Invariants of Inversive 2-structures on Groups of Labels

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Abstract

For a finite set D of nodes let $E_2(D) = \{(x, y) \mid x, y \in D, x \neq y\}$. We define an inversive Δ2-structure g as a function $g\colon E_2(D) \to \Delta$ into a given group Δ satisfying the property $g(x, y) = g(y, x)^{-1}$ for all $(x, y) \in E_2(D)$. For each function (selector) $\sigma\colon D \to \Delta$ there corresponds an inversive Δ2-structure g^σ defined by $g^\sigma(x, y) = \sigma(x) \cdot g(x, y) \cdot \sigma(y)^{-1}$. A function η mapping each g into the group Δ is called an invariant, if $\eta(g^\sigma) = \eta(g)$ for all g and σ. We study the group of free invariants η of inversive Δ2-structures, where η is defined by a word from the free monoid with involution generated by the set $E_2(D)$. In particular, if Δ is abelian, then the group of free invariants is generated by triangle words of the form $(x_0, x_1)(x_1, x_2)(x_2, x_0)$.

Contents

1 Introduction and Motivation

In this paper we shall study complete edge-labeled directed graphs without loops or multiple edges, *i.e.*, labeled 2-structures, see [2], with a group Δ of labels on the edges. Our treatment of these structures is a continuation of the study made in [3]. However, the present paper can be read independently. In particular, our definitions for the (inversive) labeled 2-structures on a group Δ of labels are simplified, but equivalent, versions of those given in [3].

In Section 1.1 we introduce the dynamic labeled 2-structures formally, and in Sections 1.2 and 1.3 we give some motivation of these systems. Invariants of dynamic labeled 2-structures will be studied from Section 3 onwards.

1.1 Group labeled 2-structures

Before motivating the dynamic labeled 2-structures we shall give its formal definition. For this let D be a finite set, and let

$$E_2(D) = \{(x, y) \mid x, y \in D,\ x \neq y\}$$

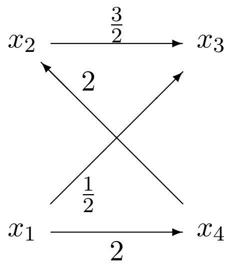

Figure 1: An $\mathbb{R}^+$2-structure g

be the complete set of (directed) *edges* between the elements of D. For an edge $e = (x, y) \in E_2(D)$ we let $e^{-1} = (y, x)$ be the *reverse edge* of e.

Let Δ be a (possibly infinite) group. The identity element of Δ is usually denoted by 1_Δ. A Δ-*labeled 2-structure* (or a Δ*2-structure*, for short) $g = (D, \lambda, \Delta)$ is an edge-labeled directed graph with the finite *domain* D as its nodes, the set $E_2(D)$ as its edges and $\lambda\colon E_2(D) \to \Delta$ as its labeling function. The group Δ may be infinite, while the domain D is always assumed to be finite. Since g is determined by its labeling function λ, we shall later identify a Δ2-structure with its labeling function. We use this convention already in the next definition.

An *inversive* Δ*2-structure* is a mapping $g\colon E_2(D) \to \Delta$ satisfying $g(e^{-1}) = g(e)^{-1}$ for all $e \in E_2(D)$. The family of inversive Δ2-structures with domain D will be denoted by $\mathcal{R}(D, \Delta)$.

In a pictorial representation of an inversive Δ2-structure g we shall usually omit the edges

- that have the label 1_Δ;
- the reverses of the drawn edges.

Example 1. Let $\Delta = (\mathbb{R}^+, \cdot)$ be the multiplicative group of positive real numbers. In Fig. 1 we have a $\mathbb{R}^+$2-structure g, where, *e.g.*, we have $g(x_2, x_1) = g(x_1, x_2)^{-1} = 1$, $g(x_2, x_3) = \frac{3}{2}$, and $g(x_3, x_2) = \frac{2}{3}$. $\square$

The group Δ of labels of a $g \in \mathcal{R}(D, \Delta)$ becomes employed by the selectors, which, in essence, label the nodes $x \in D$ by the elements of the group Δ.

A function $\sigma\colon D \to \Delta$ is called a *selector*. For each selector σ and $g \in \mathcal{R}(D, \Delta)$ define g^σ by

$$g^\sigma(x, y) = \sigma(x) \cdot g(x, y) \cdot \sigma(y)^{-1}$$

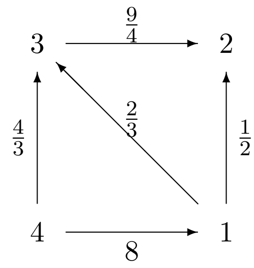

Figure 2: The image g^σ

for all $(x, y) \in E_2(D)$. The family

$$[g] = \{g^\sigma \mid \sigma \colon D \to \Delta\}$$

is a (*single axiom*) *dynamic* Δ*2-structure* (*generated by* g).

Hence a selector σ transforms each $g \colon E_2(D) \to \Delta$ into a $g^\sigma \colon E_2(D) \to \Delta$ by a direct left and an (inversive) right multiplication. The new value of an edge depends on the (values of the) nodes and on the label of the edge.

Example 2. Let g be as in Example 1, see Fig. 1. Define a selector $\sigma \colon D \to \mathbb{R}$ by $\sigma(x_i) = 5 - i$. Then, *e.g.*, we have $g^\sigma(x_1, x_2) = 4 \cdot 1 \cdot \frac{1}{3} = \frac{4}{3}$. The image g^σ is drawn in Fig. 2, where we have labeled the nodes by the values of σ. □

This paper deals with invariants of Δ2-structures. We give now a short overview of our results. The exact definitions concerning invariants are presented in Section 3.

A function η mapping the inversive Δ2-structures g into the group Δ is an *invariant*, if $\eta(g^\sigma) = \eta(g)$ for all g and σ. An invariant is thus immune to the selectors.

Each word $w = e_1 e_2 \dots e_k$ from the free monoid $M(D)$ with involution, generated by the set $E_2(D)$, defines in a natural way a mapping ψ_w, $\psi_w(g) = g(e_1)g(e_2)\dots g(e_k)$, such that $\psi_w(g) \in \Delta$ for each $g \in \mathcal{R}(D, \Delta)$. If ψ_w is an invariant, then it is called a *free invariant.*

We show that the free invariants form an abelian group $Inv\,(D \to \Delta)$ consisting of mappings into the center $Z(\Delta)$ of the group Δ of labels. Moreover, the free invariants are closely related to verbal identities of the quotient group $\Delta/Z(\Delta)$ and of Δ itself. Our main result in this respect is that for a word w, which forms a closed walk, the mapping ψ_w is a free invariant if and only if w is a verbal identity of the quotient group $\Delta/Z(\Delta)$.

If Δ is abelian, then the group of free invariants is generated by triangle words of the form $(x_0, x_1)(x_1, x_2)(x_2, x_0)$. This result generalizes a result of [14], where the case $\Delta = \mathbb{Z}_2$ is considered. In the abelian case $Inv\,(D \to \Delta)$ is independent of Δ. For the nonabelian case we give a partial characterization of the group of free invariants in terms of 'characteristic' powers of triangles and commutator words. We note that in the general case the structure of the group $Inv\,(D \to \Delta)$ depends on the verbal identities of Δ.

1.2 Connections to graph theory

Directed graphs, where the labels of the edges come from a group, are investigated in several areas of graph theory. The most notable of these is the study of *Cayley graphs*, see *e.g.* [7] or [11].

In topological graph theory the *voltage graphs* are defined as directed graphs with an (inversive) group labeling of the edges, see [6]. It is interesting to notice – especially in view of Section 1.3 – that in the theory of voltage graphs an action similar to a selector becomes defined in a natural way.

The case where Δ is the cyclic group $\mathbb{Z}_2 = \{0, 1\}$ of two elements has received much attention in literature. *Seidel switching* was defined in connection with the problem of finding equilateral n-tuples of points in elliptic geometry, see [10]. This problem gives rise to the following problem for undirected graphs: determine the equivalence classes of undirected graphs with n nodes with respect to the following operation (called *Seidel switching*). Let $G \to G'$, if there is a node x such that $G' = (D, E')$ is obtained from $G = (D, E)$ by removing all edges (x, y) and (y, x) incident with x, and adding all pairs (x, y) and (y, x) not in E. Hence $G \to G'$, if for some node x,

$$E' = (E \setminus \{(x, y), (y, x) \mid y \neq x\}) \cup \{(x, y), (y, x) \mid (x, y) \notin E\}.$$

Let then $\leftrightarrow^*$ be the equivalence relation determined by $\to$, *i.e.*, $G \leftrightarrow^* G'$ if and only if $G = G'$ or there exists a finite sequence $G_0 \to G_1 \to \cdots \to G_k$ with $\{G, G'\} = \{G_0, G_k\}$. One now asks how many equivalence classes of $\leftrightarrow^*$ are there for a set D of nodes (up to isomorphism of graphs)?

We can reformulate the above problem in terms of dynamic $\mathbb{Z}_2$2-structures as follows. Let us consider a $g \in \mathcal{R}(D, \mathbb{Z}_2)$ as an undirected graph, where $g(e) = 1$ ($g(e) = 0$, resp.) means that e is (not, resp.) an edge of g. Consider a node $x \in D$ and a selector σ, for which $\sigma(x) = 1$, and $\sigma(y) = 0$ for all other nodes $y \neq x$. Clearly, the image g^σ represents a graph, where the existing connections from x are removed and the nonexistent connections from

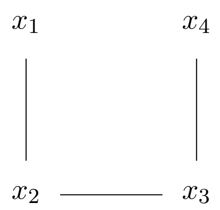

Figure 3: g with labels in $\mathbb{Z}_2$

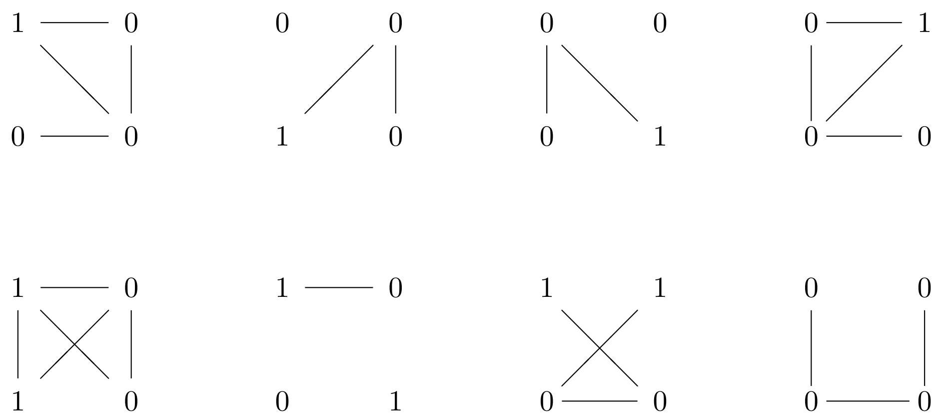

Figure 4: The images g^σ

x are created. Therefore $G \to G'$ holds if and only if for the corresponding $\mathbb{Z}_2$2-structures g and g', $g' = g^\sigma$ for such a selector σ. From this we obtain that $g \leftrightarrow^* g'$ if and only if $g' = g^\sigma$ for a selector $\sigma\colon D \to \mathbb{Z}_2$.

