

INVARIANTS OF THREE-MANIFOLDS DERIVED FROM LINKING MATRICES OF FRAMED LINKS

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Introduction. In [16], R. Kirby and P. Melvin study invariants of 3-manifolds $\tau_r (r \geq 3)$ introduced by E. Witten [38], N. Reshetikhin and V.G. Turaev [31], and W.B.R. Lickorish [25, 26, 27] (see also [18]). In particular, Kirby and Melvin calculated τ_3 and τ_4 explicitly. Let M be a closed, oriented 3-manifold obtained from an (integral) framed link L . Then $\tau_3(M)$ can be written as follows [16, §6].

$$\tau_3(M) = c^{-\sigma} \sqrt{2}^{-n} \sum_{S < L} \sqrt{-1}^{S \cdot S}.$$

Here n is the number of components of L , σ is the signature of its linking matrix, $c = \exp(\pi\sqrt{-1}/4)$, the sum is taken over all sublinks of L including the empty sublink, and $S \cdot S$ is the sum of all the entries in the linking matrix of S .

In this paper, we generalize τ_3 and define another series of invariants of 3-manifolds. Let q be a primitive N -th ($2N$ -th, resp.) root of unity for an odd (even, resp.) positive integer N . Put

$$Z_N(M; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{l \in \langle \mathbb{Z}/N\mathbb{Z} \rangle^n} q^{lAl},$$

where $G_N(q) = \sum_{h \in \mathbb{Z}/N\mathbb{Z}} q^{h^2}$ (a Gaussian sum), A is the linking matrix of L , l is regarded as a column vector, and ${}^t l$ is its transposed row vector. One can easily see that $Z_2(M; \sqrt{-1}) = \tau_3(M)$. We will show that these are all invariants for M (Theorem 1.3). As Kirby and Melvin proved for $\tau_3(M)$, $Z_N(M; q)$ is also invariant under homotopy equivalence. More precisely, it is determined by the first Betti number of M and the linking pairing on $\text{Tor } H_1(M; \mathbb{Z})$ for any N and q (Proposition 2.5, Corollary 2.6).

We will express the absolute value of $Z_N(M; q)$ in terms of the cohomology ring of M with $\mathbb{Z}/N\mathbb{Z}$ -coefficients (Theorem 3.2). When $|Z_N(M; q)| \neq 0$, we can also determine its phase (Theorem 4.5). It is a generalization of the Brown invariant $\beta(M)$ [16, §6] defined by the linking matrix using the signature and Brown's invariant [2] for $\mathbb{Z}/4\mathbb{Z}$ -valued quadratic forms on a $\mathbb{Z}/2\mathbb{Z}$ -vector space.

We can also calculate $Z_N(M; q)$ explicitly for 3-manifolds with linking pair-

ings which are members of generator system of linking pairings on finite abelian groups (Theorem 5.1). We also show that when M is a cyclic covering space of an oriented link, $Z_N(M; q)$ is essentially equivalent to the link invariant introduced by E. Date, M. Jimbo, K. Miki, and T. Miwa [4] using chiral Potts models (Proposition 6.3).

Other purpose of this paper is to describe various relationship of our invariants with quantum field theory, quantum groups, and $U(1)$ gauge theory. It is known that $Z_N(M; q)$ can be obtained from solutions to the polynomial equations associated with $\mathbf{Z}/N\mathbf{Z}$ -fusion rules [20, 21, 30]. It is also defined using a quasitriangular Hopf algebra as $\tau_r(M)$ [6, 31, 16] (§7). If N is even, the absolute values of our invariants coincide with the invariants of T. Gocho [8], which is defined via $U(1)$ gauge theory with charge N (§8). We can also prove that invariants of R. Dijkgraaf and E. Witten [5] can be described using our invariants if $G = \mathbf{Z}/N\mathbf{Z}$ (§9).

For basic concepts concerning 3-manifolds and links we refer the reader to [3, 11, 32].

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1. Definition of invariants. An oriented link in the 3-sphere S^3 is a finite collection of disjoint, smoothly embedded, oriented circles L_1, L_2, \dots , and L_n in S^3 . An (oriented, integral) *framed link* is an oriented link, each component L_i being provided with a framing f_i which is an isotopy class of a section of the projection $\partial N(L_i) \rightarrow L_i$. We can obtain a connected, closed, oriented 3-manifold M_L by surgery on S^3 along a framed link L . M_L is the result of gluing n copies of $D^2 \times S^1$ to $S^3 - \cup_{i=1}^n \text{int } N(L_i)$ so that the i -th $\partial D^2 \times \{*\}$ is identified with f_i . It is well known [24, 37] that each connected, closed, oriented 3-manifold can be obtained by surgery on S^3 along a certain framed link.

Let $A = (\lambda_{ij})$ ($1 \leq i, j \leq n$) be the linking matrix of L , that is, $\lambda_{ij} = \text{lk}(L_i, L_j)$ and $\lambda_{ii} = \text{lk}(L_i, f_i)$. Here $\text{lk}(\cdot, \cdot)$ denotes the linking number in S^3 . Denote by $\sigma(A)$ the signature of A (the number of positive eigenvalues minus the number of negative eigenvalues). Let N and d be coprime integers ($N \geq 2, d \geq 1$) with $N+d$ odd and put $q = \exp(d\pi\sqrt{-1}/N)$. Note that q is a primitive N -th root of unity if N is odd and a primitive $2N$ -th root of unity if N is even. Now we consider the following formula:

$$(1.1) \quad Z_N(M, L; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{i \in (\mathbf{Z}/N\mathbf{Z})^n} q^{i|A|},$$

where M is obtained by surgery on S^3 along L and $G_N(q) = \sum_{h \in \mathbf{Z}/N\mathbf{Z}} q^{h^2}$. $G_N(q)$ is called a Gaussian sum and its properties are well-known (see Lemma 4.4).

