

Invariant States in Statistical Mechanics

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Abstract. Mathematically we consider a C^* -algebra \mathfrak{A} , acted upon by the group \mathbf{T} of space-translations, which has an asymptotic abelian property. We analyse invariant states over \mathfrak{A} . Physically this programme can be considered as a kinematical study of equilibrium states in statistical mechanics. Each invariant state can be uniquely decomposed into elementary invariant states (E -states). These elementary states have, amongst other characteristics, the physical property that space-averages of local observables are constants in the corresponding representations. In an E -state the discrete spectrum S_D of space-translations is additive which gives rise to the classification E_I , E_{II} , and E_{III} corresponding to the three possibilities that S_D contains one point, a lattice of points, or a set with accumulation points. An E_{II} -state can be uniquely decomposed into states (L -states) having a symmetry with respect to a closed subgroup \mathbf{T}_L of space-translations (S_D and \mathbf{T}_L are reciprocal lattices). L -states have properties with respect to \mathbf{T}_L analogous to the properties of E_I -states with respect to \mathbf{T} . The decomposition into L -states is the inverse process of ‘homogenizing’ a lattice state by smearing it over a lattice distance. The mathematical methods which we employ have more general applications.

1. Introduction

In a recent paper [1] with S. DOPLICHER, which we refer to as DKR, we among other things partially analysed a class of states which we consider to be suitable as equilibrium states in statistical mechanics. It is the purpose of the present paper to generalise and extend this analysis. As our approach to statistical mechanics is rather different from the traditional one let us first briefly state our point of view. In the usual formulation one considers first the microscopic problem of a finite number N of particles in a box of volume V and then considers the macroscopic limit $V \rightarrow \infty$ with the mean density N/V held fixed. Instead of following this procedure we wish to consider directly a description of the macroscopic or infinite, system. In order to understand our procedure it is worthwhile to recall a few of the salient features of the microscopic approach. If we consider the quantum mechanical case the description of N particles in volume V is given kinematically by the Fock representation of the canonical commutation, or anti-commutation, relations (for further details see, for instance, [2], [3], [4]) and dynamically by speci-

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fication of the Hamiltonian describing the interaction of the particles. In the limit of an infinite volume the simplicity of this description is lost and other states or representations of the gauge invariant subalgebra of the commutation (anti-commutation) relations occur which are not unitarily equivalent to the Fock representation; the particular state which occurs is determined by the interaction i.e. the Hamiltonian [5]. This complication is a consequence of the fact that in the macroscopic system we have an infinite number of degrees of freedom. However to compensate this complication there is an added simplification in the description of the infinite system because the limit states have the property of invariance under the group of space translations. Thus the kinematic description of equilibrium states in quantum statistical mechanics should be given by a discussion of the translationally invariant states of the gauge invariant subalgebra of the commutation (anti-commutation) relations. In the present paper we will consider a more general problem and analyse the translationally invariant states over an algebra \mathfrak{A} which has the structure

$$\|[A(x), B]\| \xrightarrow{|x| \rightarrow \infty} 0, \quad A, B \in \mathfrak{A}. \quad (1)$$

This general approach has the advantage of not only covering the quantum mechanical case but also covering abelian algebras which are of interest in classical statistical mechanics.

The results of our analysis are as follows. Any invariant state over \mathfrak{A} has a unique decomposition (theorem 2) into invariant states of a special type which we call E -states. These E -states were already considered in DKR and they can be characterised by a number of equivalent conditions (theorem 3) of which we mention the following three. Firstly if $\pi(A)$ and $U(x)$ are the representations, determined by an E -state, of \mathfrak{A} and the group \mathbf{T} of space-translations respectively, then the set of operators $\pi(\mathfrak{A}) \cup U(\mathbf{T})$ is irreducible in the representation space \mathfrak{H} . Secondly, in \mathfrak{H} there is a unique translationally invariant vector Ω . Thirdly, we have

$$\lim_{V \rightarrow \infty} \frac{1}{V} \int dx (\psi | \pi(A(x)) | \varphi) = (\Omega | \pi(A) | \Omega) (\psi | \varphi) \quad (2)$$

for all $A \in \mathfrak{A}$ and $\psi, \varphi \in \mathfrak{H}$. This last condition expresses the physical result that the global average of a local observable exists and is a constant in the representation determined by an E -state. Further analysis (theorem 4) shows that E -states can be divided into three classes E_I , E_{II} and E_{III} -states, where the classification is with respect to the discrete eigenstates of $U(x)$. In an E_I -state there is one and only one discrete eigenstate of $U(x)$, namely Ω , but in an E_{II} -state there is an infinite number of discrete eigenstates occurring with a lattice structure. In an E_{III} -state there is also an infinite number of discrete eigenstates but

they occur in a dense manner making it impossible to define a lattice distance. Next (theorem 5), we show that an E_{II} -state has a unique decomposition into states which are invariant under a closed subgroup \mathbf{T}_L of space-translations and which we call L -states. These latter lattice type states can also be characterised by a number of equivalent conditions (theorem 6) which are similar to those characterising E_{I} -states: if $U_L(x_L)$ is the representation of \mathbf{T}_L then the set of operators $\pi(\mathfrak{A}) \cup U_L(\mathbf{T}_L)$ is irreducible, there is a unique vector Ω_L which is invariant under $U_L(\mathbf{T}_L)$, and a condition similar to (2) holds but the mean value is taken over lattice points. The L -states are analogous to E_{I} -states insofar that they give rise to only one discrete eigenstate of \mathbf{T}_L , namely Ω_L . We thus see that the analysis of invariant states delivers in a unique and natural manner states of a homogeneous nature (E_{I} -states), states of a periodic nature (L -states) in which the space-translation symmetry is broken and states of the mysterious E_{III} type.

An independent analysis of invariant states has been carried out by RUELLE [6] who has obtained results similar to our theorems 2 and 3 using rather different methods. The advantage of our methods appears to be that they are open to generalisation to more complicated symmetry groups than the abelian group of space-translations.

In section 2 we begin by developing some properties of mean values of operators in Hilbert space. In section 3 we introduce the algebraic definitions which are necessary for the sequel. Section 4 is devoted to the first decomposition theorem and the characterisation of E -states. The second decomposition theorem and the characterisation of L -states is given in section 5. We discuss possible properties of E_{I} -state and L -states in section 6 and we end with a discussion of our results and their possible generalisations in Section 7.

2. Mean values

We start our discussion by developing some properties of means of functions and means of operator-valued functions acting on Hilbert space. In order to define the notion of mean we first introduce a special class of filters, or less generally, sequences of functions over the Euclidean space $\mathbf{T} = \mathbf{R}^n$ denoting points in this space by x, y , etc. In fact all definitions and results of this, and the two following sections can be applied to a general locally compact abelian group \mathbf{T} with Haar measure dx a fact which will be of relevance in section 5.

Definition 1. An M -filter f is a family of functions f_α over \mathbf{T} , indexed by a filter α , which satisfies the following properties:

- i) $f_\alpha(x) \geq 0$
- ii) $\int dx f_\alpha(x) = 1$ for each α ,
- iii) $\lim_{\alpha} \int dx |f_\alpha(x) - f_\alpha(x + y)| = 0$ for all $y \in \mathbf{T}$.

Utilising this definition we define for each M -filter a mean \underline{h}_f of a complex function h by

$$\underline{h}_f = \lim_{\alpha} \int dx f_{\alpha}(x) h(x) \tag{4}$$

whenever this definition makes sense. The mean \underline{h}_f depends in general upon the particular choice of M -filter as the notation already implies. As an illustration we note that a particular choice of f_{α} would be the functions $V^{-1}\chi_v$ where χ_v is the characteristic function of the compact subset v of volume V i.e.

$$\begin{aligned} \chi_v(x) &= 1 & \text{if } x \in v \\ &= 0 & \text{if } x \notin v \end{aligned} \tag{5}$$

and with this choice the limit would be a limit $V \rightarrow \infty$ (cf. (2)).

If $x \rightarrow A(x)$ is an operator valued function acting on a Hilbert space \mathfrak{H} we can define means of $A(x)$ in various ways and the most useful of these are given by the following.

Definition 2. *If $x \rightarrow A(x)$ is an operator-valued function acting on a Hilbert space \mathfrak{H} , if f is an M -filter and if operators ${}_w\underline{A}_f$, ${}_s\underline{A}_f$ and ${}_u\underline{A}_f$ on \mathfrak{H} exist with the properties*

$$\begin{aligned} \lim_{\alpha} (\varphi | \int dx f_{\alpha}(x) A(x) - {}_w\underline{A}_f | \psi) &= 0 & \text{for all } \varphi, \psi \in \mathfrak{H} \\ \lim_{\alpha} \| (\int dx f_{\alpha}(x) A(x) - {}_s\underline{A}_f) \psi \| &= 0 & \text{for all } \psi \in \mathfrak{H} \\ \lim_{\alpha} \| \int dx f_{\alpha}(x) A(x) - {}_u\underline{A}_f \| &= 0 \end{aligned} \tag{6}$$

respectively, then we call ${}_w\underline{A}_f$ a weak mean, ${}_s\underline{A}_f$ a strong mean and ${}_u\underline{A}_f$ a uniform mean of $A(x)$.

It should be noted that these means have properties similar to those of limits, the existence of the uniform mean implies the existence of the strong mean which in turn implies the existence of the weak mean. If any two of the means of $A(x)$ exist it also follows that they are equal. We now wish to apply these definitions to some specific cases.

If $x \rightarrow U(x)$ is a strongly continuous unitary representation of the group T of translations in \mathbb{R}^n by operators acting on \mathfrak{H} then in general $U(x)$ has the spectral decomposition.

$$U(x) = \sum_n E(p_n) e^{i p_n x} + \int dE(p) e^{i p x} \tag{7}$$

where $d(\varphi | E(p) | \psi)$ is a continuous measure for all $\varphi, \psi \in \mathfrak{H}$ and $E(p_n)$ is the projector onto all states $|n\rangle$ such that

$$U(x) |n\rangle = e^{i p_n x} |n\rangle. \tag{8}$$

Lemma 1. *All weak means of $U(x)$ exist and are all equal to $E(0)$. More generally, for any p*

$$\lim_{\alpha} (\varphi | \int dx f_{\alpha}(x) e^{-i p x} U(x) - E(p) | \psi) = 0, \quad \varphi, \psi \in \mathfrak{H} \quad (9)$$

where f is any M -filter.

Proof. In order to prove (9) we first note that we can assume, without loss of generality, that

$$E(p) | \varphi) = 0 \quad (10)$$

and secondly we have

$$\begin{aligned} \lim_{\alpha} |(\chi | (1 - e^{i p y} U(-y)) (\int dx f_{\alpha}(x) e^{-i p x} U(x) - E(p)) | \psi)| \\ = \lim_{\alpha} |(\chi | \int dx (f_{\alpha}(x) - f_{\alpha}(x + y)) e^{-i p x} U(x) | \psi)| < \\ < \lim_{\alpha} \int dx |f_{\alpha}(x) - f_{\alpha}(x + y)| \\ = 0 \end{aligned} \quad (11)$$

for all $\chi \psi \in \mathfrak{H}$ and all $y \in \mathbb{T}$.

Now however we can, by virtue of (10), choose $\chi_i \in \mathfrak{H}$ and $y_i \in \mathbb{R}^n$ such that for $\varepsilon > 0$.

$$\|\varphi - \sum_{i=1}^n (1 - e^{-i p y_i} U(y_i)) \chi_i\| < \varepsilon. \quad (12)$$

As the operator appearing in (9) is uniformly bounded with respect to α , combining (12) and (11) gives (9).

