Pacific Journal of Mathematics

INVARIANT THEORY FOR THE FREE LEFT-REGULAR BAND AND A Q-ANALOGUE

SARAH BRAUNER, PATRICIA COMMINS AND VICTOR REINER

Volume 322 No. 2 February 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 322, No. 2, 2023
https://doi.org/10.2140/pjm.2023.322.251

INVARIANT THEORY FOR THE FREE LEFT-REGULAR BAND AND A Q-ANALOGUE

SARAH BRAUNER, PATRICIA COMMINS AND VICTOR REINER

To the memory of Georgia Benkart

We examine from an invariant theory viewpoint the monoid algebras for two monoids having large symmetry groups. The first monoid is the *free left-regular band* on *n* letters, defined on the set of all *injective* words, that is, the words with at most one occurrence of each letter. This monoid carries the action of the symmetric group. The second monoid is one of its *q*-analogues, considered by K. Brown, carrying an action of the finite general linear group. In both cases, we show that the invariant subalgebras are semisimple commutative algebras, and characterize them using *Stirling* and *q-Stirling numbers*.

We then use results from the theory of random walks and random-to-top shuffling to decompose the entire monoid algebra into irreducibles, simultaneously as a module over the invariant ring and as a group representation. Our irreducible decompositions are described in terms of *derangement symmetric functions*, introduced by Désarménien and Wachs.

1. Introduction

Motivated by results on mixing times for shuffling algorithms on permutations, Bidigare [1997] and Bidigare, Hanlon, and Rockmore [Bidigare et al. 1999] developed a complete spectral analysis for a class of random walks on chambers of a hyperplane arrangement. Their work relied heavily on the *Tits semigroup* structure on the cones of the arrangement. Later, Brown [2000] generalized their analysis to random walks coming from semigroups $\mathcal{F}$ which form a *left-regular band* (LRB), meaning that $x^2 = x$ for all x and $xyx = xy$ for all x, y in $\mathcal{F}$.

Here we study two examples of left-regular bands M, related to those discussed by Brown, having actions of large groups of monoid automorphisms G:

- the *free LRB on n letters* [Brown 2000, §1.3], denoted $\mathcal{F}_n$, with G the symmetric group $\mathfrak{S}_n$, and

MSC2020: 05E10, 16W22, 60J10.
Keywords: left-regular band, shuffle, random-to-top, random-to-random, Bidigare–Hanlon–Rockmore, Stirling number, semigroup, monoid, symmetric group, general linear group, unipotent character.

- a q-analogue $\mathcal{F}_n^{(q)}$ related to monoids in [Brown 2000], and G the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$.

For both monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, we examine the *monoid algebra* $R := \boldsymbol{k}M$ with coefficients in a commutative ring $\boldsymbol{k}$, and answer the two *main questions of invariant theory* for G acting on R:

Question 1.1. What is the structure of the invariant subalgebra R^G?

Question 1.2. What is the structure of R, simultaneously as an R^G-module and a G-representation?

Section 2 answers Question 1.1 with our first main result, using the combinatorics of *Stirling* and *q-Stirling numbers*. We paraphrase it here; see Theorem 2.9 for a more precise statement.

Theorem 1.3. *Consider either monoid $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$ with symmetry groups $G = \mathfrak{S}_n, \mathrm{GL}_n$, and assume that $\boldsymbol{k}$ is a field in which $|G|$ is invertible.*

(1) *The invariant subalgebra R^G is a commutative subalgebra of R generated by a single element*; *call this element x for $M = \mathcal{F}_n$ and $x^{(q)}$ for $M = \mathcal{F}_n^{(q)}$.*

(2) *The elements $x, x^{(q)}$ have minimal polynomials*

$$f(X) = \begin{cases} X(X-1)(X-2)\cdots(X-n), & \text{if } M = \mathcal{F}_n, \\ X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q), & \text{if } M = \mathcal{F}_n^{(q)}, \end{cases}$$

where $[m]_q := 1 + q + \cdots + q^{m-1}$ is a standard q-analogue of the integer $m \geq 0$.

(3) *In particular, $R^G \cong \boldsymbol{k}[X]/(f(X))$, and R^G acts semisimply on R, with*
   - *x-eigenvalues $0, 1, 2, \ldots, n$ on $R = \boldsymbol{k}\mathcal{F}_n$,*
   - *$x^{(q)}$-eigenvalues $[0]_q, [1]_q, \ldots, [n]_q$ on $R = \boldsymbol{k}\mathcal{F}_n^{(q)}$.*

Since the above hypothesis that $|G|$ is invertible in $\boldsymbol{k}$ also implies that $\boldsymbol{k}G$ acts semisimply by Maschke's theorem, this leads to our next goal: a complete answer to Question 1.2 above, decomposing the monoid algebra R into simple modules for the simultaneous (commuting) actions of R^G and G. The fact that R^G is generated by a single, semisimple element x (respectively, $x^{(q)}$) reduces this problem to understanding each eigenspace of x (respectively, $x^{(q)}$) as a $\boldsymbol{k}G$-module.

To describe these $\boldsymbol{k}G$-modules, recall that irreducible representations $\{\chi^\lambda\}$ of $\mathfrak{S}_n$ are indexed by partitions λ of n and let $\mathcal{C}(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n)$, where $\mathcal{C}(\mathfrak{S}_n)$ denotes the $\mathbb{Z}$-module of *virtual characters* of $\mathfrak{S}_n$. Then the classical *Frobenius characteristic map* ch is an algebra isomorphism between $\mathcal{C}(\mathfrak{S})$ and the ring of symmetric functions Λ. It has $\mathrm{ch}(\chi^\lambda) = s_\lambda$, the *Schur function*, and the trivial representation $\mathbf{1}_n$ has $\mathrm{ch}(\mathbf{1}_n) = h_n$, the *complete homogeneous* symmetric function.

There is a parallel and q-analogous story for a subset of irreducible representations $\{\chi_q^\lambda\}$ of GL_n called the *unipotent representations*, also indexed by partitions λ of n. These are the irreducible constituents of the GL_n-permutation action on the set $\mathrm{GL}_n/B = \mathcal{F}(V)$ of *complete flags of subspaces* in $V = (\mathbb{F}_q)^n$. Here, too, there is a *q-Frobenius characteristic map* ch_q that defines an algebra isomorphism between $\mathcal{C}(\mathrm{GL}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathrm{GL}_n)$ and Λ, where $\mathcal{C}(\mathrm{GL}_n)$ is the free $\mathbb{Z}$-submodule of the class functions on GL_n spanned by the unipotent characters $\{\chi_q^\lambda\}$. As one might hope, $\mathrm{ch}_q(\chi_q^\lambda) = s_\lambda$ and $\mathrm{ch}_q(\mathbf{1}_{\mathrm{GL}_n}) = h_n$, where $\mathbf{1}_{\mathrm{GL}_n}$ is the trivial representation of GL_n.

This allows us to phrase parallel answers to Question 1.2, in terms of an important family of symmetric functions introduced by Désarménien and Wachs [1988], which we will call the *Désarménien–Wachs derangement symmetric functions* $\{\mathfrak{d}_n\}_{n=0,1,2,\ldots}$, reviewed in Section 3C. Here $\mathfrak{d}_n$ is both the Frobenius image of an $\mathfrak{S}_n$-representation $\mathcal{D}_n$ that we call the *Derangement representation*, as well as the q-Frobenius image of a q-analogous GL_n-representation $\mathcal{D}_n^{(q)}$. As the name suggests, these representations have dimensions counted by the *derangement numbers* and *q-derangement numbers*, respectively[1]. They have irreducible decomposition

$$\mathcal{D}_n \cong \bigoplus_{Q} \chi^{\lambda(Q)} \quad \text{and} \quad \mathcal{D}_n^{(q)} \cong \bigoplus_{Q} \chi_q^{\lambda(Q)},$$

where Q runs through all standard Young tableaux of size n whose *first ascent* is even [Reiner and Webb 2004]. Derangement symmetric functions have connections to many well-studied objects in combinatorics such as the complex of injective words [Reiner and Webb 2004], random-to-top and random-to-random shuffling [Uyemura-Reyes 2002], higher Lie characters [Uyemura-Reyes 2002], and configuration spaces [Hersh and Reiner 2017]; see Section 3C. We add to this list by showing they form crucial building blocks for the invariant theory of $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$.

Section 4 derives the following answer to Question 1.2, paraphrased here — see Theorem 4.11 for a more precise statement:

Theorem 1.4. *Let $\boldsymbol{k}$ be a field whose characteristic does not divide $|G|$. Then when $x, x^{(q)}$ act on $\boldsymbol{k}\mathcal{F}_n, \boldsymbol{k}\mathcal{F}_n^{(q)}$, for each $j = 0, 1, 2, \ldots, n$, the j-eigenspace for x and $[j]_q$-eigenspace for $x^{(q)}$ carry G-representations with the same Frobenius map images*

$$\mathrm{ch}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) = \sum_{\ell=j}^{n} h_{n-\ell}\cdot h_j \cdot \mathfrak{d}_{\ell-j} = \mathrm{ch}_q \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big).$$

Our proofs use techniques that go back to a discussion between Michelle Wachs and Reiner in the analysis of random-to-top shuffling, and have been employed

[1]There are two natural families of $\boldsymbol{k}\mathfrak{S}_n$-modules whose dimensions are the derangement numbers, discussed in [Hersh and Reiner 2017, Theorem 1.2]. The representation $\mathcal{D}_n$ here is the one with character $\widehat{\mathrm{Lie}}_n$ in the notation of [Hersh and Reiner 2017, Equation (1)].

more recently by Dieker and Saliola [2018] and Lafrenière [2020] in the analysis of random-to-random shuffling and a generalization. The method constructs eigenvectors of $x, x^{(q)}$ acting on $\mathfrak{F}_n, \mathfrak{F}_n^{(q)}$ from null vectors associated to the analogous operators for smaller values of n. Combining these ideas with various filtrations on $\boldsymbol{k}M$ allows us to describe the eigenspaces as parabolic inductions of derangement representations in a conceptual way, avoiding character computations.

The remainder of the paper proceeds as follows: Section 2 introduces the monoid algebras of interest, $R = \boldsymbol{k}\mathfrak{F}_n, \boldsymbol{k}\mathfrak{F}_n^{(q)}$, and proves Theorem 1.3, describing in parallel the invariant subalgebras R^G for $G = \mathfrak{S}_n, \mathrm{GL}_n$. Section 3 reviews the relation between symmetric functions, representations of $\mathfrak{S}_n$ and unipotent representations of GL_n. It also introduces the derangement symmetric functions $\mathfrak{d}_n$, and describes some of their many definitions and guises. Section 4 proves Theorem 1.4, simultaneously decomposing the monoid algebra R into simple modules for R^G and $\boldsymbol{k}G$, with arguments in parallel for $R = \boldsymbol{k}\mathfrak{F}_n$ and $R = \boldsymbol{k}\mathfrak{F}_n^{(q)}$.

2. Definitions, background, and the answer to Question 1.1

We introduce the monoids $M = \mathfrak{F}_n, \mathfrak{F}_n^{(q)}$, the symmetries $G = \mathfrak{S}_n, \mathrm{GL}_n$ of the monoid algebras $R = \boldsymbol{k}M$, and analyze the invariant rings R^G. Useful references are Brown [2000] and B. Steinberg [2016].

2A. *The monoids $\mathfrak{F}_n$ and $\mathfrak{F}_n^{(q)}$.*

Definition 2.1. The *free left-regular band* (or LRB) on n letters $\mathfrak{F}_n$ (see [Brown 2000, §1.3] and [Steinberg 2016, §14.3.1]) consists, as a set, of all words $\boldsymbol{a} = (a_1, a_2, \ldots, a_\ell)$ with letters a_i from $\{1, 2, \ldots, n\}$ and no repeated letters, that is, $a_i \neq a_j$ for $1 \leq i < j \leq n$. Here the *length* $\ell(\boldsymbol{a}) := \ell$ lies anywhere in the range $0 \leq \ell \leq n$. The set $\mathfrak{F}_n$ becomes a semigroup under the following operation: if $\boldsymbol{b} = (b_1, \ldots, b_m)$ is another word in $\mathfrak{F}_n$, then their product is

$$\boldsymbol{a} \cdot \boldsymbol{b} := (a_1, \ldots, a_\ell, b_1, \ldots, b_m)^\wedge,$$

where we have borrowed the notation from Brown [2000] that for a sequence $\boldsymbol{c} = (c_1, \ldots, c_p)$, the subsequence $\boldsymbol{c}^\wedge = (c_1, \ldots, c_p)^\wedge$ is obtained by removing any letter c_i that appears already in the prefix $(c_1, c_2, \ldots, c_{i-1})$. One can check that the empty word ( ) is an identity element for this operation, and hence $\mathfrak{F}_n$ is not only a semigroup, but a *monoid*.