REMARK 1.2. For N odd, $q^{i|A|}$ is well-defined since q is an N -th root of

unity. For N even, we can also easily see that it is well-defined though q is a $2N$ -th root of unity. In both case, we can regard $l \mapsto {}^t l A l$ as a quadratic form in the following sense. If N is odd, a quadratic form on $(\mathbb{Z}/N\mathbb{Z})^n$ is a function $Q: (\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/N\mathbb{Z}$ satisfying $Q(ax) = a^2 Q(x)$ as usual. If N is even, a $\mathbb{Z}/2N\mathbb{Z}$ -valued quadratic form on $(\mathbb{Z}/N\mathbb{Z})^n$ associated to (\cdot, \cdot) is a function $Q: (\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/2N\mathbb{Z}$ satisfying $Q(ax) = a^2 Q(x) \in \mathbb{Z}/2N\mathbb{Z}$ and $Q(x+y) = Q(x) + Q(y) + 2 \cdot (x, y) \in \mathbb{Z}/N\mathbb{Z}$. Here $(\cdot, \cdot): (\mathbb{Z}/N\mathbb{Z})^n \times (\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/N\mathbb{Z}$ is a symmetric bilinear (not assumed to be non-singular) form, and $2: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{Z}/2N\mathbb{Z}$ is a homomorphism sending 1 to 2. In this case, $Q(l) = {}^t \bar{l} A \bar{l} \pmod{2N}$ with a lift $\bar{l} \in (\mathbb{Z})^n$ of l and (l, l') is ${}^t l A l' \pmod{N}$. (This definition coincides with that in [2, 10, 28] for the case that $N=2$ and (\cdot, \cdot) is non-singular.) Then the sum $\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{l A l}$ is written as $\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{Q(l)}$ and is an invariant of quadratic forms.

Theorem 1.3. $Z_N(M, L; q)$ is a topological invariant of M and does not depend on any choice of L .

Proof. Two unoriented framed links L and L' determine the same closed 3-manifold if and only if L' may be obtained from L by Kirby moves; “stabilization” and “handle sliding” (see [15]). Two framed links L and L' are related by a stabilization if they are identical except for elimination or insertion of a splitted, unknotted component L'_i with framing f'_i such that $\text{lk}(L'_i, f'_i) = \pm 1$. L and L' are related by a handle sliding if they are identical except for changing a component L_j by $L'_j = L_j \#_b f_i$ with framing f'_j such that $\text{lk}(L'_j, f'_j) = \text{lk}(L_j, f_j) + \text{lk}(L_i, f_i) \pm 2 \text{lk}(L_i, f_j)$. Here $\#_b$ means the band connected sum with b a band connecting f_i and L_j . The sign is $+$ if the orientations of f_i and L_j are coherent and $-$ otherwise.

Now from a theorem of R. Kirby [15], it suffices to verify that a stabilization, a handle sliding, and reversing of an orientation do not change $Z_N(M, L; q)$. Assume first that two framed links L and L' are related by a stabilization. We assume that L' is obtained from L by inserting a splitted, unknotted component. Then denoting by A the linking matrix of L with n components, that of L' is given by

$$A' = \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Since the size and the signature of A' is $n+1$ and $\sigma(A) \pm 1$ respectively, we have

$$Z_N(M, L'; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A) \mp 1} |G_N(q)|^{-n-1} \sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{l A l} \sum_{h \in \mathbb{Z}/N\mathbb{Z}} q^{\pm h^2}.$$

Since $\sum_{h \in \mathbb{Z}/N\mathbb{Z}} q^{\pm h^2} = G_N(q)$ or $\overline{G_N(q)}$ (the complex conjugate), we obtain $Z_N(M, L'; q) = Z_N(M, L; q)$.

Let L and L' be two framed links related by a handle sliding such that $L'_s = L_s \#_b f_i$. Then the linking matrix $A' = (\lambda'_{ij})$ of L' satisfies

$$\begin{aligned} \lambda'_{ss} &= \lambda_{ss} + \lambda_{it} \pm 2\lambda_{st}, \\ \lambda'_{is} &= \lambda_{is} \pm \lambda_{it} && (i \neq s), \\ \lambda'_{sj} &= \lambda_{sj} \pm \lambda_{tj} && (j \neq s), \\ \lambda'_{ij} &= \lambda_{ij} && (i \neq s, j \neq s). \end{aligned}$$

Hence $A' = {}^tTAT$ holds with $T_{ii} = 1$, $T_{is} = \pm 1$ and $T_{ij} = 0$ otherwise, where $T = (T_{ij})$. Putting $l' = T^{-1}l$, we have

$$\sum_{l' \in \langle \mathbb{Z} / N\mathbb{Z} \rangle^n} q^{l'A'l'} = \sum_{l \in \langle \mathbb{Z} / N\mathbb{Z} \rangle^n} q^{lAl}.$$

Since n and $\sigma(A)$ remain unchanged under this transformation, we have $Z_N(M, L'; q) = Z_N(M, L; q)$.

