We now obtain the operator-valued, number density over the spectrum of $\mathscr{Y}$ attached to the representation π_ϕ (cf. 1, § 7, Theorems 1 and 2 and III, § 3, Corollary 1 of Ref. [4]).

3.17. Theorem. *Let (X, B, μ) be a totally σ-finite measure space; and, in the preceding discussion, let $H = \mathscr{L}^2(X, B, \mu)$, $\mathscr{Y} = \mathscr{L}^\infty(X, B, \mu)$ and $\mathscr{Y}_* = \mathscr{L}^1(X, B, \mu)$. If ϕ is a $G(\mathscr{Y})$-invariant state on $\mathscr{A}(H)$, there exists a μ-a.e. unique, σ-measurable mapping $N_\phi(\ |\mathscr{Y}) : X \to D_\phi(\mathscr{Y})''$ such that*

$$\langle \psi ; \Omega_\phi(f)\rangle = \int f(x) \langle \psi ; N_\phi(x|\mathscr{Y})\rangle d\mu(x) \quad \forall \psi \in (D_\phi(\mathscr{Y})'')_* ; \forall f \in \mathscr{L}^1(X, B, \mu) .$$

Further, $0 \leq N_\phi(x|\mathscr{Y}) \leq \mathbb{1}\ \mu$-a.e.

Proof. Separability of H implies that of $\mathscr{A}(H)$, H_ϕ and hence of $(D_\phi(\mathscr{Y})'')_*$ and of $(D_\phi(\mathscr{Y})'')_*^+$ [the normal positive linear forms on $D_\phi(\mathscr{Y})''$]: let D_*^Q be a countable dense set of $(D_\phi(\mathscr{Y})'')_*$ such that $D_*^Q \cap (D_\phi(\mathscr{Y})'')_*^+$ in dense in $(D_\phi(\mathscr{Y})'')_*^+$, and let D_*^Q denote the set of all finite linear combinations of elements of D_*^Q with coefficients in $Q+iQ$ (Q = rationals). It is clear that D_*^Q is itself a countable dense set of $\{D_\phi(\mathscr{Y})'')_*$ such that $\{D_*^Q \cap (D_\phi(\mathscr{Y})'')_*^+\}^- = (D_\phi(\mathscr{Y})'')_*^+$. We now prove uniqueness. Suppose that $\bar{N}_\phi(\ |\mathscr{Y})$, $N_\phi(\ |\mathscr{Y})$ are two such mappings. It follows from the uniqueness of Lemma 3.14 that for each $\psi \in (D_\phi(\mathscr{Y})'')_*$, $K_\phi(\psi)_x = \langle \psi; N_\phi(x|\mathscr{Y})\rangle = \langle \psi; \bar{N}_\phi(x|\mathscr{Y})\rangle$ μ-*a.e.* There exists, then, a measurable set $X_0 \subset X$ such that $\mu(X_0)=0$ and such that $X \bar{\in} U_0 \Rightarrow$

$$\langle \psi; N_\phi(x|\mathscr{Y})\rangle = \langle \psi; \bar{N}_\phi(x|\mathscr{Y})\rangle \ \forall \ \psi \in D_*^Q .$$

Since D_*^Q is dense in $(D_\phi(\mathscr{Y})'')_*$, it follows that $x \bar{\in} X_0 \Rightarrow N_\phi(x|\mathscr{Y}) = \bar{N}_\phi(x|\mathscr{Y})$. This proves essential uniqueness of $N_\phi(\ |\mathscr{Y})$. We now prove existence. There exists by Lemma 3.14 for *each* $\psi \in (D_\phi(\mathscr{Y})'')_*$ a μ-*a.e.* unique essentially bounded function $K_\phi(\psi)$ such that

$$\langle \psi; \Omega_\phi(f)\rangle = \int_X f(x) K_\phi(\psi)_x d\mu(x) \ \forall \ f \in \mathscr{L}^1(X, B, \mu) .$$

We have, for $\lambda, \gamma \in \mathbb{R}$ and $\psi, \psi' \in (D_\phi(\mathscr{Y})'')_*$.