We next deduce the existence of all strong means of $U(x)$.

Lemma 2. *All strong means of $U(x)$ exist and are all equal to $E(0)$. More generally, for any p ,*

$$\lim_{\alpha} \|(\int dx f_{\alpha}(x) e^{-i p x} U(x) - E(p)) \varphi\| = 0, \quad \varphi \in \mathfrak{H} \quad (13)$$

where f is any M -filter.

Proof. The above argument actually establishes that

$$\lim_{\alpha} \{ \int dx f_{\alpha}(x) e^{-i p x} U(x) - E(p) \} \varphi = 0 \quad (14)$$

for φ in the total set

$$E(p) \mathfrak{H} \cup \{ (1 - e^{-i p y} U(y)) \chi; \chi \in \mathfrak{H} \}. \quad (15)$$

Again due to uniform boundedness with respect to α this is enough to deduce the Lemma.

Although we do not wish to consider uniform means of $e^{-i p x} U(x)$ we mention, without proof, that such means exist if and only if the spectral measure $dE(p)$ defined by (7), vanishes in some open neighbourhood of p -space containing the point p . In this case the means are of course all equal to $E(p)$.

The next step in our development is to use the information concerning the means of $U(x)$ to derive properties of the means of other operator-valued functions. If $A_1, A_2 \dots A_n$ are bounded operators acting on \mathfrak{H} we define the respective translates $A_1(x) \dots A_n(x)$ to be the operators given by

$$A_i(x) = U(x) A_i U(x)^{-1} \quad \text{for } i = 1, 2, \dots, n. \tag{16}$$

Our next result concerns means of products of operator-valued functions $A_i(x)$ defined in this manner.

Lemma 3. *If f^i are n M -filters, $i = 1, 2, \dots, n$, then*

$$\lim_{\alpha_1} \dots \lim_{\alpha_n} \langle \chi | \int dx_1 \dots dx_n f_{\alpha_1}^1(x_1) \dots f_{\alpha_n}^n(x_n) A_1(x_1) \dots A_n(x_n) - E(p) A_1 E(p) \dots E(p) A_n E(p) | \varphi \rangle = 0 \tag{17}$$

and

$$\lim_{\alpha_1} \dots \lim_{\alpha_n} \| \langle \chi | \int dx_1 \dots dx_n f_{\alpha_1}^1(x_1) \dots f_{\alpha_n}^n(x_n) A_1(x_1) \dots A_n(x_n) - E(p) A_1 E(p) \dots E(p) A_n E(p) \varphi \| = 0 \tag{18}$$

for all $\chi, \varphi, \psi \in \mathfrak{H}$, for all p , and for any order or manner of the limits i.e. the limit may be taken with respect to the product filter.

Proof. We note that.

$$\begin{aligned} & | \langle \chi | \int dx_1 \dots dx_n f_{\alpha_1}^1(x_1) \dots f_{\alpha_n}^n(x_n) A_1(x_1) \dots A_n(x_n) - E(p) A_1 E(p) \dots A_n E(p) | \varphi \rangle | \\ &= | \sum_{r=1}^n \langle \chi | \int dx_1 \dots dx_r f_{\alpha_1}^1(x_1) \dots f_{\alpha_r}^r(x_r) A_1(x_1) \dots A_r(x_r) E(p) A_{r+1} \dots A_n E(p) | \varphi \rangle - \\ & \quad - \langle \chi | \int dx_1 \dots dx_r f_{\alpha_1}^1(x_1) \dots f_{\alpha_{r-1}}^{r-1}(x_{r-1}) A_1(x_1) \dots A_{r-1}(x_{r-1}) E(p) A_r \dots A_n E(p) | \varphi \rangle | \tag{19} \\ &= | \sum_{r=1}^n \langle \chi | \int dx_1 \dots dx_r f_{\alpha_1}^1(x_1) \dots f_{\alpha_r}^r(x_r) A_1(x_1) \dots A_{r-1}(x_{r-1}) (e^{-i p x} U(x) - E(p)) \cdot A_r E(p) \dots A_n E(p) | \varphi \rangle | \leq \\ & \leq \sum_{r=1}^n \| \chi \| \cdot \| A_1 \| \dots \| A_{r-1} \| \| \int dx_r f_{\alpha_r}^r(x_r) (e^{-i p x} U(x) - E(p)) \times \\ & \quad \times A_r E(p) \dots A_n E(p) | \varphi \| . \end{aligned}$$

Now combining inequality (19) and (13) we can deduce (17). In an analogous manner we may deduce (18).

Finally we note that even if the spectrum of $U(x)$ is purely continuous lemmas 1 and 2 provide non-trivial results but lemma 3 is an empty statement. In this case it is therefore necessary to develop more

powerful methods for investigating means of the translates $A_i(x)$. In the sequel we will apply the above results to certain algebras of operators and we prepare the ground for these applications by introducing the required algebraic definitions and notations in the next section.

3. Algebraic definitions

In the following we are interested in examining a C^* -algebra \mathfrak{A} on which the group \mathbf{T} of space-translations acts as a group of automorphisms (for general information on C^* -algebras see [7] or, for a short resumé, the appendix of [8]). We use the notation $A \in \mathfrak{A} \rightarrow A(x) \in \mathfrak{A}$ to describe the action of these automorphisms and assume strong continuity i.e. we assume

$$\|A(x) - A\| \xrightarrow{|x|=0} 0 \tag{20}$$

where $\| \cdot \|$ denotes the algebraic norm. We will limit our discussion to the consideration of invariant states over \mathfrak{A} . Explicitly a state Φ over \mathfrak{A} is a positive linear form over \mathfrak{A} and an invariant state is specified by the additional property

$$\Phi(A(x)) = \Phi(A) \tag{21}$$

for all $x \in \mathbf{R}^n$ and $A \in \mathfrak{A}$. It is well known that from each state one can construct a cyclic representation of \mathfrak{A} by operators $\pi(\mathfrak{A})$ acting in a Hilbert space \mathfrak{H} and that from an invariant state it is also possible to construct a unitary representation $U(x)$ of the group \mathbf{T} for which the cyclic vector Ω is invariant. The connection between these quantities is given by

- i) $\Phi(A) = (\Omega | \pi(A) | \Omega) ,$
- ii) $\pi(A(x)) = U(x) \pi(A) U(x)^{-1} ,$ (22)
- iii) $U(x)\Omega = \Omega .$

Due to property ii) we call π a \mathbf{T} -covariant representation of \mathfrak{A} . We have previously shown in DKR that all \mathbf{T} -covariant representations of \mathfrak{A} , of which the above are a special subclass, can be best studied by considering an algebra $\mathfrak{A}_1^{\mathbf{T}}$ which is constructed from \mathfrak{A} and \mathbf{T} in such a manner that its $*$ -representations $\hat{\pi}$ are in one-to-one correspondence with the \mathbf{T} -covariant representations (π, U) of $(\mathfrak{A}, \mathbf{T})$. The algebra $\mathfrak{A}_1^{\mathbf{T}}$, which we call a covariance algebra, is of use in the ensuing analysis and we refer to DKR for details of its construction etc.; for our present purposes it suffices to note the following facts. If $\hat{\pi}$ is a representation of $\mathfrak{A}_1^{\mathbf{T}}$ then it is connected to representations π and U , of \mathfrak{A} and \mathbf{T} respectively, by

$$\hat{\pi}(X) = \int du \pi(X_u) U(u) \tag{23}$$

where $X \in \mathfrak{Q}_1^T$ is an L_1 -integrable function from the space \mathbb{T} to \mathfrak{Q} i.e. where $u \in \mathbb{T} \rightarrow X_u \in \mathfrak{Q}$ is such that

$$\int du \|X_u\| < \infty . \tag{24}$$

Further, it is possible to extend the \mathbb{T} -automorphism of \mathfrak{Q} to automorphisms of \mathfrak{Q}_1^T by defining for each $X \in \mathfrak{Q}_1^T$ the translate $X(x)$ through

$$\{X(x)\}_u = X_u(x) \tag{25}$$

and thus in the representation $\hat{\pi}$ we have

$$\hat{\pi}(X(x)) = U(x) \hat{\pi}(X) U(x)^{-1} = \int du \pi(X_u) U(u) . \tag{26}$$

Finally it is possible to extend an invariant state Φ over \mathfrak{Q} to an invariant state $\hat{\Phi}$ over \mathfrak{Q}_1^T through the definition

$$\hat{\Phi}(X) = \int du \Phi(X_u) \tag{27}$$

and the representations (π, U) and $\hat{\pi}$ associated with Φ and $\hat{\Phi}$ respectively act in the same Hilbert space \mathfrak{H} . The invariant vector Ω is cyclic for both $\pi(\mathfrak{Q})$ and $\hat{\pi}(\mathfrak{Q}_1^T)$. We will call $\hat{\Phi}$ the *canonical extension* of Φ .

In the following section we will also need the concept of an enveloping von Neumann algebra (see [7] Chapter 12) as applied specifically to a covariance algebra \mathfrak{Q}_1^T . The enveloping von Neumann algebra \mathfrak{Q}_1^{T**} of \mathfrak{Q}_1^T is the dual of the Banach space \mathfrak{Q}_1^{T*} , which consists of the continuous linear forms on \mathfrak{Q}_1^T i.e. \mathfrak{Q}_1^{T**} is the bi-dual of \mathfrak{Q}_1^T .

An alternative definition is given by identifying \mathfrak{Q}_1^{T**} with the weak closure of the set of representatives of \mathfrak{Q}_1^T contained in the ‘‘universal representations’’ $\hat{\tau}$ which is in turn defined to be the direct sum of all cyclic representations of \mathfrak{Q}_1^T . From this latter definition it is clear that \mathfrak{Q}_1^{T**} is a von Neumann algebra containing \mathfrak{Q}_1^T in the sense that it contains the set $\hat{\tau}(\mathfrak{Q}_1^T)$ which is in one-to-one correspondence with \mathfrak{Q}_1^T because $\hat{\tau}$ is a faithful representation. As a general feature of the theory of von Neumann enveloping algebras an arbitrary representation $\hat{\pi}$ of \mathfrak{Q}_1^T (a state $\hat{\Phi}$ over \mathfrak{Q}_1^T) can be uniquely extended to a W^* -representation of \mathfrak{Q}_1^{T**} (a weakly continuous state over \mathfrak{Q}_1^{T**}). Due to the definition of \mathfrak{Q}_1^T the representation $\hat{\tau}$ determines representations (τ, U_τ) of $(\mathfrak{Q}, \mathbb{T})$ and we note (DKR end of Theorem 3) that both $\tau(\mathfrak{Q})$ and $U_\tau(\mathbb{T})$ are contained in \mathfrak{Q}_1^{T**} . Furthermore, $\tau(\mathfrak{Q})$ and $U_\tau(\mathbb{T})$ are faithful representations of \mathfrak{Q} and \mathbb{T} as a consequence of $\hat{\tau}$ being faithful and thus \mathfrak{Q} and \mathbb{T} are embedded in \mathfrak{Q}_1^{T**} as a sub-algebra and a group of unitaries respectively. We can also consider the group algebra $L_1(\mathbb{T})$ of \mathbb{T} as a sub- * -algebra of \mathfrak{Q}_1^{T**} by assigning to every $f \in L_1(\mathbb{T})$ the operator $\hat{\tau}(f)$ where

$$\hat{\tau}(f) = \int dx f(x) U_\tau(x) .$$

The embedding follows because the definition $f \rightarrow \hat{\tau}(f)$ is easily seen to provide a faithful representation of $L_1(\mathbb{T})$ such that $\hat{\tau}(f)$ commutes with

the commutant of $U_\tau(\mathbf{T})$. Finally all the spectral projectors of the representation $U_\tau(\mathbf{T})$ will be in the weak closure of the sub-algebra $\mathbf{L}_1(\mathbf{T})$ of $\mathfrak{A}_1^{\mathbf{T}^{**}}$ and in particular the associated projector $E_\tau(0)$ onto the invariant states is an element of $\mathfrak{A}_1^{\mathbf{T}^{**}}$. This last fact will be of importance in the sequel.