If L' is a framed link which is obtained from L by reversing orientation of a component L_k , then the linking matrix of L' is tSAS , where $S = (S_{ij})$ with $S_{ij} = 0$ ($i \neq j$), $S_{ii} = 1$ ($i \neq k$), and $S_{kk} = -1$. So $Z_N(M, L'; q) = Z_N(M, L; q)$ by a similar way as above.

This completes the proof. ■

By Theorem 1.3 we have topological invariants of M .

DEFINITION 1.4. *Let M be a connected, closed, compact 3-manifold obtained by surgery on S^3 along a framed link L . Then we put $Z_N(M; q) = Z_N(M, L; q)$.*

2. Fundamental properties. In this section we study fundamental properties of the invariant $Z_N(M; q)$.

First of all, we note that $Z_N(S^3; q) = 1$ for any N and q . If M is obtained from a framed link L , the mirror image of L gives $-M$, M with the opposite orientation. Since the linking matrix of the mirror image of L is $-A$ with A the linking matrix of L , we have

Proposition 2.1. *For a closed, oriented 3-manifold M ,*

$$Z_N(-M; q) = \overline{Z_N(M; q)}.$$

The split union of two framed links gives the connected sum of the corresponding 3-manifolds. So we have

Proposition 2.2. *If M_1 and M_2 are closed, oriented 3-manifolds, then*

$$Z_N(M_1 \# M_2; q) = Z_N(M_1; q) Z_N(M_2; q).$$

$Z_N(M; q)$ also factors associated with a factorization of N .

Proposition 2.3. *If $N=N_1 N_2$ with coprime integers N_1 and N_2 , then*

$$Z_N(M; q) = Z_{N_1}(M; q^{N_2}) Z_{N_2}(M; q^{N_1}).$$

Proof. $l \in (\mathbf{Z}/N\mathbf{Z})^n$ is uniquely expressed as $l=N_2 l_1+N_1 l_2$ for $l_1 \in (\mathbf{Z}/N_1 \mathbf{Z})^n$ and $l_2 \in (\mathbf{Z}/N_2 \mathbf{Z})^n$. Hence we have

$$\begin{aligned} \sum_{l \in (\mathbf{Z}/N\mathbf{Z})^n} q^{lA} &= \sum_{l_1 \in (\mathbf{Z}/N_1\mathbf{Z})^n, l_2 \in (\mathbf{Z}/N_2\mathbf{Z})^n} q^{N_2^2 \cdot l_1 A_1 + N_1^2 \cdot l_2 A_2 + 2N_1 N_2 \cdot l_1 A_2} \\ &= \sum_{l_1 \in (\mathbf{Z}/N_1\mathbf{Z})^n} q^{N_2^2 \cdot l_1 A_1} \sum_{l_2 \in (\mathbf{Z}/N_2\mathbf{Z})^n} q^{N_1^2 \cdot l_2 A_2}, \end{aligned}$$

where the second equality holds since $q^{2N_1 N_2} = 1$. In a similar way, we obtain $G_N(q) = G_{N_1}(q^{N_2}) G_{N_2}(q^{N_1})$. Therefore $Z_N(M; q)$ factors as above. ■

As R. Kirby and P. Melvin state for $\tau_3(M)$ [16, 6.2 Remark], $Z_N(M; q)$ is also a homotopy invariant (see Corollary 2.6 below) for every N and q . To prove this, we review results of M. Kneser and P. Puppe [17], A.H. Durfee [7], and R.H. Kyle [22].

Let B and B' be symmetric integral matrices. B and B' are said to be *stably equivalent* (or closely related in [22]) if they are equivalent under the equivalence relation generated by the following Q_1 and Q_2 :

$$\begin{aligned} Q_1: B &\leftrightarrow {}^tSBS \text{ with } S \text{ integral and unimodular,} \\ Q_2: B &\leftrightarrow \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix}. \end{aligned}$$

As in the previous section, let M be a 3-manifold obtained by surgery on S^3 along a framed link L and A its linking matrix. Summarizing results in [17, 7, 22], we can conclude that stable equivalence class is determined by the first Betti number of M and the linking pairing on $\text{Tor } H_1(M; \mathbf{Z})$. More precisely, the following proposition holds.

Proposition 2.4. *Stable equivalence class of linking matrices of framed links is determined by the first Betti number of the 3-manifold M obtained from it and $(\text{Tor } H_1(M; \mathbf{Z}), \lambda)$, that is, two linking matrices A and A' are stably equivalent if and only if M and M' satisfy (1) and (2) below, where M (M' , resp.) are obtained from framed link L (L' , resp.) with linking matrix A (A' , resp.).*

- (1) *The first Betti numbers of M M' are equal.*
- (2) *There exists an isomorphism between $\text{Tor } H_1(M; \mathbf{Z})$ and $\text{Tor } H_1(M'; \mathbf{Z})$ which induces an isomorphism between the linking pairings λ and λ' .*