(1) $K(\lambda\psi + \gamma\psi')_x = \lambda K_\phi(\psi)_x + \gamma K_\phi(\psi')_x \mu$-*a.e.*

(2) $|K_\phi(\psi)_x| \leqq \|\psi\| \mu$-*a.e.*

(3) $\psi \geqq 0 \Rightarrow K_\phi(\psi)_x \geqq 0 \mu$-*a.e.*

There exists, then, a measurable set $X_0 \subset X$ such that $\mu(X_0)=0$ and such that (1), (2), and (3) hold *everywhere* on $X - X_0$ for $\gamma, \lambda \in Q+iQ$ and for all $\psi, \psi' \in D_*^Q$. Define $\bar{K}_\phi(\psi)_x = K_\phi(\psi)_x X(X-X_0)$ where $X(X-X_0)$ is the characteristic function of $X - X_0$. $\bar{K}_\phi$, mapping D_*^Q into the Banach space of bounded B-measurable functions $M(X, B)$, extends by continuity to a complex-linear mapping $\hat{K}_\phi$ of $(D_\phi(\mathscr{Y})'')_*$ into $M(X, B)$. Thus, for each $x \in X$, $\psi \to \hat{K}_\phi(\psi)_x$ is a positive, linear mapping of $(D_\phi(\mathscr{Y})'')_*$ into $\mathbb{C}$ bounded by 1. There exists therefore, for each $x \in X$, a unique operator $N_\phi(x|\mathscr{Y}) \in D_\phi(\mathscr{Y})''$ such that $\langle \psi; N_\phi(x|\mathscr{Y})\rangle = \hat{K}_\phi(\psi)_x \forall \psi \in (D_\phi(\mathscr{Y})'')_*$. One easily verifies that $N_\phi(\ |\mathscr{Y})$ satisfies the requirements of the theorem. □

3.18. Theorem. *Let ϕ be a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$, and let $P_\phi^{G(\mathscr{Y})}$ denote the cone of $G(\mathscr{Y})$-invariant normal positive linear forms on $\pi_\phi(\mathscr{A})''$. Then, with the above notation and definitions, the restriction map $R: P_\phi^{G(\mathscr{Y})} \to (D_\phi(\mathscr{Y})'')_*^+$ of $P_\phi^{G(\mathscr{Y})}$ into the cone of normal positive linear forms on $D_\phi(\mathscr{Y})''$ is a bijection.*

Proof. Let $\tilde{\psi}$ be a normal positive linear form on $D_\phi(\mathscr{Y})''$. The form ψ defined for each $S \in \pi_\phi(\mathscr{A})''$ by $\langle \psi; S\rangle = \langle \tilde{\psi}; \mathscr{E}_\phi(S|\mathscr{Y})\rangle$ is clearly a $G(\mathscr{Y})$-invariant normal positive linear form on $\pi_\phi(\mathscr{A})''$ extending $\tilde{\psi}$. R is therefore surjective. Now let $\psi \neq \psi' \in P_\phi^{G(\mathscr{Y})}$ with $R_\psi = R_{\psi'}$. Then, there exists $S \in \mathscr{A}$ such that $0 \neq \langle \psi - \psi'; \pi_\phi(S)\rangle$. This, however, results in the contradiction:

$$\begin{aligned} 0 &= \langle R_\psi - R_{\psi'}; \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y})\rangle = \langle \psi - \psi'; \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y})\rangle \\ &= \lim_{\Gamma(\mathscr{Y})} \langle \psi - \psi'; \mathscr{E}_\phi(\pi_\phi(S)|\tilde{E})\rangle = \langle \psi - \psi'; \pi_\phi(S)\rangle . \end{aligned}$$

R is therefore injective. □

3.19. Remark. It follows from the proof of 3.18 that if ϕ is a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$, and if ψ is a $G(\mathscr{Y})$-invariant normal positive linear form on $\pi_\phi(\mathscr{A})''$, then $\langle \psi; \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y})\rangle = \langle \psi; \pi_\phi(S)\rangle \ \forall \ S \in \mathscr{A}$.

4. Global Theory

In the present section, we focus on the global consistency of the local results obtained in the preceding section. We first show (4.2) that the set of all $G(\mathscr{Y})$-invariant states on $\mathscr{A}$ is a simplex and give a characterization of the set of extreme points, showing it to be weak* compact. The set $S(\mathscr{Y})$ of extremal $G(\mathscr{Y})$-invariant states supports the global extension we are looking for (4.3–4.5).