Example 3. Let $D = \{x_1, x_2, x_3, x_4\}$ and consider the $\mathbb{Z}_2$2-structure g from Fig. 3, where a line denotes value 1 of $\mathbb{Z}_2$. There are $2^{|D|} = 16$ different selectors $\sigma\colon D \to \Delta$, but some of them have the same image g^σ. In fact, there are only 8 different images g^σ as depicted in Fig. 4, where again the nodes are labeled by the values of a selector σ which applied to g yields g^σ. □

Seidel switching is closely connected to *signed graphs* and *two-graphs*. We refer to [5, 8, 15, 16, 17] for these topics.

Let then $\Delta = \mathbb{Z}_3$ be the cyclic group of three elements $0, 1, 2$. An inversive $\mathbb{Z}_3$2-structure g can be identified with an *oriented graph*, *i.e.*, with a

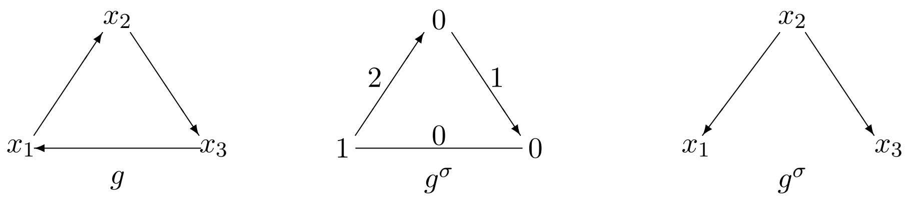

Figure 5: $\mathbb{Z}_3$2-structure g and its image g^σ

directed graph $G = (D, E)$, where $e \in E$ implies $e^{-1} \notin E$. Indeed, we can choose $E = \{e \in E_2(D) \mid g(e) = 1\}$, and interpret $g(e) = 0$ as $e, e^{-1} \notin E$, $g(e) = 2$ as $e \notin E$, $e^{-1} \in E$, and $g(e) = 1$ as $e \in E$, $e^{-1} \notin E$.

Example 4. Let g be the directed cycle of Fig. 5, where an arrow denotes the value $1 \in \mathbb{Z}_3$. The second inversive $\mathbb{Z}_3$2-structure of Fig. 5 is obtained from g using the selector σ, for which $\sigma(x_1) = 1$, $\sigma(x_2) = 0 = \sigma(x_3)$. The third directed graph of Fig. 5 is a redrawing of g^σ using our conventions. □

Let us choose $\Delta = \mathbb{Z}_4$. *All directed graphs can be represented as inversive $\mathbb{Z}_4$2-structures.* Indeed, if $G = (D, E)$ is a directed graph, then we define the representing $\mathbb{Z}_4$2-structure g by

$$g(e) = \begin{cases} 0, & \text{if } e, e^{-1} \notin E\ , \\ 1, & \text{if } e \in E,\ e^{-1} \notin E\ , \\ 2, & \text{if } e, e^{-1} \in E\ , \\ 3, & \text{if } e \notin E,\ e^{-1} \in E\ . \end{cases}$$

We would also like to point out that, as discussed in [3], dynamic labeled 2-structures are closely related to *graph transformations* as considered in the area of graph grammars, see *e.g.* [4]. Providing techniques for proving or disproving that an edge-labeled graph can be derived from another one, is important also from the point of view of graph transformations.

1.3 Evolution of networks

The dynamic labeled 2-structures were motivated in [3] by evolutionary processes of networks. We shall now briefly describe this motivation.

Assume we are given a finite network of processors D in which each pair $\{x, y\}$ of processors communicates through the two channels $e = (x, y)$ and $e^{-1} = (y, x)$ directed in opposite ways. The states of the channels

Figure 6: Concurrent actions

are (coded by) elements of a set Δ, which need not have group structure. Each processor $x \in D$ has two sets of actions, output actions O_x and input actions I_x, by which it can change the states of the channels from and to x, respectively. The actions of x are thus transformations of Δ, *i.e.*, each $\varphi_x \in O_x$ or $\gamma_x \in I_x$ is a function $\Delta \to \Delta$. An action $\varphi_x \in O_x$ will change the contents of the outgoing channels: if the value of a channel (x, y) is $a_y \in \Delta$, then it will be changed to $\varphi_x(a_y)$ by this action. Similarly, an action $\gamma_x \in I_x$ changes the contents of the incoming channels: the value b_y of the channel (y, x) will be changed to $\gamma_x(b_y)$ by this action.

Notice that for each (x, y) there are two processors, x and y, that change the state of this channel; x changes it by a transformation $\varphi_x \in O_x$ and y changes it by a $\gamma_y \in I_y$.

At any stage the locally determined choices for the actions $\varphi_x \in O_x$ and $\gamma_x \in I_x$ of the processors $x \in D$ are represented by a *selector*, that is, by a function σ that for each x gives its actions, $\sigma(x) = (\varphi_x, \gamma_x) \in O_x \times I_x$.

The network as a whole is modified by the concurrent actions of the processors. It is assumed that a network satisfies the following natural axioms.

In order to avoid unnecessary sequencing of actions it is assumed that
(1) the composition of two actions from O_x (I_x, resp.) is again an action, *i.e.*, the sets O_x and I_x are transformation semigroups on Δ.

In order to assure that the effect of concurrent actions in different processors is well-defined it is assumed that
(2) the combined action of $\varphi_x \in O_x$ and $\gamma_y \in I_y$ for $x \neq y$ should be independent of the order in which they are taken, *i.e.*, the transformation semigroups O_x and I_y must permute: $\varphi_x\gamma_y = \gamma_y\varphi_x$, see Fig. 6.

In order to assure a minimal freedom of the actions for each processor it is assumed that
(3) for each $a, b \in \Delta$ and $x \in D$ there exist $\varphi \in O_x$ and $\gamma \in I_x$ such that $\varphi(a) = b$ and $\gamma(a) = b$, *i.e.*, the semigroups O_x and I_x are transitive.

When these axioms are assumed, the situation in the network becomes simple. Indeed, it was shown in [3], see also [1], that if $|D| \geq 3$, then for all $x, y \in D$, $O_x = O_y$ and $I_x = I_y$, and hence, in a network satisfying the above axioms, the actions come from two semigroups O and I that are independent of the processors. Moreover, as shown in [3], the transformation semigroups O and I are isomorphic (simply transitive) groups of permutations. (In essence, O and I become defined by left and (inversive) right multiplication of a group). Further, we can define an operation on the alphabet Δ such that Δ becomes a group isomorphic to O (and I).

A global state of (all the channels of) a network will be represented by a Δ2-structure. Hence an evolution of a network becomes represented by a family of Δ2-structures, each of which represents a possible global state of the network. The transitions from one labeled 2-structure to another (hence from one global state to another) are the transformations induced by the selectors.

2 Preliminaries on Monoids and Groups

Let $A = \{a_1, a_2, \dots, a_n\}$ be a (finite) set of symbols, called *letters* or *variables*. The set A of letters is called an *alphabet*. Each finite sequence $w = (a_{i_1}, a_{i_2}, \dots, a_{i_m})$ of letters is called a *word* over A. The empty sequence is the *empty word*, and it will be denoted by 1. The word w is usually written as a concatenation of the letters, $w = a_{i_1} a_{i_2} \dots a_{i_m}$. We denote by A^* the set of all words (including the empty word 1) over A. The set A^* forms a monoid under the operation of catenation: for all words $w_1, w_2 \in A^*$, $w_1 \cdot w_2 = w_1 w_2$. This monoid A^* is the *free* (*word*)*monoid* generated by A, and it satisfies the following basic extension property, see [9].

Theorem 1. *Let A be an alphabet and M a monoid. Then each function $\alpha\colon A \to M$ can be extended to a unique (monoid) homomorphism $\alpha^\flat\colon A^* \to M$.*

A word u is a *subword* of a word $w \in A^*$, if $w = w_1 u w_2$ for some words $w_1, w_2 \in A^*$. We say also that a letter $a \in A$ *occurs* in a word w, if a is a subword of w. The number of occurrences of a letter a in a word w is denoted by $|w|_a$. Hence $|w|_a = 0$ in case a does not occur in w, and in general

$$|w_1 w_2|_a = |w_1|_a + |w_2|_a.$$

Let M be a monoid. A bijection $\delta\colon M \to M$ is called an *involution* of M, if δ is an antiautomorphism of order two, *i.e.*, if $\delta(x \cdot y) = \delta(y) \cdot \delta(x)$ and $\delta^2(x) = x$ for all $x, y \in M$. In this case (M, δ) is called a *monoid with involution*.

Let (M_1, δ_1) and (M_2, δ_2) be two monoids with involution. A function $\alpha\colon M_1 \to M_2$ is a *homomorphism* (*between monoids with involution*), denoted $\alpha\colon (M_1, \delta_1) \hookrightarrow (M_2, \delta_2)$, if α is a monoid homomorphism, $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in M_1$, and, $\alpha(\delta_1(x)) = \delta_2(\alpha(x))$ for all $x \in M_1$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an alphabet, and let $A^{-1} = \{a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\}$ be another alphabet disjoint from A. The alphabet $B = A \cup A^{-1}$ is called an *alphabet with involution.*

Consider the free monoid B^* generated by the letters $a_i \in A$ and $a_i^{-1} \in A^{-1}$, and define a function $\delta\colon B^* \to B^*$ by $\delta(1) = 1$, and

$$\delta(b_1 b_2 \ldots b_k) = \delta(b_k)\delta(b_{k-1}) \ldots \delta(b_1) \qquad (b_i \in B),$$

where $\delta(a_i) = a_i^{-1}$ and $\delta(a_i^{-1}) = a_i$ for each $a \in A$. Clearly, δ is an involution of the monoid B^*. This involution of B^* is denoted simply by the inverse notation: $\delta(b) = b^{-1}$ with $(b^{-1})^{-1} = b$ for all $b \in B$. The monoid $(B^*, ^{-1})$ with involution is denoted by $M(B)$, and it is called the *free monoid with involution over* B (or over A). Clearly, for all words $w_1, w_2, \ldots, w_k \in M(B)$,

$$(w_1 w_2 \ldots w_k)^{-1} = w_k^{-1} w_{k-1}^{-1} \ldots w_1^{-1}.$$

An analogy of Theorem 1 holds for monoids with involutions, see [12]:

Theorem 2. *Let $B = A \cup A^{-1}$ be an alphabet with involution and (M, δ) any monoid with involution. Then each function $\alpha\colon A \to M$ can be extended to a unique homomorphism $\alpha^\flat\colon M(B) \hookrightarrow (M, \delta)$.*

By Theorem 2 each homomorphism $\alpha\colon M(B) \hookrightarrow (M, \delta)$ is uniquely determined by its images $\alpha(a)$ on the generators $a \in A$. We say that a homomorphism $\alpha\colon M(B) \hookrightarrow M$ *becomes defined by the mapping* $\alpha|A\colon A \to M$.