Here the linking pairing on $\text{Tor } H_1(M; \mathbf{Z})$ is defined as follows. An exact sequence of coefficient groups

$$0 \rightarrow \mathbf{Z} \xrightarrow{i} \mathbf{Q} \xrightarrow{\eta} \mathbf{Q}/\mathbf{Z} \rightarrow 0$$

gives rise to a long exact sequence of homology groups of M :

$$\rightarrow H_2(M; \mathbf{Q}) \xrightarrow{\eta_*} H_2(M; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta_*} H_1(M; \mathbf{Z}) \xrightarrow{i_*} H_1(M; \mathbf{Q}) \rightarrow ,$$

where δ_* is the connecting homomorphism. The *linking pairing*

$$\lambda: \text{Tor } H_1(M; \mathbf{Z}) \times \text{Tor } H_1(M; \mathbf{Z}) \rightarrow \mathbf{Q}/\mathbf{Z}$$

is defined by $\lambda(\alpha, \beta) = \alpha \cdot \hat{\beta}$ where $\delta_* \hat{\beta} = \beta$ and a dot means the intersection product

$$H_1(M; \mathbf{Z}) \times H_2(M; \mathbf{Q}/\mathbf{Z}) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

One can easily check that λ is well-defined.

By the above proposition, we immediately have the following proposition.

Proposition 2.5. *If M and M' satisfy the conditions (1) and (2) in Proposition 2.4, then $Z_N(M; q) = Z_N(M'; q)$.*

Proof. Since the corresponding linking matrices A and A' are stably equivalent, we have $Z_N(M; q) = Z_N(M'; q)$ as in the proof of Theorem 1.4. ■

Clearly two 3-manifolds which are homotopy equivalent satisfy the conditions (1) and (2). So we have

Corollary 2.6. *If M and M' are homotopy equivalent, then $Z_N(M; q) = Z_N(M'; q)$.*

3. Absolute value. In this section we calculate the absolute value of $Z_N(M; q)$ and give its topological meaning.

First of all we prepare a lemma which will be used frequently in this paper. A proof is an easy exercise.

Lemma 3.1. *Let z be a primitive N -th root of unity. Then*

$$\sum_{x \in (\mathbf{Z}/N\mathbf{Z})^n} z^{xy} = \begin{cases} N^n & \text{if } y = 0 \in (\mathbf{Z}/N\mathbf{Z})^n, \\ 0 & \text{if } y \neq 0 \in (\mathbf{Z}/N\mathbf{Z})^n, \end{cases}$$

where we regard x and y as column vectors.

Now $|Z_N(M; q)|$ is given as follows. This generalizes [16, Theorem 6.3].

Theorem 3.2. *If there exists α in $H^1(M; \mathbf{Z}/N\mathbf{Z})$ with $\alpha \cup \alpha \cup \alpha \neq 0$, then $Z_N(M; q) = 0$. Otherwise $|Z_N(M; q)| = |H^1(M; \mathbf{Z}/N\mathbf{Z})|^{1/2}$ where $|\cdot|$ in the right hand side is the order of the set.*

Proof. Let M be a 3-manifold obtained by surgery on S^3 along an n -

component framed link L . From (1.1), we have

$$|Z_N(M; q)| = |G_N(q)|^{-n} \cdot \left| \sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{lAl} \right|.$$

We first calculate $|G_N(q)|^2 = N$.

$$\begin{aligned} |G_N(q)|^2 &= \sum_{h, h' \in \mathbb{Z}/N\mathbb{Z}} q^{h'^2 - h^2} \\ &= \sum_{h''} q^{h''^2} \sum_h q^{2h''h} \quad (h' = h'' + h) \\ &= N. \end{aligned}$$

The last equality follows from Lemma 3.1 putting $n=1$, $x=h''$, $y=h$, $z=q^2$ since q^2 is a primitive N -th root of unity.

Next we calculate the absolute value of $\sum q^{lAl}$. In a similar way as above, we have

$$\begin{aligned} \left| \sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{lAl} \right|^2 &= \sum_{l', l} q^{l'A'l' - lAl} \\ &= \sum_{l''} q^{l''A'l''} \sum_l q^{2lAl''} \quad (l' = l'' + l) \\ &= N^n \sum_{l'' \in \ker L_A} q^{l''A'l''}, \end{aligned}$$

where L_A is a linear map $L_A: (\mathbb{Z}/N\mathbb{Z})^n \rightarrow (\mathbb{Z}/N\mathbb{Z})^n, l \mapsto Al$. The last equality follows from Lemma 3.1 putting $x=l$ and $y=Al''$. Therefore we have

$$|Z_N(M; q)|^2 = \left| \sum_{l \in \ker L_A} q^{lAl} \right|.$$

Now there are two cases to consider.

Case 1: N is odd. Recall that q is an N -th root of unity. For $l \in \ker L_A$, we have $lAl=0$ in $\mathbb{Z}/N\mathbb{Z}$ and $q^{lAl}=1$. Hence $|Z_N(M; q)|^2$ is equal to the order of $\ker L_A$. By Lemma 3.3 below, we have

$$|Z_N(M; q)| = |H^1(M; \mathbb{Z}/N\mathbb{Z})|^{1/2}.$$

In this case $\alpha \cup \alpha = 0$ holds for any α in $H^1(M; \mathbb{Z}/N\mathbb{Z})$, because the cup product is skew-symmetric and the order of $H^3(M; \mathbb{Z}/N\mathbb{Z})$ is odd. Hence we obtain Theorem 4.1 for N odd.