The next lemma follows easily from Moore's theorem.

4.0. Lemma. *Let $A \in B(H)$ be such that $0 \leqq A \leqq \mathbb{1}$. Let ω_A denote the unique gauge invariant generalized free state on $\mathscr{A}$ such that $\langle \omega_A; a^*(f)a(g)\rangle = (g, Af)\ \forall f, g \in H$. Then, the map $\omega: B(H)_1^+ \to F(H)$ is a homeomorphism when the former is equipped with the σ-topology, and the latter with the weak*-topology.*

4.1. Definition. Let $\mathscr{Y}$ be a maximal abelian von Neumann algebra on H. Let $\bar{S}(\mathscr{Y})$ denote the set of all $A \in \mathscr{Y}$ satisfying $0 \leqq A \leqq \mathbb{1}$ and $A^2 Y_A = A Y_A$. Let $S(\mathscr{Y}) \equiv \omega(\bar{S}(\mathscr{Y}))$. We remark that $\bar{S}(\mathscr{Y})$ is a σ-compact subset of $\mathscr{Y}$; therefore $S(\mathscr{Y})$ is w^*-compact.

4.2. Theorem. *The set $\mathfrak{S}^{G(\mathscr{Y})}$ of all $G(\mathscr{Y})$-invariant states on $\mathscr{A}$ is a (Choquet) simplex, and $S(\mathscr{Y})$ is the set of extremal $G(\mathscr{Y})$-invariant states on $\mathscr{A}$. For each $G(\mathscr{Y})$-invariant state ϕ on $\mathscr{A}$, there exists, by Choquet's theorem* [cf. 12, A.5], *a unique regular probability measure μ_ϕ on $\mathfrak{S}^{G(\mathscr{Y})}$ (or $\mathfrak{S}$) such that*

(1) $\mu_\phi(S(\mathscr{Y})) = 1$.

(2) $\langle \phi; A\rangle = \int \langle \sigma; A\rangle d\mu_\phi(\sigma) \quad \forall A \in \mathscr{A}$.

Proof. We show that the cone $P^{G(\mathscr{Y})}$ of $G(\mathscr{Y})$-invariant positive linear forms on $\mathscr{A}$ is a lattice. For this it is sufficient to show that, for each $G(\mathscr{Y})$-invariant state ϕ, the cone $P_\phi^{G(\mathscr{Y})}$ of all $G(\mathscr{Y})$-invariant normal positive linear forms on $\pi_\phi(\mathscr{A})''$ is a lattice. Indeed, if $\psi, \psi' \in P^{G(\mathscr{Y})}$, then $\psi, \psi' \in P_\phi^{G(\mathscr{Y})}$ for $\phi = (\psi + \psi')/\langle \psi + \psi'; \mathbb{1}\rangle$. Since $P_\phi^{G(\mathscr{Y})}$ is, by Theorem 3.18, order isomorphic to the set of normal positive linear forms on the abelian von Neumann algebra $D_\phi(\mathscr{Y})''$, and since this last set is a lattice, it follows that $P_\phi^{G(\mathscr{Y})}$ is a lattice. Hence, $\mathfrak{S}^{G(\mathscr{Y})}$ is a simplex. We now characterize the set of extremal $G(\mathscr{Y})$-invariant states. Since $P_\phi^{G(\mathscr{Y})}$ is order isomorphic to $(D_\phi(\mathscr{Y})'')_*^+$, a $G(\mathscr{Y})$-invariant state ϕ on $\mathscr{A}$ is extremal $G(\mathscr{Y})$-invariant if and only if $\phi|_{D_\phi(\mathscr{Y})''}$ is a character (the distinction between ϕ as a state on $\mathscr{A}$ and its extension to a normal state on $\pi_\phi(\mathscr{A})''$ is not made explicit). It follows from Theorem 3.2 and Remark 3.19 that if $\phi|_{D_\phi(\mathscr{Y})''}$ is a character, then ϕ is a gauge invariant generalized free state. Since every gauge invariant generalized free state is a factor [11, 5.1], it follows from Lemma 3.10 that $D_\phi(\mathscr{Y}_A)'' = D_\phi(\mathscr{Y})''$. Now, if $\phi|_{D_\phi(\mathscr{Y})''}$ is a character, we have (with the notation of Corollary 3.16),

$$\begin{aligned}(f_i, \omega^{-1}(\phi)f_i) &= \langle \phi; \pi_\phi(a^*(f_i)a(f_i))\rangle = \langle \phi; \pi_\phi(a^*(f_i)a(f_i))^2\rangle \\ &= (f_i, \omega^{-1}(\phi)f_i)^2 = 0 \quad \text{or} \quad 1\,.\end{aligned}$$