Definition 2.2. The q-analogue of $\mathfrak{F}_n$ that we will consider will be denoted $\mathfrak{F}_n^{(q)}$. As a set, it consists of all partial flags of subspaces $\boldsymbol{A} = (A_1, A_2, \ldots, A_\ell)$, where A_i is an i-dimensional $\mathbb{F}_q$-linear subspace of $(\mathbb{F}_q)^n$, and $A_1 \subset A_2 \subset \cdots \subset A_\ell$. Again the length $\ell(\boldsymbol{A}) := \ell$ lies in the range $0 \leq \ell \leq n$. The set $\mathfrak{F}_n^{(q)}$ becomes a semigroup under the following operation: if $\boldsymbol{B} = (B_1, \dots, B_m)$ is another such flag in $\mathcal{F}_n^{(q)}$, then

$$\boldsymbol{A} \cdot \boldsymbol{B} := (A_1, \dots, A_\ell, A_\ell + B_1, A_\ell + B_2, \dots, A_\ell + B_m)^\wedge$$

using a similar notation as before: for a sequence $\boldsymbol{C} = (C_1, \dots, C_p)$ of nested subspaces $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_p$, the subsequence $\boldsymbol{C}^\wedge$ is obtained by removing any subspace C_i that appears already in the prefix $(C_1, C_2, \dots, C_{i-1})$. As above, $\mathcal{F}_n^{(q)}$ is not only a semigroup, but a *monoid*, since the empty flag ( ) is an identity element.

Remark 2.3. *Warning*: Brown [2000, §1.4 and §5] introduced two other monoids $\mathcal{F}_{n,q}$ and $\overline{\mathcal{F}}_{n,q}$, closely related to $\mathcal{F}_n^{(q)}$. All three are different q-analogues of $\mathcal{F}_n$, related as follows:

Considered as a set, Brown's first q-analogue $\mathcal{F}_{n,q}$ consists of all sequences $\boldsymbol{v} = (v_1, v_2, \dots, v_\ell)$ of linearly independent vectors in $(\mathbb{F}_q)^n$. For another sequence $\boldsymbol{v}' = (v'_1, v'_2, \dots, v'_m)$, one defines their product

$$\boldsymbol{v} \cdot \boldsymbol{v}' := (v_1, v_2, \dots, v_\ell, v'_1, v'_2 \dots, v'_m)^\wedge,$$

where $(u_1, \dots, u_p)^\wedge$ is obtained by removing any u_i which is dependent upon the preceding vectors $(u_1, \dots, u_{i-1})$. One may regard the monoid $\mathcal{F}_n^{(q)}$ as a quotient monoid of $\mathcal{F}_{n,q}$ via the surjection

$$\mathcal{F}_{n,q} \twoheadrightarrow \mathcal{F}_n^{(q)}, \quad (v_1, v_2, \dots, v_\ell) \mapsto (A_1, A_2, \dots, A_\ell),$$

where $A_i := \mathbb{F}_q v_1 + \mathbb{F}_q v_2 + \cdots + \mathbb{F}_q v_i$.

Brown's second q-analogue $\overline{\mathcal{F}}_{n,q}$ turns out to be a further quotient of either $\mathcal{F}_{n,q}$ or $\mathcal{F}_n^{(q)}$, whose motivation he explains in [Brown 2000, §5.1 and §5.2]. It is q-analogous to a certain quotient monoid of $\mathcal{F}_n$ that he denotes $\overline{\mathcal{F}}_n$, which one could define as follows: the monoid quotient map $\mathcal{F}_n \twoheadrightarrow \overline{\mathcal{F}}_n$ identifies the longest words, those of length n, with their prefix word of length $n-1$,

$$\overline{(a_1, a_2, \dots, a_{n-1}, a_n)} = \overline{(a_1, a_2, \dots, a_{n-1})}.$$

One can then define Brown's second q-analogue $\overline{\mathcal{F}}_{n,q}$ as a quotient of $\mathcal{F}_n^{(q)}$, where the monoid quotient map $\mathcal{F}_n^{(q)} \twoheadrightarrow \overline{\mathcal{F}}_{n,q}$ identifies a complete flag of length n with the flag of length $n-1$ that omits the (improper) subspace $(\mathbb{F}_q)^n$ at the end:

$$\overline{(A_1, A_2, \cdots, A_{n-1}, (\mathbb{F}_q)^n)} = \overline{(A_1, A_2, \dots, A_{n-1})}.$$

2B. ***Symmetries of the monoid algebras.*** Let $\boldsymbol{k}$ be a commutative ring with 1. For any finite monoid M (such as $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$), the *monoid algebra* $R = \boldsymbol{k}M$ is the free $\boldsymbol{k}$-module with basis elements given by the elements $\boldsymbol{a}$ of M, and multiplication extended $\boldsymbol{k}$-linearly from the monoid operation on the basis elements

$$\Big(\sum_{\boldsymbol{a}} p_{\boldsymbol{a}}\, \boldsymbol{a}\Big)\Big(\sum_{\boldsymbol{b}} q_{\boldsymbol{b}}\, \boldsymbol{b}\Big) = \sum_{\boldsymbol{a},\boldsymbol{b}} p_{\boldsymbol{a}} q_{\boldsymbol{b}}\, \boldsymbol{a} \cdot \boldsymbol{b} = \sum_{\boldsymbol{c}} \Big(\sum_{\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{c}} p_{\boldsymbol{a}} q_{\boldsymbol{b}}\Big) \boldsymbol{c}.$$

Note that any group G of monoid automorphisms of M acts as ring automorphisms on $R = \boldsymbol{k}M$. In particular, the symmetric group $\mathfrak{S}_n$ permuting letters $\{1, 2, \ldots, n\}$ acts on $\mathcal{F}_n$ via

$$w(a_1, \ldots, a_\ell) = (w(a_1), \ldots, w(a_\ell)).$$

Similarly, the finite general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$ acts on $\mathcal{F}_n^{(q)}$ by

$$g(A_1, \ldots, A_\ell) = (g(A_1), \ldots, g(A_\ell)).$$

Our first goal is to analyze the G-invariant subalgebras R^G in both cases.

2C. *The invariant subalgebras R^G and Question 1.1.* Since the groups G permute the monoid elements M, the monoid algebra $R = \boldsymbol{k}M$ becomes a permutation representation of G. Therefore, the invariant subalgebra R^G has as a $\boldsymbol{k}$-basis the orbit sums $\left\{\sum_{\boldsymbol{a}\in\mathcal{O}} \boldsymbol{a}\right\}$ as one runs through all G-orbits $\mathcal{O}$ on M. For both of the monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, one can easily identify the G-orbits, since the groups $G = \mathfrak{S}_n$ and GL_n act transitively on the subsets

$$\mathcal{F}_{n,\ell} := \{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) = \ell\},$$
$$\mathcal{F}_{n,\ell}^{(q)} := \{\boldsymbol{A} \in \mathcal{F}_n^{(q)} : \ell(\boldsymbol{A}) = \ell\}.$$

Thus the G-invariant subalgebras R^G have $\boldsymbol{k}$-bases $\{x_\ell\}_{\ell=0,1,\ldots,n}$, and $\{x_\ell^{(q)}\}_{\ell=0,1,\ldots,n}$, defined by

$$x_\ell := \sum_{\boldsymbol{a}\in\mathcal{F}_{n,\ell}} \boldsymbol{a} \quad \text{and} \quad x_\ell^{(q)} := \sum_{\boldsymbol{A}\in\mathcal{F}_{n,\ell}^{(q)}} \boldsymbol{A}. \tag{1}$$

Example 2.4. Let $q = 2$, $n = 3$, $\ell = 1$, and let e_1, e_2, e_3 be standard basis vectors for $V = (\mathbb{F}_2)^3$. Using the notation $\langle v_1, v_2, \ldots, v_m\rangle$ for the $\mathbb{F}_q$-span of the vectors $\{v_1, v_2, \ldots, v_m\}$ in V, one has

$$x_1^{(2)} = (\langle e_1\rangle)+(\langle e_2\rangle)+(\langle e_3\rangle)+(\langle e_1+e_2\rangle)+(\langle e_1+e_3\rangle)+(\langle e_2+e_3\rangle)+(\langle e_1+e_2+e_3\rangle).$$

It will be convenient to adopt the convention that $x_{n+1} := 0 =: x_{n+1}^{(q)}$.

Using the $\boldsymbol{k}$-bases in (1) for $(\boldsymbol{k}\mathcal{F}_n)^{\mathfrak{S}_n}$ and $(\boldsymbol{k}\mathcal{F}_n^{(q)})^{\mathrm{GL}_n}$, there is a simple rule for multiplication by the elements

$$x := x_1 = \sum_{i=1}^{n} (i) = (1) + (2) + \cdots + (n),$$
$$x^{(q)} := x_1^{(q)} = \sum_{\text{lines } L\subset(\mathbb{F}_q)^n} (L).$$

To state the rule, recall a standard q-analogue of nonnegative integers

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

Lemma 2.5. *Inside R^G for the monoid algebras $R = \boldsymbol{k}M$ with $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, the elements x and $x^{(q)}$ act on the (ordered) $\boldsymbol{k}$-bases (1) as follows: for $\ell = 0, 1, \dots, n$,*

$$x \cdot x_\ell = \ell x_\ell + x_{\ell+1},$$
$$x^{(q)} \cdot x_\ell^{(q)} = [\ell]_q \, x_\ell^{(q)} + q^\ell \, x_{\ell+1}^{(q)}.$$

In other words, x and $x^{(q)}$ act on R^G, in the ordered bases above, via the matrices:

$$x = \begin{bmatrix} 0 & & & & & \\ 1 & 1 & & & & \\ & 1 & 2 & & & \\ & & 1 & \ddots & & \\ & & & & n-1 & \\ & & & & 1 & n \end{bmatrix} \quad \text{and} \quad x^{(q)} = \begin{bmatrix} [0]_q & & & & & \\ q^0 & [1]_q & & & & \\ & q^1 & [2]_q & & & \\ & & q^2 & \ddots & & \\ & & & & [n-1]_q & \\ & & & & q^{n-1} & [n]_q \end{bmatrix}.$$

Proof. Note that the product $x \cdot x_\ell$ is G-invariant, and is a sum of terms $\boldsymbol{a}$ of length ℓ or $\ell+1$, so it must have the form $c \cdot x_\ell + d \cdot x_{\ell+1}$ for some constants c, d in $\boldsymbol{k}$. The constant $d = 1$, since any word $\boldsymbol{a} = (a_1, a_2, \dots, a_{\ell+1})$ of length $\ell+1$ arises uniquely as $(a_1) \cdot (a_2, \dots, a_{\ell+1})$. The constant $c = \ell$, since any word $(a_1, a_2, \dots, a_\ell)$ of length ℓ arises in ℓ ways, from these products:

$$(a_1) \cdot (\underline{a_1}, a_2, a_3, a_4, \dots, a_\ell),$$
$$(a_1) \cdot (a_2, \underline{a_1}, a_3, a_4, \dots, a_\ell),$$
$$(a_1) \cdot (a_2, a_3, \underline{a_1}, a_4, \dots, a_\ell),$$
$$\vdots$$
$$(a_1) \cdot (a_2, a_3, a_4, \dots, a_\ell, \underline{a_1}).$$

For the q-analogous formula, one argues similarly that

$$x^{(q)} \cdot x_\ell^{(q)} = c \cdot x_\ell^{(q)} + d \cdot x_{\ell+1}^{(q)}$$

for some constants c, d in $\boldsymbol{k}$. We first show that the constant $d = q^\ell$. Any flag $\boldsymbol{A} = (A_1, A_2, \dots, A_{\ell+1})$ of length $\ell+1$ arises from products of the form $(A_1) \cdot (B_1, B_2 \dots, B_\ell)$, where the flag $B_1 \subset B_2 \subset \cdots \subset B_\ell$ satisfies $A_1 + B_i = A_{i+1}$ for $i = 1, 2, \dots, \ell$. If one picks $B_1, B_2, \dots, B_\ell$ sequentially, then having chosen B_{i-1}, one must choose B_i so that B_i/B_{i-1} is any line inside the 2-dimensional quotient space A_{i+1}/B_{i-1} other than the line $(A_1 + B_{i-1})/B_{i-1}$. Since there are $q+1$ lines in A_{i+1}/B_{i-1}, this gives q choices for B_i, and q^ℓ sequential choices in total for $B_1, B_2, \dots, B_\ell$.

We next argue that the constant $c = [\ell]_q$. Any flag $\boldsymbol{A} = (A_1, A_2, \dots, A_\ell)$ of length ℓ arises from products of the form $(A_1) \cdot (B_1, B_2 \dots, B_\ell)$ in which the flag $B_1 \subset B_2 \subset \cdots \subset B_\ell$ has $A_1 \subseteq B_\ell$ (else, $(A_1) \cdot (B_1, B_2 \dots, B_\ell)$ has length $\ell+1$, not ℓ).

Letting i_0 be the smallest index for which $A_1 \subseteq B_{i_0}$, one finds that $1 \le i_0 \le \ell$. Having fixed i_0, the B_i for i in the range $i_0 \le i \le \ell$ are completely determined by $B_i = A_1 + B_i = A_i$. Meanwhile, for i in the range $1 \le i \le i_0 - 1$, as in the argument for the constant $d = q^\ell$ above, one can sequentially choose each of $B_1, B_2, \dots, B_{i_0-1}$ in q ways so that they satisfy $A_1 + B_i = A_{i+1}$. This gives q^{i_0-1} choices, which when summed over $i_0 = 1, 2, \dots, \ell$ gives $1 + q + q^2 + \cdots + q^{\ell-1} = [\ell]_q$ sequential choices in total. □

Lemma 2.5 allows us to connect R^G to the *Stirling* and *q-Stirling numbers*, briefly reviewed here.