Case 2: N is even. In this case q is a $2N$ -th root of unity. As in Remark 1.3 we regard $l \mapsto lAl$ as a map $(\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/2N\mathbb{Z}$. We denote the restriction of this map to $\ker L_A$ by $\varphi: \ker L_A \rightarrow \{0, N\} \subset \mathbb{Z}/2N\mathbb{Z}$. Then φ is a homomorphism because

$${}^t(\tilde{l} + \tilde{l}') A(\tilde{l} + \tilde{l}') = {}^t\tilde{l}A\tilde{l} + {}^t\tilde{l}'A\tilde{l}' + 2 \cdot {}^t\tilde{l}A\tilde{l}'$$

and $2 \cdot {}^t\tilde{l}A\tilde{l}'$ can be divided by $2N$. Therefore we have

$$|Z_N(M; q)| = \begin{cases} |\ker L_A|^{1/2} & \varphi \equiv 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemmas 3.3 and 3.4 below, we obtain

$$\begin{aligned} & |Z_N(M; q)| \\ &= \begin{cases} |H^1(M; \mathbf{Z}/N\mathbf{Z})|^{1/2} & \text{if } \alpha \cup \alpha \cup \alpha = 0 \text{ for any } \alpha \in H^1(M; \mathbf{Z}/N\mathbf{Z}), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This completes the proof. ■

Lemma 3.3. *ker L_A is isomorphic to $H^1(M; \mathbf{Z}/N\mathbf{Z})$.*

Proof. Since M is a union of S^3 -int $N(L)$ and n copies of $D^2 \times S^1$, we have the Mayer-Vietoris exact sequence below.

$$\begin{aligned} \dots \rightarrow & H^1(M; \mathbf{Z}/N\mathbf{Z}) \rightarrow H^1(S^3\text{-int } N(L); \mathbf{Z}/N\mathbf{Z}) \oplus \bigoplus^n H^1(D^2 \times S^1; \mathbf{Z}/N\mathbf{Z}) \\ & \xrightarrow{f} \bigoplus^n H^1(T^2; \mathbf{Z}/N\mathbf{Z}) \rightarrow \dots \end{aligned}$$

Hence $H^1(M; \mathbf{Z}/N\mathbf{Z})$ is isomorphic to $\ker f$. Moreover f corresponds to a matrix $\begin{pmatrix} 1_n & 1_n \\ A & 0 \end{pmatrix}$ as a map $f: (\mathbf{Z}/N\mathbf{Z})^n \oplus (\mathbf{Z}/N\mathbf{Z})^n \rightarrow (\mathbf{Z}/N\mathbf{Z})^{2n}$, where 1_n is the $n \times n$ identity matrix. Since $\ker f$ is isomorphic to $\ker L_A$, we obtain Lemma 3.3. ■

Lemma 3.4. *Let N be even. With the isomorphism ι in Lemma 3.3, the next diagram commutes:*

$$\begin{array}{ccc} \ker L_A & \xrightarrow{\varphi} & \{0, N\} \subset \mathbf{Z}/2N\mathbf{Z} \\ \iota \downarrow & & \downarrow \times \frac{1}{2} \\ H^1(M; \mathbf{Z}/N\mathbf{Z}) & \xrightarrow{\psi} & \left\{0, \frac{N}{2}\right\} \subset H^3(M; \mathbf{Z}/N\mathbf{Z}) = \mathbf{Z}/N\mathbf{Z}, \end{array}$$

where ψ is defined by $\psi(\alpha) = \alpha \cup \alpha \cup \alpha$.

Proof. Let l be an element in $\ker L_A$, and put $\alpha = \iota(l)$. We calculate $\alpha \cup \alpha \cup \alpha$ in the Poincaré dual and we will show that $\alpha \cup \alpha \cup \alpha$ is equal to $\varphi(l)/2$.

Let S be a branched surface representing the Poincaré dual modulo $\mathbf{Z}/N\mathbf{Z}$ of α in $M = (S^3 - \text{int } N(L)) \cup \bigcup_{i=1}^n D^2 \times S^1$ such that branch locus of S is a union of disjoint circles in $S^3 - N(L)$ and the number of sheets meeting along each circle is a multiple of N . Since $[S]$ is the Poincaré dual of $\iota(l)$, $S \cap \partial N(L_i)$ is a union of \tilde{l}_i circles in $\partial N(L_i)$, each of which is parallel to the framing f_i , where $\tilde{l}_i \in \mathbf{Z}$ is a lift of $l_i \in \mathbf{Z}/N\mathbf{Z}$ with $\iota l = (l_1, \dots, l_n)$. Let m_i be a meridian of L_i in $S^3 - N(L)$. Since $[m_i]$'s generate $H_1(M; \mathbf{Z})$, we may assume that branch locus

of S is a union of m_i 's. Let $a_i N$ be the number of sheets of S meeting along m_i .