Let $B = A \cup A^{-1}$ be an alphabet with an involution. We say that the words $w_1, w_2 \in B^*$ are *freely equal*, denoted $w_1 =_F w_2$, if w_1 can be obtained from w_2 by a finite number of additions or deletions of subwords of the form ww^{-1} and $w^{-1}w$, where $w \in B^*$. Clearly, the relation $=_F$ is an equivalence relation on B^*. Moreover, it satisfies the following *congruence condition*:

$$w_1 =_F w_2 \ \text{ and } \ w_3 =_F w_4 \quad \Longrightarrow \quad w_1 w_3 =_F w_2 w_4. \tag{1}$$

The equivalence classes

$$[w]_F = \{u \mid u \in B^*,\ u =_F w\}$$

form the *free group* $F(B)$ over the alphabet B, when the product is defined by

$$[w_1]_F \cdot [w_2]_F = [w_1 w_2]_F \qquad (w_1, w_2 \in B^*).$$

By (1) this operation is well defined. The inverse $[w]_F^{-1}$ of an element $[w]_F$ is $[w^{-1}]_F$, and the identity element of $F(B)$ is $[1]_F$.

A group Δ is a monoid $(\Delta, ^{-1})$ with its inverse function as its involution. Therefore we have the following result.

Lemma 1. *Let $\alpha\colon M(B) \hookrightarrow \Delta$ be a homomorphism for a group Δ. If $w_1 =_F w_2$, then $\alpha(w_1) = \alpha(w_2)$.*

In particular, if $\alpha\colon A \to \Delta$ is any function from a set A to a group Δ, then, by Theorem 2, α has an extension to a monoid homomorphism $\alpha^\flat\colon M(B) \hookrightarrow \Delta$, where $B = A \cup A^{-1}$, and $w_1 =_F w_2$ implies $\alpha^\flat(w_1) = \alpha^\flat(w_2)$.

Let again $B = A \cup A^{-1}$. A group Δ is said to satisfy a *verbal identity* $u \in B^*$, if for all homomorphisms $\alpha\colon M(B) \hookrightarrow \Delta$, $\alpha(u) = 1_\Delta$.

Example 5.

1. The free group $F(B)$ has only the *trivial verbal identities*, *i.e.*, if $u \in B^*$ is a verbal identity of $F(B)$, then u is freely equal to the empty word, $u =_F 1$.

2. If Δ is an Abelian group, then it satisfies the verbal identity $x^{-1}y^{-1}xy$ for $x, y \in A$.

3. A finite group Δ of order $k = |\Delta|$ satisfies the verbal identity x^k, where $x \in A$.

4. The free Burnside group $B(d, n)$ (of exponent d and rank $n \geq 1$) is defined so that it satisfies the verbal identity x^d, where $x \in A$. Note that here $B(d, n)$ is nonabelian if $d \geq 3$ and $n \geq 2$. However, $B(2, n)$ is always Abelian, see [11].

□

3 Free Invariants

We shall consider now the problem how to recognize that a Δ2-structure g is obtainable from another Δ2-structure h by an application of a selector. In order to answer this question we study invariant properties of Δ2-structures, and show that the invariants are shown to be closely connected to the verbal identities of the group Δ of labels.

3.1 General invariants

The inversive Δ2-structures in $\mathcal{R}(D,\Delta)$ are naturally divided into dynamic labeled 2-structures, *i.e.*, the family $\{[g] \mid g \in \mathcal{R}(D,\Delta)\}$ forms a partition of $\mathcal{R}(D,\Delta)$. A function $\eta\colon \mathcal{R}(D,\Delta) \to \Delta$ is called an *invariant*, if it maps the elements of each $[g]$ into the same element,

$$\eta(g_1) = \eta(g_2) \qquad \text{if} \quad [g_1] = [g_2].$$

Hence an invariant is immune to the selectors, and, in the terminology of networks, the study of invariants is the study of those properties of a network that remain unchanged during its evolution.

If η is a constant mapping, $\eta(g) = a$ (for some $a \in \Delta$) for all $g \in \mathcal{R}(D,\Delta)$, then clearly η is an invariant. In particular, if the group Δ is trivial, $\Delta = \{1_\Delta\}$, then the only function $\eta\colon \mathcal{R}(D,\Delta) \to \Delta$ there is, the constant function $\eta(g) = 1_\Delta$, is an invariant. The case $|D| = 1$ is also trivial, for in this case we have $E_2(D) = \emptyset$ and therefore there exists only one labeled 2-structure g in $\mathcal{R}(D,\Delta)$.

3.2 Free invariants

In general an invariant is a function that can be independent of the specific properties of the inversive Δ2-structures. In order to reflect these properties, we shall restrict ourselves to those invariants that are more faithful to the labeled 2-structures in the sense that they are defined by variables corresponding to the edges. We use free monoids with involution to formulate this correspondence.

We denote by $M(D)$ the *free monoid with involution* $(E_2(D), ^{-1})$. Hence $M(D)$ consists of the free monoid $E_2(D)^*$ of words generated by the elements of $E_2(D)$ together with a naturally defined involution,

$$(e_1 e_2)^{-1} = e_2^{-1} e_1^{-1},$$

where e^{-1} is as before the reverse edge of e.

In order to clarify the distinction between the edges of a Δ2-structure and the generators of the free monoid $M(D)$ with involution, the generators $e \in M(D)$ will be called *variables*.

By Theorem 2, each inversive Δ2-structure $g\colon E_2(D) \to \Delta$ extends to a unique homomorphism $g^\flat\colon M(D) \hookrightarrow \Delta$ such that $g^\flat(e) = g(e)$ and $g^\flat(e^{-1}) = g(e)^{-1}$ for all $e \in E_2(D)$. We note also that if $\alpha\colon M(D) \hookrightarrow \Delta$ is any homomorphism, then $\alpha = g^\flat$, where $g \in \mathcal{R}(D,\Delta)$ is defined by $g(e) = \alpha(e)$. Hence we have the following simple result.

Theorem 3. *The mappings* $g^\flat$ *for* $g \in \mathcal{R}(D,\Delta)$ *are exactly the homomorphisms* $M(D) \hookrightarrow \Delta$.

3.3 Variable functions and free invariants

Each word $w = e_1 e_2 \dots e_k \in M(D)$ defines a function ψ_w from $\mathcal{R}(D,\Delta)$ into Δ as follows:

$$\psi_w(g) = g(e_1)g(e_2)\dots g(e_k) \qquad (g \in \mathcal{R}(D,\Delta)).$$

We call the function ψ_w the *variable function represented by* w, and we let

$$Var\,(D \to \Delta) = \{\psi_w \colon \mathcal{R}(D,\Delta) \to \Delta \mid w \in M(D)\}$$

be the set of all variable functions represented by the words in $M(D)$. We shall also write ψ_w^Δ instead of ψ_w, when the group Δ is not clear from the context.

Further, two words w_1 and w_2 from $M(D)$ are said to be *equivalent* (*over* Δ), denoted $w_1 \equiv w_2$, if they represent the same variable function: $\psi_{w_1} = \psi_{w_2}$.

The relation $\equiv$ is clearly an equivalence relation on words. Moreover, the following lemma is easy to prove.

Lemma 2. *If the words* $w_1, w_2 \in M(D)$ *are freely equal then they are equivalent:* $w_1 =_F w_2$ *implies* $w_1 \equiv w_2$.

An invariant $\psi_w \in Var\,(D \to \Delta)$ is called a *free invariant* of $\mathcal{R}(D,\Delta)$. We denote by $Inv\,(D \to \Delta)$ the set of all free invariants of $\mathcal{R}(D,\Delta)$.

If w is freely equal to the empty word 1, then $w \equiv 1$, and consequently in this case $\psi_w = \psi_1 \in Inv\,(D \to \Delta)$, since the constant function ψ_1, $\psi_1(g) = 1_\Delta$, is a free invariant.

Example 6. (1) Let $\Delta = \mathbb{Z}_3 = \{0,1,2\}$ be the cyclic group of order three (with the additive notation), and let $D = \{x_1, x_2, x_3\}$. We write $e_{ij} = (x_i, x_j)$ for all $i \neq j$. The word $w_1 = e_{12}e_{23}e_{31}$ represents an invariant, *i.e.*, $\psi_{w_1} \in Inv\,(D \to \Delta)$, since for all $g \in \mathcal{R}(D,\Delta)$ and $\sigma \colon D \to \Delta$ with $\sigma(x_i) = a_i$, $\psi_{w_1}(g) = g(e_{12}) + g(e_{23}) + g(e_{31})$ and

$$\begin{aligned}
\psi_{w_1}(g^\sigma) &= g^\sigma(e_{12}) + g^\sigma(e_{23}) + g^\sigma(e_{31}) \\
&= (a_1 + g(e_{12}) - a_2) + (a_2 + g(e_{23}) - a_3) + (a_3 + g(e_{31}) - a_1) \\
&= g(e_{12}) + g(e_{23}) + g(e_{31}) = \psi_{w_1}(g)\ ,
\end{aligned}$$

because Δ is abelian.

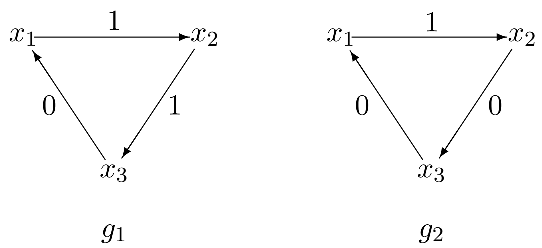

Figure 7: Inversive $\mathbb{Z}_3$2-structures

For the inversive Δ2-structures g_1 and g_2 from Fig. 7, we have $\psi_{w_1}(g_1) = 2 \neq 1 = \psi_{w_1}(g_2)$, which implies that for all selectors σ, $g_2 \neq g_1^\sigma$, since $\psi_{w_1}(g_1^\sigma) = 2$ but $\psi_{w_1}(g_2) = 1$ for each σ. Hence in this case $[g_1] \neq [g_2]$.

On the other hand, let $w_2 = e_{12}e_{23}^{-1}e_{31}$. Now, $\psi_{w_2}(g) = g(e_{12}) + g(e_{32}) + g(e_{31})$ and

$$\psi_{w_2}(g^\sigma) = g^\sigma(e_{12}) + g^\sigma(e_{32}) + g^\sigma(e_{31}) = 2a_3 - 2a_2 + \psi_{w_2}(g) ,$$

and thus $\psi_{w_2}(g^\sigma) = \psi_{w_2}(g)$ if and only if $2a_3 - 2a_2 = 0$ in $\mathbb{Z}_3$. It follows that ψ_{w_2} is not an invariant of $\mathcal{R}(D, \Delta)$, since the selector may have chosen $a_2 = 1$ and $a_3 = 2$.

(2) Suppose the group Δ satisfies the verbal identity e_1^n, *i.e.*, $a^n = 1_\Delta$ for all $a \in \Delta$. Then obviously the word $w = e^n$ represents an invariant for each variable $e \in M(D)$. In fact, w is equivalent to the empty word 1. □

The following example shows that if the domain D has at most two elements, then for any group Δ, $Inv\,(D \to \Delta) = \{\psi_1\}$ for the *trivial free invariant* ψ_1 represented by the empty word 1.