Definition 2.6. The classical *Stirling numbers of the second kind* $(S(n,k))_{k,n=0,1,\dots}$ have two closely related families of q-analogues $S_q(n,k)$, $\tilde{S}_q(n,k)$, introduced by Carlitz [1933, §4] and studied by many others, e.g., Cai, Ehrenborg, and Readdy [Cai et al. 2018], Garsia and Remmel [1986], Gould [1961], de Médicis and Leroux [1993], Milne [1978; 1982], Sagan and Swanson [2022], Wachs and White [1991], among others. Using the notation[2] in [Milne 1978], all three are doubly indexed triangles defined for (n,k) with $n, k \ge 0$, having initial conditions that set them all equal to 1 when $(n,k) = (0,0)$, and vanishing whenever $n + k \ge 1$ but either $k = 0$ or $n = 0$. When both $n, k \ge 1$, they are then defined by the recursions

$$\begin{aligned} S(n,k) &= S(n-1,k-1) + k \cdot S(n-1,k), \\ \tilde{S}_q(n,k) &= \tilde{S}_q(n-1,k-1) + [k]_q \cdot \tilde{S}_q(n-1,k), \\ S_q(n,k) &= q^{k-1} \cdot S_q(n-1,k-1) + [k]_q \cdot S_q(n-1,k). \end{aligned} \tag{2}$$

An easy induction using the recursion lets one check that, for all n and k, one has the relation

$$S_q(n,k) = q^{\binom{k}{2}} \tilde{S}_q(n,k),$$

and for $n \ge 1$, one has

$$S(n,1) = S_q(n,1) = \tilde{S}_q(n,1) = 1, \quad S(n,n) = \tilde{S}_q(n,n) = 1, \quad S_q(n,n) = q^{\binom{n}{2}}. \tag{3}$$

Remark 2.7. Alternatively, one can consider $S(n,k)$, $\tilde{S}_q(n,k)$, $S_q(n,k)$ as change-of-basis matrices in the polynomial rings $\boldsymbol{k}[t]$ with $\boldsymbol{k} = \mathbb{Z}, \mathbb{Z}[q], \mathbb{Z}[q,q^{-1}]$, respectively. Consider the obvious ordered $\boldsymbol{k}$-basis of $\boldsymbol{k}[t]$ given by the powers $(t^n)_{n=0}^\infty = (1, t, t^2, \dots)$, versus these ($q$-)*falling factorial* $\boldsymbol{k}$-bases,

$$\begin{aligned} (t)_n &:= t(t-1)(t-2)\cdots(t-(n-1)), && \text{in } \mathbb{Z}[t], \\ (t)_{n,q} &:= t(t-[1]_q)(t-[2]_q)\cdots(t-[n-1]_q), && \text{in } \mathbb{Z}[q][t] \text{ or } \mathbb{Z}[q,q^{-1}][t]. \end{aligned}$$

[2]Notational conflicts are unavoidable. E.g., our $S_q(n,k)$, $\tilde{S}_q(n,k)$ here equal $\bar{S}[n,k]$, $S[n,k]$, respectively, in [Sagan and Swanson 2022].

Then one has these change-of-basis formulas (see[3] Gould [1961, §3], Milne [1978, Equation (1.14)], and [Garsia and Remmel 1986, Equation (I.17)]):

$$
\begin{aligned}
t^n &= \sum_k S(n,k)\cdot(t)_k, && \text{in } \mathbb{Z}[t],\\
t^n &= \sum_k \tilde{S}_q(n,k)\cdot(t)_{k,q}, && \text{in } \mathbb{Z}[q][t],\\
t^n &= \sum_k S_q(n,k)q^{-\binom{k}{2}}\cdot(t)_{k,q}, && \text{in } \mathbb{Z}[q,q^{-1}][t].
\end{aligned} \tag{4}
$$

We next show that $S(n,k)$, $S_q(n,k)$ also mediate a natural change-of-basis within R^G.

Corollary 2.8. *Let $\boldsymbol{k}$ be a commutative ring with 1, and let $R=\boldsymbol{k}M$ with $M=\mathcal{F}_n$ or $\mathcal{F}_n^{(q)}$. Then the (q-)Stirling numbers $S(m,k)$ and $S_q(m,k)$ are the expansion coefficients for the powers $\{x^m\}_{m=0,1,\dots,n}$ and $\{(x^{(q)})^m\}_{m=0,1,\dots,n}$ in the orbit-sum $\boldsymbol{k}$-bases $\{x_k\}_{k=0,1,\dots,n}$ and $\{x_k^{(q)}\}_{k=0,1,\dots,n}$ of R^G:*

$$x^m=\sum_k S(m,k)\,x_k \quad \textit{and} \quad (x^{(q)})^m=\sum_k S_q(m,k)\,x_k^{(q)}.$$

Thus unitriangularity of $\{S(m,k)\}$ shows $\{x^k\}_{k=0,1,\dots,n}$ always gives a $\boldsymbol{k}$-basis for R^G, while triangularity of $\{S_q(m,k)\}$ shows $\{(x^{(q)})^k\}_{k=0,1,\dots,n}$ is a $\boldsymbol{k}$-basis for R^G if and only if q lies in $\boldsymbol{k}^\times$.

Proof. Both expansions follow by induction on m. Here is the inductive step calculation in the q-Stirling case, applying induction, Lemma 2.5, and (2) for equalities $(*)$, $(**)$, and $(***)$, respectively:

$$
\begin{aligned}
(x^{(q)})^m = x^{(q)}\cdot(x^{(q)})^{m-1} &\overset{(*)}{=} x^{(q)}\cdot\sum_k S_q(m-1,k)\;x_k^{(q)}\\
&= \sum_k S_q(m-1,k)\;x^{(q)}\cdot x_k^{(q)}\\
&\overset{(**)}{=} \sum_k S_q(m-1,k)\big([k]_q x_k^{(q)}+q^k x_{k+1}^{(q)}\big)\\
&= \sum_k\big([k]_q S_q(m-1,k)+q^{k-1}S_q(m-1,k-1)\big)x_k^{(q)}\\
&\overset{(***)}{=} \sum_k S_q(m,k)x_k^{(q)}.
\end{aligned}
$$

The q-expansion is invertible only when q lies in $\boldsymbol{k}^\times$ due to triangularity and $S_q(m,m)=q^{\binom{m}{2}}$. ☐

This leads to our answer for Question 1.1.

[3]The formulas as discussed by Milne [1978, (1.14)] use the notation $[x]=(y-1)/(q-1)$, where $y=q^x$ is regarded as an indeterminate. To agree with notation and (4) here, one should substitute $t=[x]=(y-1)/(q-1)$, so that $y=1+t(q-1)$.

Theorem 2.9. *Let $\boldsymbol{k}$ be any commutative ring with* 1, *and let $R = \boldsymbol{k}M$ for either of the monoids $M = \mathfrak{F}_n, \mathfrak{F}_n^{(q)}$, with symmetry groups $G = \mathfrak{S}_n, \mathrm{GL}_n$. If $M = \mathfrak{F}_n^{(q)}$, assume further that q is in $\boldsymbol{k}^\times$.*

(i) *The unique $\boldsymbol{k}$-algebra map $\boldsymbol{k}[X] \xrightarrow{\gamma} R$ defined by*

$$X \mapsto \begin{cases} x, & \text{if } M = \mathfrak{F}_n, \\ x^{(q)}, & \text{if } M = \mathfrak{F}_n^{(q)}, \end{cases}$$

induces an algebra isomorphism $\boldsymbol{k}[X]/(f(X)) \cong R^G$, where

$$f(X) := \begin{cases} X(X-1)(X-2)\cdots(X-n), & \text{if } M = \mathfrak{F}_n, \\ X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q), & \text{if } M = \mathfrak{F}_n^{(q)}. \end{cases}$$

Hence, R^G is commutative and generated by x or $x^{(q)}$.

(ii) *If $\boldsymbol{k}$ is a field, where $|G|$ is invertible, then x or $x^{(q)}$ acts semisimply on any finite-dimensional R^G-module, with eigenvalues contained in the lists*

$$\begin{cases} 0, 1, 2, \ldots, n, & \text{if } M = \mathfrak{F}_n, \\ [0]_q, [1]_q, [2]_q, \ldots, [n]_q, & \text{if } M = \mathfrak{F}_n^{(q)}. \end{cases}$$

Proof. For (i), note that Lemma 2.5 shows that x or $x^{(q)}$ acts on R^G with characteristic polynomial $f(X)$. Consequently, the kernel of the algebra map $\boldsymbol{k}[X] \xrightarrow{\gamma} R^G$ contains $f(X)$, and γ descends to a map on the quotient $\boldsymbol{k}[X]/(f(X)) \xrightarrow{\gamma} R^G$. Moreover, since $f(X)$ is monic of degree $n+1$, the quotient $\boldsymbol{k}[X]/(f(X))$ has $\boldsymbol{k}$-basis $(1, X, X^2, \ldots, X^n)$, and Corollary 2.8 shows that γ maps this onto a $\boldsymbol{k}$-basis of powers $\{x^k\}_{k=0}^n$ or $\{(x^{(q)})^k\}_{k=0}^n$ for R^G. Hence, γ is an algebra isomorphism.

For (ii), assume that $\boldsymbol{k}$ is a field where the roots of the characteristic polynomial $f(X)$ of x or $x^{(q)}$ acting on R^G are all distinct. This means that $f(X)$ must also be the minimal polynomial for x, or $x^{(q)}$ acting on R^G, and that it acts semisimply in any finite dimensional R^G-module, with eigenvalues contained in that set of roots. Lastly, note the groups G have cardinalities

$$|G| = \begin{cases} |\mathfrak{S}_n| = n!, & \text{for } M = \mathfrak{F}_n, \\ |\mathrm{GL}_n| = q^{\binom{n}{2}}(q-1)^n[n]!_q \\ \qquad = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1}), & \text{for } M = \mathfrak{F}_n^{(q)}, \end{cases}$$

where the q-factorial $[n]!_q$ is defined by

$$[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q. \tag{5}$$

One can then check that the invertibility of $n!$ in $\boldsymbol{k}$ and distinctness of $0, 1, 2, \ldots, n$ are both equivalent to $\boldsymbol{k}$ having characteristic zero or a prime $p > n$, while invertibility of $|\mathrm{GL}_n|$ in $\boldsymbol{k}$ and distinctness of $[0]_q, [1]_q, [2]_q, \ldots, [n]_q$ are both equivalent to $\boldsymbol{k}$ having characteristic zero or characteristic coprime to q and to $[m]_q$ for $m = 1, 2, \ldots, n$. ∎

We close this section with some remarks on Brown's other q-analogues of $\mathcal{F}_n$.

Remark 2.10. The analysis in Lemma 2.5 can be lifted to an analogous (and even simpler) computation in Brown's first q-analogue $\mathcal{F}_{n,q}$. Denoting the orbit sum $\boldsymbol{k}$-basis in $\boldsymbol{k}\mathcal{F}_{n,q}$ by $y_0, y_1, \dots, y_n$, multiplication by the element $y := y_1 = \sum_{v \in (\mathbb{F}_q)^n \setminus \{\mathbf{0}\}} (v)$ acts on that basis as follows:

$$y \cdot y_\ell = (q^\ell - 1) y_\ell + y_{\ell+1}. \tag{6}$$

Bearing in mind that the monoid surjection $\mathcal{F}_{n,q} \xrightarrow{\pi} \mathcal{F}_n^{(q)}$ described in Remark 2.3 has exactly

$$(q-1)(q^2-q)\cdots(q^\ell - q^{\ell-1}) = (q-1)^\ell q^{\binom{\ell}{2}}$$

preimages $(v_1, v_2, \dots, v_\ell)$ for every flag $\boldsymbol{A} = (A_1, A_2, \dots, A_\ell)$, one can check that (6) maps under the linearization $\boldsymbol{k}\mathcal{F}_{n,q} \xrightarrow{\pi} \boldsymbol{k}\mathcal{F}_n^{(q)}$ to a formula consistent with the second formula in Lemma 2.5.

Remark 2.11. It is also easy to check that Lemma 2.5 gives similar computations in the other monoids $\overline{\mathcal{F}}_n$ and $\overline{\mathcal{F}}_{n,q}$ considered by Brown, discussed in Remark 2.3. Specifically, in $\boldsymbol{k}\overline{\mathcal{F}}_n$, one has

$$\bar{x} \cdot \bar{x}_\ell = \begin{cases} \ell \bar{x}_\ell + \bar{x}_{\ell+1}, & \text{if } 0 \le \ell < n-1, \\ n \bar{x}_{n-1}, & \text{if } \ell = n-1, \end{cases}$$

and in $\boldsymbol{k}\overline{\mathcal{F}}_{n,q}$, one has

$$\bar{x}^{(q)} \cdot \bar{x}_\ell^{(q)} = \begin{cases} [\ell]_q \bar{x}_\ell^{(q)} + q^\ell \bar{x}_{\ell+1}^{(q)}, & \text{if } 0 \le \ell < n-1, \\ [n]_q \bar{x}_{n-1}^{(q)}, & \text{if } \ell = n-1. \end{cases}$$

The point is that when one $\boldsymbol{k}$-linearizes the monoid surjection $\mathcal{F} \to \overline{\mathcal{F}}_n$ it maps $x_\ell \longmapsto \bar{x}_\ell$ for $i \le n-2$, and maps $x_{n-1}, x_n \longmapsto \bar{x}_{n-1}$. An analogous statement holds for $\mathcal{F}^{(q)} \to \overline{\mathcal{F}}_{n,q}$. One can then check that applying these linearized surjections to Lemma 2.5 gives the above formulas.