Since the boundary of $S \cap (S^3 - \text{int } N(L))$ consists of \tilde{l}_i copies of f_i in $\partial N(L_i)$, we have $\sum a_i N [m_i] = \sum \tilde{l}_i [f_i]$ in $H_1(S^3 - \text{int } N(L); \mathbf{Z})$. Moreover the classes $[f_i]$'s are determined by

$$\begin{pmatrix} [f_1] \\ \vdots \\ [f_n] \end{pmatrix} = A \begin{pmatrix} [m_1] \\ \vdots \\ [m_n] \end{pmatrix}.$$

Hence we obtain a relation between a_i 's and \tilde{l}_i 's:

$$\begin{pmatrix} a_1 N \\ \vdots \\ a_n N \end{pmatrix} = A \begin{pmatrix} \tilde{l}_1 \\ \vdots \\ \tilde{l}_n \end{pmatrix}.$$

Now we calculate the self-intersection of S . Since $S - \cup m_i$ is orientable, we can push S in a normal direction. There are self-intersections near m_i , as in Figure 3.1. Hence we have

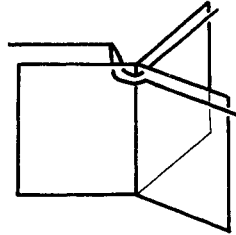


Figure 3.1.

$$\begin{aligned} [S] \cdot [S] &= \sum (1 + 2 + \dots + (a_i N - 1)) [m_i] \\ &= \sum \frac{a_i N}{2} [m_i] \in H_1(M; \mathbf{Z}/N\mathbf{Z}). \end{aligned}$$

Since $[S] \cdot [m_i] = \tilde{l}_i$, we obtain

$$\begin{aligned} [S] \cdot [S] \cdot [S] &= \sum \frac{a_i N}{2} \tilde{l}_i \\ &= \frac{1}{2} \tilde{l} A \tilde{l} = \frac{1}{2} \varphi(l). \end{aligned}$$

This is the required formula. ■

REMARK 3.4. The above lemma also follows algebraically from [35, Theorem I], which states that

$$\alpha \cup \alpha \cup \beta = \frac{N^2}{2} \lambda(\alpha, \beta) \in \mathbf{Z}/N\mathbf{Z}$$

for $\alpha, \beta \in H^1(M; \mathbf{Z}/N\mathbf{Z})$. Here $\bar{\alpha}, \bar{\beta} \in \text{Tor } H_1(M; \mathbf{Z})$ satisfy $\lambda(\bar{\alpha}, x) = \alpha(x)/n \in \mathbf{Q}/\mathbf{Z}$ and $\lambda(\bar{\beta}, x) = \alpha(x)/n \in \mathbf{Q}/\mathbf{Z}$ for any $x \in \text{Tor } H_1(M; \mathbf{Z})$.

4. Phase. Now we study the phase of $Z_N(M; q)$.

We use the following notations for an odd integer x : (cf. [33])

$$(4.1) \quad \begin{aligned} \varepsilon(x) &= (-1)^{(x-1)/2} = \begin{cases} 1 & x \equiv 1 \pmod{4}, \\ -1 & x \equiv 3 \pmod{4}, \end{cases} \\ \omega(x) &= (-1)^{(x^2-1)/8} = \begin{cases} 1 & x \equiv \pm 1 \pmod{8}, \\ -1 & x \equiv \pm 3 \pmod{8}. \end{cases} \end{aligned}$$

Note that $\varepsilon: (\mathbf{Z}/4\mathbf{Z})^\times \rightarrow \{1, -1\}$ and $\omega: (\mathbf{Z}/8\mathbf{Z})^\times \rightarrow \{1, -1\}$ are homomorphisms.

By Theorem 3.1, $Z_N(M; q) \neq 0$ if $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \mathbf{Z}/N\mathbf{Z})$. So we assume this in the following of this section.

We put

$$Z_N(A; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} \sqrt{N}^{-n} \sum_{I \in (\mathbf{Z}/N\mathbf{Z})^n} q^{|IA|},$$

for an integral symmetric $n \times n$ matrix A . $Z_N(M; q) = Z_N(A; q)$ if M is obtained from a framed link L with linking matrix A . We note that if N is odd, A may be regarded as a matrix in $\mathbf{Z}/N\mathbf{Z}$ and if N is even, the diagonal entries in A may be regarded as integers modulo $2N$ and the off-diagonal entries modulo N . We will try to diagonalize A to calculate the phase.

From Proposition 2.3 we will restrict ourselves to the case $N = p^m$ with p prime for a while.

If p is odd, we can diagonalize A as a matrix in $\mathbf{Z}/N\mathbf{Z}$, that is, there exists a matrix $S \in SL(n, \mathbf{Z})$ such that

$$(4.2) \quad {}^tSAS \equiv \bigoplus_{j=1}^n (a_j) \pmod{p^m}.$$

If $p=2$, we cannot diagonalize A itself in general, but it is proved that one can diagonalize the block sum of A and $(1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \dots \oplus (2^{m-1}) \oplus (-2^{m-1})$, that is, there exists a matrix $S \in SL(n+2m, \mathbf{Z})$ such that

$$(4.3) \quad {}^tS(A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \dots \oplus (2^{m-1}) \oplus (-2^{m-1})) S \equiv \bigoplus_{j=1}^{n+2m} (a_j),$$

where the diagonal entries are considered modulo 2^{m+1} and the off-diagonal entries modulo 2^m . (In fact, it can be proved that $A \oplus (1) \oplus (2) \oplus \dots \oplus (2^{m-1})$ is diagonalizable, using the technique to diagonalize $\begin{pmatrix} 0 & 2^j \\ 2^j & 0 \end{pmatrix} \oplus (2^j)$.) Note that the phase of $Z_N(A; q)$ remains unchanged by replacing A with $A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2)$

$\oplus \dots \oplus (2^{m-1}) \oplus (-2^{m-1})$ since easy calculations show that $Z_N((2^j); q) \neq 0$ and $Z_N((2^j); q) = \overline{Z_N((-2^j); q)}$ for $j < m$.