Next we recall that a state Φ over \mathfrak{A} is said to be *strongly clustering* if

$$|\Phi(A(x)B) - \Phi(A(x))\Phi(B)| \xrightarrow{|x|=\infty} 0 \quad \text{for all } A, B \in \mathfrak{A} \quad (28)$$

and we will call a state *weakly clustering* (cf. DKR, section VI) if

$$\lim_\alpha \int dx f_\alpha(x) \{\Phi(A(x)B) - \Phi(A(x))\Phi(B)\} = 0 \quad (29)$$

for all $A, B \in \mathfrak{A}$ and for some M -filter f . These definitions are principally motivated by physical considerations. If, for example $A(x)$ represents a measurement at the point x and $\Phi(A(x))$ denotes the value obtained by making this measurement in the state Φ then (28) implies that the result of the product measurement $A(x)B$ is equal to the product of the results of the individual measurements in the limit $|x| \rightarrow \infty$. This interpretation implies the absence of infinite range correlations in the state Φ . Alternatively the weak clustering definition (29) implies that any long range correlations in the state Φ have at most an oscillatory nature. However the latter definition also has a direct mathematical motivation which is provided by

Theorem 1. *Let \mathfrak{A} be a C^* -algebra on which the group \mathbf{T} acts as a group of automorphisms, Φ be an invariant state over \mathfrak{A} and $\hat{\Phi}$ the canonical extension of Φ to $\mathfrak{A}_1^{\mathbf{T}}$. If (π, U) and $\hat{\pi}$ are the representations of $(\mathfrak{A}, \mathbf{T})$ and $\mathfrak{A}_1^{\mathbf{T}}$ on the Hilbert space \mathfrak{H} and Ω the invariant vector, in \mathfrak{H} , determined by Φ , or $\hat{\Phi}$, then the following are equivalent:*

- i) Φ is a weakly clustering state
- ii) Ω is the only invariant vector in \mathfrak{H}
- iii) $\lim_\alpha \int dx f_\alpha(x) (\psi | U(x) - E_\Omega | \varphi) = 0, \psi, \varphi \in \mathfrak{H}$
- iv) $\lim_\alpha \|\int dx f_\alpha(x) (U(x) - E_\Omega)\psi\| = 0, \psi \in \mathfrak{H}$

where E_Ω is the projector on Ω , and f is any M -filter (thus (29) is valid for all M -filters).

Further, it follows from each of the equivalent conditions i)–iv) that (π, U) is irreducible i.e. $\hat{\pi}$ is irreducible or, equivalently, $\hat{\Phi}$ is a pure state over $\mathfrak{A}_1^{\mathbf{T}}$.

Proof. It is clear that iv) implies iii) but the proof of lemma 2 shows that the converse is also true. Further Lemma 1 shows that ii) and iii) are equivalent. If we next rewrite the definition of weak clustering, with the aid of (22), in the form

$$\lim_\alpha \int dx f_\alpha(x) \{(\Omega | \pi(A) (U(x)^{-1} - E_\Omega) \pi(B) | \Omega)\} = 0 \quad (30)$$

we immediately see that iii) implies i). However as Ω is cyclic for π we also deduce from (30) that i) implies iii). This demonstrates the equivalence of statements i)—iv). Finally we remark that (30) establishes that E_Ω is in the weak closure of (π, U) but as Ω is cyclic for π it must follow that (π, U) is irreducible.

Up to this stage we have considered quite general C^* -algebras acted on by the group of automorphisms \mathbf{T} and we have placed no structure requirements upon these algebras. In this broad framework theorem 1 appears to be an optimal result. However, in physics, one is interested in algebras with properties reflecting local structure and therefore in the following we will restrict our attention to a special class of algebras which are characterised by an ‘asymptotically local’ property. These algebras, which were first introduced in DKR under the name of asymptotically abelian algebras, are, as discussed in the introduction, of special interest in Statistical Mechanics.

Definition 3. *An asymptotically abelian algebra \mathfrak{A} is a C^* -algebra acted on by the group of automorphism \mathbf{T} with the property*

$$\| [A(x), B] \| \xrightarrow{|x|=\infty} 0 \tag{31}$$

for all $A, B \in \mathfrak{A}$.

In the ensuing sections we will analyse the structure of invariant states over such algebras and will see that the result of theorem 1 can be considerably strengthened in this case (cf. theorem 3).

4. Invariant States

The major aims of this section are, firstly, to show that any invariant state over an asymptotically abelian algebra can be uniquely decomposed into invariant weakly clustering states, secondly, to characterise and, thirdly, to classify these latter states. As a preliminary, and also an aid, to this investigation we first derive the two following lemmas.

Lemma 4. *If Φ is an invariant state over an asymptotically abelian algebra and (π, U) are the representations of $(\mathfrak{A}, \mathbf{T})$ associated with Φ then, for any p_l, p_n, p_m and all $A, B \in \mathfrak{A}$,*

$$E(p_l) \pi(A) E(p_l + p_m) \pi(B) E(p_n) = E(p_l) \pi(B) E(p_n - p_m) \pi(A) E(p_n) \tag{32}$$

where the $E(p_n)$ are the projectors onto discrete eigenstates of $U(x)$ defined by the spectral decomposition (7). In particular for any p_n and all $A, B \in \mathfrak{A}$

$$[E(p_n) \pi(A) E(p_n), E(p_n) \pi(B) E(p_n)] = 0. \tag{33}$$

Proof. As \mathfrak{A} is an asymptotically abelian algebra we may deduce from (31) that

$$e^{ip_m x} (\varphi | E(p_l) [\pi(A(x)), \pi(B)] E(p_n) | \varphi) \xrightarrow{|x|=\infty} 0 \tag{34}$$

for any p_l, p_m, p_n , all $A, B \in \mathfrak{A}$ and for all vectors ψ, φ in the representation space \mathfrak{S} . However (34) implies that

$$\lim_{\alpha} \int dx f_{\alpha}(x) e^{ip_m x} (\psi | E(p_l) [\pi(A(x)), \pi(B)] E(p_n) | \varphi) = 0 \quad (35)$$

for any M -filter f . Next, using the \mathbf{T} -covariance property

$$\pi(A(x)) = U(x) \pi(A) U(x)^{-1} \quad (36)$$

and the fact that

$$U(x) E(p_n) = e^{ip_n x} E(p_n) \quad (37)$$

we can rewrite (35) in the form

$$\begin{aligned} \lim_{\alpha} \int dx f_{\alpha}(x) (\psi | E(p_l) \{ \pi(A) e^{i(p_l + p_m)x} U(x)^{-1} \pi(B) - \\ - \pi(B) e^{-i(p_n - p_m)x} U(x) \pi(A) \} E(p_n) | \varphi) = 0. \end{aligned} \quad (38)$$

Application of the result (9) of Lemma 1 to (38) delivers (32). In order to deduce (33) from (32) it is only necessary to choose $p_l = p_n$ and $p_m = 0$.

Lemma 5. *If Φ is an invariant state over an asymptotically abelian algebra \mathfrak{A} and (π, U) are the representations of $(\mathfrak{A}, \mathbf{T})$ associated with Φ then all strong means of $\pi(A(x))$ exist, are independent of the choice of M -filter f , are in the centre of the von Neumann algebra generated by (π, U) and, denoting the strong means by ${}_s\pi(A)$ we have*

$${}_s\pi(A) \pi(B) \Omega = \pi(B) (E(0) \pi(A) E(0)) \Omega. \quad (39)$$

Thus if $\hat{\Phi}$ is the canonical extension of Φ to \mathfrak{A}_1^T and $\hat{\pi}$ is the associated representation of \mathfrak{A}_1^T then ${}_s\pi(A)$ is in the centre $\mathfrak{B}_{\hat{\pi}}$ of the von Neumann algebra generated by $\hat{\pi}$. Further $\mathfrak{B}_{\hat{\pi}}$ is the smallest von Neumann algebra containing all strong means ${}_s\pi(A)$ and the mapping $Z \in \mathfrak{B}_{\hat{\pi}} \rightarrow E(0) Z E(0)$ defines an isomorphism between von Neumann algebras. Moreover $\mathfrak{B}_{\hat{\pi}} = \hat{\pi}(\mathfrak{A}_1^T)$.

Proof. Recalling the definition (6) of strong means of operator-valued functions we see that the existence of ${}_s\pi(A)$ its independence of the choice of M -filter f , and its action (39), are all established if we can prove that

$$\lim_{\alpha} \|\int dx f_{\alpha}(x) \pi(A(x)) \pi(B) \Omega - \pi(B) (E(0) \pi(A) E(0)) \Omega\| = 0 \quad (40)$$

because $\{\pi(B) \Omega; B \in \mathfrak{A}\}$ is a dense set of vectors and

$$\|\int dx f_{\alpha}(x) \pi(A(x))\| \leq \|A\|.$$

However

$$\begin{aligned} \|\int dx f_{\alpha}(x) (\pi(A(x)) \pi(B) - \pi(B) E(0) \pi(A) E(0)) \Omega\| &\leq \\ &\leq \|\int dx f_{\alpha}(x) [\pi(A(x)), \pi(B)] \Omega\| + \\ &+ \|\pi(B) \int dx f_{\alpha}(x) (\pi(A(x)) - E(0) \pi(A)) \Omega\| \leq \quad (41) \\ &\leq \int dx f_{\alpha}(x) \|\pi(A(x)), \pi(B)\| + \\ &+ \|\pi(B)\| \|\int dx f_{\alpha}(x) (U(x) - E(0)) \pi(A) \Omega\| \end{aligned}$$

and in the filter limit the first term of (41) tends to zero due to (31) and the second term due to Lemma 2. Now that we have established that ${}_s\pi(A)$ exists and acts in the manner described by (39) it is easy to demonstrate that it is in the centre of the von Neumann algebra generated by (π, U) . In particular we have by definition that ${}_s\pi(A)$ is in the weak closure of π so it is only necessary to show that it commutes with (π, U) . This latter fact is, however, a consequence of (39) e.g.

$$\begin{aligned} {}_s\pi(A) \pi(C) \pi(B) \Omega &= \pi(C) \pi(B) E(0) \pi(A) E(0) \Omega \\ &= \pi(C) {}_s\pi(A) \pi(B) \Omega . \end{aligned} \quad (42)$$

The second statement of the Lemma follows from the fact that the von Neumann algebras generated by (π, U) and $\hat{\pi}$ are identical (DKR end of the theorem 3). We note that the abelian character of the set $\{{}_s\pi(A); A \in \mathfrak{A}\}$ is a consequence of (39) and (33). The final statement of the Lemma is established by noting that $Z \in \mathfrak{B}_A$ has the property $[Z, \pi(B)] = 0 = [Z, E(0)]$, hence

$$Z \pi(B) \Omega = \pi(B) E(0) Z E(0) \Omega$$

and therefore

$$(Z - {}_s\pi(A)) \pi(B) \Omega = \pi(B) E(0) (Z - \pi(A)) E(0) \Omega$$

from which the stated facts immediately follow.