3. Representation-theoretic preliminaries

Having answered Question 1.1 by describing the structure of R^G, the next few subsections collect and review some facts regarding representations of $G = \mathfrak{S}_n$ and $G = \mathrm{GL}_n$ that will help us answer Question 1.2 in Section 4 on the structure of R, simultaneously as an R^G-module and a G-representation.

3A. ***Semisimplicity, filtrations, and eigenspaces.*** In what follows, we will be examining various modules V over the monoid algebra $R = \boldsymbol{k}M$ for the two monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, carrying $\boldsymbol{k}G$-module structures for the automorphism groups $G = \mathfrak{S}_n, \mathrm{GL}_n$. In all cases, the G-actions on R and V will be *compatible* in the sense that

$$g(r \cdot v) = g(r) \cdot g(v) \quad \text{for all } r \in R,\ v \in V,\ g \in G.$$

Note that in this setting, V carries commuting actions of R^G and of $\boldsymbol{k}G$, and we will wish to describe it simultaneously as a module over both.

Henceforth, assume that $\boldsymbol{k}$ is a field in which $|G|$ is invertible, and take V to be finite-dimensional over $\boldsymbol{k}$. This implies that V is semisimple both as an R^G-module due to Theorem 2.9 (ii), and as a $\boldsymbol{k}G$-module by Maschke's Theorem.

In order to answer Question 1.2, we will utilize two important features of our setting:

(1) Semisimplicity implies that given a filtration by R^G-submodules and $\boldsymbol{k}G$-submodules V_i

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V,$$

one actually has an R^G-module and $\boldsymbol{k}G$-module isomorphism

$$V \cong \bigoplus_i V_i / V_{i-1}.$$

This will play a crucial role in Section 4B (specifically, in our proof of Theorem 1.4), where we will define filtrations on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$ that significantly simplify the analysis.

(2) By Theorem 2.9 (ii), we have that R^G is generated by the single element x or $x^{(q)}$, which acts diagonalizably with certain eigenvalues λ all lying in $\boldsymbol{k}$. It follows that in order to understand the R^G and $\boldsymbol{k}G$-module structure of any module V, it suffices to decompose the eigenspaces $\ker((x-\lambda)|_V)$ as $\boldsymbol{k}G$-modules.

Hence, we will answer Question 1.2 by describing the j-eigenspaces of $\boldsymbol{k}\mathcal{F}_n$ as $\mathfrak{S}_n$-representations and the $[j]_q$-eigenspaces of $\boldsymbol{k}\mathcal{F}_n^{(q)}$ as GL_n- representations for $j = 0, 1, \ldots, n$.

3B. ***Symmetric functions, $\mathfrak{S}_n$-representations, and unipotent* GL_n*-representations.*** We review here the relation between the *ring of symmetric functions* Λ and representations of $\mathfrak{S}_n$; see Sagan [1991] and Stanley [1999] as references, and for undefined terminology. We then review the parallel story for R. Steinberg's *unipotent representations* of GL_n; see [Grinberg and Reiner 2014, §4.2, §4.6, and §4.7] as a reference.

The *ring of symmetric functions* Λ (of bounded degree, in infinitely many variables) may be viewed as a polynomial algebra $\mathbb{Z}[h_1, h_2, \ldots] = \mathbb{Z}[e_1, e_2, \ldots]$, where h_n and e_n are the *complete homogeneous* and *elementary* symmetric functions of degree n. One may view Λ as a graded $\mathbb{Z}$-algebra $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$, which we wish to relate to the direct sum

$$\mathcal{C}(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n),$$

where $\mathcal{C}(\mathfrak{S}_n)$ denotes the $\mathbb{Z}$-module of *virtual characters* of $\mathfrak{S}_n$. That is, $\mathcal{C}(\mathfrak{S}_n)$ is the free $\mathbb{Z}$-module on the basis of irreducible characters $\{\chi^\lambda\}$ indexed by the partitions λ of n, or alternatively, the $\mathbb{Z}$-submodule of class functions on $\mathfrak{S}_n$ of the form $\chi - \chi'$ for genuine characters χ, χ'. One makes $\mathcal{C}(\mathfrak{S})$ into a graded algebra via the *induction product* defined by

(7) $$\mathcal{C}(\mathfrak{S}_{n_1}) \times \mathcal{C}(\mathfrak{S}_{n_2}) \to \mathcal{C}(\mathfrak{S}_{n_1+n_2}), \quad (f_1, f_2) \mapsto f_1 * f_2 := (f_1 \otimes f_2) \uparrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}},$$

where $(-) \uparrow_H^G$ is the usual *induction* of class functions on a subgroup H to class functions on G.

For later use, we note that since $[\mathfrak{S}_{n_1+n_2} : \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}] = \binom{n_1+n_2}{n_1}$, whenever f_1, f_2 are genuine characters, one has the formula for the degree of $f_1 * f_2$:

(8) $$\deg(f_1 * f_2) = \binom{n_1+n_2}{n_1} \deg(f_1) \deg(f_2).$$

One then has the *Frobenius characteristic isomorphism* of $\mathbb{Z}$-algebras $\mathcal{C}(\mathfrak{S}) \xrightarrow{\text{ch}} \Lambda$, mapping

$$\mathcal{C}(\mathfrak{S}) \xrightarrow{\text{ch}} \Lambda, \quad \mathbf{1}_{\mathfrak{S}_n} \mapsto h_n, \ \text{sgn}_{\mathfrak{S}_n} \mapsto e_n, \ \chi^\lambda \mapsto s_\lambda.$$

Here, s_λ is the *Schur function*. For a composition $\alpha = \alpha_1, \alpha_2, \dots, \alpha_\ell$, we use the standard shorthand

$$h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell}.$$

For later use, we note that one can express the regular representation $\boldsymbol{k}\mathfrak{S}_n = \mathbf{1}_{\mathfrak{S}_1} * \mathbf{1}_{\mathfrak{S}_1} * \cdots * \mathbf{1}_{\mathfrak{S}_1}$, implying

(9) $$\text{ch}\ \boldsymbol{k}\mathfrak{S}_n = h_1^n = h_{1^n}.$$

There is a parallel story for a certain subset of GL_n-representations. Specifically, there is a collection of irreducible GL_n-representations $\{\chi_q^\lambda\}$, indexed by partitions λ of n, which are the irreducible constituents occurring within the GL_n-permutation action on the set GL_n / B of *complete flags of subspaces* $\mathcal{F}(V)$ in $V = (\mathbb{F}_q)^n$. They were studied by R. Steinberg [1951], and are now called the *unipotent characters* of GL_n. Letting $\mathcal{C}(\text{GL}_n)$ represent the free $\mathbb{Z}$-submodule of the class functions on GL_n with unipotent characters $\{\chi_q^\lambda\}$ as a basis, one can define the *parabolic* or *Harish–Chandra induction* product on the direct sum $\mathcal{C}(\text{GL}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\text{GL}_n)$ as follows:

$$\mathcal{C}(\text{GL}_{n_1}) \times \mathcal{C}(\text{GL}_{n_2}) \to \mathcal{C}(\text{GL}_{n_1+n_2}),$$
$$(f_1, f_2) \mapsto f_1 * f_2 := \left((f_1 \otimes f_2) \Uparrow_{\text{GL}_{n_1} \times \text{GL}_{n_2}}^{P_{n_1,n_2}}\right) \uparrow_{P_{n_1,n_2}}^{\text{GL}_{n_1+n_2}}.$$

Here, P_{n_1,n_2} is the maximal parabolic subgroup of $\mathrm{GL}_{n_1+n_2}$ setwise stabilizing the $\mathbb{F}_q$-span of the first n_1 standard basis vectors, and $(-)\Uparrow_{\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2}}^{P_{n_1,n_2}}$ is the *inflation* operation that creates a $\mathrm{GL}\,n_1 \times \mathrm{GL}_{n_2}$-representation from a P_{n_1,n_2}-representation, by precomposing with the surjective homomorphism $P_{n_1,n_2} \twoheadrightarrow \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ sending $\left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right]$. For later use, we note that since the inflation operation does not change the degree of a representation, and since

$$[\mathrm{GL}_{n_1+n_2} : P_{n_1,n_2}] = \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_q = \frac{[n_1+n_2]!_q}{[n_1]!_q[n_2]!_q},$$

(with $[n]!_q$ as in (5)) when f_1, f_2 are genuine characters, one has this degree formula for $f_1 * f_2$:

$$\deg(f_1 * f_2) = \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_q \deg(f_1)\deg(f_2). \tag{10}$$

This parabolic induction operation turns out to make $\mathcal{C}(\mathrm{GL})$ into an associative, commutative $\mathbb{Z}$-algebra. One then has a q-analogue of the Frobenius isomorphism $\mathcal{C}(\mathrm{GL})\xrightarrow{\mathrm{ch}_q}\Lambda$ sending[4]

$$\mathcal{C}(\mathrm{GL})\xrightarrow{\mathrm{ch}_q}\Lambda, \quad \mathbf{1}_{\mathrm{GL}_n} \mapsto h_n, \ \chi_q^\lambda \mapsto s_\lambda.$$

Note that the permutation representation $\boldsymbol{k}[\mathrm{GL}_n/B]$ of GL_n on the complete flags can be expressed as $\mathbf{1}_{\mathrm{GL}_1} * \mathbf{1}_{\mathrm{GL}_1} * \cdots * \mathbf{1}_{\mathrm{GL}_1}$, and therefore one has this q-analogue of (9):

$$\mathrm{ch}_q\ \boldsymbol{k}[\mathrm{GL}_n/B] = h_1^n = h_{1^n}. \tag{11}$$

3C. ***(q-)derangement numbers and representations.*** A central role in this story is played by the classical *derangement numbers* d_n and the *q-derangement numbers* $d_n(q)$ of Wachs [1989]:

$$\begin{aligned} d_n &:= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} = n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+\frac{(-1)^n}{n!}\right), \\ d_n(q) &:= [n]!_q\sum_{k=0}^{n}\frac{(-1)^k}{[k]!_q}. \end{aligned} \tag{12}$$

There are two well-known combinatorial models for d_n counting permutations in $\mathfrak{S}_n$:

- *derangements*, which are the fixed-point free permutations, or

[4]One might wonder which GL_n-character maps under ch_q to the elementary symmetric function e_n; it is the *Steinberg representation*, in which GL_n acts on the top homology of the *Tits building*, which is the simplicial complex of flags of nonzero proper subspaces in $(\mathbb{F}_q)^n$.

- *desarrangements*, which are permutations $w = (w_1, w_2, \dots, w_n)$ whose first ascent position i with $w_i < w_{i+1}$ (using $w_{n+1} = n+1$ by convention) occurs for an even position i.

Wachs [1989], and later Désarménien and Wachs [1993], gave various interpretations for $d_n(q)$. In particular, $d_n(q)$ is still closely related to derangements and desarrangements. Letting D_n and E_n denote the derangements and desarrangements in S_n, and defining the *major index* statistic of a permutation $w = (w_1, \dots, w_n)$ as $\mathrm{maj}(\sigma) = \sum_{i : w_i > w_{i+1}} i$, one has

$$d_n(q) = \sum_{\sigma \in D_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in E_n} q^{\mathrm{maj}(\sigma^{-1})}.$$

These d_n and $d_n(q)$ are the dimensions for a pair of representations of $\mathfrak{S}_n$ and GL_n, which we call the *derangement representation* $\mathcal{D}_n$ and its (unipotent) q-analogue $\mathcal{D}_n^{(q)}$. Both have the same symmetric function image $\mathfrak{d}_n$ under the Frobenius maps ch and ch_q, a symmetric function with many equivalent descriptions. For the reader's convenience, and for future use, we will compile these descriptions in Proposition 3.1, after first reviewing terminology.

Define for a permutation $w = (w_1, w_2, \dots, w_n)$ in $\mathfrak{S}_n$ its *descent set*

$$\mathrm{Des}(w) := \big\{ i \in \{1, 2, \dots, n-1\} : w_i > w_{i+1} \big\}.$$

For example, $w = (6, 3, 5, 2, 1, 4)$ has $\mathrm{Des}(w) = \{1, 3, 4\}$. Note that the definition of a *desarrangement* given above may be rephrased as a permutation w in $\mathfrak{S}_n$ for which the smallest element of $\{1, 2, \dots, n\} \setminus \mathrm{Des}(w)$ is even. Thus $w = (6, 3, 5, 2, 1, 4)$ is a desarrangement, since $\min(\{1, 2, 3, 4, 5, 6\} \setminus \{1, 3, 4\}) = 2$ is even.

Given a standard Young tableau Q with n cells written in English notation, its *descent set* is

$$\mathrm{Des}(w) := \{ i \in \{1, 2, \dots, n-1\} : i+1 \text{ appears south and weakly west of } i \text{ in } Q \}.$$

For example,

$$Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

has $\mathrm{Des}(Q) = \{1, 3, 4\}$. Define a *desarrangement tableau* to be a standard Young tableau Q with n cells for which the smallest element of $\{1, 2, \dots, n\} \setminus \mathrm{Des}(Q)$ is even. Thus, the example tableau Q given above is a desarrangement tableau.