Now the phase of $Z_{p^m}(A, q)$ is equal to that of $(G_{p^m}(q))^{-\sigma(A)} \prod_j \sum_{h \in \mathbf{Z}/p^m\mathbf{Z}} q^{ah^2}$. Thus we only need to calculate the sum

$$G_N(a; q) = \sum_{h \in \mathbf{Z}/N\mathbf{Z}} q^{ah^2}$$

for an integer a and a prime-power N . Note that a Gaussian sum $G_N(q)$ is equal to $G_N(1; q)$. Let $q = \exp(d\pi\sqrt{-1}/p^m)$ with $(d, p) = 1$ and $d + p$ odd, and $a = p^k c$ with $(p, c) = 1$. If p is odd, we also write q as $\exp(2b\pi\sqrt{-1}/p^m)$ putting $d = 2b$. We can describe the above sum as follows.

Lemma 4.4.

(1) p is odd. Let $\left(\frac{x}{p}\right)$ be Legendre's symbol, that is, $\left(\frac{x}{p}\right) = 1$ if there exists an integer l such that $l^2 \equiv x \pmod{p}$, and $\left(\frac{x}{p}\right) = -1$ otherwise. Then

$$G_{p^m}(a; q) = \begin{cases} p^m & \text{if } k - m \geq 0, \\ \sqrt{p}^{m+k} & \text{if } k - m < 0 \text{ and even,} \\ \left(\frac{c}{p}\right) \left(\frac{b}{p}\right) \sqrt{p}^{m+k} & \text{if } k - m < 0 \text{ and odd, and } p \equiv 1 \pmod{4}, \\ \left(\frac{c}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p}^{m+k} & \text{if } k - m < 0 \text{ and odd, and } p \equiv 3 \pmod{4}. \end{cases}$$

(2) $p = 2$. Put $\zeta = \exp(\pi\sqrt{-1}/4)$. Then

$$G_{2^m}(a; q) = \begin{cases} 2^m & \text{if } k - m > 0, \\ 0 & \text{if } k - m = 0, \\ \zeta^{cd} \sqrt{2}^{m+k} & \text{if } k - m < 0 \text{ and even,} \\ \zeta^{2(c)\mathfrak{z}(e)} \sqrt{2}^{m+k} & \text{if } k - m < 0 \text{ and odd.} \end{cases}$$

Proof. For the case that $k - m > 0$ or the case that $k - m = 0$ and p is odd, the formulas follow since $q^a = 1$. If $k - m = 0$ and p is even, $G_N(a, q) = 0$ since $q^a = -1$. The case that $p = 2$ and $k - m < 0$, the formula follows from $G_{2^m}(a, q) = 2G_{2^{m-2}}(a, q)$ ($m \geq k + 3$) and direct computations for $m = k + 1$ and $k + 2$. The case that p is odd and $k - m < 0$ is well-known. For a proof, see for example [23, Chapter IV, §3]. (There are some errors in [23], which one can easily fix.) The proof is complete. ■

From this lemma, we know that the phase of $Z_N(A; q)$ takes only eight values. So we define $\phi_N(A; q) \in \mathbf{Z}/8\mathbf{Z}$ as follows.

We first consider the case that $N = p^m$ for an odd prime p . Let a_j 's be diagonal entries when A is diagonalized as in (4.2). Let $a_j = p^{k_j} c_j$ with $(p, c_j) = 1$

for $a_j \neq 0$. Note that we can assume $k_j - m$ is always negative. We put n_+ and n_- as follows.

$$n_+ = \#\{a_j \mid k_j - m \text{ is odd and } \left(\frac{c_j}{p}\right) = 1\}$$

$$n_- = \#\{a_j \mid k_j - m \text{ is odd and } \left(\frac{c_j}{p}\right) = -1\}$$

Here $\#\{\cdot\}$ means the number of elements in $\{\cdot\}$. Then $\phi_{p^m}(A; q) \in \mathbf{Z}/8\mathbf{Z}$ is defined as follows.

$$\phi_{p^m}(A; q) = \begin{cases} 2\left(\left(\frac{b}{p}\right) - 1\right)n_+ - 2\left(\left(\frac{b}{p}\right) + 1\right)n_- & \text{if } p \equiv 1 \pmod{4} \text{ and } m \text{ is even,} \\ 2\left(\left(\frac{b}{p}\right) - 1\right)n_+ - 2\left(\left(\frac{b}{p}\right) + 1\right)n_- - 2\left(\left(\frac{b}{p}\right) - 1\right)\sigma(A) & \text{if } p \equiv 1 \pmod{4} \\ & \text{and } m \text{ is odd,} \\ 2\left(\frac{b}{p}\right)n_+ - 2\left(\frac{b}{p}\right)n_- & \text{if } p \equiv 3 \pmod{4} \text{ and } m \text{ is even,} \\ 2\left(\frac{b}{p}\right)n_+ - 2\left(\frac{b}{p}\right)n_- - 2\left(\frac{b}{p}\right)\sigma(A) & \text{if } p \equiv 3 \pmod{4} \text{ and } m \text{ is odd.} \end{cases}$$

Then from Lemma 4.4 and $\left(\frac{x}{p}\right) = \zeta^{2\left(\frac{x}{p}\right)-1}$, it follows that $\phi_{p^m}(A; q) \pi \sqrt{-1}/4$ is the phase of $Z_{p^m}(A; q)$.