Thus, $Y_A\omega^{-1}(\phi) = \Sigma\lambda_i A_i$ with $\lambda_i = 0$ or 1, proving that $Y_A\omega^{-1}(\phi)$ is a projector. Conversely if $T \in \mathscr{Y}_1^+$ we have: ω_T is $G(\mathscr{Y})$-invariant, $D_{\omega_T}(\mathscr{Y})'' = D_{\omega_T}(\mathscr{Y}_A)''$ and, for $i \neq j$,

$$\begin{aligned}\langle \omega_T; \pi_{\omega_T}(a^*(f_i)a(f_i)a^*(f_j)a(f_j))\rangle &= \langle \omega_T; \pi_{\omega_T}(a^*(f_i)a(f_i))\rangle \\ &\quad \cdot \langle \omega_T; \pi_{\omega_T}(a^*(f_j)a(f_j))\rangle\,.\end{aligned}$$

If, further, $Y_A T$ is a projector we have, by reversing the above argument, that

$$\langle \omega_T; \pi_{\omega_T}(a^*(f_i)a(f_i))\rangle^2 = \langle \omega_T; \pi_{\omega_T}(a^*(f_i)a(f_i))^2\rangle \quad \forall\, i \in I .$$

Since $D_{\omega_T}(\mathscr{Y}_A)'' = \{\pi_{\omega_T}(a^*(f_i)a(f_i))\}''_{i\in I}$, ω_T is a character on $D_{\omega_T}(\mathscr{Y})''$ and is therefore an extremal $G(\mathscr{Y})$-invariant state on $\mathscr{A}$. Assertions 1 and 2 follow immediately from Choquet's theorem, bearing in mind that $S(\mathscr{Y})$ is compact and $\mathfrak{S}$ is metrizable. □

4.3. (a) Definition. Let $D(\mathscr{Y})$ denote the C^*-algebra of continuous, complex-valued functions on the compact Hausdorff space $S(\mathscr{Y})$. Let $\mathscr{E}(\ |\mathscr{Y})$ denote the canonical map of $\mathscr{A}$ into $D(\mathscr{Y})$. That is, for each $T \in \mathscr{A}$, $\mathscr{E}(T|\mathscr{Y})(\sigma) \equiv \langle \sigma; T\rangle\ \forall\, \sigma \in S(\mathscr{Y})$.

We remark that since (3.19 and 3.15) for $f, g \in H$ and $\sigma \in S(\mathscr{Y})$ $\langle \sigma; a^*(f)a(g)\rangle = \langle \sigma; \Omega_\sigma(\Lambda(g, f))\rangle$, and since $\mathscr{Y}_*$ consists of vector-forms, we can add:

4.3. (b) Definition. Let $\Omega: \mathscr{Y}_* \to D(\mathscr{Y})$ be defined, for each $\sigma \in S(\mathscr{Y})$ and $\Lambda \in \mathscr{Y}_*$, by $\Omega(\Lambda)(\sigma) = \langle \sigma; \Omega_\sigma(\Lambda)\rangle$.

4.4. Remark. It is clear that $\mathscr{E}(\ |\mathscr{Y})$ is a positive, *-preserving, linear map of norm 1: we show in 4.5 that it is "almost" a conditional expectation.

It is also clear that $\Omega: \mathscr{Y}_* \to D(\mathscr{Y})$ is a positive, linear, contractive injection which satisfies the formula:

$$\mathscr{E}(A_{NM}(f_1, \ldots, f_N; g_1, \ldots, g_M)|\mathscr{Y}) = \delta_{MN}\det\{\Omega(\Lambda(g_m, f_n))\} .$$

4.5. Theorem. *Let ϕ be a $G(\mathscr{Y})$-invariant state on $\mathscr{A}$. There exists a unique *-representation $\hat{\pi}_\phi: D(\mathscr{Y}) \to B(H_\phi)$ such that*

$$\mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y}) = \hat{\pi}_\phi \mathscr{E}(S|\mathscr{Y}) \quad \forall\, S \in \mathscr{A} .$$

Further, $\hat{\pi}_\phi(\Omega(\Lambda)) = \Omega_\phi(\Lambda) \quad \forall\, \Lambda \in \mathscr{Y}_*$.