Finally we deduce that $\mathfrak{B}_A = \hat{\pi}(\mathfrak{A}_1^T)'$ by showing that this commutant is abelian. Now $\hat{\pi}(\mathfrak{A}_1^T)'$ is isomorphic to $E(0) \hat{\pi}(\mathfrak{A}_1^T)' E(0)$ because $E(0) \in \hat{\pi}(\mathfrak{A}_1^T)''$ and $\Omega \in E(0)\mathfrak{H}$ is cyclic. But

$$E(0) \hat{\pi}(\mathfrak{A}_1^T)' E(0) \subseteq E(0) (E(0) \hat{\pi}(\mathfrak{A}_1^T)' E(0))' E(0) = (E(0) \hat{\pi}(\mathfrak{A}_1^T)' E(0))''$$

where the last equality follows because Ω is cyclic, in $E(0)\mathfrak{H}$, for the abelian algebra $E(0) \hat{\pi}(\mathfrak{A}_1^T)' E(0)$ (see [10] page 89). Hence $\hat{\pi}(\mathfrak{A}_1^T)'$ is isomorphic to an abelian algebra and is thus abelian. (Quite generally this is the case for any von Neumann algebra whose commutant contains an abelian projector with a cyclic vector in its range.)

We are now in a position to derive the main decomposition theorem for invariant states.

Theorem 2. *If Φ is an invariant state over an asymptotically abelian algebra \mathfrak{A} and the associated representation π acts in a separable Hilbert space \mathfrak{H} then there exists a unique decomposition of Φ*

$$\Phi = \int d\mu(\lambda) \Phi_\lambda \quad (43)$$

into invariant weakly clustering states Φ_λ over \mathfrak{A} i.e. for each $A \in \mathfrak{A}$ one has the Radon integral

$$\Phi(A) = \int d\mu(\lambda) \Phi_\lambda(A) , \quad (44)$$

and the states Φ_λ generate representations π_λ which are mutually inequivalent.

Further, if $\hat{\Phi}$ and $\hat{\Phi}_\lambda$ are the canonical extensions of Φ and Φ_λ to \mathfrak{A}_1^T , respectively, then the corresponding decomposition

$$\hat{\Phi}(X) = \int d\mu(\lambda) \hat{\Phi}_\lambda(X), \quad X \in \mathfrak{A}_1^T \tag{45}$$

is the central decomposition of $\hat{\Phi}$ over \mathfrak{A}_1^T and this decomposition is into pure states.

Corresponding to the above decompositions we have the associated decompositions of the representations e.g.

$$\hat{\pi}(X) = \int d\mu(\lambda) \hat{\pi}_\lambda(X), \quad X \in \mathfrak{A}_1^T. \tag{46}$$

Proof. The existence of the decomposition (43) will be deduced from the existence of the decomposition (46). Since $\hat{\pi}$ acts on a separable Hilbert space \mathfrak{H} it has a unique central decomposition which, according to Lemma 5, is a decomposition into irreducibles.

Specifically \mathfrak{H} is the direct ingegral of Hilbert spaces \mathfrak{H}_λ

$$\mathfrak{H} = \int d\mu(\lambda) \mathfrak{H}_\lambda$$

and the operators $B \in \hat{\pi}(\mathfrak{A}_1^T)''$ are the decomposable operators

$$B = \int d\mu(\lambda) B_\lambda \quad B \in \hat{\pi}(\mathfrak{A}_1^T)'' \tag{47}$$

whilst the operators T of the center $\mathfrak{B}_\hat{\pi} = \hat{\pi}(\mathfrak{A}_1^T)'$ are the diagonalizable operators

$$T = \int d\mu(\lambda) T(\lambda), \quad T(\lambda) \in \mathbb{C}, \quad T \in \mathfrak{B}_\hat{\pi}.$$

If we consider (47) for $B = \hat{\pi}(X)$, $X \in \mathfrak{A}_1^T$ we obtain the central decomposition of $\hat{\pi}$ into irreducible representation $\hat{\pi}_\lambda$ of \mathfrak{A}_1^T :

$$\hat{\pi}(X) = \int d\mu(\lambda) \hat{\pi}_\lambda(X) \quad X \in \mathfrak{A}_1^T \tag{48}$$

where $\hat{\pi}_\lambda(X)$ is defined to be equal to $\{\hat{\pi}(X)\}_\lambda$.

On the other hand the cyclic vector $\Omega \in \mathfrak{H}$ associated to the positive form $\hat{\Phi}$ is decomposed as

$$\Omega = \int d\mu(\lambda) \Omega_\lambda \quad \Omega_\lambda \in \mathfrak{H}_\lambda \tag{49}$$

where $\Omega_\lambda \neq 0$ for almost all λ (see [7], 8.8.1). Hence

$$\hat{\Phi}(X) = \int d\mu(\lambda) \hat{\Phi}_\lambda(X) \tag{50}$$

where $\hat{\Phi}_\lambda$ is defined by

$$\hat{\Phi}_\lambda(X) = (\Omega_\lambda | \hat{\pi}_\lambda(X) | \Omega_\lambda)_{\mathfrak{H}_\lambda} \quad X \in \mathfrak{A}_1^T. \tag{51}$$

Now let (π, U) and (π_λ, U_λ) be the covariant representations of $(\mathfrak{A}, \mathfrak{T})$ corresponding to $\hat{\pi}$ and $\hat{\pi}_\lambda$ respectively. We have for $A \in \mathfrak{A}$, $x \in \mathfrak{T}$ and

$X \in \mathfrak{A}_T^T$, (see [1] theorem 3)

$$\pi(A) \hat{\pi}(X) = \pi(\varrho(A) X) = \int d\mu(\lambda) \hat{\pi}_\lambda(\varrho(A) X) = \int d\mu(\lambda) \pi_\lambda(A) \hat{\pi}_\lambda(X)$$

$$U(x) \hat{\pi}(X) = \pi(V(x) X) = \int d\mu(\lambda) \hat{\pi}_\lambda(V(x) X) = \int d\mu(\lambda) U_\lambda(x) \hat{\pi}_\lambda(X)$$

from which, using cyclicity of the vector (49), one deduces

$$\pi(A) = \int d\mu(\lambda) \pi_\lambda(A) \tag{52a}$$

$$U(x) = \int d\mu(\lambda) U_\lambda(x) . \tag{52b}$$

Furthermore, for all $x \in \mathbf{T}$

$$U(x)\Omega = \int d\mu(\lambda) U_\lambda(x)\Omega_\lambda = \Omega = \int \Omega_\lambda d\mu(\lambda)$$

whence, for almost all λ

$$U_\lambda(x)\Omega_\lambda = \Omega_\lambda \text{ for all } x \in \mathbf{T} \tag{53}$$

(this is first proven for all rational x and follows for all x by the strong continuity of $U_\lambda(x)$). This shows that the pure state $\hat{\Phi}_\lambda$ is an invariant state. The decomposition for the state Φ over \mathfrak{A} resulting from (52 a) and (49):

$$\Phi(A) = (\Omega | \pi(A) | \Omega) = \int d\mu(\lambda) \Phi_\lambda(A) \quad A \in \mathfrak{A} \tag{54}$$

where Φ_λ is defined by

$$\Phi_\lambda(A) = (\Omega_\lambda | \pi_\lambda(A) | \Omega_\lambda)_{\mathfrak{B}_\lambda}$$

is therefore a decomposition into invariant states generating covariant representations (π_λ, U_λ) such that the set $\pi_\lambda(\mathfrak{A}) \cup U(\mathbf{T})$ is irreducible. We have shown in [1] theorem 7 (cf. theorem 3 below) that this property is equivalent to weak clustering for those states Φ_λ .

To demonstrate the inequivalence of the π_λ generated by the Φ_λ we first deduce from (32) that

$$E(p_i) \pi(A) E(p_i) \pi(B) E(p_n) = E(p_i) \pi(B) E(p_n) \pi(A) E(p_n) .$$

However, if (π, U) is a covariant representation associated with a weakly clustering state Φ we will prove in Theorem 4 that the projectors $E(p_i), E(p_n)$, onto discrete eigenstates $|l\rangle, |n\rangle$ of $U(x)$, are at most one dimensional. Thus we have

$$\langle l | \pi(A) | l \rangle = \langle n | \pi(A) | n \rangle$$

and hence Φ is the only invariant state in the representation it generates. Thus if $\Phi_{\lambda_1}, \Phi_{\lambda_2}$ generate equivalent representations then $\Phi_{\lambda_1} = \Phi_{\lambda_2}$ which is contrary to the construction for $\lambda_1 \neq \lambda_2$.

The general idea of the uniqueness proof is to reduce a given decomposition into invariant weakly clustering states to a decomposition into pure states over an abelian C^* -algebra. In this latter case the decomposition is known to be unique. The abelian C^* -algebra which we

consider is the smallest norm-closed sub *-algebra of the enveloping von Neumann algebra \mathfrak{A}_1^{T**} containing all operators of the form $E_\tau(0) \tau(\mathfrak{A}) E_\tau(0)$. We will denote this algebra by \mathfrak{E} . It is abelian as a consequence of (33) of Lemma 4.

Let us first define a linear one-to-one mapping of the invariant states Φ over \mathfrak{A} to states ϕ over \mathfrak{E} . This mapping is obtained by first extending Φ over \mathfrak{A} to a $\hat{\Phi}$ over \mathfrak{A}_1^T by the canonical procedure (27), then extending $\hat{\Phi}$ to a weakly continuous state $\bar{\Phi}$ over \mathfrak{A}_1^{T**} and finally taking the restriction ϕ of $\bar{\Phi}$ to the subalgebra \mathfrak{E} . This procedure is well defined because each step is uniquely specified, it is obviously linear and it remains to be shown that it is one-to-one. To demonstrate this property we first recall that $\hat{\Phi}$ is an invariant state over \mathfrak{A}_1^T and as an immediate consequence $\bar{\Phi}$ is such that

$$\bar{\Phi}(U_\tau(x) X U_\tau(-x)) = \bar{\Phi}(X) \quad X \in \mathfrak{A}_1^{T**}$$

thus ϕ will have the property

$$\begin{aligned} \phi(E_\tau(0) \tau(A) E_\tau(0)) &= \bar{\Phi}(E_\tau(0) \tau(A) E_\tau(0)) \\ &= \bar{\Phi}(\tau(A)) = \Phi(A) . \end{aligned} \tag{55}$$

Therefore ϕ determines Φ and the mapping is one-to-one.

Next we demonstrate that the weak clustering property of Φ over \mathfrak{A} is equivalent to the purity of ϕ over \mathfrak{E} . This is established by noting that

$$\begin{aligned} \Phi(A(x) B) - \Phi(A) \Phi(B) &= \phi(E_\tau(0) \tau(A) U_\tau(-x) \tau(B) E_\tau(0)) - \\ &\quad - \phi(E_\tau(0) \tau(A) E_\tau(0)) \phi(E_\tau(0) \tau(B) E_\tau(0)) \end{aligned} \tag{56}$$

and taking means of both sides and using Lemma 1 we see that the weak clustering of Φ is equivalent to the property

$$\begin{aligned} \phi(E_\tau(0) \tau(A) E_\tau(0) \tau(B) E_\tau(0)) \\ = \phi(E_\tau(0) \tau(A) E_\tau(0)) \phi(E_\tau(0) \tau(B) E_\tau(0)) \end{aligned} \tag{57}$$

for ϕ . But this last condition amounts to multiplicativity of ϕ over \mathfrak{E} or, since \mathfrak{E} is abelian, purity of ϕ .