Finally, for integers $n \geq 1$ and $D \subseteq [n]$, define *Gessel's fundamental quasisymmetric function*

$$L_{n,D} := \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

which is a formal power series in $x_1, x_2, \dots$ and is homogeneous of degree n. For w in $\mathfrak{S}_n$, let $\lambda(w)$ denote its cycle type partition of n. For any partition λ of n, the *higher Lie character* of Thrall [1942] or the *Gessel–Reutenauer symmetric function* $\mathfrak{L}_\lambda$ (see [Gessel and Reutenauer 1993], [Grinberg and Reiner 2014, §6.6], and [Stanley 1999, Exercise 7.89]) can be defined as

$$\mathfrak{L}_\lambda := \sum_{\substack{w \in \mathfrak{S}_n: \\ \lambda(w)=\lambda}} L_{n,\mathrm{Des}(w)}.$$

Proposition 3.1. *With the convention that* $\mathfrak{d}_0 := 1$, *the following definitions of a sequence of symmetric functions* $\{\mathfrak{d}_n\}_{n=0,1,2,\dots}$ *are all equivalent*:

(A) $\mathfrak{d}_n = h_1 \mathfrak{d}_{n-1} + (-1)^n e_n$ *for* $n \geq 1$;

(B) $\mathfrak{d}_n = \sum_{k=0}^{n} (-1)^k e_k \cdot h_{1^{n-k}}$;

(C) $\mathfrak{d}_n = h_{1^n} - \sum_{j=0}^{n-1} \mathfrak{d}_j h_{n-j}$ $\big($*or equivalently*, $h_{1^n} = \sum_{j=0}^{n} \mathfrak{d}_j h_{n-j}\big)$ *for* $n \geq 1$;

(D) $\mathfrak{d}_n = \sum_Q s_{\lambda(Q)}$, *where Q runs through the desarrangement tableaux of size* n;

(E) $\mathfrak{d}_n = \sum_w L_{n,\mathrm{Des}(w)}$, *where w runs through all desarrangements in* $\mathfrak{S}_n$;

(F) $\mathfrak{d}_n = \sum_w L_{n,\mathrm{Des}(w)}$, *where w runs through all derangements in* $\mathfrak{S}_n$;

(G) $\mathfrak{d}_n = \sum_w \mathfrak{L}_{\lambda(w)}$, *where w runs through all derangements in* $\mathfrak{S}_n$.

We will mainly need definition (C) for $\mathfrak{d}_n$. However, we wish to point out that part (D) decomposes $\mathfrak{d}_n$ very explicitly into Schur functions, illustrated in Table 1 for $n = 0, 1, 2, 3, 4$.

<table>
<tr><th>n</th><th>desarrangement tableaux Q</th><th>$\mathfrak{d}_n$</th></tr>
<tr><td>0</td><td>$\varnothing$</td><td>1</td></tr>
<tr><td>1</td><td>none</td><td>0</td></tr>
<tr><td>2</td><td>$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$</td><td>$s_{(1,1)}$</td></tr>
<tr><td>3</td><td>$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$</td><td>$s_{(2,1)}$</td></tr>
<tr><td>4</td><td>$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$</td><td>$s_{(1,1,1,1)} + s_{(2,1,1)} + s_{(2,2)} + s_{(3,1)}$</td></tr>
</table>

Table 1. Decomposition of $\mathfrak{d}_n$ into Schur functions for $n = 0, 1, 2, 3, 4$.

Sketch proof of Proposition 3.1.. We sketch some of the equivalences here. The equivalence of (A) and (B) is straightforward. Defining $\{\mathfrak{d}_n\}$ by (A), note they satisfy definition (C) by induction on n:

$$
\begin{aligned}
(13)\qquad \sum_{j=0}^{n} \mathfrak{d}_j h_{n-j} = \Big(\sum_{j=1}^{n} \mathfrak{d}_j h_{n-j}\Big) + h_n \;&= \Big(\sum_{j=1}^{n} (h_1 \mathfrak{d}_{j-1} + (-1)^j e_j)\cdot h_{n-j}\Big) + h_n \\
&= h_1 \sum_{j=1}^{n} \mathfrak{d}_{j-1} h_{n-j} \overset{(*)}{=} \sum_{j=0}^{n} (-1)^j e_j h_{n-j} \\
&\overset{(**)}{=} h_1 \cdot h_{1^{n-1}} + 0 = h_{1^n}.
\end{aligned}
$$

Here, equality $(*)$ used $\sum_{j=0}^{n}(-1)^j e_j h_{n-j} = 0$ for $n \geq 1$, and equality $(**)$ used induction. Consequently, (A) and (C) define the same sequence of polynomials $\{\mathfrak{d}_n\}$, and so (A), (B), and (C) coincide.

Defining $\{\mathfrak{d}_n\}$ by the explicit sum (D), let us check that they also satisfy the recursive definition (A) by induction on n. In the base case $n = 0$, both have $\mathfrak{d}_0 = 1$, since the unique (empty) tableau of size 0 is a desarrangement tableau. In the inductive step, using the Pieri formula shows that $h_1 \cdot \mathfrak{d}_{n-1}$ is the sum over all standard tableaux of size n obtained from a desarrangement tableau Q of size $n-1$ by adding n in any corner cell. This produces all desarrangement tableaux of size n, except the single column tableau Q_0 which:

- is produced for n odd, but is *not* a desarrangement tableaux, and
- is not produced for n even, but *is* a desarrangement tableau.

These exceptions are corrected by $(-1)^n e_n$ in the formula $\mathfrak{d}_n = h_1 \mathfrak{d}_{n_1} + (-1)^n e_n$ in (A). Consequently, (A) and (D) define the same sequence of polynomials $\{\mathfrak{d}_n\}$.

The equivalence of (D) and (E) uses two facts. First, applying the Robinson–Schensted bijection to w to obtain a pair of standard Young tableaux (P, Q), one has $\mathrm{Des}(w) = \mathrm{Des}(Q)$; see [Stanley 1999, Lemma 7.23.1]. Thus, w is a desarrangement if and only if Q is a desarrangement tableau[5]. Second, $s_\lambda = \sum_P L_{\mathrm{Des}(P)}$, where P runs over standard Young tableaux of shape λ, by [Stanley 1999, Theorem 7.19.7].

The equivalence of (E) and (F) was proven by Désarménien and Wachs [1988], where they showed that both families of symmetric functions defined in (E) and (F) satisfy the recursive definition (C). Their proof also used the equivalence of (F) and (G) that follows from the definition of $\mathfrak{L}_\lambda$. □

Note that part (B) of Proposition 3.1 generalizes the formulas in (12), upon taking dimensions of the various representations and using (8) and (10). Similarly,

[5]Our earlier examples $w = (6, 3, 5, 2, 1, 4)$ and Q also exemplify this, as $w \mapsto (P, Q)$ with $Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$ and $P = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$.

part (C) corresponds to the formulas:

$$\begin{aligned} \dim_{\boldsymbol{k}} \boldsymbol{k}\mathfrak{S}_n &= n! = \sum_{j=0}^{n} d_{n-j}\binom{n}{j}, \\ \dim_{\boldsymbol{k}} \boldsymbol{k}[\mathrm{GL}_n/B] &= [n]!_q = \sum_{j=0}^{n} d_{n-j}(q)\begin{bmatrix} n \\ j \end{bmatrix}_q, \end{aligned} \tag{14}$$

after taking into account (9) and (11).

We conclude this section with some further historical remarks and context on the derangement representations $\mathcal{D}_n$ and symmetric functions $\mathfrak{d}_n$.

Remark 3.2. We are claiming no originality in Proposition 3.1. As mentioned in its proof, the equivalence of (C), (E), (F), and (G) is work of Désarménien and Wachs [1988]. In [Reiner and Webb 2004, Propositions 2.2, 2.1, and 2.3], it is noted that one can repackage their results to include part (D). It was also noted there that the tensor product $\mathrm{sgn} \otimes \mathcal{D}_n$ of $\mathcal{D}_n$ with the one-dimensional sign representation sgn of $\mathfrak{S}_n$, carries the same $\boldsymbol{k}\mathfrak{S}_n$-module as the homology of the *complex of injective words* on n letters. Therefore, after tensoring with the sign character of $\mathfrak{S}_n$ or applying the fundamental involution ω on symmetric functions, parts (A), (C), and (D) above correspond to [Reiner and Webb 2004, Propositions 2.2, 2.1, and 2.3].

Remark 3.3. It was noted in [Hersh and Reiner 2017] that $\mathcal{D}_n$ occurs naturally in the *representation stability and FI-module structure* (as in Church, Ellenberg, and Farb [Church et al. 2015]) on the cohomology of the configuration space of n labeled points in $\mathbb{R}^d$ for d odd. Specifically, $\mathcal{D}_n$ is the $\boldsymbol{k}\mathfrak{S}_n$-module on the subspace of *FI-module generators* for this cohomology, denoted $\widehat{\mathrm{Lie}}_n$ in [Hersh and Reiner 2017, Theorems 1.2 and 1.3].

Remark 3.4. As hinted at in Section 1, $\mathcal{D}_n$ also occurs as the $\boldsymbol{k}\mathfrak{S}_n$-module on the kernel of two shuffling operators on $\boldsymbol{k}\mathfrak{S}_n$, both studied by Uyemura-Reyes: *random-to-top* shuffles [2002, §1.1.7, §3.2.2, and §4.5.3] (also known as the *Tsetlin library*) and *random-to-random* shuffles [2002, Chapter 5]; see also [Steinberg 2016, Proposition 14.5] and Section 4A below. More generally, Uyemura-Reyes [2002, Theorem 4.1] described the $\boldsymbol{k}\mathfrak{S}_n$-module structure on the eigenspaces for *all* Bidigare–Hanlon–Rockmore shuffling operators that carry $\mathfrak{S}_n$-symmetry. Among these are random-to-top shuffles, whose eigenvalue multiplicities had previously been computed by Phatarfod [1991], ignoring the $\boldsymbol{k}\mathfrak{S}_n$-module structure. See also the discussion by Hanlon and Hersh [2004, §3] and by Saliola, Welker, and Reiner [Reiner et al. 2014, §VI.9].

Remark 3.5. In unpublished notes, Garsia [2012] (see also Tian [2016]), studied the *top-to-random* shuffling operator, which is adjoint or transpose to the random-to-top operator. There he sketched a proof that its minimal polynomial is $X(X-1)(X-2)\cdots(X-n)$. The element x acts as (rescaled) random-to-top on the chamber space of $\mathcal{F}_n$ (see 4A). In light of the fact that an operator and its transpose have the same minimal polynomial, Garsia's sketch is closely related to the part of our proof of Theorem 2.9 dealing with $M = \boldsymbol{k}\mathcal{F}_n$.

4. Answering Question 1.2

Our goal here is to answer Question 1.2, by describing the $\boldsymbol{k}G$-module decompositions on the eigenspaces of $x, x^{(q)}$ as they act on $\boldsymbol{k}M$ for $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$.

Recall the ($\boldsymbol{k}$-vector space) direct sum decompositions by length:

$$\boldsymbol{k}\mathcal{F}_n = \bigoplus_{\ell=0}^{n} \boldsymbol{k}\mathcal{F}_{n,\ell}, \quad \text{where } \mathcal{F}_{n,\ell} := \{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) = \ell\},$$
$$\boldsymbol{k}\mathcal{F}_n^{(q)} = \bigoplus_{\ell=0}^{n} \boldsymbol{k}\mathcal{F}_{n,\ell}^{(q)}, \quad \text{where } \mathcal{F}_{n,\ell}^{(q)} := \{\boldsymbol{A} \in \mathcal{F}_n^{(q)} : \ell(\boldsymbol{A}) = \ell\}.$$

Following Brown [2000], we call the monoid elements of $\mathcal{F}_{n,n}$ and $\mathcal{F}_{n,n}^{(q)}$ of maximum length *chambers*. Their $\boldsymbol{k}$-spans $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$, which we call the *chamber spaces*, form submodules for the action of both the monoid algebras $\boldsymbol{k}M$ and the group algebras $\boldsymbol{k}G$. We first analyze the structure of these chamber spaces in Section 4A, and then use this to analyze the entire semigroup algebra $\boldsymbol{k}M$ in Section 4B.

4A. ***The chamber spaces.*** The chamber space $\boldsymbol{k}\mathcal{F}_{n,n}$ consists of all words of length n. Thus, as a $\boldsymbol{k}\mathfrak{S}_n$ module it is isomorphic to the left regular-representation $\boldsymbol{k}\mathfrak{S}_n$. Similarly, $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ has as a $\boldsymbol{k}$-basis the set $\mathcal{F}(V) = \{\boldsymbol{A} = (A_1, \dots, A_n)\}$ of all complete flags $A_1 \subset \cdots \subset A_{n-1} \subset A_n (= V)$, and is isomorphic to the coset representation of GL_n on $\boldsymbol{k}[\mathrm{GL}_n / B]$.

We start with an old observation: multiplication by x acts on $\mathcal{F}_{n,n}$ as a (rescaled) version of the random-to-top operator on $\boldsymbol{k}\mathfrak{S}_n$; see, for instance, B. Steinberg [2016, Proposition 14.5].

Example 4.1. If $n = 4$ and $w = (3, 1, 4, 2)$ in $\mathcal{F}_{4,4}$, then

$$\begin{aligned} x \cdot w &= ((1) + (2) + (3) + (4)) \cdot (3, 1, 4, 2) \\ &= (1, 3, 4, 2) + (2, 3, 1, 4) + (3, 1, 4, 2) + (4, 3, 1, 2), \end{aligned}$$

which $\left(\text{after scaling by } \frac{1}{4}\right)$ is the result of random-to-top shuffling on w as an element of $\boldsymbol{k}\mathfrak{S}_4$.