Proof. Uniqueness would follow by continuity from norm density of $\mathscr{E}(\mathscr{A}|\mathscr{Y})$ in $D(\mathscr{Y})$. Since $\mathscr{E}(\mathscr{A}|\mathscr{Y})$ is a self-adjoint, linear subspace of $D(\mathscr{Y})$, separating $S(\mathscr{Y})$ and containing the unit, density will follow from the Stone-Weierstrass theorem if $\mathscr{E}(\mathscr{A}|\mathscr{Y})^{-N}$ can be proven to be an algebra. It is sufficient to show that $U, V \in \mathscr{A} \Rightarrow \mathscr{E}(V|\mathscr{Y})\mathscr{E}(U|\mathscr{Y}) \in \mathscr{E}(\mathscr{A}|\mathscr{Y})^{-N}$. We have, for arbitrary $\sigma \in S(\mathscr{Y})$,

$$\begin{aligned} &\mathscr{E}(V|\mathscr{Y}) \cdot \mathscr{E}(U|\mathscr{Y})(\sigma) \\ &= \langle \sigma; \mathscr{E}_\sigma(\pi_\sigma(V)|\mathscr{Y})\mathscr{E}_\sigma(\pi_\sigma(U)|\mathscr{Y})\rangle = \langle \sigma; \pi_\sigma(V)\mathscr{E}_\sigma(\pi_\sigma(U)|\mathscr{Y})\rangle \\ &= \lim_{n\to\infty} \langle \sigma; V\mathscr{E}(U|\tilde{E}^n)\rangle = \lim_{n\to\infty} \mathscr{E}(V\mathscr{E}(U|\tilde{E}^n)|\mathscr{Y})(\sigma), \end{aligned}$$

where $\{\tilde{E}^n\}$ is a dense increasing sequence of $\mathscr{Y}$-partitions of H [by 3.2(2), 3.13, 3.19, and the fact that σ is a character on $D_\sigma(\mathscr{Y})''$].

Consequently, [6, IV.6.4] $\mathscr{E}(V|\mathscr{Y})\mathscr{E}(U|\mathscr{Y}) = \text{weak-}\lim_{n\to\infty} \mathscr{E}(V\mathscr{E}(U|\tilde{E}^n)|\mathscr{Y})$ (the *continuous* function on the left is the point-wise limit of a bounded sequence of *continuous* functions). By [6, V.3.13] $\mathscr{E}(V|\mathscr{Y})\mathscr{E}(U|\mathscr{Y}) \in \mathscr{E}(\mathscr{A}|\mathscr{Y})^{-N}$.

This completes the proof of uniqueness; we now prove existence. Let $S \in \mathscr{A}$, $\psi \in \mathfrak{S}_\phi$ and $\mu_{\mathscr{E}^*_\phi(\psi|\mathscr{Y})}$ be the unique maximal measure on $\mathfrak{S}^{G(\mathscr{Y})}$ representing $\mathscr{E}^*_\phi(\psi|\mathscr{Y})$, where $\mathscr{E}^*_\phi(\psi|\mathscr{Y})$ is the $G(\mathscr{Y})$-invariant, normal state on $\pi_\phi(\mathscr{A})''$ defined as $\langle \mathscr{E}^*_\phi(\psi|\mathscr{Y}); S\rangle \equiv \langle \psi; \mathscr{E}_\phi(S|\mathscr{Y})\rangle$. We have

$$\langle \psi; \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y})\rangle \equiv \langle \mathscr{E}^*_\phi(\psi|\mathscr{Y}); S\rangle = \int \mathscr{E}(S|\mathscr{Y})(\sigma)\, d\mu_{\mathscr{E}^*_\phi(\psi|\mathscr{Y})}(\sigma) .$$