Now let us assume that we are given a decomposition of Φ into weakly clustering states

$$\Phi(A) = \int d\nu(\xi) \Phi_\xi(A) \quad A \in \mathfrak{A} \tag{58}$$

where the integral is a Radon integral taken over some locally compact space. We will prove that this decomposition induces a corresponding decomposition

$$\phi(M) = \int d\nu(\xi) \phi_\xi(M) \quad M \in \mathfrak{E} \tag{59}$$

where the ϕ and ϕ_ξ are obtained from the Φ and Φ_ξ as explained above.

Thus the ϕ_ξ will be pure states over \mathfrak{C} and the uniqueness of decomposition into pure states for an abelian C^* -algebra, together with the fact that the correspondence $\Phi \rightarrow \phi$ is one-to-one will entail uniqueness of the decomposition (58).

The passage from (58) to (59) is effected in the following manner. We first note that due to the definition (27) of the canonical extension we can deduce from (58) the decomposition

$$\Phi(X) = \int d\nu(\xi) \Phi_\xi(X), \quad X \in \mathfrak{A}_1^T \tag{60}$$

or identically,

$$\bar{\Phi}(X) = \int d\nu(\xi) \bar{\Phi}_\xi(X), \quad X \in \mathfrak{A}_1^T \tag{61}$$

where we have used Fubini's theorem to interchange the order of integrations. Now to complete the deduction of (59) we establish that (61) is valid not only for $X \in \mathfrak{A}_1^T$ but also for $M \in \mathfrak{C}$. To demonstrate this, we show that an $M \in \mathfrak{C}$ can be approximated in the weak topology of \mathfrak{A}_1^{T**} by a sequence of elements of \mathfrak{A}_1^T and then we can use Lebesgue's dominated convergence theorem to interchange the integration in (61) with the weak limit. As \mathfrak{C} is the norm closure of linear combinations of the form

$$E_\tau(0) \tau(A_1) E_\tau(0) \tau(A_2) \dots E_\tau(0) \tau(A_n) E_\tau(0), \quad A_i \in \mathfrak{A} \quad i = 1, 2 \dots n$$

it is only necessary to show that such elements may be weakly approximated by sequences. However it follows directly from Lemma 5 that they can even be approximated strongly.

We notice that this uniqueness proof does not depend upon a separability assumption for the representation induced by the state Φ .

Now that we have seen that all invariant states over an asymptotically abelian algebra can be uniquely expressed in terms of invariant weakly clustering states we will denote those latter basic states by the special name *E-states*:

Definition 4. *If \mathfrak{A} is an asymptotically abelian algebra then the invariant weakly clustering states over \mathfrak{A} are called E-states.*

Theorem 1 has already shown that invariant weakly clustering states over a general algebra \mathfrak{A} have a number of very special properties; the assumption of an asymptotically abelian character for \mathfrak{A} increases this number considerably. The next theorem (a generalization of theorem 7 of DKR) gives alternative properties which may be used to characterize E-states.

Theorem 3. *Let Φ be an invariant state over an asymptotically abelian algebra \mathfrak{A} and $\bar{\Phi}$ its canonical extension to \mathfrak{A}_1^T . If (π, U) and $\hat{\pi}$ are the associated representations, on the Hilbert space \mathfrak{H} , of $(\mathfrak{A}, \mathfrak{T})$ and \mathfrak{A}_1^T and Ω is the corresponding invariant cyclic vector then the following are equivalent*

- i) Φ is a weakly clustering state,
- ii) Ω is the only invariant vector in \mathfrak{H} ,

- iii) (π, U) is irreducible i.e. $\hat{\mathcal{A}}$ is irreducible or $\hat{\Phi}$ is pure,
- iv) $\hat{\mathcal{A}}(\mathcal{Q}_1^f)''$ is a factor i.e. $\hat{\Phi}$ is a primary state,
- v) $\hat{\Phi}(\hat{\Phi})$ is an extremal element of the convex set of invariant states over $\mathcal{A}(\mathcal{Q}_1^f)$ i.e. $\hat{\Phi}(\hat{\Phi})$ cannot be further decomposed in terms of invariant states,
- vi) $\lim_{\alpha} \int dx f_{\alpha}(x) (\psi | U(x) - E_{\Omega} | \varphi) = 0, \psi, \varphi \in \mathfrak{H}$,
- vii) $\lim_{\alpha} \| \int dx f_{\alpha}(x) (U(x) - E_{\Omega}) \psi \| = 0, \psi \in \mathfrak{H}$,
- viii) $\lim_{\alpha} \int dx f_{\alpha}(x) (\psi | \pi(A(x)) - (\Omega | \pi(A) | \Omega) | \chi) = 0, \psi, \chi \in \mathfrak{H}, A \in \mathcal{A}$,
- ix) $\lim_{\alpha} \| \int dx f_{\alpha}(x) (\pi(A(x)) - (\Omega | \pi(A) | \Omega)) \psi \| = 0, \psi \in \mathfrak{H}, A \in \mathcal{A}$.

In the above f is any M -filter and E_{Ω} is the projector onto Ω .

Proof. Most of the equivalences cited in the theorem have already been proved in the foregoing. In particular i) \Leftrightarrow ii) \Leftrightarrow vi) \Leftrightarrow vii) \Rightarrow iii) is given by theorem 1, iii) \Rightarrow iv) by definition and iv) \Rightarrow iii) due to Corollary 2 of theorem 2. Further ix) \Rightarrow viii) by definition and that viii) \Rightarrow vi) is easily seen by choosing $\chi = \Omega$ and using the fact that Ω is cyclic for π . Next if $\hat{\Phi}$ is a pure state it is an extremal element of the set of all states and we clearly have iii) \Rightarrow v). We complete the proof of the theorem by demonstrating that v) \Rightarrow iii) \Rightarrow ii) and vii) \Rightarrow ix).

v) \Rightarrow iii). If we assume that $\hat{\mathcal{A}}$ is reducible we can decompose it into irreducibles and there will be a corresponding decomposition of $\hat{\Phi}$. However this latter decomposition will be a decomposition into invariant states because the commutant of $\hat{\mathcal{A}}$ commutes with $U(x)$. Thus we have a contradiction.

iii) \Rightarrow ii). Let us assume that there are many invariant states in \mathfrak{H} and we denote the projector on all invariant states by $E(0)$. However due to lemma 4 $E(0) \pi(\mathcal{A}) E(0)$ is an abelian set and this is in contradiction with the irreducibility of (π, U) unless $E(0)$ is one-dimensional i.e. Ω is unique.

vii) \Rightarrow ix). We have straightforwardly that

$$\begin{aligned} & \| \int dx f_{\alpha}(x) (\pi(A(x)) - (\Omega | \pi(A) | \Omega)) \pi(B) \Omega - \\ & \quad - \int dx f_{\alpha}(x) \pi(B) (U(x) - E_{\Omega}) \pi(A) \Omega \| \qquad (62) \\ & = \| \int dx f_{\alpha}(x) [\pi(A(x)), \pi(B)] \Omega \|. \end{aligned}$$

However in the filter limit the right hand side tends to zero due to (31). Remembering that Ω is cyclic for π and that all operators occurring in (62) are bounded we have immediately the complete equivalence vii) \Leftrightarrow ix).

As we have now reduced the discussion of invariant states to a discussion of E -states it is perhaps at this point appropriate to comment upon the physical significance of these latter states in the light of Theorem 3. We would like particularly to comment upon condition viii)

of this theorem which shows that

$$\lim_{\alpha} \int dx f_{\alpha}(x) (\psi | \pi(A(x)) | \psi) = (\Omega | \pi(A) | \Omega), \quad \|\psi\| = 1 \quad (63)$$

Now if the E -state described an equilibrium situation in statistical mechanics and Ω was the vector representing the ground state of the system in equilibrium then ψ would be a vector representing a state of the system which was locally disturbed from equilibrium. We then see from (63) that the average of all measurements of a local observable A at the various points of space would be a global quantity whose value is independent of the vector state in which it is measured. Further from condition ix) we have

$$\lim_{\alpha} \int dx dy f_{\alpha}(x) f_{\alpha}(y) \{ (\psi | \pi(A(x)) \pi(A(y)) | \psi) - (\psi | \pi(A(x)) | \psi) (\psi | \pi(A(y)) | \psi) \} = 0$$

for $A \in \mathfrak{A}$, $\psi \in \mathfrak{H}$ and $\|\psi\| = 1$. This latter condition expresses the fact that the fluctuations of the values of the space averages of all local observables A are small in all vector states, a characteristic of single-phase equilibrium.

Obviously the essential next step in an analysis of invariant states is to classify the E -states. One classification which appears to be of physical interest is based upon the analysis of the discrete spectrum of the space-translation operators. The necessary mathematical information concerning this spectrum is given by the following theorem.

Theorem 4. *Let Φ be an E -state and (π, U) the associated representations of $(\mathfrak{A}, \mathfrak{T})$. The group representatives U have the general spectral decomposition*

$$U(x) = \sum_{p_n \in \mathfrak{S}_D} E(p_n) e^{ip_n x} + \int_{\mathfrak{S}_C} dE(p) e^{ipx} \quad (64)$$

where the sum and integral correspond to the discrete spectrum \mathfrak{S}_D and the continuous spectrum \mathfrak{S}_C of U respectively. It follows that the discrete spectrum is subgroup i.e. if $p_n, p_m \in \mathfrak{S}_D$ then $p_n + p_m \in \mathfrak{S}_D$ and $-p_n \in \mathfrak{S}_D$ and further it is simple i.e. for each $p_n \in \mathfrak{S}_D$ the corresponding projector $E(p_n)$ is one-dimensional.

Proof. We have already seen that in an E -state (π, U) is irreducible but from lemma 4 we know that $E(p_n) \pi(\mathfrak{A}) E(p_n)$ is abelian for $p_n \in \mathfrak{S}_D$. Therefore we have a contradiction unless $E(p_n)$ is one-dimensional. This proves the simplicity of \mathfrak{S}_D . Next we note that the eigenstate of $E(p_n)$ is cyclic for (π, U) , due to irreducibility, and hence cyclic for π . Now from (32) of lemma 4 we have, in particular, that

$$E(p_n) \pi(A) E(p_n + p_m) \pi(B) E(p_m) = E(p_n) \pi(B) E(0) \pi(A) E(p_m). \quad (65)$$

For all $A, B \in \mathfrak{A}$. Now if we assume that $p_n, p_m \in \mathfrak{S}_D$ but $p_n + p_m \notin \mathfrak{S}_D$ we

must conclude that

$$E(p_n) \pi(B) E(0) \pi(A) E(p_m) = 0 \tag{66}$$

for all $A, B \in \mathfrak{A}$. But as $E(p_n), E(p_m)$ project onto states cyclic under π this would entail $E(0) = 0$ which is a contradiction because $E(0)$ projects onto the invariant vector Ω associated with Φ . This proves the additivity of S_D . The symmetry of S_D follows similarly from

$$E(0) \pi(A) E(p_n) \pi(B) E(0) = E(0) \pi(B) E(-p_n) \pi(A) E(0) \tag{67}$$

which is a special case of (32).