In this sense, the results in this section for the chamber space $\boldsymbol{k}\mathcal{F}_{n,n}$ are repackaging previously mentioned results on random-to-top shuffling and the $\mathfrak{S}_n$-action on its eigenspaces, due to Uyemura-Reyes [2002, Theorem 4.19], building on the computation of Phatarfod [1991] of the eigenvalue multiplicities. On the other hand, as far as we aware, our results for the q-analogue $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ in Theorem 4.2 are new.

We record here the action of $x^{(q)}$ on a complete flag $\boldsymbol{A}$ in $V = (\mathbb{F}_q)^n$, using Definition 2.2:

$$x^{(q)} \cdot \boldsymbol{A} = \sum_{\text{lines } L \in V} (L) \cdot \boldsymbol{A} = \sum_{\text{lines } L \in V} (L, L+A_1, L+A_2, \dots, L+A_{n-1}, L+A_n)^{\wedge}.$$

For $j = 0, 1, \dots n$, we will write the j- and $[j]_q$-eigenspaces of the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ as

$$\ker\big((x - j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big) \quad \text{and} \quad \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big).$$

In Theorem 4.2 below, we relate these j- and $[j]_q$-eigenspaces to $\mathcal{D}_{n-j}$ and $\mathcal{D}_{n-j}^{(q)}$. Our proof depends crucially on Proposition 4.5, Proposition 4.7, and Lemma 4.8 (all proved in Section 4A1) wherein we explicitly construct eigenvectors for the action of x and $x^{(q)}$ on the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ from the null vectors of the same operators for smaller n.

Theorem 4.2. *When x and $x^{(q)}$ act on $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$, for each $j = 0, 1, 2, \dots, n$, their eigenspaces carry representations with the same Frobenius map images*

$$\operatorname{ch} \ker\big((x - j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big) = h_j \cdot \mathfrak{d}_{n-j} = \operatorname{ch}_q \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big).$$

In other words, one has $\boldsymbol{k}G$-module isomorphisms:

$$\ker\big((x - j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big) \cong \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{n-j},$$
$$\ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big) \cong \mathbf{1}_{\mathrm{GL}_j} * \mathcal{D}_{n-j}^{(q)}.$$

Proof. Lemma 4.8 below exhibits G-equivariant injections

$$\begin{aligned} \mathbf{1}_{\mathfrak{S}_j} * \ker(x|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}}) &\hookrightarrow \ker\big((x - j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big), \\ \mathbf{1}_{\mathrm{GL}_j} * \ker\big(x^{(q)}|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}^{(q)}}\big) &\hookrightarrow \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big). \end{aligned} \tag{15}$$

We now use facts proven by Phatarfod [1991] for $q = 1$ and by Brown [2000, §5.2] for the q-analogue[6]:

$$\dim_{\boldsymbol{k}} \ker(x|_{\boldsymbol{k}\mathcal{F}_{n,n}}) = d_n \quad \text{and} \quad \dim_{\boldsymbol{k}} \ker\big(x^{(q)}|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big) = d_n(q).$$

Hence, the spaces on the left sides in (15) have dimensions $d_{n-j}\binom{n}{j}$ and $d_{n-j}(q)\left[{n \atop j}\right]_q$, respectively. However, since eigenspaces for distinct eigenvalues are always linearly independent, and since

$$\boldsymbol{k}\mathcal{F}_{n,n} \cong \boldsymbol{k}\mathfrak{S}_n \quad \text{and} \quad \boldsymbol{k}\mathcal{F}_{n,n}^{(q)} \cong \boldsymbol{k}[\mathrm{GL}_n / B]$$

[6]A minor discrepancy here is that Brown analyzes the action of $x^{(q)}$ not on the chamber space of $\boldsymbol{k}\mathcal{F}_n^{(q)}$ itself, but rather on the chamber space of the quotient $\boldsymbol{k}\overline{\mathcal{F}}_n^{(q)}$ discussed in Remark 2.3 above. However, just as Brown points out for $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$ in [Brown 2000, Remark, p. 888], the bijection $(A_1, A_2, \dots, A_{n-1}, V) \mapsto (A_1, A_2, \dots, A_{n-1})$ between chambers of $\mathcal{F}_n^{(q)}$ and those of $\boldsymbol{k}\overline{\mathcal{F}}_n^{(q)}$ will commute with both the action of GL_n and with multiplication by $x^{(q)}$.

have dimensions $n!$ and $[n]!_q$, the equations in (14) imply that the injections in (15) must all be isomorphisms.

It also follows from the above analysis, or from Theorem 2.9 (ii), that

$$\boldsymbol{k}\mathfrak{F}_{n,n} = \bigoplus_{j=0}^{n} \ker\big((x-j)|_{\boldsymbol{k}\mathfrak{F}_{n,n}}\big) \quad \text{and} \quad \boldsymbol{k}\mathfrak{F}^{(q)}_{n,n} = \bigoplus_{j=0}^{n} \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathfrak{F}^{(q)}_{n,n}}\big).$$

Then using (9) and (11) and comparing with Proposition 3.1 (C), the theorem follows. □

4A1. *Constructing eigenvectors from null vectors: proof of Lemma 4.8.* The goal of this subsection is to prove Lemma 4.8. It relies on parallel constructions[7] of eigenvectors for x and $x^{(q)}$ acting on the spaces $\boldsymbol{k}\mathfrak{F}_{n,n}$ and $\boldsymbol{k}\mathfrak{F}^{(q)}_{n,n}$ from null vectors for the same operators for smaller n.

Definition 4.3. Let $[n] := \{1, 2, \ldots, n\}$, and fix a j-element subset U of $\{1, 2, \ldots, n\}$. Let $\mathfrak{S}_{[n]\setminus U}$ denote the permutations $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n-j})$ of the complementary subset $[n] \setminus U$, written in one-line notation. On the $\boldsymbol{k}$-vector space $\boldsymbol{k}[\mathfrak{S}_{[n]\setminus U}]$ having these permutations as a $\boldsymbol{k}$-basis, define two maps $\Psi_U, \Phi_U : \boldsymbol{k}[\mathfrak{S}_{[n]\setminus U}] \to \boldsymbol{k}[\mathfrak{S}_n]$ by extending these rules $\boldsymbol{k}$-linearly:

$$\Psi_U(\boldsymbol{a}) := \sum_{\boldsymbol{b}\in\mathfrak{S}_U} (b_1, b_2, \ldots, b_j, a_1, a_2, \ldots, a_{n-j}),$$

$$\Phi_U(\boldsymbol{a}) := \sum_{\boldsymbol{b}\in\mathfrak{S}_U} (a_1, b_1, b_2, \ldots, b_j, a_2, \ldots, a_{n-j}),$$

where the summation indices $\boldsymbol{b}$ run over all permutations $\boldsymbol{b} = (b_1, b_2, \ldots, b_j)$ in $\mathfrak{S}_U$.

Example 4.4. Let $n = 5$ and $U = \{4, 5\}$. Then

$$\Psi_U((1,2,3)) = (\mathbf{4}, \mathbf{5}, 1, 2, 3) + (\mathbf{5}, \mathbf{4}, 1, 2, 3),$$

$$\Phi_U((1,2,3)) = (1, \mathbf{4}, \mathbf{5}, 2, 3) + (1, \mathbf{5}, \mathbf{4}, 2, 3).$$

To state the next proposition, introduce for $U \subseteq [n]$ the *free left-regular band* $\mathfrak{F}_U$ *on* U, having an obvious isomorphism $\mathfrak{F}_U \cong \mathfrak{F}_j$ if $j = |U|$. Also let $x_U := \sum_{i\in U}(i)$ inside $\boldsymbol{k}\mathfrak{F}_U$.

Proposition 4.5. *Fix a j-element subset U of $[n]$ and a permutation $\boldsymbol{a}$ in $\mathfrak{S}_{[n]\setminus U}$. Then*

$$x \cdot \Psi_U(\boldsymbol{a}) = j \cdot \Psi_U(\boldsymbol{a}) + \Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a}).$$

Consequently, if v in $\boldsymbol{k}\mathfrak{F}_{[n]\setminus U, n-j}$ has $x_{[n]\setminus U} \cdot v = 0$, then $\Psi_U(v)$ is a j-eigenvector for x on $\boldsymbol{k}\mathfrak{F}_{n,n}$:

$$x \cdot \Psi_U(v) = j \cdot \Psi_U(v).$$

[7]Reiner is grateful to Michelle Wachs for explaining to him the $\boldsymbol{k}\mathfrak{F}_n$ version of this construction (the operator Ψ_U) in 2002, in the context of random-to-top shuffling.

Proof. One can calculate that

$$x \cdot \Psi_U(\boldsymbol{a}) = \sum_{i=1}^{n} (i) \cdot \Psi_U(\boldsymbol{a}) = \sum_{i \in U} (i) \cdot \Psi_U(\boldsymbol{a}) + \sum_{i \in [n] \setminus U} (i) \cdot \Psi_U(\boldsymbol{a})$$
$$= j \cdot \Psi_U(\boldsymbol{a}) + \Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a}),$$

where we explain here the two substitutions in the last equality. The fact that the left sum equals $j \cdot \Psi_U(\boldsymbol{a})$ follows from the last equation $x \cdot x_j = j \cdot x_j$ in Lemma 2.5 applied to $\boldsymbol{k}\mathcal{F}_U \cong \boldsymbol{k}\mathcal{F}_j$. The fact that the right sum is $\Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a})$ follows via direct calculation from the definitions. □

We next introduce two q-analogous maps $\Psi_U^{(q)}$ and $\Phi_U^{(q)}$.

Definition 4.6. Fix U a j-dimensional $\mathbb{F}_q$-linear subspace of $V = (\mathbb{F}_q)^n$. Let $\mathcal{F}(V/U)$ denote the set of maximal flags in the quotient space V/U

$$\boldsymbol{A} = (A_1, A_2 \dots, A_{n-j-1}, \underbrace{A_{n-j}}_{=V/U}).$$

On the space $\boldsymbol{k}[\mathcal{F}(V/U)]$ with these flags as a $\boldsymbol{k}$-basis, we define the maps $\Psi_U^{(q)}, \Phi_U^{(q)} : \boldsymbol{k}[\mathcal{F}(V/U)] \to \boldsymbol{k}[\mathcal{F}(V)]$ by extending the following rules $\boldsymbol{k}$-linearly:

$$\Psi_U^{(q)}(\boldsymbol{A}) := \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (B_1, B_2, \dots, B_{j-1}, U, A_1 + U, A_2 + U, \dots, A_{n-j-1} + U, V),$$

$$\Phi_U^{(q)}(\boldsymbol{A}) := \sum_{\substack{\text{lines } L: \\ L \subset U + A_1, \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L, L + B_1, \dots, L + B_{j-1}, \overbrace{L + U}^{=L+U+A_1}, L + U + A_2, \dots, L + U + A_{n-j-1}, V),$$

where the summation indices $\boldsymbol{B}$ run over all complete flags $\boldsymbol{B} = (B_1, \dots, B_{j-1}, U)$ in $\mathcal{F}(U)$.

To state the next proposition, introduce for any $\mathbb{F}_q$-vector space U of dimension j the monoid $\mathcal{F}_U^{(q)} \cong \mathcal{F}_j^{(q)}$ by identifying $U \cong \mathbb{F}_q^j$. Also introduce the element of the monoid algebra $\boldsymbol{k}\mathcal{F}_U^{(q)}$

$$x_U^{(q)} := \sum_{\text{lines } L \text{ in } U} (L).$$

Proposition 4.7. *For a j-dimensional subspace U of $V = (\mathbb{F}_q)^n$ and complete flag $\boldsymbol{A}$ in $\mathcal{F}(V/U)$,*

$$x^{(q)} \cdot \Psi_U^{(q)}(\boldsymbol{A}) = [j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \Phi_U^{(q)}\big(x_{V/U}^{(q)} \cdot \boldsymbol{A}\big).$$

Hence if v in $\boldsymbol{k}\mathcal{F}_{V/U,n-j}^{(q)}$ has $x_{V/U}^{(q)} \cdot v = 0$, then $\Psi_U^{(q)}(v)$ is a $[j]_q$-eigenvector for $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$:

$$x^{(q)} \cdot \Psi_U^{(q)}(v) = [j]_q \cdot \Psi_U^{(q)}(v).$$

Proof. One can calculate that

$$x^{(q)} \cdot \Psi_U^{(q)}(\boldsymbol{A}) = \sum_{\substack{\text{lines } L \\ \text{in } V}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}) = \sum_{\substack{\text{lines } L \\ \text{in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \sum_{\substack{\text{lines } L \\ \text{not in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A})$$
$$= [j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \Phi_U^{(q)}(x_{V/U}^{(q)} \cdot \boldsymbol{A}),$$

where we explain here the two substitutions in the last equality. The fact that the left sum equals $[j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A})$ follows from the last equation $x^{(q)} \cdot x_j^{(q)} = [j]_q \cdot x_j^{(q)}$ in Lemma 2.5 applied to $\boldsymbol{k}\mathcal{F}_U^{(q)} \cong \boldsymbol{k}\mathcal{F}_j^{(q)}$. To check the substitution made for the right sum, one calculates directly that

$$\begin{aligned}
&\Phi_U^{(q)}(x_{V/U}^{(q)} \cdot \boldsymbol{A}) \\
&\quad= \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \Phi_U^{(q)}((\bar{L}) \cdot \boldsymbol{A}) \\
&\quad= \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \Phi_U^{(q)}\big((\bar{L}, \bar{L}+A_1, \bar{L}+A_2, \dots, \bar{L}+A_{n-j-1}, V/U)^\wedge\big) \\
&\quad= \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \sum_{\substack{\text{lines } L \text{ in } V \\ L \subset U+\bar{L} \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L, L+B_1, \dots, L+B_{j-1}, \overbrace{L+B_j}^{=L+U}, \\
&\qquad\qquad\qquad L+U+A_1, \dots, L+U+A_{n-j-1}, V)^\wedge \\
&\quad= \sum_{\substack{\text{lines } L \text{ in } V \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L) \cdot (B_1, \dots, B_{j-1}, \overbrace{B_j}^{=U}, A_1+U, A_2+U, \dots, A_{n-j-1}+U, V) \\
&\quad= \sum_{\substack{\text{lines } L \\ \text{not in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}). \qquad \square
\end{aligned}$$

We are at last ready to prove Lemma 4.8.