Example 7. Assume $D = \{x, y\}$, and write $e = (x, y)$. then each invariant ψ_w is represented by a word e^k for an integer k, that is, $w \equiv e^k$, because $(x, y)(y, x) = (x, y)(x, y)^{-1} \equiv 1$. Assume that ψ_w is an invariant for $w = e^k$ for some $k \neq 0$. Define $g \in \mathcal{R}(D, \Delta)$ to be such that $g(e) = 1_\Delta$. Let $a \in \Delta$, and let σ be the selector for which $\sigma(x) = a$ and $\sigma(y) = 1_\Delta$. We have now that $g^\sigma(e) = a$ and hence $\psi_w(g^\sigma) = \psi_w(g)$ implies $a^k = g(e)^k = 1_\Delta$. Consequently, for all $a \in \Delta$, we have that $a^k = 1_\Delta$. Therefore ψ_w is a constant mapping: for all $g \in \mathcal{R}(D, \Delta)$, $\psi_w(g) = \psi_1(g) = 1_\Delta$. Hence the cases where $|D| \leq 2$ are trivial. □

The constant functions $\eta\colon \mathcal{R}(D, \Delta) \to \Delta$ are invariants of $\mathcal{R}(D, \Delta)$, but as verified in the next example only the trivial constant function is a free invariant.

Example 8. Let $g \in \mathcal{R}(D,\Delta)$ be defined by $g(e) = 1_\Delta$ for all $e \in E_2(D)$. Hence the homomorphism $g^\flat$ is a constant, $g^\flat(e) = 1_\Delta$ for each $e \in E_2(D)$, and thus if $w \in M(D)$, then $\psi_w(g) = 1_\Delta$. This implies that the only constant invariant $\eta\colon \mathcal{R}(D,\Delta) \to \Delta$ that is represented by a word, is the identity ψ_1 of $Var\,(D \to \Delta)$. $\square$

From the definition of $Var\,(D \to \Delta)$ we obtain immediately that for all $w, w_i \in M(D)$ $(i = 1, 2)$, and for all $g \in \mathcal{R}(D,\Delta)$:

$$\psi_{w_1 w_2}(g) = \psi_{w_1}(g) \cdot \psi_{w_2}(g), \quad \psi_{w^{-1}}(g) = \psi_w(g)^{-1} \quad \text{and} \quad \psi_1(g) = 1_\Delta.$$

From this observation we obtain the following result.

Theorem 4. *The variable functions $Var\,(D \to \Delta)$ from a group under the operation*

$$\psi_{w_1} \cdot \psi_{w_2} = \psi_{w_1 w_2} \qquad (w_1, w_2 \in M(D)).$$

The identity of the group $Var\,(D \to \Delta)$ is ψ_1 for the empty word 1. Furthermore, we have

$$\psi_{w^k}(g) = \psi_w(g)^k \qquad \text{for all } w \in M(D),\ k \in \mathbb{Z},\ g \in \mathcal{R}(D,\Delta).$$

Hence we can write $\psi_{w^k} = \psi_w^k$ for a word w and an integer k. In particular, $\psi_{w^{-1}} = \psi_w^{-1}$.

It is clear that $Var\,(D \to \Delta)$ is generated by the variable functions ψ_e represented by the variables, $e \in E_2(D)$. Since $E_2(D)$ is always finite, the group $Var\,(D \to \Delta)$ is finitely generated.

4 Group Properties of $Inv\,(D \to \Delta)$

4.1 $Inv\,(D \to \Delta)$ is an abelian group

In the following result we prove that each free invariant $\psi_w \in Inv\,(D \to \Delta)$ is, in fact, a mapping $\psi_w\colon \mathcal{R}(D,\Delta) \to Z(\Delta)$ into the center

$$Z(\Delta) = \{a \in \Delta \mid ab = ba \ \text{ for all } \ b \in \Delta\}$$

of the group Δ. It follows from this that $Inv\,(D \to \Delta)$ is an abelian group.

Theorem 5. *1. For all $\psi_w \in Inv\,(D \to \Delta)$ and $g \in \mathcal{R}(D,\Delta)$, $\psi_w(g) \in Z(\Delta)$.*

2. $Inv\,(D \to \Delta)$ is a subgroup of $Z(Var\,(D \to \Delta))$. In particular, $Inv\,(D \to \Delta)$ is an abelian group.

Proof. Let $\psi_w \in Inv\,(D \to \Delta)$ be a free invariant for the word $w = e_1 e_2 \dots e_k$, and let $g \in \mathcal{R}(D, \Delta)$. Define for each $a \in \Delta$ a selector σ_a by $\sigma_a(x) = a$ for all $x \in D$. Hence $\psi_e(g^{\sigma_a}) = a \cdot g(e) \cdot a^{-1}$ for all $e \in E_2(D)$. We have now that

$$\begin{aligned}\psi_w(g^{\sigma_a}) &= ag(e_1)a^{-1}a \dots a^{-1}ag(e_k)a^{-1} \\ &= ag(e_1)g(e_2)\dots g(e_k)a^{-1} = a\psi_w(g)a^{-1},\end{aligned}$$

from which it follows that $a \cdot \psi_w(g) = \psi_w(g) \cdot a$, since $\psi_w(g^{\sigma_a}) = \psi_w(g)$. This shows that $\psi_w(g)$ commutes with every element of Δ and thus $\psi_w(g) \in Z(\Delta)$ as was required in Case (1).

Let then $\psi_{w_i} \in Inv\,(D \to \Delta)$, for $i = 1, 2$, and let σ be a selector. Hence $\psi_{w_i}(g^\sigma) = \psi_{w_i}(g)$ for all $g \in \mathcal{R}(D, \Delta)$. Now, for all $g \in \mathcal{R}(D, \Delta)$,

$$(\psi_{w_1}\psi_{w_2^{-1}})(g^\sigma) = \psi_{w_1}(g^\sigma) \cdot \psi_{w_2^{-1}}(g^\sigma) = \psi_{w_1}(g) \cdot \psi_{w_2}(g)^{-1} = (\psi_{w_1}\psi_{w_2^{-1}})(g),$$

which shows that $Inv\,(D \to \Delta)$ is a subgroup of $Var\,(D \to \Delta)$.

On the other hand, if $\psi_w \in Inv\,(D \to \Delta)$ and $\psi_{w_1} \in Var\,(D \to \Delta)$, then, by the previous case, $\psi_w(g) \in Z(\Delta)$ for all $g \in \mathcal{R}(D, \Delta)$, and thus $\psi_{w_1}(g) \cdot \psi_w(g) = \psi_w(g) \cdot \psi_{w_1}(g)$, which implies that $\psi_{w_1}\psi_w = \psi_w\,\psi_{w_1}$. Therefore, $Inv\,(D \to \Delta)$ is a subgroup of the center $Z(Var\,(D \to \Delta))$. □

By Theorem 5, if the center $Z(\Delta)$ is trivial, $Z(\Delta) = \{1_\Delta\}$, then also the group $Inv\,(D \to \Delta)$ is trivial, $Inv\,(D \to \Delta) = \{\psi_1\}$. See also Example 9 below.

4.2 Invariants and group constructions

The following theorem relates invariants for a group Δ to the invariants for some basic constructions of groups.

Theorem 6. *Let $w \in M(D)$ with $\psi_w^\Delta \in Inv\,(D \to \Delta)$ for a group Δ.*

1. *For each subgroup N of Δ, $\psi_w^N \in \mathrm{Inv}(D \to N)$.*

2. *If $\alpha\colon \Delta \to \Sigma$ is a group homomorphism onto Σ, then $\psi_w^\Sigma \in \mathrm{Inv}(D \to \Sigma)$.*

3. *For a normal subgroup N of Δ, $\psi_w^{\Delta/N} \in \mathrm{Inv}(D \to \Delta/N)$.*

Proof. The first claim is obvious because if N is a subgroup of Δ, then each $g \in \mathcal{R}(D, N)$ belongs to $\mathcal{R}(D, \Delta)$ and each selector $\sigma\colon D \to N$ is also a selector $\sigma\colon D \to \Delta$.

Let $w = e_1 e_2 \dots e_k$ with $e_i \in E_2(D)$, and assume then that $\alpha\colon \Delta \to \Sigma$ is a surjective group homomorphism. Let $g_1 \in \mathcal{R}(D, \Sigma)$ and $\sigma_1\colon D \to \Sigma$ be arbitrary. Consider a $g \in \mathcal{R}(D, \Delta)$ that satisfies the equalities $\alpha(g(e)) = g_1(e)$ for all $e \in E_2(D)$. Such a g exists because α is onto Σ. Similarly, let $\sigma\colon D \to \Delta$ be a selector such that $\alpha(\sigma(x)) = \sigma_1(x)$ for all $x \in D$. Now,

$$\alpha(\psi_w^{\Delta}(g)) = \alpha(g(e_1)g(e_2)\dots g(e_k)) = g_1(e_1)g_1(e_2)\dots g_1(e_k) = \psi_w^{\Sigma}(g_1)$$

and, similarly,

$$\alpha(\psi_w^{\Delta}(g^{\sigma})) = g_1^{\sigma_1}(e_1)\dots g_1^{\sigma_1}(e_k) = \psi_w^{\Sigma}(g_1^{\sigma_1})\,.$$

By assumption ψ_w^{Δ} is a free invariant, and thus $\psi_w^{\Delta}(g) = \psi_w^{\Delta}(g^{\sigma})$, which implies that $\psi_w^{\Sigma}(g_1) = \psi_w^{\Sigma}(g_1^{\sigma_1})$ as required in Case (2).

Case (3) follows from Case (2), since every quotient Δ/N with respect to a normal subgroup N is a homomorphic image of the group Δ. □

4.3 Graphs of words

In this section we present some general results connecting free invariants to graphs of words.

A variable function ψ_w need not follow the paths of Δ2-structures. We define now variable functions that are faithful also to the graphical presentation of inversive Δ2-structures.

A word $w \in M(D)$ is a *closed walk of length* n, if $w = e_1 e_2 \dots e_n$, where $e_i = (x_i, x_{i+1})$ for $i = 1, 2, \dots, n-1$, and $e_n = (x_n, x_1)$ for some nodes $x_i \in D$. A closed walk

$$t(x_0, x_1, x_2) = (x_0, x_1)(x_1, x_2)(x_2, x_0)$$

is called a *triangle at* x_0, if $x_i \neq x_j$ for each $i \neq j$. We also say that the empty word $1 \in M(D)$ is a (*trivial*) *walk*. For a fixed node $x_0 \in D$, the set

$$T_D^{x_0} = \{t(x_0, y, z) \mid x_0, y, z \in D \text{ are distinct }\}$$

is the *bucket of triangles at* x_0.