The result of the above theorem provides an immediate and natural classification of E -states for the case $\mathbf{T} = \mathbf{R}^n$.

Definition 5. *If Φ is an E -state and Ω is the associated invariant vector and U the corresponding representation of \mathbf{T} then*

1. Φ is called an E_I -state if Ω is the only discrete eigenvector of U .
2. Φ is called an E_{II}^m -state if the discrete spectrum S_D of U spans an m -dimensional subspace of \mathbf{R}^n and if there exists a minimum nonzero distance between all pairs of points of S_D .
3. Φ is called an E_{III} -state if it does not fall into the two foregoing categories.

As S_D is additive the definition of E_{II} -states is equivalent to the requirement that S_D contains no accumulation points. On the other hand in an E_{III} -state accumulation points of S_D will occur and due to the additivity property each point in some subset of S_D will be an accumulation point. Thus in E_{III} -state S_D will be dense in some subspace of \mathbf{R}^n that it spans.

At this point we end our general analysis of E_I -states and in the following section we will analyse E_{II} -states. We are unable to analyse E_{III} -states and will not consider their structure further. In section 6 we return a discussion of possible properties of E_I -states.

5. Lattice states

The aim of this section is to present an analysis of E_{II} -states. The major feature of our results is that an arbitrary E_{II} -state can be uniquely decomposed into states which are invariant under a subgroup of translations and which are characterised by a number of equivalent conditions similar to those characterising E_I -states. The methods which we use closely parallel those of previous sections; our general tactics are the following. For each E_{II} -state we define a certain subgroup T_L of the group \mathbf{T} (L for lattice). This subgroup consists of all translations which leave the corresponding discrete eigenvectors of \mathbf{T} invariant. T_L acts as a group of automorphisms on the algebra \mathfrak{A} and the E_{II} -state is an invariant state with respect to this group. Now we apply to the pair

$(\mathcal{Q}, \mathbf{T}_L)$, or equivalently to $\mathcal{Q}_1^{\mathbf{T}_L}$ our previous results. As we mentioned earlier, these results are valid for any locally compact abelian group and are thus applicable to \mathbf{T}_L with appropriate notational changes.

We begin by giving abstract definitions of \mathbf{T}_L and related concepts and then illustrating these concepts by examples. Now the discrete spectrum S_D of $U(x)$ in an E_{Π} -state is, according to the definition, a discrete closed sub-group of the dual group $\tilde{\mathbf{T}}$ of \mathbf{T} ($\tilde{\mathbf{T}} \equiv p$ -space). We next define the closed subgroup \mathbf{T}_L of \mathbf{T} by the requirement that

$$p_n x = 0 \pmod{2\pi}, \quad p_n \in S_D. \tag{68}$$

This amounts to our prescription of selecting all translations which leave invariant the discrete eigenvectors of $U(x)$. The subgroup \mathbf{T}_L is commonly referred to in the mathematical literature as the annihilator of $S_D \subset \tilde{\mathbf{T}}$ (see [11] chapter 2,2-1). We next introduce the quotient group $\mathbf{K} = \mathbf{T}/\mathbf{T}_L$. \mathbf{K} is the group of translations of \mathbf{T} modulo translations of \mathbf{T}_L . Generally \mathbf{T}_L and \mathbf{K} can be identified with the dual groups of $\tilde{\mathbf{T}}/S_D$ and S_D respectively ([11] theorem 2.1.2) and thus, since S_D is discrete, \mathbf{K} is compact. The subgroup \mathbf{T}_L will be discrete if, and only if, $\tilde{\mathbf{T}}/S_D$ is compact i.e. if S_D spans $\tilde{\mathbf{T}}$ as a vector space, which is the case of E_{Π}^n -states. In order to apply our previous results to the group \mathbf{T}_L we must describe the Haar measure on \mathbf{T}_L . We obtain the simplest description for an E_{Π}^m -state in the following way. Let us call $\tilde{\mathbf{T}}^m$ the m -dimensional vectorial subspace spanned by S_D in $\tilde{\mathbf{T}}$ and let us choose a complementary subspace $\tilde{\mathbf{T}}^{n-m}$, then $\tilde{\mathbf{T}}$ is the product group $\tilde{\mathbf{T}}^m \times \tilde{\mathbf{T}}^{n-m}$. Consequently

$$\tilde{\mathbf{T}}/S_D = \tilde{\mathbf{T}}^m/S_D \times \tilde{\mathbf{T}}^{n-m} \tag{69}$$

where $\tilde{\mathbf{T}}^m/S_D$ is compact and $\tilde{\mathbf{T}}^{n-m}$ is isomorphic to \mathbf{R}^{n-m} . Thus the dual group \mathbf{T}_L of $\tilde{\mathbf{T}}/S_D$ is equal to the product of the duals,

$$\mathbf{T}_L = \mathbf{T}_L^m \times \mathbf{T}^{n-m} \tag{70}$$

where \mathbf{T}_L^m is a discrete group and \mathbf{T}^{n-m} is again isomorphic to \mathbf{R}^{n-m} (\mathbf{T}_L^m and S_D are "reciprocal lattices"). Thus the Haar measure on \mathbf{T}_L is the product of a discrete sum over \mathbf{T}_L^m and the Lebesgue measure on \mathbf{T}^{n-m} . In the event that $m = n$ (E_{Π}^n -states) \mathbf{T}_L is discrete and the Haar measure is given by discrete sums.

Let us illustrate the above definitions in the case of two dimensions where we have the possibility of E_{Π}^1 -states or E_{Π}^2 -states. Firstly we consider an E_{Π}^1 -state where, by definition, the discrete spectrum S_D of U consists of points equally spaced on a line. Choosing cartesian coordinates (x_1, x_2) in $\mathbf{T} = \mathbf{R}^2$ and the corresponding coordinates (p_1, p_2) in the dual space $\tilde{\mathbf{T}}$ we can assume, without loss of generality, that the

spectrum S_D consists of the points $(a_1 N, 0)$ where a_1 is a fixed scale factor and N takes integer values; in this case T_L^1 is the discrete group of translations through distances $2\pi M/a_1$ where M takes integer values, in the x_1 -direction. The group T_L consists of translations whose component in the x_1 -direction is an integer multiple of $2\pi/a_1$, and whose component in the x_2 -direction is arbitrary. The quotient group K consists of translations in the x_1 -direction modulo the translations of T_L^1 and is isomorphic to the rotations (modulo 2π) on a circle. A general function f on T_L can be viewed as a sequence of functions $f_M(x_2)$ and the integral of this function with respect to the Haar measure on T_L is given by

$$\sum_{M=-\infty}^{\infty} \int dx_2 f_M(x_2). \tag{71}$$

Secondly if we consider an E_{II}^2 -state T_L will consist of translations of the type $(2\pi M_1/a_1, 2\pi M_2/a_2)$ where a_1 and a_2 are both fixed by the spectrum S_D and where M_1, M_2 , take integer values. In this case K is isomorphic to the product of two groups both describing rotations (modulo 2π) on a circle. A function f on T_L is then a double array of constants f_{M_1, M_2} and the integral of the function with respect to the Haar measure is the double sum

$$\sum_{M_1=-\infty}^{\infty} \sum_{M_2=-\infty}^{\infty} f_{M_1, M_2}. \tag{72}$$

Now we proceed to the analysis of E_{II} -states and we begin by introducing a number of notational definitions which will be used in the remainder of this section.

Definitions. Let Φ be an E_{II} -state and (π, U) the associated representations of (\mathfrak{A}, T) on the space \mathfrak{H} with the cyclic invariant vector Ω . We denote by U_L the restriction of U to the subgroup T_L of T and by E the projector in \mathfrak{H} on states invariant under T_L so that

$$U_L(x_L) = E + \int dE_L(p) e^{ipx_L} \quad x_L \in T_L \tag{73}$$

and

$$E = \sum_{p_n \in S_D} E(p_n). \tag{74}$$

On the space $E\mathfrak{H}$ we define the algebra \mathfrak{M} to be the C^* -algebra generated by $E\pi(\mathfrak{A})E$ and we denote by U_E the restriction of U to $E\mathfrak{H}$. We note that for $x_L \in T_L$ the restriction U_E reduces to the identity (in $E\mathfrak{H}$) so that U_E is in fact a representation of the compact quotient group $K = T/T_L$.

With the above notation we now have

Lemma 6. *The C^* -algebra \mathfrak{M} is abelian and countably generated. Further, the set of operators \mathfrak{M} and $U_E(K)$ is irreducible in $E\mathfrak{H}$ and Ω is cyclic for \mathfrak{M} .*

Proof. As lemma 4 is valid for any locally compact abelian group we obtain immediately, by replacing \mathbf{T} by \mathbf{T}_L in that lemma, the fact that \mathfrak{M} is abelian. As a consequence of (74) \mathfrak{M} is generated by the $E(p_n) \pi(\mathfrak{A}) E(p_m)$ but as $E(p_n)$ is one-dimensional for all p_n we have that \mathfrak{M} is countably generated. Finally since the set of operators $\pi(\mathfrak{A})$ and U are, by theorem 3, irreducible in \mathfrak{H} and Ω is cyclic for $\pi(\mathfrak{A})$ it follows that \mathfrak{M} and $U_E(\mathbf{K})$ are irreducible in $E\mathfrak{H}$ and Ω is cyclic for \mathfrak{M} .

After these preliminaries we now wish to give the major result of this section concerning the decomposition of $E_{\mathbf{II}}$ -states. It is a consequence of our earlier result, theorem 2, that there is a unique decomposition of an $E_{\mathbf{II}}$ -state into states which are invariant with respect to \mathbf{T}_L and weakly clustering with respect to this subgroup. As one would expect this decomposition ‘breaks’ the symmetry with respect to the group \mathbf{K} and this is explicitly displayed by

Theorem 5. *If Φ is an $E_{\mathbf{II}}$ -state then there is a unique decomposition of Φ into states Φ_ξ which are invariant and weakly clustering with respect to the subgroup \mathbf{T}_L . Furthermore this decomposition can be written as an integral with respect to the Haar measure $dm(\xi)$ of \mathbf{K} and one has*

$$\Phi = \int_{\mathbf{K}} dm(\xi) \Phi_\xi \tag{75}$$

in such a way that the states Φ_ξ are transforms of each other under the action of \mathbf{K} i.e.

$$\Phi_{\xi+\eta}(A) = \Phi_\xi(A(y)), \quad y \in \eta. \tag{76}$$

The representations π and π_ξ associated with Φ and Φ_ξ , respectively, are simultaneously faithful, but are inequivalent for $\xi \neq \eta$. However, the representations of \mathbf{T}_L associated with the states Φ_ξ are all unitary equivalent.

Corresponding to the decomposition (75) there is a decomposition of the canonical extension $\hat{\Phi}$ of Φ ;

$$\hat{\Phi} = \int_{\mathbf{K}} dm(\xi) \hat{\Phi}_\xi \tag{77}$$

where

$$\hat{\Phi}_{\xi+\eta}(X) = \hat{\Phi}_\xi(X(y)), \quad y \in \eta \tag{78}$$

and the decomposition (77) is the central decomposition of $\hat{\Phi}$ over $\mathfrak{A}_1^{T_L}$ which is into pure states. The representations $\hat{\pi}$ and $\hat{\pi}_\xi$ of $\mathfrak{A}_1^{T_L}$ generated by $\hat{\Phi}$ and $\hat{\Phi}_\xi$ are simultaneously faithful.