We can now define, without ambiguity, $\hat{\pi}_\phi^0 : \mathscr{E}(\mathscr{A}|\mathscr{Y}) \to B(H_\phi)$ by $\hat{\pi}_\phi^0(\mathscr{E}(S|\mathscr{Y})) \equiv \mathscr{E}_\phi(\pi_\phi(S)|\mathscr{Y}) \forall S \in \mathscr{A}$. It is clear that $\hat{\pi}_\phi^0$ is linear and *-preserving. Let again $U, V \in \mathscr{A}$; then, by the weak convergence proven above, we have

$$\langle \psi; \mathscr{E}_\phi(\pi_\phi(U)|\mathscr{Y})\mathscr{E}_\phi(\pi_\phi(V)|\mathscr{Y})\rangle = \lim_{n\to\infty} \langle \mathscr{E}_\phi^*(\psi|\mathscr{Y}); U\mathscr{E}(V|\tilde{E}^n)\rangle$$

$$= \lim_{n\to\infty} \int \mathscr{E}(U\mathscr{E}(V|\tilde{E}^n)|\mathscr{Y})(\sigma)\, d\mu_{\mathscr{E}_\phi^*(\psi|\mathscr{Y})}(\sigma)$$

$$= \int \mathscr{E}(U|\mathscr{Y})\mathscr{E}(V|\mathscr{Y})(\sigma)\, d\mu_{\mathscr{E}_\phi^*(\psi|\mathscr{Y})}(\sigma).$$

Thus, for each $V \in \mathscr{A}$ we have,

$$\|\mathscr{E}_\phi(\pi_\phi(V)|\mathscr{Y})\|^2$$

$$= \sup_{\|\xi\|=1} (\xi, \mathscr{E}_\phi(\pi_\phi(V)|\mathscr{Y})\mathscr{E}_\phi(\pi_\phi(V)|\mathscr{Y})\xi) \leqq \|\mathscr{E}(V|\mathscr{Y})\|^2_{\text{sup-norm}}.$$

Therefore, $\hat{\pi}_\phi^0$ extends by continuity to a continuous, *-preserving, linear map $\hat{\pi}_\phi$ of $D(\mathscr{Y}) = \mathscr{E}(\mathscr{A}|\mathscr{Y})^{-N}$ into $G_\phi(\mathscr{Y})$. It is clear that, for each normal form ψ on $\pi_\phi(\mathscr{A})''$, $\langle \psi; \hat{\pi}_\phi(T)\rangle = \int T(\sigma) d\mu_{\mathscr{E}_\phi^*(\psi|\mathscr{Y})}(\sigma) \ \forall\, T \in D(\mathscr{Y})$.

Hence, for $U, V \in \mathscr{A}$, we have

$$\hat{\pi}_\phi(\mathscr{E}(U|\mathscr{Y})\mathscr{E}(V|\mathscr{Y})) = \hat{\pi}_\phi(\mathscr{E}(U|\mathscr{Y})) \cdot \hat{\pi}_\phi(\mathscr{E}(V|\mathscr{Y})).$$

Hence, $\hat{\pi}_\phi$ is a *-representation of $D(\mathscr{Y})$. The remaining assertion is immediate. □

4.6. Corollary. *Let $\mathfrak{S}(\mathscr{Y})$ denote the set of regular probability measures on $S(\mathscr{Y})$. Then, the map $\mathscr{E}^*(\ |\mathscr{Y}) : \mathfrak{S}(\mathscr{Y}) \to \mathfrak{S}^{G(\mathscr{Y})}$ defined for each $\mu \in \mathfrak{S}(\mathscr{Y})$ by*

$$\langle \mathscr{E}^*(\mu|\mathscr{Y}); S\rangle = \int_{\mathfrak{S}(\mathscr{Y})} \mathscr{E}(S|\mathscr{Y})(\sigma)\, d\mu(\sigma) \ \forall\, S \in \mathscr{A}$$

is an affine, bijective map.

Discussion

In Theorem 3.2, the net of conditional expectations is proven to converge only on $\pi_\phi(\mathscr{A})$ (not on $\pi_\phi(\mathscr{A})''$). If ϕ is the Fock state and if $\mathscr{Y}$ is non-atomic, there exist elements of $\pi_\phi(\mathscr{A})''$ on which the net does not converge. By restricting our attention to $\pi_\phi(\mathscr{A})$, we have been able to treat those cases when $\pi_\phi(\mathscr{A})''$ is not $G(\mathscr{Y})$-finite.