Lemma 4.8. *With our usual notation of $G = \mathfrak{S}_n, \mathrm{GL}_n$ acting on $\boldsymbol{k}M$ for $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, one has G-equivariant injections for $j = 0, 1, 2, \dots, n$:*

$$\mathbf{1}_{\mathfrak{S}_j} * \ker(x|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}}) \hookrightarrow \ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big),$$
$$\mathbf{1}_{\mathrm{GL}_j} * \ker(x^{(q)}|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}^{(q)}}) \hookrightarrow \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big).$$

Proof. We give the proof for $\mathcal{F}_n^{(q)}$; the proof for $\mathcal{F}_n$ is analogous, but easier.

For each j-dimensional subspace U of $V = (\mathbb{F}_q)^n$, define a subspace $E(U)$ of $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ as the image under $\Psi_U^{(q)}$ of the nullspace for $x^{(q)} = x_{V/U}^{(q)}$ acting on $\boldsymbol{k}\mathcal{F}_{V/U,n-j}^{(q)} \cong \boldsymbol{k}\mathcal{F}_{n-j,n-j}^{(q)}$:

$$E(U) := \Psi_U^{(q)}\big(\ker x^{(q)}|_{\boldsymbol{k}\mathcal{F}_{V/U,n-j}^{(q)}}\big).$$

According to Proposition 4.7, each $E(U)$ is a subspace of the $[j]_q$-eigenspace $\ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\big)$. Note that vectors in $E(U)$ are sums of complete flags $\boldsymbol{A} = (A_1, \dots, A_n)$ that pass through $A_j = U$, and hence for $U \neq U'$, they are supported on basis elements of $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ indexed by disjoint sets of complete flags. Therefore, the subspace sum of all $E(U)$ is a direct sum $\bigoplus_U E(U)$ inside this $[j]_q$-eigenspace for x. It only remains to produce an isomorphism of GL_n-representations

$$\bigoplus_U E(U) \cong \mathbf{1}_{\mathrm{GL}_j} * \ker\bigl(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}\bigr). \tag{16}$$

Recall that GL_n acts transitively on j-subspaces U. Fix the particular subspace U_0 spanned by the first j standard basis vectors in $V = (\mathbb{F}_q)^n$, whose GL_n-stabilizer is the maximal parabolic subgroup $P_{j,n-j}$. It follows (see, e.g., Webb [2016, Proposition 4.3.2]) that $\bigoplus_U E(U)$ carries the GL_n-representation induced from $P_{j,n-j}$ acting on $E(U_0)$. However, because elements in $E(U_0)$ are supported on flags $\boldsymbol{A}$ in $E(U_0)$ that all pass through $A_j = U_0$, this $P_{j,n-j}$-action is inflated through the surjection $P_{j,n-j} \twoheadrightarrow \mathrm{GL}_j \times \mathrm{GL}_{n-j}$. Furthermore, the definition of $\Psi^{(q)}_{U_0}(-)$ as a symmetrized sum over complete flags in U_0 shows that GL_j fixes elements of $E(U_0)$ pointwise, while elements of GL_{n-j} act as they do on $\ker\bigl(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}\bigr)$. Comparing with (7) proves the desired isomorphism (16). □

4B. ***The entire semigroup algebra.*** Having described the eigenspaces of the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}$ as G-representations, we now turn to the entire semigroup algebras $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}^{(q)}_n$.

Our strategy here will be to introduce filtrations on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}^{(q)}_n$, and study the action of x and $x^{(q)}$ on the associated graded modules with respect to these filtrations. (Recall from the discussion in Section 3A that by semisimplicity, this is an equivalent way to understand the R^G and $\boldsymbol{k}G$-module structures on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}^{(q)}_n$.)

Recall that for $\boldsymbol{a} \in \mathcal{F}_n$ and $\boldsymbol{A} \in \mathcal{F}^{(q)}_n$ the length of $\boldsymbol{a}$ is $\ell(\boldsymbol{a})$ and the length of $\boldsymbol{A}$ is $\ell(\boldsymbol{A})$.

Definition 4.9. Define

$$\boldsymbol{k}\mathcal{F}_{n,\geq \ell} := \mathrm{span}_{\boldsymbol{k}}\{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) \geq \ell\},$$
$$\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell} := \mathrm{span}_{\boldsymbol{k}}\{\boldsymbol{A} \in \mathcal{F}^{(q)}_n : \ell(\boldsymbol{A}) \geq \ell\}.$$

In other words, $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}$ and $\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}$ are the $\boldsymbol{k}$-spans of the monoid elements of length at least ℓ.

We then introduce filtrations $\{\boldsymbol{k}\mathcal{F}_{n,\geq \ell}\}_{\ell=0,1,\dots,n,n+1}$ and $\{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}\}_{\ell=0,1,\dots,n,n+1}$:

$$\begin{aligned}\{0\} &= \boldsymbol{k}\mathcal{F}_{n,\geq n+1} \subset \boldsymbol{k}\mathcal{F}_{n,\geq n} \subset \cdots \subset \boldsymbol{k}\mathcal{F}_{n,\geq 1} \subset \boldsymbol{k}\mathcal{F}_{n,\geq 0} = \boldsymbol{k}\mathcal{F}_n,\\ \{0\} &= \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq n+1} \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq n} \subset \cdots \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq 1} \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq 0} = \boldsymbol{k}\mathcal{F}^{(q)}_n.\end{aligned} \tag{17}$$

Since $\ell(\boldsymbol{a} \cdot \boldsymbol{b}) \geq \ell(\boldsymbol{b})$, it is easily seen that each $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}$ is a $\boldsymbol{k}\mathcal{F}_n$-submodule, and a $\boldsymbol{k}\mathfrak{S}_n$-submodule. Analogously, $\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}$ is a $\boldsymbol{k}\mathcal{F}^{(q)}_n$-submodule, and a $\boldsymbol{k}\,\mathrm{GL}_n$-submodule.

Recall that for $U \subset [n]$ of size j one has $\mathcal{F}_U \cong \mathcal{F}_j$ and $x_U = \sum_{i \in U}(i)$. Analogously, recall that for U a j-dimensional subspace of V, one has $\mathcal{F}_U^{(q)} \cong \mathcal{F}_j^{(q)}$ and

$$x_U^{(q)} = \sum_{\text{lines } L \in U} (L).$$

Both $\boldsymbol{k}\mathcal{F}_U$ and $\boldsymbol{k}\mathcal{F}_U^{(q)}$ have $\boldsymbol{k}$-vector space direct sum decompositions defined by length of words, so that one can identify $\boldsymbol{k}\mathcal{F}_{U,\ell} \cong \boldsymbol{k}\mathcal{F}_{j,\ell}$ and $\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)} \cong \boldsymbol{k}\mathcal{F}_{j,\ell}^{(q)}$ for $\ell = 0, 1, \ldots, j$.

As $\boldsymbol{k}$-vector spaces, one has a direct sum decomposition for the filtration factors

$$\begin{aligned} \boldsymbol{k}\mathcal{F}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1} &= \bigoplus_{\substack{U \subseteq \{1,2,\ldots,n\} \\ |U| = \ell}} \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}, \\ \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell+1} &= \bigoplus_{\substack{\mathbb{F}_q\text{-subspaces } U \subseteq (\mathbb{F}_q)^n \\ \dim(U) = \ell}} \overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}, \end{aligned} \tag{18}$$

where $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}$ and $\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}$ denote the image of the subspaces $\boldsymbol{k}\mathcal{F}_{U,\ell}$ and $\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}$ within the quotient on the left. The next proposition is a simple but crucial observation about these summands in (18) that is used in the proof of Theorem 1.4.

Proposition 4.10. *Consider the summands on the right sides of* (18).

- *Each* $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}$ *is a* $\boldsymbol{k}\mathcal{F}_n$*-submodule of* $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1}$, *annihilated by* (j) *for* $j \notin U$.
- *Each* $\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}$ *is a* $\boldsymbol{k}\mathcal{F}^{(q)}_n$*-submodule of* $\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell+1}$, *annihilated by* (L) *for lines* $L \not\subset U$.

Consequently, one has

$$\begin{aligned} x \cdot \bar{\boldsymbol{a}} &= x_U \cdot \bar{\boldsymbol{a}}, \quad &&\textit{for } \bar{\boldsymbol{a}} \textit{ in } \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}, \\ x^{(q)} \cdot \overline{\boldsymbol{A}} &= x_U^{(q)} \cdot \overline{\boldsymbol{A}}, \quad &&\textit{for } \overline{\boldsymbol{A}} \textit{ in } \overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}. \end{aligned}$$

Proof by example. Consider $n = 3$, with $\ell = 2$ and $U = \{1, 2\}$. Then working in the quotient $\overline{\boldsymbol{k}\mathcal{F}_{U,2}}$, because $3 \notin U$, the element (3) of $\boldsymbol{k}\mathcal{F}_3$ will annihilate the element $\overline{(1, 2)}$ of $\boldsymbol{k}\mathcal{F}_{3,\geq 2}/\boldsymbol{k}\mathcal{F}_{3,\geq 3}$. One has

$$(3) \cdot \overline{(1, 2)} = \overline{(3, 1, 2)} = 0 \quad \text{in } \boldsymbol{k}\mathcal{F}_{3,\geq 2}/\boldsymbol{k}\mathcal{F}_{3,\geq 3},$$

because $\ell(3, 1, 2) = 3 > 2 = \ell$. Thus, $x = (1) + (2) + (3)$ acts on $\overline{(1, 2)}$ as

$$\begin{aligned} x \cdot \overline{(1, 2)} &= ((1) + (2) + (3)) \cdot \overline{(1, 2)} \\ &= \overline{(1, 2)} + \overline{(2, 1)} + \overline{(3, 1, 2)} = \overline{(1, 2)} + \overline{(2, 1)} = x_U \cdot \overline{(1, 2)}. \end{aligned}$$

The proof for $\mathcal{F}_n^{(q)}$ is analogous: one has $\ell((L) \cdot \boldsymbol{A}) > \ell(\boldsymbol{A}) = \ell$ for lines $L \not\subset U$ and $\boldsymbol{A} \in \mathcal{F}^{(q)}_{U,\ell}$. □

We now prove our main result of this section, encompassing Theorem 1.4 from Section 1.

Theorem 4.11. *Let $\boldsymbol{k}$ be a field in which $|G|$ is invertible. Then x and $x^{(q)}$ act diagonalizably on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$, and for each $j = 0, 1, 2, \ldots, n$, their eigenspaces carry representations with the same Frobenius map images*

$$\operatorname{ch}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) = \sum_{\ell=j}^{n} h_{(n-\ell,j)} \cdot \mathfrak{d}_{\ell-j} = \operatorname{ch}_q \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big).$$

In other words, one has $\boldsymbol{k}G$-module isomorphisms

$$\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j},$$
$$\ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \mathbf{1}_{\mathrm{GL}_j} * \mathcal{D}^{(q)}_{\ell-j}.$$

Proof. The filtrations in (17) show that

$$\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) \cong \bigoplus_{\ell=0}^{n} \ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}_{n,\geq\ell+1}}\big),$$
(19)
$$\ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big) \cong \bigoplus_{\ell=0}^{n} \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell+1}}\big).$$

It remains to analyze each summand on the right.

We have seen that (18) is also a direct sum decomposition as $\boldsymbol{k}M$-modules for $M = \boldsymbol{k}\mathcal{F}_n, \boldsymbol{k}\mathcal{F}^{(q)}$. For $G = \mathfrak{S}_n, \mathrm{GL}_n$, the action of $\boldsymbol{k}M$ and $\boldsymbol{k}G$ on both sides in (18) commute.

In the case of $M = \mathcal{F}_n$, this leads to the following equalities and isomorphisms of $\boldsymbol{k}\mathfrak{S}_n$-modules, explained below. Let $U_0 := \{1, 2, \ldots, \ell\}$. Then:

$$\begin{aligned}
\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}_{n,\geq\ell+1}}\big) &\overset{\text{(i)}}{=} \bigoplus_{\substack{U\subseteq\{1,2,\ldots,n\}:\\|U|=\ell}} \ker((x-j)|_{\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}})\\
&\overset{\text{(ii)}}{=} \bigoplus_{\substack{U\subseteq\{1,2,\ldots,n\}:\\|U|=\ell}} \ker((x_U-j)|_{\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}})\\
&\overset{\text{(iii)}}{\cong} \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \ker\big((x_{U_0}-j)|_{\boldsymbol{k}\mathcal{F}_{\ell,\ell}}\big)\\
&\overset{\text{(iv)}}{\cong} \begin{cases} 0, & \text{if } \ell < j\\ \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j}, & \text{if } \ell \geq j.\end{cases}
\end{aligned}$$

- Equality (i) is the restriction of the $\boldsymbol{k}\mathfrak{S}_n$-module isomorphism (18) to j-eigenspaces for x.
- Equality (ii) arises because x acts the same as x_U on $\overline{\mathcal{F}_{U,\ell}}$, by Proposition 4.10.