Proof. Before giving details we first outline the main features of the proof. Firstly we know from theorem 2, on replacing \mathbf{T} by \mathbf{T}_L , that there exists a unique decomposition of Φ into states which are weakly clustering and invariant with respect to \mathbf{T}_L . We will construct this decomposition by the ‘‘diagonalization’’ of the abelian algebra \mathfrak{M} acting in $E\mathfrak{H}$ (a slight variant of the construction in the proof of theorem 2). Secondly we demonstrate that this decomposition can be rewritten in the form (75)

by using the fact that the set of operators $\mathfrak{M} \cup U_E$ is irreducible in $E\mathfrak{H}$ and that \mathbf{T} acts on \mathfrak{M} in the covariant manner

$$U_E(x) E\pi(A) E U_E(x)^{-1} = E\pi(A(x))E. \tag{79}$$

These facts, together with the remark that \mathbf{T}_L is characterised as the set of $x \in \mathbf{T}$ acting trivially on \mathfrak{M} , allow us to establish an identification between the quotient group $\mathbf{K} = \mathbf{T}/\mathbf{T}_L$ and the spectrum \mathfrak{S} of \mathfrak{M} . We now proceed to the details.

We recall that the abelian C^* -algebra \mathfrak{M} can be identified with the set $\mathbf{C}_0(\mathfrak{S})$ of continuous functions on \mathfrak{S} vanishing at infinity, through the well known Gelfand isomorphism defined by

$$\widehat{M}(s) = s(M) \tag{80}$$

where $M \in \mathfrak{M}$, $s \in \mathfrak{S}$ and $\widehat{M} \in \mathbf{C}_0(\mathfrak{S})$. The positive form on \mathfrak{M} defined by $\Omega \in E\mathfrak{H}$ is then represented by a Radon measure m on \mathfrak{S} i.e.

$$(\Omega | M | \Omega) = \int dm(s) \widehat{M}(s). \tag{81}$$

Since Ω is cyclic for \mathfrak{M} in $E\mathfrak{H}$ the latter can be identified with the space $\mathbf{L}_2(\mathfrak{S}, m)$ in such a way that if

$$\psi \in E\mathfrak{H} \leftrightarrow \widehat{\psi} \in \mathbf{L}_2(\mathfrak{S}, m) \tag{82}$$

we have a “diagonalisation” of \mathfrak{M} in the sense that (see [12] § 17.4)

$$\widehat{M\psi}(s) = \widehat{M}(s) \widehat{\psi}(s), \quad \psi \in \mathfrak{H}, M \in \mathfrak{M}. \tag{83}$$

This diagonalisation effects the decomposition of Φ ,

$$\Phi(A) = (\Omega | E\pi(A)E | \Omega) = \int_{\mathfrak{S}} dm(s) \Phi_s(A) \tag{84}$$

where the states Φ_s are defined by

$$\Phi_s(A) = s(E\pi(A)E) = \widehat{E\pi(A)E}(s) \tag{85}$$

which implies that they are invariant and weakly clustering with respect to \mathbf{T}_L through an argument identical to that given in the proof of theorem 2.

We next wish to examine the action of $U_E(x)$, $x \in \mathbf{T}$, on $\mathbf{L}_2(\mathfrak{S}, m) = E\mathfrak{H}$. From (79) we see that \mathbf{T} acts on \mathfrak{M} as a group of automorphisms

$$M \in \mathfrak{M} \rightarrow M_x \in \mathfrak{M} \tag{86}$$

of the algebra \mathfrak{M} . We define the action of \mathbf{T} on the spectrum \mathfrak{S} by

$$s \in \mathfrak{S} \rightarrow s_x \in \mathfrak{S} \tag{87}$$

with

$$s_x(M) = s(M_{-x}) \tag{88}$$

so that

$$\widehat{M}_x(s) = \widehat{M}(s_{-x}). \tag{89}$$

From this we easily deduce the action of $U_E(x)$ on $L_2(\mathfrak{S}, m)$ and we have

$$\widehat{U_E(x) \psi} = \widehat{\psi}(s_{-x}) \tag{90}$$

in accordance with (79). The fact that $U_E(x)$ is unitary reflects itself in the property that the measure m is invariant for the mapping (87). Note that the mapping (87) is such that $(s, x) \rightarrow s_x$ is continuous form $\mathfrak{S} \times \mathbf{T}$ to \mathfrak{S} . Indeed we have for arbitrary $M_i \in \mathfrak{M} \ i = 1 \dots n$

$$\begin{aligned} |s_x(M_i) - s'_{x'}(M_i)| &\leq |s_x(M_i) - s'_x(M_i)| + |s'_x(M_i) - s'_{x'}(M_i)| \leq \\ &\leq |s(M_{i-x}) - s'(M_{i-x})| + \|M_{i-x} - M_{i-x'}\| \end{aligned} \tag{91}$$

where

$$|s(M_{i-x}) - s'(M_{i-x})| \leq \frac{\epsilon}{2} \quad i = 1 \dots n \tag{92}$$

defines a weak neighbourhood of s in \mathfrak{S} and

$$\|M_{i-x} - M_{i-x'}\| \leq \frac{\epsilon}{2} \tag{93}$$

for $x - x'$ small enough by our assumption (20).

Now let u be an element of \mathfrak{S} and consider the orbit \mathcal{O}_u of u under \mathbf{T} defined as the set

$$\mathcal{O}_u = \{u_x; x \in T\}. \tag{94}$$

\mathcal{O}_u is a Borel subset of \mathfrak{S} and, as a consequence of (83) and (90), $L_2(\mathcal{O}_u, m)$ is a subspace of $E\mathfrak{H}$ invariant under the combined action of \mathfrak{M} and U_E . Thus as the set of operators $\mathfrak{M} \cup U_E$ is irreducible $L_2(\mathcal{O}_u, m)$ is either the null subspace or the whole $L_2(\mathfrak{S}, m)$ space and hence in the latter case the complementary set of \mathcal{O}_u is of zero m -measure. Irreducibility implies that the invariant measure m is ergodic and on the other hand \mathfrak{M} is countably generated i.e. separable and consequently the spectrum \mathfrak{S} admits a countable basis of open sets. Therefore ergodicity of m implies transitivity (see [13], Theorem 3.9) and thus we can choose a $u \in \mathfrak{S}$ such that \mathcal{O}_u has an m -negligeable complement. Thus the integral in (87) can be equivalently taken over \mathcal{O}_u i.e.

$$\Phi(A) = \int_{\mathcal{O}_u} dm(s) \Phi_s(A) = \int_{\mathcal{O}_u} dm(s) s(E\pi(A)E). \tag{95}$$

As the action of \mathbf{T} on \mathcal{O}_u is now strictly transitive we can, due to the continuity of the mapping $(s, x) \rightarrow s_x$, conclude that \mathcal{O}_u is a homogeneous space for \mathbf{T} . Consequently, since \mathbf{T} is a separable group and \mathbf{T}_L is the stabiliser of the points in \mathcal{O}_u i.e.

$$\mathbf{T}_L = \{x \in T; s_x = s\}, \quad s \in \mathcal{O}_u \tag{96}$$

we have a homeomorphic mapping of \mathcal{O}_u onto the quotient group $\mathbf{K} = \mathbf{T}/\mathbf{T}_L$. Explicitly, we have

$$\xi \in \mathbf{K} \leftrightarrow u_\xi \in \mathcal{O}_u \tag{97}$$

where u_ξ is defined by

$$u_\xi = u_x \quad \text{with } x \text{ in the class } \xi \text{ of } \mathbf{T} \text{ modulo } \mathbf{T}_L. \tag{98}$$

Furthermore the mapping is such that for $\eta, \xi \in \mathbf{K}$

$$\eta + \xi \in \mathbf{K} \leftrightarrow u_{\eta+\xi} \in \mathcal{O}_u \tag{99}$$

and the image of the measure m on \mathcal{O}_u is the Haar measure (which we also denote by m) on \mathbf{K} because the Haar measure is the only quasi-invariant measure on \mathbf{K} .

We can now rewrite the decomposition (95) replacing \mathcal{O}_u by the isomorphic \mathbf{K}

$$\Phi(A) = \int_{\mathbf{K}} dm(\eta) \eta(E\pi(A)E). \tag{100}$$

Now using the invariance of the Haar measure and taking account of (97) (99) we have

$$\begin{aligned} \Phi(A) &= \int_{\mathbf{K}} dm(\xi) (\xi + \eta) (E\pi(A)E) \\ &= \int_{\mathbf{K}} dm(\xi) \xi(E\pi(A(y)E), \quad y \in \eta \\ &= \int_{\mathbf{K}} dm(\xi) \Phi_\xi(A) \end{aligned} \tag{101}$$

where the states Φ_ξ defined by

$$\Phi_\xi(A) = \xi(E\pi(A(y)E), \quad y \in \eta \tag{102}$$

obviously satisfy (76).

The above proof invokes general theorems of the theory of ergodic measures for the replacement of \mathcal{S} by \mathbf{K} in the integral (84). In fact we can use the compactness of $\mathbf{K} = \mathbf{T}/\mathbf{T}_L$ to deduce in an elementary way the even stronger result that there exists a $u \in \mathcal{S}$ for which \mathcal{O}_u is homeomorphic to \mathbf{K} (without a zero-measure difference set). Since \mathcal{S} has a countable basis of open sets it is known that one can easily prove the existence of a $u \in \mathcal{S}$ such that the closure $\bar{\mathcal{O}}_u$ of \mathcal{O}_u is the support of the ergodic measure m , i.e. coincides with the whole spectrum \mathcal{S} since \mathfrak{M} is by definition faithfully represented in $\mathbf{L}_2(\mathcal{S}, m)$. The mapping of \mathbf{K} onto \mathcal{O}_u defined by (97) and (98) stems from the continuous mapping

$$x \in \mathbf{T} \rightarrow u_x \in \mathcal{O}_u$$

constant on the classes modulo \mathbf{T}_L : it is therefore continuous from the compact space \mathbf{K} to the Hausdorff space \mathcal{S} and since it is one-to-one it is homeomorphic. We thus conclude that $\mathcal{O}_u = \bar{\mathcal{O}}_u = \mathcal{S}$.

We now prove the rest of the theorem. Now that we have established (76) it is a direct consequence that π_ξ and π_η are simultaneously faithful. More generally (theorem 1.7 [14]) we have that

$$\text{Ker } \Phi = \bigcap_{\xi \in \mathbf{K}} \text{Ker } \Phi_\xi. \tag{103}$$

The inequivalence of π_ξ and π_η follows as in theorem 2. To demonstrate the unitary equivalence of the representations of \mathbf{T}_L associated with $\Phi_{\xi+\eta}$ and Φ_η we first note that the mapping

$$\pi_{\xi+\eta}(A) \Omega_{\xi+\eta} \in \mathfrak{H}_{\xi+\eta} \rightarrow \pi_\xi(A(y)) \Omega_\xi \in \mathfrak{H}_\xi \quad y \in \eta \tag{104}$$

extends to an isometric linear mapping of $\mathfrak{H}_{\xi+\eta}$ onto \mathfrak{H}_ξ due to (76). The same equation also shows that

$$\begin{aligned} (\pi_{\xi+\eta}(A) \Omega_{\xi+\eta}, U_{\xi+\eta}(x_L) \pi_{\xi+\eta}(B) \Omega_{\xi+\eta}) \\ = (\pi_\xi(A(y)) \Omega_\xi, U_\xi(x_L) \pi_\xi(B(y)) \Omega_\xi), \quad y \in \eta \end{aligned}$$

which demonstrates the unitary equivalence.