We remark that Corollary 4.6 is the natural generalization of the work of Shale and Stinespring on states symmetric about a basis [14]. Araki [1] has generalized Theorem 4.2.

On the physical side, we have isolated a classical field of number densities on the spectrum of an arbitrary, complete one-partical observable $\mathcal{O}$. When $\mathcal{O}$ has discrete spectrum, the field is simply the lattice gas [3]. When $\mathcal{O}$ has continuous spectrum, the field is macroscopic (i.e. centre-valued [10]). This results from the coarseness of the CAR-algebra description of a Fermi field on a continuous physical space; coarsness which is preserved by normality of the projector upon the classical field.

Appendix

Proof of Theorem 0.2. We give a sketch. See 16.I for details and notation.

1) Existence of the extension follows from amenability of G and the assumption $\mathscr{A}^M \subseteqq \mathscr{A}^G$ (e.g. Ruelle [12, 6.2.13]): Uniqueness is immediate.

2) Since each state on $\mathscr{A}^G$ can be extended (e.g. Ruelle) to a G-invariant state on $\mathscr{A}$, $\mathscr{A}^M$ separates the set of all states on $\mathscr{A}^G$. Therefore, by an Extension of the Stone-Weierstrass theorem [5, 11.3.1], $\mathscr{A}^M = \mathscr{A}^G$.

3) Let $\mathscr{B}$ be the eneveloping von-Neumann algebra of $\mathscr{A}$. $\mathscr{A}$ is isomorphic to a σ-dense C^*-subalgebra of $\mathscr{B}$, and G extends to a group of automorphisms of $\mathscr{B}$. If $S \in \mathscr{B}$ (resp. $\phi \in \mathscr{B}^*$), and if m is a mean on $B(G)$, define $mS \in \mathscr{B}$ (resp. $m^*\phi \in \mathscr{B}^*$) by $\langle mS; \mu \rangle = m\langle \dot{g}S; \mu \rangle \ \forall \mu \in \mathscr{B}_*$ (resp. $\langle m^*\phi; S \rangle = m\langle \phi; \dot{g}S \rangle \ \forall S \in \mathscr{B}$). If ϕ is a state on $\mathscr{B}$ and if η is an *invariant* mean, then it is easily seen that $\phi \circ \eta$ and $\eta^*\phi$ are two G-invariant states on $\mathscr{B}$ which coincide on $\mathscr{A}^G$. Consequently, by 1, $\langle \phi; \eta S \rangle = \langle \eta^*\phi; S \rangle \ \forall S \in \mathscr{A}$ (though not necessarily for all $S \in \mathscr{B}$). Choose a net $\{M_\beta\}_{\beta \in I}$ of finite means weak*-convergent to η. Then, for each state ϕ on $\mathscr{B}$ and $S \in \mathscr{A}$, $\langle \phi; \eta S \rangle = \langle \eta^*\phi; S \rangle = \eta\langle \phi; \dot{g}S \rangle = \lim M_\beta \langle \phi; \dot{g}S \rangle = \lim \langle \phi; M_\beta S \rangle$. Thus, by linearity, for each $S \in \mathscr{A}$, $\eta S = \text{weak-}\lim M_\beta S$. We conclude that $\eta S \in \mathrm{Co}\{gS | g \in G\}^{-\text{weak}}$ and by Mazur's theorem [6, V.3.13], that $\eta S \in \mathrm{Co}\{gS | g \in G\}^{-N}$. Consequently, $\mathrm{Co}\{gS | g \in G\}^{-N} \cap \mathscr{A}^G$ is not empty. To prove that $\{\eta S\} = \mathrm{Co}\{gS | g \in G\}^{-N} \cap \mathscr{A}^G$ suppose that $\bar{S} \in \mathrm{Co}\{gS | g \in G\}^{-N} \cap \mathscr{A}^G$ with $\bar{S} \neq \eta S$. There exists, therefore, $\phi \in \mathscr{B}^*$ such that $\langle \phi; \bar{S} - \eta S \rangle \neq 0$; thus $\langle \eta^*\phi; \bar{S} - \eta S \rangle \neq 0$. But $\eta^*\phi$ is, by continuity, and linearity, constant over $\mathrm{Co}\{gS | g \in G\}^{-N}$. Define $S^G = \eta S$.

4) These properties follow immediately from those of the *invariant* mean η. Uniqueness is immediate. □

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