Theorem 7. *Let* $w \in M(D)$ *a closed walk. Then* $\psi_w \in Inv\,(D \to \Delta)$ *if and only if* $\psi_w(g) \in Z(\Delta)$ *for all* $g \in \mathcal{R}(D, \Delta)$. *In particular, if* Δ *is abelian, then* ψ_w *is a free invariant of* $\mathcal{R}(D, \Delta)$.

Proof. First of all, if $\psi_w \in Inv\,(D \to \Delta)$, then by Theorem 5(1), $\psi_w(g) \in Z(\Delta)$ for all $g \in \mathcal{R}(D, \Delta)$.

Let then $w = (x_1, x_2) \cdot \dots \cdot (x_n, x_1) \in M(D)$ be any closed walk, $g \in \mathcal{R}(D, \Delta)$ and let σ be a selector. Hence

$$\psi_w(g) = g(x_1, x_2)g(x_2, x_3) \dots g(x_{n-1}, x_n)g(x_n, x_1)$$

and

$$g^\sigma(x_i, x_{i+1(n)}) = \sigma(x_i) \cdot g(x_i, x_{i+1(n)}) \cdot \sigma(x_{i+1(n)})^{-1}$$

for each $i = 1, 2, \dots, n$. After the reductions $\sigma(x_i)\sigma(x_i)^{-1} = 1_\Delta$ for $i = 2, \dots, n$, we obtain that $\psi_w(g^\sigma)$ is a conjugate of $\psi_w(g)$: $\psi_w(g^\sigma) = \sigma(x_1) \cdot \psi_w(g) \cdot \sigma(x_1)^{-1}$. Clearly, if $\psi_w(g) \in Z(\Delta)$, then $\psi_w(g^\sigma) = \psi_w(g)$. This proves the claim. □

Example 9. Consider the *dihedral group* D_{2n} of $2n$ elements ($n \geq 3$),

$$D_{2n} = \{1, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\},$$

where 1 is the identity element of D_{2n}. The group D_{2n} is the symmetry group of a regular n-gon of the Euclidean plane, and it is generated by a rotation a (with an angle of $2\pi/n$) together with a reflection b with respect to a diagonal of the n-gon. These generators satisfy the following defining relations

$$a^n = 1, \qquad b^2 = 1, \qquad ab = ba^{-1}.$$

It is rather immediate that if n is odd, then $Z(D_{2n})$ is trivial, and hence in this case the group $Inv\,(D \to \Delta)$ of free invariants is also trivial by Theorem 5.

On the other hand, if n is even, then $Z(D_{2n})$ contains two elements, *i.e.*, it is isomorphic to the cyclic group $\mathbb{Z}_2$. Let us consider the case $n = 4$. In this case $Z(D_8) = \{1, a^2\}$. Assume that the domain D has at least three nodes, and let w be any closed walk. We claim that the variable function ψ_{w^2} is a free invariant. Here w^2 is the closed walk that traverses w twice around. Indeed, let g be an inversive Δ2-structure. Then

$$\psi_{w^2}(g) = (\psi_w(g))^2,$$

and it is easy to check that for all $c \in D_8$, $c^2 = 1$ or a^2. Therefore $\psi_{w^2}(g) \in Z(D_8)$ for all $g \in \mathcal{R}(D, \Delta)$, and thus ψ_{w^2} is a free invariant by Theorem 7. □

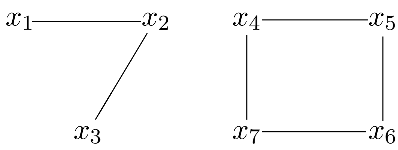

Figure 8: The supporting graph $G(w)$ of $w = e_{12}e_{45}e_{67}e_{32}e_{23}e_{56}e_{74}$

Let $w = (x_1, y_1) \dots (x_n, y_n) \in M(D)$ be a word. The *supporting graph* of w is the undirected graph $G(w) = (D, E(w))$, where the set of edges is

$$E(w) = \{(x_i, y_i),\ (y_i, x_i) \mid i = 1, 2, \dots, n\}.$$

For a connected component C of $G(w)$ let $\alpha_C \colon M(D) \hookrightarrow M(D)$ be the homomorphism such that

$$\alpha_C((x,y)) = \begin{cases} (x,y), & \text{if } \ x, y \in C \ , \\ 1, & \text{otherwise} \ . \end{cases}$$

The word $w_C = \alpha_C(w)$ is called a *connected component* of w. If $w = w_C$ for a connected component w_C of w, then w is said to be *connected.*

Example 10. Let $D = \{x_1, \dots, x_7\}$, and denote again $e_{ij} = (x_i, x_j)$ for each $i \neq j$. If $w = e_{12}e_{45}e_{67}e_{32}e_{23}e_{56}e_{74}$, then the graph $G(w)$ of w is in Fig. 8. The connected components of w are $w_1 = e_{12}e_{32}e_{23}$ ($\equiv e_{12}$) and $w_2 = e_{45}e_{67}e_{56}e_{74}$. □

Note that the graph $G(w)$ clearly depends on the word $w \in M(D)$ and hence the above definition does not define graphs for the functions $\psi_w \in Var\,(D \to \Delta)$.

The connected components of words representing invariants provide smaller words for invariants:

Theorem 8. *If ψ_w is a free invariant, then so is ψ_{w_C} for each connected component w_C of w.*

Proof. Assume that $\psi_w \in Inv\,(D \to \Delta)$ for $w \in M(D)$, and let w_C be a connected component of w. For a selector σ define a new selector σ_C by

$$\sigma_C(x) = \begin{cases} \sigma(x) & \text{if } x \in C \ , \\ 1_\Delta & \text{if } x \notin C \ . \end{cases}$$

Clearly, $\psi_{w_C}(g^\sigma) = \psi_{w_C}(g^{\sigma_C})$, since only the nodes $x \in C$ occur in the variables of w_C. Moreover, let $g_1 \colon E_2(D) \to \Delta$ be defined by

$$g_1(x,y) = \begin{cases} g(x,y) & \text{if } x, y \in C \ , \\ 1_\Delta & \text{if } x \notin C \text{ or } y \notin C \ . \end{cases}$$

Clearly, $\psi_{w_C}(g) = \psi_w(g_1)$ and hence $\psi_{w_C}(g) = \psi_w(g_1^{\sigma_C})$, since ψ_w is a free invariant. Finally, $\psi_w(g_1^{\sigma_C}) = \psi_{w_C}(g^{\sigma_C})$ by the definition of σ_C, and thus $\psi_{w_C}(g^\sigma) = \psi_{w_C}(g)$ as required. □

For an abelian group Δ we can prove also the converse of Theorem 8. Indeed, each $\psi_w \in Var\,(D \to \Delta)$ is a product $\psi_w = \psi_{w_{C_1}}\psi_{w_{C_2}} \dots \psi_{w_{C_k}}$, where $w_{C_1}, w_{C_2}, \dots, w_{C_k}$ are the connected components of w. Since $Inv\,(D \to \Delta)$ is a group, we have shown

Theorem 9. *Let Δ be an abelian group and $w \in M(D)$. Then $\psi_w \in Inv\,(D \to \Delta)$ if and only if $\psi_{w_C} \in Inv\,(D \to \Delta)$ for all connected components w_C of w.*

4.4 Verbal identities

We note first that if $w \in M(D)$ is a verbal identity of a group Δ, then $\alpha(w) = 1_\Delta$ for all homomorphisms $\alpha \colon M(D) \hookrightarrow \Delta$, and hence, by Theorem 3, $g^\flat(w) = 1_\Delta$ for all $g \in \mathcal{R}(D, \Delta)$. Therefore $w \equiv 1$, and $\psi_w = \psi_1$ is the trivial free invariant.

The following result is a straightforward corollary to Theorem 6(3) and the fact that for all $\psi_w \in Inv\,(D \to \Delta)$, $\psi_w(g) \in Z(\Delta)$.

Theorem 10. *If $\psi_w \in Inv\,(D \to \Delta)$, then the word $w \in M(D)$ is a verbal identity of the quotient $\Delta/Z(\Delta)$.*

The variable functions represented by closed walks are graphically most interesting, and for these also the converse of Theorem 10 holds.

Theorem 11. *Let $w \in M(D)$ be a closed walk. Then $\psi_w \in Inv\,(D \to \Delta)$ if and only if w is a verbal identity of $\Delta/Z(\Delta)$.*

Proof. Let us denote $Z = Z(\Delta)$ for short, and assume w is a verbal identity of the quotient Δ/Z. Hence $\alpha(w) = 1_{\Delta/Z}$ for all homomorphisms $\alpha \colon M(D) \to \Delta/Z$.

Let $g \in \mathcal{R}(D, \Delta)$, and define $\alpha \colon M(D) \hookrightarrow \Delta/Z$ by $\alpha(u) = g^\flat(u)Z$. Clearly, α is a homomorphism, and $Z = 1_{\Delta/Z} = \alpha(w) = g^\flat(w)Z$. Hence $\psi_w(g) = g^\flat(w) \in Z$, and so $\psi_w \in Inv\,(D \to \Delta)$ by Theorem 7.

In the other direction the claim follows directly from Theorem 10. □

Theorem 11 is an improvement of the statement for abelian groups in Theorem 7, since if Δ is abelian, then $Z(\Delta) = \Delta$ and $\Delta/Z(\Delta)$ is a trivial group, for which all words are verbal identities; in particular, the closed walks are verbal identities.

Example 11. Consider the *quaternion group* Q_8 generated by two elements a and b, and which is subject to the relations

$$a^4 = 1,\ b^2 = a^2,\ ba = a^3 b.$$

The group Q_8 consists of the following eight elements: $1, a, a^2, a^3, b, ab, a^2b$ and a^3b, and it is a nonabelian group. Usually, Q_8 is represented as a group of unit coordinate vectors (in a four-dimensional vector space), in which the elements are $-1, 1, i, -i, j, -j, k, -k$ and they satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -kj,$$

or Q_8 is represented as a group of matrices generated by

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

over complex numbers. The center of Q_8 consists of two elements: $Z(Q_8) = Z = \{1, a^2\}$, and the quotient Q_8/Z is a four-element abelian group. The right cosets are Z, aZ, bZ and abZ. Further, the elements of Q_8/Z are all of order two. Therefore for any closed walk w, ψ_{w^2} is a free invariant. $\square$

In the next theorem we relate the free invariants ψ_w to special verbal identities of the group Δ itself. These results, although more general and more restrictive, are less appealing than the simple characterization in Theorem 11 for closed walks.