- Isomorphism (iii) arises because the summands indexed by U, with $|U| = \ell$, are permuted transitively by $\mathfrak{S}_n$ with the typical summand for $U_0 = \{1, 2, \ldots, \ell\}$ stabilized by the subgroup $\mathfrak{S}_{U_0} \cong \mathfrak{S}_\ell$. Thus, this is an induced $\boldsymbol{k}\mathfrak{S}_n$-module, e.g., by applying [Webb 2016, Proposition 4.3.2].

- Isomorphism (iv) comes from applying Theorem 4.2 to $\boldsymbol{k}\mathcal{F}_\ell$.

The argument for $M = \mathcal{F}_n^{(q)}$ is similar. In particular, setting U_0 to be the $\mathbb{F}_q$-span of the first ℓ standard basis vectors $e_1, e_2, \ldots, e_\ell$ in $(\mathbb{F}_q)^n$, one has equalities and isomorphisms of $\boldsymbol{k}\,\mathrm{GL}_n$-modules:

$$\begin{aligned}
\ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell+1}}\big) &\overset{\text{(i)}}{=} \bigoplus_{\substack{U\subseteq(\mathbb{F}_q)^n:\\ \dim(U)=\ell}} \ker\big((x^{(q)} - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big)\\
&\overset{\text{(ii)}}{=} \bigoplus_{\substack{U\subseteq(\mathbb{F}_q)^n:\\ \dim(U)=\ell}} \ker\big((x_U^{(q)} - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big)\\
&\overset{\text{(iii)}}{\cong} \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \ker\big((x_{U_0}^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big)\\
&\overset{\text{(iv)}}{\cong} \begin{cases} 0, & \text{if } \ell < j,\\ \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \mathbf{1}_{\mathrm{GL}_j} * \mathcal{D}^{(q)}_{\ell-j}, & \text{if } \ell \geq j,\end{cases}
\end{aligned}$$

where isomorphisms (i), (ii), and (iv) are justified exactly as in the proof of $q = 1$ above. For isomorphism (iii), note (as in the proof of Lemma 4.8) that GL_n acts transitively on ℓ-subspaces U, and that U_0 has GL_n-stabilizer subgroup $P_{\ell,n-\ell,}$, so that by [Webb 2016, Proposition 4.3.2],

$$\bigoplus_{\substack{U\subseteq(\mathbb{F}_q)^n:\\ \dim(U)=\ell}} \ker\big((x_U^{(q)} - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big)$$

has the GL_n-representation induced from the $P_{\ell,n-\ell}$-action on $\ker\big((x_{U_0}^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big)$. Since every $A \in \boldsymbol{k}\mathcal{F}^{(q)}_{U_0,\ell} \cong \boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}$ is a flag $(A_1, \ldots, A_\ell)$, with $A_\ell = U_0$, it follows that this $P_{\ell,n-\ell}$-action is inflated through the surjection $P_{\ell,n-\ell} \twoheadrightarrow \mathrm{GL}_\ell \times \mathrm{GL}_{n-\ell}$, where the action of GL_ℓ is as $\ker\big((x_{U_0}^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big)$ and the action of $\mathrm{GL}_{n-\ell}$ is trivial. □

Example 4.12. We illustrate Theorem 1.4 by computing the Frobenius map image for each j-eigenspace of x on $\boldsymbol{k}\mathcal{F}_n$, or equivalently the q-Frobenius map image for each $[j]_q$-eigenspace of $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_n^{(q)}$. For $n = 2, 3$, Tables 2 and 3 show these symmetric functions in their j-th row, decomposed into columns labeled by ℓ, indexing each filtration factor from (18) that contributes a term.

	$\ell=0$	$\ell=1$	$\ell=2$
$j=0$	$h_2\cdot\mathfrak{d}_0 = h_2\cdot s_{()} = s_{(2)}$	$h_1\cdot\mathfrak{d}_1 = h_1\cdot 0 = 0$	$h_0\cdot\mathfrak{d}_2 = h_0\cdot s_{(1,1)} = s_{(1,1)}$
$j=1$		$h_{(1,1)}\cdot\mathfrak{d}_0 = h_{(1,1)}\cdot s_{()} = s_{(1,1)}+s_{(2)}$	$h_1\cdot\mathfrak{d}_1 = h_1\cdot 0 = 0$
$j=2$			$h_2\cdot\mathfrak{d}_0 = h_2\cdot s_{()} = s_{(2)}$

Table 2. Frobenius map images for eigenspaces of x and $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_2$ and $\boldsymbol{k}\mathcal{F}_2^{(q)}$.

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$
$j=0$	$h_3\cdot\mathfrak{d}_0 = h_3\cdot s_{()} = s_{(3)}$	$h_2\cdot\mathfrak{d}_1 = h_2\cdot 0 = 0$	$h_1\cdot\mathfrak{d}_2 = h_1\cdot s_{(1,1)} = s_{(2,1)}+s_{(1,1,1)}$	$h_0\cdot\mathfrak{d}_3 = h_0\cdot s_{(2,1)} = s_{(2,1)}$
$j=1$		$h_{(2,1)}\cdot\mathfrak{d}_0 = h_{(2,1)}\cdot s_{()} = s_{(3)}+s_{(2,1)}$	$h_{(1,1)}\cdot\mathfrak{d}_1 = h_{(1,1)}\cdot 0 = 0$	$h_1\cdot\mathfrak{d}_2 = h_1\cdot s_{(1,1)} = s_{(2,1)}+s_{(1,1,1)}$
$j=2$			$h_{(2,1)}\cdot\mathfrak{d}_0 = h_{(2,1)}\cdot s_{()} = s_{(3)}+s_{(2,1)}$	$h_2\cdot\mathfrak{d}_1 = h_2\cdot 0 = 0$
$j=3$				$h_3\cdot\mathfrak{d}_0 = h_3\cdot s_{()} = s_{(3)}$

Table 3. Frobenius map images for eigenspaces of x and $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_3$ and $\boldsymbol{k}\mathcal{F}_3^{(q)}$.

Acknowledgements

The authors are very grateful to Darij Grinberg for helpful references and conversations, to Franco Saliola for useful discussions, and to Peter Webb for organizing a reading seminar on Benjamin Steinberg's text [2016] that helped spark this project. Reiner thanks Michelle Wachs for helpful discussions on random-to-top shuffling. Brauner and Commins were supported by NSF Graduate Research Fellowships and Reiner by NSF grant DMS-2053288.

References

[Bidigare 1997] T. P. Bidigare, *Hyperplane arrangement face algebras and their associated Markov chains*, Ph.D. thesis, University of Michigan, 1997, https://www.proquest.com/docview/304356095.

[Bidigare et al. 1999] T. P. Bidigare, P. Hanlon, and D. Rockmore, "A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements", *Duke Math. J.* **99**:1 (1999), 135–174. MR Zbl

[Brown 2000] K. S. Brown, "Semigroups, rings, and Markov chains", *J. Theoret. Probab.* **13**:3 (2000), 871–938. MR Zbl

[Cai et al. 2018] Y. Cai, R. Ehrenborg, and M. Readdy, "q-Stirling identities revisited", *Electron. J. Combin.* **25**:1 (2018), art. id. 1.37. MR Zbl

[Carlitz 1933] L. Carlitz, "On abelian fields", *Trans. Amer. Math. Soc.* **35**:1 (1933), 122–136. MR Zbl

[Church et al. 2015] T. Church, J. S. Ellenberg, and B. Farb, "FI-modules and stability for representations of symmetric groups", *Duke Math. J.* **164**:9 (2015), 1833–1910. MR Zbl

[Désarménien and Wachs 1988] J. Désarménien and M. L. Wachs, "Descentes des dérangements et mots circulaires", *Sém. Lothar. Combin.* **19** (1988), art. id. B19a. Zbl

[Désarménien and Wachs 1993] J. Désarménien and M. L. Wachs, "Descent classes of permutations with a given number of fixed points", *J. Combin. Theory Ser. A* **64**:2 (1993), 311–328. MR Zbl

[Dieker and Saliola 2018] A. B. Dieker and F. V. Saliola, "Spectral analysis of random-to-random Markov chains", *Adv. Math.* **323** (2018), 427–485. MR Zbl

[Garsia 2012] A. M. Garsia, "On the powers of top to random shuffling", unpublished lecture notes, Univ. California San Diego, 2012.

[Garsia and Remmel 1986] A. M. Garsia and J. B. Remmel, "Q-counting rook configurations and a formula of Frobenius", *J. Combin. Theory Ser. A* **41**:2 (1986), 246–275. MR Zbl

[Gessel and Reutenauer 1993] I. M. Gessel and C. Reutenauer, "Counting permutations with given cycle structure and descent set", *J. Combin. Theory Ser. A* **64**:2 (1993), 189–215. MR Zbl

[Gould 1961] H. W. Gould, "The q-Stirling numbers of first and second kinds", *Duke Math. J.* **28** (1961), 281–289. MR Zbl

[Grinberg and Reiner 2014] D. Grinberg and V. Reiner, "Hopf algebras in combinatorics", preprint, 2014. arXiv 1409.8356

[Hanlon and Hersh 2004] P. Hanlon and P. Hersh, "A Hodge decomposition for the complex of injective words", *Pacific J. Math.* **214**:1 (2004), 109–125. MR Zbl

[Hersh and Reiner 2017] P. Hersh and V. Reiner, "Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$", *Int. Math. Res. Not.* **2017**:5 (2017), 1433–1486. MR Zbl

[Lafrenière 2020] N. Lafrenière, "Eigenvalues of symmetrized shuffling operators", *Sém. Lothar. Combin.* **82B** (2020), art. id. 78. MR Zbl

[de Médicis and Leroux 1993] A. de Médicis and P. Leroux, "A unified combinatorial approach for q-(and p,q-) Stirling numbers", *J. Statist. Plann. Inference* **34**:1 (1993), 89–105. Correction in **35**:2 (1993), 267. MR Zbl

[Milne 1978] S. C. Milne, "A q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials", *Trans. Amer. Math. Soc.* **245** (1978), 89–118. MR Zbl

[Milne 1982] S. C. Milne, "Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers", *Adv. Math.* **43**:2 (1982), 173–196. MR Zbl

[Phatarfod 1991] R. M. Phatarfod, "On the matrix occurring in a linear search problem", *J. Appl. Probab.* **28**:2 (1991), 336–346. MR Zbl

[Reiner and Webb 2004] V. Reiner and P. Webb, "The combinatorics of the bar resolution in group cohomology", *J. Pure Appl. Algebra* **190**:1-3 (2004), 291–327. MR Zbl

[Reiner et al. 2014] V. Reiner, F. Saliola, and V. Welker, *Spectra of symmetrized shuffling operators*, Mem. Amer. Math. Soc. **1072**, Amer. Math. Soc., Providence, RI, 2014. MR Zbl

[Sagan 1991] B. E. Sagan, *The symmetric group*: *representations*, *combinatorial algorithms*, *and symmetric functions*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991. MR Zbl

[Sagan and Swanson 2022] B. E. Sagan and J. P. Swanson, "q-Stirling numbers in type B", preprint, 2022. arXiv 2205.14078

[Stanley 1999] R. P. Stanley, *Enumerative combinatorics, II*, Cambridge Stud. Adv. Math. **62**, Cambridge Univ. Press, 1999. MR Zbl

[Steinberg 1951] R. Steinberg, "A geometric approach to the representations of the full linear group over a Galois field", *Trans. Amer. Math. Soc.* **71** (1951), 274–282. MR Zbl

[Steinberg 2016] B. Steinberg, *Representation theory of finite monoids*, Springer, 2016. MR Zbl

[Thrall 1942] R. M. Thrall, "On symmetrized Kronecker powers and the structure of the free Lie ring", *Amer. J. Math.* **64** (1942), 371–388. MR Zbl

[Tian 2016] R. Tian, "Generalizations of an expansion formula for top to random shuffles", *Ann. Comb.* **20**:4 (2016), 899–916. MR Zbl

[Uyemura-Reyes 2002] J.-C. Uyemura-Reyes, *Random walk, semi-direct products, and card shuffling*, Ph.D. thesis, Stanford University, 2002, https://www.proquest.com/docview/305519817.

[Wachs 1989] M. L. Wachs, "On q-derangement numbers", *Proc. Amer. Math. Soc.* **106**:1 (1989), 273–278. MR Zbl

[Wachs and White 1991] M. L. Wachs and D. White, "p, q-Stirling numbers and set partition statistics", *J. Combin. Theory Ser. A* **56**:1 (1991), 27–46. MR Zbl

[Webb 2016] P. Webb, *A course in finite group representation theory*, Cambridge Stud. Adv. Math. **161**, Cambridge Univ. Press, 2016. MR Zbl

Received July 20, 2022.

SARAH BRAUNER
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES
braun622@umn.edu

PATRICIA COMMINS
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES
commi010@umn.edu

VICTOR REINER
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES
reiner@umn.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 322 No. 2 February 2023