The properties of the decomposition of the canonical extension $\hat{\Phi}$ of Φ to $\mathfrak{A}_1^{T_L}$ follow immediately from theorem 2 and the above. Theorem 5 has shown that the representations U_ξ of \mathbf{T}_L associated with the states Φ_ξ occurring in the decomposition (75) are unitarily equivalent and we can also deduce the result,

Theorem 6. *If Φ_ξ is a state occurring in the decomposition (75) of an $E_{\mathbb{I}\mathbb{I}}$ -state and U_ξ is the associated representation of \mathbf{T}_L on \mathfrak{H}_ξ then U_ξ has one, and only one, discrete eigenvector in \mathfrak{H}_ξ . This unique eigenvector is the invariant vector Ω_ξ associated with Φ_ξ .*

Proof. Instead of considering the Φ_ξ we consider the canonical extensions over $\mathfrak{A}_1^{T_L}$ or rather their continuous extensions to the enveloping C^* -algebra $\mathfrak{A}_1^{\overline{T_L}}$ (see DKR Section 4). As $\hat{\Phi}_\xi$ is a pure state Kadison's transitivity theorem ([7] 2.8.4) assures us that any discrete eigenstate ψ of $U_\xi(x_L)$ in \mathfrak{H}_ξ with eigenvalue e^{iqx_L} can be written

$$\psi = \hat{\mathcal{A}}_\xi(X) \Omega_\xi \tag{105}$$

for some $\eta \in \overline{\mathfrak{A}_1^{T_L}}$. Therefore we have

$$\hat{\Phi}_\xi(X^* X(x_L)) = \hat{\Phi}_\xi(X^* X) e^{iqx_L}. \tag{106}$$

But from (78) we conclude that

$$\begin{aligned} \hat{\Phi}_{\xi+\eta}(X^*(-y_\eta) X(-y_\eta + x_L)) \\ = \hat{\Phi}_{\xi+\eta}(X^*(-y_\eta) X(-y_\eta)) e^{iqx_L}, \quad y_\eta \in \mathfrak{H}_\eta. \end{aligned} \tag{107}$$

This last relation shows that the vector

$$\int_K^\oplus d m(\eta) \hat{\mathcal{A}}_{\xi+\eta}(X(-y_\eta)) \Omega_{\xi+\eta} \tag{108}$$

(where the choice of $y_\eta \in \mathfrak{H}_\eta$ is a continuous section) in the direct integral space

$$\int_K^\oplus d m(\eta) \mathfrak{H}_{\xi+\eta} \tag{109}$$

is a discrete eigenstate of $U_L(x_L)$ with eigenvalue e^{iqx_L} and we therefore

conclude from (73) that $q = 0$. Thus all discrete eigenvectors of U_ξ are invariant vectors but as Φ_ξ is weakly clustering it follows from theorem 1 (applied with \mathbf{T}_L) that they all coincide with Ω_ξ .

We note that the use of direct integrals in the above proof requires separability of the representation π generated by the E_{II} -state Φ , however this restriction can probably be removed.

The study of E_{II} -states has been reduced at this point to the study of states occurring in the decomposition (75) and from theorems 4 and 5 we see that these states have properties analogous to E -states. We will give them the special name L -states.

Definition 6. *If \mathfrak{A} is an asymptotically abelian algebra and \mathbf{T}_L is a closed subgroup of the group \mathbf{T} of space-translations then we call a state Φ over \mathfrak{A} an L -state if Φ is invariant with respect to \mathbf{T}_L and if the associated invariant vector Ω_L is the only discrete eigenstate of the corresponding representation U_L of \mathbf{T}_L .*

In analogy to the definition of E_I -states we have in the above definition brought forward the fact that the discrete spectrum of U_L is a point and the corresponding eigenvector unique. This implies, due to theorem 1, that L -states are weakly clustering and in fact they enjoy all of the properties listed in Theorem 4.

Theorem 7. *If Φ is an L -state then it has the equivalent properties i)—ix) of theorem 3 with the proviso that \mathbf{T} should be replaced by \mathbf{T}_L in the statement of these properties. (The replacement of \mathbf{T} by \mathbf{T}_L naturally entails replacing dx by dx_L , the Haar measure on \mathbf{T}_L , and the interpretation of f as an M -filter over \mathbf{T}_L as described above.)*

Proof. The proof of this theorem follows, in the same manner that its statement does, by replacing \mathbf{T} by \mathbf{T}_L in theorem 3 and the appropriate foregoing lemmas.

The result of this, and the previous, section have reduced the study of invariant states to the analysis of the very similar E_I and L -states together with an examination of E_{III} -states. In the following section we give a short discussion of properties, and conjectured properties, of the former states.

6. E_I -states and L -states

As E_I -states and L -states are so similar we will only discuss the former; to apply our discussion to the latter it is merely necessary to replace \mathbf{T} by \mathbf{T}_L . In theorem 3 we showed that the weak and strong means of representatives $\pi(A(x))$ of \mathfrak{A} in an E -state Φ are equal to $\Phi(A)I$ and our first remark is that the product means of $\pi(A_1(x_1)) \dots \pi(A_n(x_n))$ also exist in the weak and strong sense and are equal to $\Phi(A_1) \dots \Phi(A_n)I$. This follows straightforwardly from lemma 3 which also precisely states what we understand by product means. Our next remark is that an

E_1 -state, which is by definition weakly clustering, is possibly already strongly clustering. The representation U of \mathbf{T} associated with such a state has the form

$$U(x) = E_\Omega + \int_{s_L} dE(p) e^{i p x} \tag{110}$$

so the strong clustering property (28) is equivalent to the condition

$$\int_{s_L} d(\psi | E(p) | \varphi) e^{i p x} \xrightarrow{|x|=\infty} 0 \tag{111}$$

on the continuous measure $d(\psi | E(p) | \varphi)$ where ψ, φ are vectors in the representation space \mathfrak{H} . If this condition is not automatically fulfilled it should be a consequence of taking stronger ‘local commutativity’ conditions than (31) for the algebra \mathfrak{A} . The strong clustering property would be deducible if the representation $\pi(\mathfrak{A})$, associated with an E_1 -state, were a factor representation as the next theorem (essentially due to ARAKI [15] and BORCHERS [16]) shows.

Theorem 8. *If Φ is a state (not necessarily invariant) over an asymptotically abelian algebra \mathfrak{A} and if the associated representation π of \mathfrak{A} on \mathfrak{H} , is factorial then*

$$|\Phi(A(x)B) - \Phi(A(x))\Phi(B)| \xrightarrow{|x|=\infty} 0. \tag{112}$$

Proof. Firstly we note that Φ and π are connected by

$$\Phi(A) = (\xi | \pi(A) | \xi), \quad A \in \mathfrak{A} \tag{113}$$

where $\xi \in \mathfrak{H}$ is cyclic for π . Next we define \mathfrak{L} to be the C^* -algebra of operators generated by π and its commutant π' . With this definition we have firstly that

$$\| [B, \pi(A(x))] \| \xrightarrow{|x|=\infty} 0 \tag{114}$$

for each $A \in \mathfrak{A}$ and $B \in \mathfrak{L}$ due to asymptotical abelianness and secondly \mathfrak{L} acts irreducibly in \mathfrak{H} due to π being factorial i.e. $\{\pi'' \cap \pi'\} = \{\lambda I\}$.

However, due to this irreducibility we can use Kadison’s transitivity theorem (see [7], 2.9.1) to conclude that every $B \in \mathfrak{L}$ has a decomposition

$$B = (\xi | B | \xi) + B_1 + B_2 \tag{115}$$

such that

$$B_1^* \xi = 0, \quad B_2 \xi = 0 \quad \text{and} \quad B_1, B_2 \in \mathfrak{L}. \tag{116}$$

In particular for $A, B \in \mathfrak{A}$

$$\begin{aligned} \Phi(A(x)B) - \Phi(A(x))\Phi(B) &= (\xi | \pi(A(x)) B_1 | \xi) \\ &= -(\xi | [\pi(A(x)), B_1] | \xi), \quad B_1 \in \mathfrak{L}. \end{aligned} \tag{117}$$

Now (112) is an immediate consequence of (117) and (114).

We note that in the above theorem it is not necessary that there should be a unitary representation of \mathbf{T} associated with Φ . It is also

interesting to note that the decomposition (115) is an abstract mathematical version of the splitting into “creation” and “annihilation” operators.

Let us now show that the converse of the above theorem is not true i.e. strong clustering does not imply that π is factorial, and thus deduce that an E_1 -state is not necessarily a primary state over \mathfrak{A} . To demonstrate this we will consider an abelian algebra \mathfrak{A} generated by unitary operators $U(f)$, acting on the Fock space \mathfrak{H}_F with vacuum Ω , where

$$U(f) = \exp\{i(a(f) + a^+(f))\} \quad f \text{ real} \in \mathbf{L}_2(\mathbf{T})$$

and

$$[a(g), a^+(h)] = \int dx dy \overline{g(x)} h(y) \varrho(x - y) \quad g, h \in \mathbf{L}_2(\mathbf{T}).$$

For consistency we take ϱ to be real and of positive type, now we have

$$(\Omega | U(f) U(g_z) | \Omega) = \exp\left\{-\frac{1}{2} \int dx dy (f(x) + g(x - z)) \times \right. \\ \left. \times (f(y) + g(y - z)) \varrho(x - y)\right\} \quad (118)$$

where the action of \mathbf{T} is

$$g \rightarrow g_z; \quad g_z(x) = g(x - z).$$

It is straightforwardly checked from (118), and the definition of \mathfrak{A} that, the state over \mathfrak{A} is strongly clustering if we assume that

$$p(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

However (118) also shows that this state is not a primary state over \mathfrak{A} because as \mathfrak{A} is abelian a primary state is pure and hence multiplicative over the algebra.

We conclude from the above observations that in our general framework of asymptotic abelian algebras, which also includes abelian algebras, E_1 -states are not necessarily primary. However it is possible that they are strongly clustering.

7. Conclusion

In the present paper we have focused our attention on invariant states over an asymptotically abelian algebra as these states are of interest in statistical mechanics. We have shown how one is led to a study of particular “elementary” states and how states of a lower “crystalline” symmetry naturally arise through the breaking of translation symmetry.

The methods which we have developed have more general applications than those considered and it would be possible to analyse invariant states when the group \mathbf{T} is replaced by a more general locally compact group \mathbf{G} . The decomposition theorem (theorem 2) hinges essentially on two results only 1) the weak mean of $U(x)$ is the projector $E(0)$, and

2) the set $E(0) \pi(\cdot) E(0)$ is abelian. The first of these results is dependent only on the existence of M -filters and the second then depends on having an appropriate asymptotic abelian condition. It is worth mentioning that even in the context of the present paper it is not necessary to have the asymptotic abelian property for all directions to obtain most of our results. If one had this property in all space-like directions, as in relativistic field theory, it would also be possible to obtain similar results.

A final remark is that the methods of mean values developed here might be used in relativistic field theory to replace the methods of big translations as has also been suggested privately by R. HAAG.

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Note added in proof. The proof of the assertion in Lemma 5 that the strong means of elements of \mathfrak{A} are weakly dense in $\mathfrak{B}_{\mathfrak{A}}$ is not satisfactory. However this assertion is correct as will be shown at a later occasion.

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