We denote $D^{\pm 1} = D \cup D^{-1}$, where $D^{-1} = \{x^{-1} \mid x \in D\}$ is disjoint from D, and let

$$W(D) = E_2(D) \cup D \cup D^{-1}$$

be an alphabet with the natural involution. Consider the free monoid $M(W(D))$ with (the natural) involution generated by $W(D)$. Further, let $\gamma\colon M(D) \hookrightarrow M(W(D))$ be the homomorphism, which comes defined by

$$\gamma((x, y)) = x \cdot (x, y) \cdot y^{-1}.$$

Theorem 12. *Let Δ be a group.*

1. *Two words $w_1, w_2 \in M(D)$ are equivalent over Δ if and only if $w_1 w_2^{-1}$ is a verbal identity of Δ.*

2. *For a word $w \in M(D)$, $\psi_w \in Inv\,(D \to \Delta)$ if and only if the word $\gamma(w)w^{-1}$ is a verbal identity of Δ.*

3. *If $\psi_w \in Inv\,(D \to \Delta)$ for a word $w = e_1 e_2 \ldots e_n$ with $e_i = (x_i, y_i)$, then the word $x_1 y_1^{-1} x_2 y_2^{-1} \ldots x_n y_n^{-1}$ over $D^{\pm 1}$ is a verbal identity of Δ.*

Proof. By definition and Theorem 3, $w_1 \equiv w_2$ if and only if $\alpha(w_1) = \alpha(w_2)$, *i.e.*, $\alpha(w_1 w_2^{-1}) = 1_\Delta$, for all homomorphisms $\alpha\colon M(D) \hookrightarrow \Delta$. Hence, by the definition of verbal identity, Case (1) of the claim follows.

Let then $w = e_1 e_2 \ldots e_n \in M(D)$ with $e_i = (x_i, y_i)$. Hence $\gamma(w) = x_1 e_1 y_1^{-1} \cdot \cdots \cdot x_n e_n y_n^{-1}$. Now, $\gamma(w)w^{-1}$ is a verbal identity of Δ if and only if for all homomorphisms $\alpha\colon M(W(D)) \hookrightarrow \Delta$, $\alpha(\gamma(w)w^{-1}) = 1_\Delta$ if and only if $\alpha\gamma(w) = \alpha(w)$. Here

$$\begin{aligned} \alpha(\gamma(w)) &= \alpha(x_1)\alpha(e_1)\alpha(y_1)^{-1} \cdot \cdots \cdot \alpha(x_n)\alpha(e_n)\alpha(y_n)^{-1} \\ &= \sigma(x_1)g(e_1)\sigma(y_1)^{-1} \cdot \cdots \cdot \sigma(x_n)g(e_n)\sigma(y_n)^{-1} = \psi_w(g^\sigma); \\ \alpha(w) &= \alpha(e_1)\alpha(e_2) \cdot \cdots \cdot \alpha(e_n) = g(e_1)g(e_2)\ldots g(e_n) = \psi_w(g), \end{aligned}$$

where the selector σ and the inversive g are defined by

$$\sigma = \alpha | D \qquad \text{and} \qquad g = \alpha | E_2(D). \tag{2}$$

Claim (2) follows when we observe that each selector σ and $g \in \mathcal{R}(D, \Delta)$ define a homomorphism $\alpha\colon M(W(D)) \hookrightarrow \Delta$ by the conditions (2).

If $u \in M(W(D))$ is a verbal identity of a group Δ and $\alpha\colon M(W(D)) \hookrightarrow M(W(D))$ is an endomorphism, then $\alpha(u)$ is also a verbal identity of Δ, because for each homomorphism $\beta\colon M(W(D)) \hookrightarrow \Delta$, also $\beta\alpha$ is a homomorphism $M(W(D)) \hookrightarrow \Delta$. Letting $\alpha\colon M(W(D)) \hookrightarrow M(W(D))$ be the endomorphism such that $\alpha(x) = x$ for all $x \in D^{\pm 1}$, and $\alpha(e) = 1_\Delta$ for all $e \in E_2(D)$, we obtain Case (3) of the claim using Case (2). □

5 Invariants on Abelian Groups

5.1 Independency of free invariants

We shall show that in the abelian case the invariants represented by the triangles at x_0 form a 'complete' set of invariants.

We start with some general remarks on abelian invariants.

Clearly, in the abelian case, for $w_1, w_2 \in M(D)$ we have $w_1 w_2 \equiv w_2 w_1$. Hence the occurrences of the variables in $w \in M(D)$ can be freely permuted without violating invariant properties.

For an abelian group Δ the group $Inv\,(D \to \Delta)$ of free invariants has properties that are independent of the 2-structures. To see this let $w = (x_1, y_1)(x_2, y_2) \dots (x_n, y_n)$ be a word and $\sigma \colon D \to \Delta$ a selector. Then we have for all $g \in \mathcal{R}(D, \Delta)$,

$$\begin{aligned} \psi_w(g^\sigma) &= \sigma(x_1)g(x_1, y_1)\sigma(y_1)^{-1} \cdot \dots \cdot \sigma(x_n)g(x_n, y_n)\sigma(y_n)^{-1} \\ &= \sigma(x_1)\sigma(y_1)^{-1} \cdot \dots \cdot \sigma(x_n)\sigma(y_n)^{-1} \cdot \psi_w(g) \ , \end{aligned}$$

and thus ψ_w is a free invariant if and only if $\sigma(x_1)\sigma(y_1)^{-1} \dots \sigma(x_n)\sigma(y_n)^{-1} = 1_\Delta$ for all selectors σ. Here the latter condition does not depend on g. Let us define for each selector σ a homomorphism $\overline{\sigma} \colon M(D) \to \Delta$ such that

$$\overline{\sigma}((x, y)) = \sigma(x)\sigma(y)^{-1} \qquad \text{for all} \ \ (x, y) \in E_(D) \ .$$

By the above observations we have then

Theorem 13. *Let Δ be an abelian group. The following conditions are equivalent for a variable function ψ_w:*

1. *ψ_w is a free invariant of $\mathcal{R}(D, \Delta)$.*
2. *For each $g \in \mathcal{R}(D, \Delta)$ and for each selector σ, $\psi_w(g^\sigma) = \psi_w(g)$.*
3. *For each selector σ, $\overline{\sigma}(w) = 1_\Delta$.*
4. *There exists a $g \in \mathcal{R}(D, \Delta)$ such that for all selectors σ, $\psi_w(g^\sigma) = \psi_w(g)$.*

By the condition (3) of Theorem 13, the free invariants are independent from the inversive Δ2-structures. Indeed, in order to verify that a variable function ψ_w is a free invariant, one needs only to check that for all selectors σ the corresponding homomorphism $\overline{\sigma}$ gives the identity on w.

5.2 Complete sets of invariants

Each triangle $t \in T_D^{x_0}$ is a closed walk, and so, by Theorem 7, t represents a free invariant $\psi_t \in Inv\,(D \to \Delta)$ for abelian Δ. We shall show next that these invariants generate $Inv\,(D \to \Delta)$.

Theorem 14. *Let Δ be an abelian group and x_0 an element of the domain D. Then $Inv\,(D \to \Delta)$ is generated by the free invariants represented by the triangles at x_0.*

Proof. Let $\psi_w \in Inv\,(D \to \Delta)$ for a word $w = e_1 e_2 \dots e_n \in M(D)$.

For each variable $e = (y, z)$ with $y \neq x_0$ and $z \neq x_0$ we have

$$e \equiv (x_0, y)^{-1} \cdot (x_0, y)(y, z)(z, x_0) \cdot (z, x_0)^{-1} = (x_0, y)^{-1} \cdot t(x_0, y, z) \cdot (x_0, z) \;,$$

and because Δ is abelian, $w \equiv w_0 w_1$, where w_0 is a product of triangles at x_0 and w_1 consists of variables from the set $W = \{(x_0, y), (y, x_0) \mid y \neq x_0\}$. By Theorem 7, $\psi_{w_0} \in Inv\,(D \to \Delta)$. Since $\psi_{w_1} = \psi_w \psi_{w_0}^{-1}$ and $\psi_w \in Inv\,(D \to \Delta)$, also $\psi_{w_1} \in Inv\,(D \to \Delta)$. For the claim it is enough to show that $w_1 \equiv 1$, since this implies that $w \equiv w_0$, *i.e.*, $\psi_w = \psi_{w_0}$.

Using commutativity of Δ, w_1 can be written in an equivalent form

$$w_1 \equiv e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_k^{\epsilon_k} \;,$$

where for each $i = 1, 2, \dots, k$, $\epsilon_i \in \mathbb{Z}$ and $e_i = (x_0, y_i)$ with $y_i \neq y_j$ for $i \neq j$.

Let $g_1 \in \mathcal{R}(D, \Delta)$ be such that $g_1(e) = 1_\Delta$ for each edge e. Hence $\psi_{w_1}(g_1) = 1_\Delta$. For each $a \in \Delta$ and $i = 1, 2, \dots, k$ define a selector $\sigma_{i,a}$ by $\sigma_{i,a}(y_i) = a^{-1}$ and $\sigma_{i,a}(y) = 1_\Delta$ for $y \neq y_i$. We have then $\psi_{w_1}(g_1^{\sigma_{i,a}}) = a^{\epsilon_i}$ for all i and a. Since $\psi_{w_1} \in Inv\,(D \to \Delta)$, we have that $a^{\epsilon_i} = 1_\Delta$ for all $a \in \Delta$. This implies that $e_i^{\epsilon_i} \equiv 1$, and, consequently, $w_1 \equiv 1$, which proves the claim. □

Next we show that the triangles (at x_0) not only represent generators of $Inv\,(D \to \Delta)$ but form a large enough set to characterize the equivalence relation $[g] = [h]$ between the inversive labeled 2-structures on an abelian Δ.

A set W of invariants for $\mathcal{R}(D, \Delta)$ is said to be a *complete*, if W satisfies for all $g_1, g_2 \in \mathcal{R}(D, \Delta)$ the condition: $[g_1] = [g_2]$ if and only if $\eta(g_1) = \eta(g_2)$ for all $\eta \in W$.

In the above definition the converse implication is always valid, that is, if $g_1^\sigma = g_2$ for a selector σ, then for every invariant η of $\mathcal{R}(D, \Delta)$, $\eta(g_2) = \eta(g_1^\sigma) = \eta(g_1)$. On the other hand, if a set of invariants W is complete and $[g_1] \neq [g_2]$ for two elements $g_1, g_2 \in \mathcal{R}(D, \Delta)$, then there exists an invariant $\eta \in W$ such that $\eta(g_1) \neq \eta(g_2)$.

Theorem 15. *Let Δ be an abelian group. For each $x_0 \in D$ the bucket of triangles $T_D^{x_0}$ a complete set of invariants for $\mathcal{R}(D, \Delta)$.*

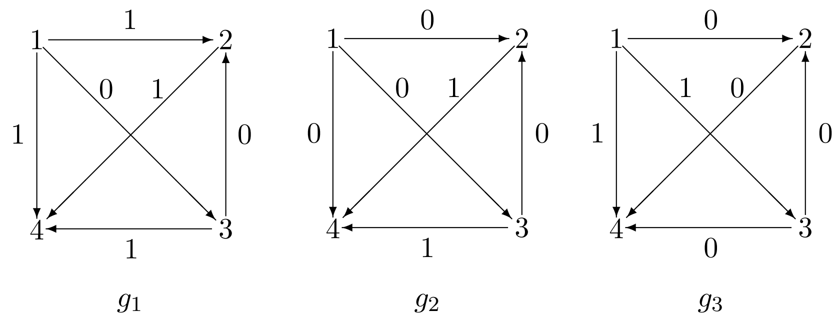

Figure 9: $\Delta = \mathbb{Z}_2$

Proof. Let $g_i \in \mathcal{R}(D, \Delta)$ for $i = 1, 2$ be such that for all triangles $t = t(x_0, y, z)$, $\psi_t(g_1) = \psi_t(g_2)$. We have to show that $[g_1] = [g_2]$.

Define the selectors σ_i, $i = 1, 2$, as follows:

$$\sigma_i(x_0) = 1_\Delta \quad \text{and} \quad \sigma_i(y) = g_i(x_0, y) \quad \text{for all } \ y \neq x_0 \ .$$

It follows that $g_i^{\sigma_i}(x_0, y) = 1_\Delta$ for all $y \neq x_0$. On the other hand, each $t \in T_D^{x_0}$ represents an invariant and thus $\psi_t(g_1^{\sigma_1}) = \psi_t(g_1) = \psi_t(g_2) = \psi_t(g_2^{\sigma_2})$. However, if $t = t(x_0, y, z)$, then $\psi_t(g_i^{\sigma_i}) = g_i^{\sigma_i}(y, z)$ for $i = 1, 2$, and thus, by above, $g_1^{\sigma_1}(y, z) = g_2^{\sigma_2}(y, z)$ for all $(y, z) \in E_2(D)$ with $y \neq x$ and $z \neq x$. Consequently, $g_1^{\sigma_1} = g_2^{\sigma_2}$, and hence $[g_1] = [g_2]$ as claimed. □

Theorem 15 allows the use of 'local' triangles for checking whether or not $[g_1] = [g_2]$. Indeed, if $[g_1] \neq [g_2]$, then we can find a (common) triangle X in g_1 and g_2, for which $[sub_{g_1}(X)] \neq [sub_{g_2}(X)]$.

Example 12. Let $\Delta = \mathbb{Z}_2$ and g_1, g_2, g_3 the structures from Fig. 9.

Consider the triangle $t = t(1, 2, 3)$. We have $\psi_t(g_1) = 1 \neq 0 = \psi_t(g_2)$, and hence by the above theorem there does not exist a selector σ for which $g_2 = g_1^\sigma$.

On the other hand, we observe that for the triangles $t = t(1, 2, 3)$ and $t = t(1, 3, 4)$, $\psi_t(g_1) = \psi_t(g_3)$, and hence $\psi_t(g_1) = \psi_t(g_3)$ for all $t \in T_D^1$. Hence, by Theorem 15, $[g_1] = [g_3]$. □

6 Invariants on Nonabelian Groups

6.1 Commutators

In this section Δ need not be abelian. We shall refine some of the techniques of [11] for verbal subgroups in order to prove that $Inv\,(D \to \Delta)$ is generated by the variable functions represented by certain characteristic powers of the triangles $t(x_0, y, z)$ at a fixed $x_0 \in D$ together with the invariants represented by commutator words.

In order to simplify the statements and proofs of our results, we will assume in this section that $|D| > 2$.

We start by recalling some group theoretical preliminaries on commutators.

A word $[u, v] = u^{-1}v^{-1}uv$ is a *commutator* of the words $u, v \in V^*$, and the submonoid of V^* generated by the commutators is called the *commutator monoid* of V^* and it is denoted by $[V]^*$. The elements of $[V]^*$ are *commutator words*. Hence each commutator word $w \in [V]^*$ is a finite catenation, $[u_1, v_1][u_2, v_2] \dots [u_n, v_n]$, of commutators.

From the definition of a commutator we obtain that for all $u, v \in V^*$, $uv =_F vu[u, v]$ and hence

$$w_1 \cdot uv \cdot w_2 =_F w_1 \cdot vu \cdot w_2[[u, v], w_2] \qquad (w_1, u, v, w_2 \in V^*)\ ,$$

which implies the following result.

Lemma 3. *Let $w = w_1w_2 \dots w_n$ be a word with subwords $w_i \in V^*$, $i = 1, 2, \dots, n$, and let π be a permutation of the index set $\{1, 2, \dots, n\}$. Then there exists a word $u_\pi \in [V]^*$ such that $w =_F w_{\pi(1)}w_{\pi(2)} \dots w_{\pi(n)} \cdot u_\pi$.*

For a group Δ the element $[a, b] = a^{-1}b^{-1}ab$ is a *commutator* of the elements a and b of Δ. Evidently, $[a, b] = 1_\Delta$ if and only if $ab = ba$ in Δ. The subgroup generated by the commutators is called the *commutator subgroup* (or *derived group*) of Δ.

As shown below in Example 13, unlike in the abelian case, in the general case the group $Inv\,(D \to \Delta)$ of free invariants depends on the structure of the group Δ, *i.e.*, $Inv\,(D \to \Delta)$ depends on the (special) identities that are satisfied in Δ. This was also witnessed by Theorems 12 and 5. In particular, if the center of Δ is trivial, then there are no nontrivial free invariants.

Example 13. Let V be a set of variables with an involution, and assume G is a group which satisfies a verbal identity $u \in V^*$. Let A be an abelian group, and $\Delta = G \times A$ the direct product of G and A with projections $\pi_1 \colon \Delta \to G$ and $\pi_2 \colon \Delta \to A$.

Further, let $\beta\colon V^* \to M(D)$ be any homomorphism which maps the variables to triangles (at a fixed node x_0), $\beta(v_i) = t_i \in T_D^{x_0}$ for all $v_i \in V$. Define a word $w \in M(D)$ by $w = \beta(u)$. Hence w is a product of triangles. We show that $\psi_w \in Var\,(D \to \Delta)$ is a free invariant.

First of all, for each $g \in \mathcal{R}(D, \Delta)$ we have

$$\psi_w(g) = g^\flat(w) = g^\flat\beta(u) = (\pi_1 g^\flat\beta(u), \pi_2 g^\flat\beta(u))\,,$$

where $\pi_1 g^\flat\beta$ is a homomorphism from V^* into G and hence $\pi_1 g^\flat\beta(u) = 1_G$ by our assumption on u. Clearly, $\psi_w(g) \in Z(\Delta)$ for all $g \in \mathcal{R}(D, \Delta)$.

For a selector σ we have $\psi_{t_i}(g^\sigma) = \sigma(x_0)\psi_{t_i}(g)\sigma(x_0)^{-1}$ and, consequently, $\psi_w(g^\sigma) = \sigma(x_0)\psi_w(g)\sigma(x_0)^{-1}$, where, by above, $\pi_1\psi_w(g^\sigma) = 1_G$, and $\pi_2\psi_w(g^\sigma) = \pi_2\psi_w(g)$, since A is abelian and hence $\psi_w(g^\sigma) = \psi_w(g)$ by Theorem 14. It follows that $\psi_w(g^\sigma) = \psi_w(g)$ as required. □

6.2 Central characters of $Inv\,(D \to \Delta)$

Next we shall show that if $\mathcal{R}(D, \Delta)$ has nontrivial free invariants, then it is restricted in the following sense: there is a specific nonnegative integer $d = d_\Delta$, called the central character of Δ, for which $\Delta^d \subseteq Z(\Delta)$. Here Δ^d is the subgroup of Δ generated by the elements of $\{a^d \mid a \in \Delta\}$.

Recall that the number of occurrences of a variable e in a word w is denoted by $|w|_e$. Hence $|w|_e = 0$ in case v does not occur in w.

The *exponent number of* $w \in M(D)$ *on* $e \in E_2(D)$ is defined to be the integer

$$\varepsilon_e(w) = |w|_e - |w|_{e^{-1}}\ .$$

In particular, if e and e^{-1} do not occur in the word w, then $\varepsilon_e(w) = 0$. It is also immediate that for all words w, $\varepsilon_e(w) = -\varepsilon_{e^{-1}}(w)$.

Let $w \in M(D)$, and define $d_w = 0$, if $\varepsilon_e(w) = 0$ for all $e \in E_2(D)$. Otherwise, let,

$$d_w = \gcd(\varepsilon_e(w) \mid e \in E_2(D),\ \varepsilon_e(w) > 0)$$

be the greatest common divisor of the positive exponent numbers of w.

Example 14. Let $D = \{x_1, x_2, x_3\}$ and denote again $e_{ij} = (x_i, x_j)$. For the word $w = e_{12}e_{23}e_{23}e_{21}$, $\varepsilon_{e_{12}}(w) = 0$, $\varepsilon_{e_{23}}(w) = 2$, $\varepsilon_{e_{13}}(w) = 0$. In this case $d_w = 2$. □

If u is a commutator word, then clearly $d_u = 0$. For the proof of the following result we refer again to [11, p.79].

Lemma 4. *Let $w \in M(D)$ be a word. Then $w =_F u$ for a commutator word u if and only if $d_w = 0$.*

We define now $d_\Delta = 0$, if for all $\psi_w \in Inv\,(D \to \Delta)$, $d_w = 0$. Otherwise, let

$$d_\Delta = \gcd(d_w \mid d_w \geq 1,\ \psi_w \in Inv\,(D \to \Delta)).$$

The integer d_Δ is called the *central character* of $Inv\,(D \to \Delta)$.

Clearly, the central character is well defined, and $d_\Delta = 0$ if and only if each free invariant $\psi_w \in Inv\,(D \to \Delta)$ is represented by commutator words only. It is also immediate that if $d_\Delta \geq 1$, then there exists a finite set, $w_1, w_2, \dots, w_r$, of words such that $d_\Delta = \gcd(d_{w_1}, d_{w_2}, \dots, d_{w_r})$.

If Δ is a finite group, then there exists a word, *e.g.* $w = e^{|\Delta|}$ for an $e \in E_2(D)$, representing an invariant (in fact, $w \equiv 1$), for which $|\Delta| \geq d_w > 0$, and thus in this case $1 \leq d_\Delta \leq |\Delta|$.

The central character is related to the center of the group Δ as follows.

Theorem 16. *For all groups Δ, $\Delta^{d_\Delta} \subseteq Z(\Delta)$. Moreover, if for each $a \in \Delta$, $a^k = 1_\Delta$ for some $k \geq 1$, then d_Δ divides k. In particular, if Δ is finite, then d_Δ divides the order $|\Delta|$ of Δ.*

Proof. If $d_\Delta = 0$, then there is nothing to prove. Suppose then that $d_\Delta \geq 1$, and let ψ_w be a free invariant. Assume that $E_2(D) = \{e_1, \dots, e_n\}$.

We shall first show that $a^{d_w} \in Z(\Delta)$ for each $a \in \Delta$. First of all, by the definition of d_w, there are integers $m_i \in \mathbb{Z}$ such that $d_w = \sum_{i=1}^n m_i \cdot \varepsilon_{e_i}(w)$. Moreover, for each $a \in \Delta$ and $i = 1, 2, \dots, n$, define an inversive Δ2-structure $g_{a,i}$ by

$$g_{a,i}(x,y) = \begin{cases} a & \text{if } e_i = (x,y)\ , \\ a^{-1} & \text{if } e_i^{-1} = (y,x)\ , \\ 1 & \text{otherwise}\ . \end{cases}$$

Then obviously $\psi_w(g_{a,i}) = a^{\varepsilon_{e_i}(w)}$ and, by Theorem 5, $a^{\varepsilon_{e_i}(w)} \in Z(\Delta)$. Consequently,

$$a^{d_w} = a^{\sum_{i=1}^n m_i \cdot \varepsilon_{e_i}(w)} \in Z(\Delta)\ .$$

By assumption, $d_\Delta > 0$, and hence there are a finite number of words w_i, $i = 1, 2, \dots, r$, with $\psi_{w_i} \in Inv\,(D \to \Delta)$ such that $d_\Delta = \gcd(d_1, \dots, d_r)$, where $d_i = d_{w_i}$, for short. Hence there are integers s_i such that

$$d_\Delta = \sum_{i=1}^r s_i \cdot d_i\ .$$

By above, for all $a \in \Delta$, $a^{d_i} \in Z(\Delta)$ and thus also $a^{d_\Delta} \in Z(\Delta)$. Hence $\Delta^{d_\Delta} \subseteq Z(\Delta)$.

For the second claim we need only to note that if $a^k = 1_\Delta$ for all $a \in \Delta$, then $w = e^k \equiv 1$ represents an invariant for all $e \in E_2(D)$, and thus, by the definition of d_Δ, d_Δ divides k. □

In particular, if $d_w = 1$ for a free invariant ψ_w, then Δ is necessarily an abelian group.

Theorem 17. *For the central character d_Δ of $Inv\,(D \to \Delta)$, $d_\Delta = 1$ if and only if Δ is an abelian group.*

Proof. The claim follows from the preceding theorem and from the fact that a triangle word $t = (x_1, x_2)(x_2, x_3)(x_3, x_1)$ represents an invariant for which $d_t = 1$, whenever $|D| \geq 3$, and hence $d_\Delta = 1$ for all abelian groups Δ. □

For a triangle t and a positive integer d, t^d is a *d-triangle*, where t^d is a catenation of t with itself d times. By Theorem 7 and Theorem 16 we have the following lemma.

Lemma 5. *Let $d = d_\Delta$ be the central character of $Inv\,(D \to \Delta)$ for the group Δ. Then ψ_t^d $(= \psi_{t^d})$ is an invariant for all triangles t.*

Proof. If $d = 0$, then $\psi_t^d = \psi_1$ and the claim is true. Assume then that $d \geq 1$.

Let $g \in \mathcal{R}(D, \Delta)$ and a selector σ be arbitrary. For a triangle $t = t(x, y, z)$ we have $\psi_t(g^\sigma) = \sigma(x)\psi_t(g)\sigma(x)^{-1}$ and hence $\psi_t^d(g^\sigma) = \sigma(x)\psi_t^d(g)\sigma(x)^{-1}$. By Theorem 16, $\psi_t^d(g) \in Z(\Delta)$ and thus $\psi_t^d(g^\sigma) = \psi_t^d(g)$, which proves the claim. □

6.3 A characterization theorem

The following theorem gives our main characterization result on invariants for nonabelian groups. It reduces the free invariants into products of d-triangles and commutator words. This theorem is weak in the sense that we do not characterize the free invariants ψ_u of the commutator words u.

Theorem 18. *Let Δ be a group and $d = d_\Delta$ be the central character of $Inv\,(D \to \Delta)$. For a word $w \in M(D)$, $\psi_w \in Inv\,(D \to \Delta)$ if and only if $\psi_w = \psi_s \cdot \psi_u$ for a product s of d-triangles and a commutator word u representing an invariant.*

Proof. If $\psi_w = \psi_s \cdot \psi_u$, where ψ_s and ψ_u are free invariants, then ψ_w is a free invariant, because the group $Inv\,(D \to \Delta)$ is closed under products.

In the other direction, let $\psi_w \in Inv\,(D \to \Delta)$ for a word $w = e_1 e_2 \dots e_n \in M(D)$, and let $x_0 \in D$ be a fixed node.

If $d = 0$, then by Lemma 4, $w \equiv u$ for a commutator word u, and the claim is obvious in this case. Let us assume that $d \geq 1$. We write $T = T_D^{x_0}$, for short.

For each $e = (y, z)$ with $y \neq x_0$ and $z \neq x_0$, we have

$$e \equiv (x_0, y)^{-1} \cdot (x_0, y)(y, z)(z, x_0) \cdot (z, x_0)^{-1} \equiv (y, x_0) \cdot t(x_0, y, z) \cdot (x_0, z) \ .$$

Let w_0 be a word obtained from w by substituting each (y, z) by the sequence $(y, x_0)t(x_0, y, z)(x_0, z)$ for $y \neq x_0$ and $z \neq x_0$. Consequently, $w \equiv w_0$, and the word w_0 will be written as $w_0 = u_1 u_2 \dots u_m \in M(D)$, where for each $i = 1, 2, \dots, m$ either $u_i \in W = \{(x_0, y), (y, x_0) \mid y \in D, y \neq x_0\}$ or $u_i \in T$.

Consider, for a while, w_0 as a word over the alphabet $W^{\pm 1} \cup T^{\pm 1}$, where the inverse of a triangle $t = t(x_0, y, z)$ is the triangle $t^{-1} = t(x_0, z, y)$. Let $\{y_1, \dots, y_k\}$ be a strict linear ordering of the set $D \setminus \{x_0\}$, and denote

$$\{t_1, t_2, \dots, t_r\} = \{t(x_0, y_i, y_j) \mid i < j\} \ ,$$

and $w_i = (x_0, y_i)$ for $i = 1, 2, \dots, k$.

Further, let ε_i be the exponent number of w_0 on $t_i \in T$, and γ_i the exponent number of w_0 on $w_i = (x_0, y_i)$ with respect to this new set $W^{\pm 1} \cup T^{\pm 1}$ of variables.

By the formation of the triangles $t_i = t(x_0, y, z)$, it is evident that ε_i equals the exponent number of w on $(y, z) \in E_2(D)$. Hence, by Lemma 5, the words $t_i^{\varepsilon_i}$ represent invariants of $\mathcal{R}(D, \Delta)$, because for each i either $\varepsilon_i = 0$ or d divides ε_i.

Let π be a permutation of the index set $\{1, 2, \dots, m\}$ of $w_0 = u_1 u_2 \dots u_m$ such that

$$u_{\pi(1)} u_{\pi(2)} \dots u_{\pi(m)} = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_r^{\varepsilon_r} \cdot w_1^{\gamma_1} w_2^{\gamma_2} \dots w_k^{\gamma_k} \ .$$

and denote $s_0 = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_r^{\varepsilon_r} \cdot w_1^{\gamma_1} w_2^{\gamma_2} \dots w_k^{\gamma_k}$ and $s = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_r^{\varepsilon_r}$. By above, $\psi_{t_i}^{\varepsilon_i} \in Inv\,(D \to \Delta)$, and hence $\psi_s \in Inv\,(D \to \Delta)$ as a product of d-triangles.

By Lemma 3, there exists a commutator word $u \in [W^{\pm 1} \cup T^{\pm 1}]^*$ such that

$$w_0 \equiv s \cdot w_1^{\gamma_1} w_2^{\gamma_2} \dots w_k^{\gamma_k} \cdot u \equiv s_0 \cdot u \ .$$

Now, $\psi_s \in Inv\,(D \to \Delta)$, and hence the word $s_1 = w_1^{\gamma_1} w_2^{\gamma_2} \dots w_k^{\gamma_k} \cdot u$ represents an invariant, since $\psi_{s_1} = \psi_s^{-1} \psi_{w_0}$.

Let $g \in \mathcal{R}(D, \Delta)$ be defined by $g(e) = 1_\Delta$ for all $e \in E_2(D)$. Now, trivially, $\psi_{s_1}(g) = 1_\Delta$. Furthermore, for each $i = 1, 2, \ldots, k$ and $a \in \Delta$, define a selector $\sigma_{i,a}$ by $\sigma_{i,a}(y_i) = a$ and $\sigma_{i,a}(z) = 1_\Delta$ for all other nodes $z \in D$. We have immediately that for the commutator word u, $\psi_u(g^{\sigma_{i,a}}) = 1_\Delta$, since only the element a is involved in the labels of $g^{\sigma_{i,a}}$. Further, for $w_i = (x_0, y_i)$ we have $\psi_{w_i}^{\gamma_i}(g^{\sigma_{i,a}}) = a^{-\gamma_i}$ and $\psi_{w_j}^{\gamma_j}(g^{\sigma_{i,a}}) = 1_\Delta$ for $j \neq i$. By combining these results we obtain that $\psi_{s_1}(g^{\sigma_{i,a}}) = a^{-\gamma_i}$. Since ψ_{s_1} is an invariant, $a^{-\gamma_i} = \psi_{s_1}(g) = 1_\Delta$. In conclusion, we have shown that $a^{\gamma_i} = 1_\Delta$ for all $i = 1, 2, \ldots, k$ and $a \in \Delta$, from which it follows that $w_i^{\gamma_i} \equiv 1$ for $i = 1, 2, \ldots, k$, and thus, by the definition of s_1, $s_1 \equiv u$, that is, $\psi_u = \psi_{s_1}$ is an invariant represented by a commutator word u. Finally, $w \equiv w_0 \equiv su$, which completes the proof. □

The main characterization result for abelian groups, Theorem 14, is a special case of Theorem 18, because if Δ is abelian, then $d_\Delta = 1$ by Lemma 17, and every commutator word is equivalent to the empty word.

Note that, by Example 8.1, the d_Δ-triangles (or the commutator words representing invariants) do not suffice alone to produce the free invariants in $Inv\,(D \to \Delta)$.

We end this section by showing that the free invariants can be specified in a balanced form.

Denote $V(x) = \{(x, y) \mid y \neq x\}$ the set of all variables which contain the node $x \in D$ in the first position. Note that $(x, y)^{-1} \notin V(x)$ for all $y \in D$. For a word $w \in M(D)$ define the *exponent number of w on the node $x \in D$* by

$$\varepsilon_x(w) = \sum_{e \in V(x)} \varepsilon_e(w) \ .$$

Theorem 19. *Let ψ_w is a free invariant for a word $w \in M(D)$. Then $a^{\varepsilon_x(w)} = 1_\Delta$ holds for each $a \in \Delta$ and $x \in D$. Moreover, there exists a word w_0 such that $w_0 \equiv w$ and $\varepsilon_x(w_0) = 0$ for each $x \in D$.*

Proof. Let us fix a node $x \in D$. Consider the selectors σ_a, $a \in \Delta$, for which $\sigma_a(x) = a$ and $\sigma_a(z) = 1_\Delta$ for all other nodes $z \in D$. Let $g \in \mathcal{R}(D, \Delta)$ be such that $g(e) = 1_\Delta$ for all edges $e \in E_2(D)$. Clearly, $\psi_w(g) = 1_\Delta$ and $\psi_w(g^{\sigma_a}) = a^{\varepsilon_x(w)}$. Since w is an invariant, $a^{\varepsilon_x(w)} = \psi_w(g^{\sigma_a}) = \psi_w(g) = 1_\Delta$, and the first claim follows.

The second claim follows from Theorem 18, because for each triangle t, $\varepsilon_x(t) = 0$, and for each commutator word u, $\varepsilon_x(u) = 0$. □

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