Next we consider the case $N=2^m$. Let a_j 's be diagonal entries when A is diagonalized as in (4.3). Let $a_j = 2^{k_j} c_j$ with c_j odd for $a_j \neq 0$. Here we assume $k_j - m < 0$ as before. Then $\phi_{2^m}(A; q)$ is defined by

$$\phi_{2^m}(A; q) = \begin{cases} d \sum_{k_j - m : \text{even}} c_j + \varepsilon(d) \sum_{k_j - m : \text{odd}} \varepsilon(c_j) - d\sigma(A) & \text{if } m \text{ is even,} \\ d \sum_{k_j - m : \text{even}} c_j + \varepsilon(d) \sum_{k_j - m : \text{odd}} \varepsilon(c_j) - \varepsilon(d) \sigma(A) & \text{if } m \text{ is odd.} \end{cases}$$

From Lemma 4.4, the phase of $Z_{2^m}(A; q)$ is $\phi_{2^m}(A; q) \pi \sqrt{-1}/4$.

According to Proposition 2.3, we define $\phi_N(A; q)$ for an arbitrary N by using

$$\phi_N(A; q) = \phi_{N_1}(A; q^{N_2}) + \phi_{N_2}(A; q^{N_1}) \in \mathbf{Z}/8\mathbf{Z},$$

where $N = N_1 N_2$ with coprime integers N_1 and N_2 .

For a closed, oriented 3-manifold M , we define $\phi_N(M; q) = \phi_N(A; q)$ for the linking matrix A of a framed link which gives M . Summarizing the above argument we have the next proposition.

Theorem 4.5. *If $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \mathbf{Z}/N\mathbf{Z})$, then*

$$Z_N(M, q) = \exp\left(\frac{\pi \sqrt{-1}}{4} \phi_N(M; q)\right) |H^1(M; \mathbf{Z}/N\mathbf{Z})|^{1/2},$$

where $\phi_N(M; q) \in \mathbf{Z}/8\mathbf{Z}$ is defined above. In particular $\phi_N(M; q)$ is a topological

invariant of M.

REMARK 4.6. By definition, $\beta(M) = -\phi_2(M; \sqrt{-1})$ is the Brown invariant [16, §6]. See [2, 10, 28] for Brown's invariant of $\mathbf{Z}/4\mathbf{Z}$ -valued quadratic forms on a $\mathbf{Z}/2\mathbf{Z}$ -vector space.

As applications of Theorem 4.5, we calculate $Z_N(M; q)$ for $\mathbf{Z}/p\mathbf{Z}$ -homology spheres. (A closed, oriented 3-manifold M is called a $\mathbf{Z}/p\mathbf{Z}$ -homology sphere if $H_i(M; \mathbf{Z}/p\mathbf{Z}) = H_i(S^3; \mathbf{Z}/p\mathbf{Z})$ for all i .)

Corollary 4.7. *Let $N = 2^m$ and $q = \exp(d\pi\sqrt{-1}/N)$. If M is a $\mathbf{Z}/2\mathbf{Z}$ -homology sphere, then the value of $Z_N(M; q)$ is as follows.*

$$Z_N(M; q) = \begin{cases} \zeta^{-d\mu(M)} & \text{if } m \text{ is even,} \\ \omega(|H_1(M; \mathbf{Z})|) \zeta^{-\varepsilon(d)\mu(M)} & \text{if } m \text{ is odd.} \end{cases}$$

where $\zeta = \exp(\pi\sqrt{-1}/4)$ and $\mu(M)$ is the μ -(or Rochlin) invariant of M (the signature modulo 16 of a spin 4-manifold with boundary M).

Proof. Since $H^1(M; \mathbf{Z}/N\mathbf{Z}) = 0$, we calculate the phase. After a change of basis we may assume that A is diagonal (mod $2N$) with diagonal entries a_j . Since M is a $\mathbf{Z}/2\mathbf{Z}$ -homology sphere, a_j is always odd. We also assume that $a_j = 1, 3, 5$, or 7 because there exists an odd integer l such that $cl^2 = 1, 3, 5$, or $7 \pmod{2N}$ for any odd integer c . Let n_c be the number of c 's in these diagonal entries ($c = 1, 3, 5$, or 7).

For m even, by the definition of $\phi_N(M; q)$, we have

$$\phi_N(M; q) \equiv d(n_1 + 3n_3 + 5n_5 + 7n_7 - \sigma(A)) \pmod{8}.$$

Since $\mu(M) \equiv \sigma(A) - (n_1 + 3n_3 + 5n_5 + 7n_7) \pmod{8}$ (see [16, Appendix C]), we obtain the required formula.

For m odd, we have

$$\phi_N(M; q) = \varepsilon(d)(n_1 - n_3 + n_5 - n_7 - \sigma(A))$$

Thus $\phi_N(M; q) + \varepsilon(d)\mu(M) \equiv -4\varepsilon(d)(n_3 + n_5) \pmod{8}$. Since $\varepsilon(d) = \pm 1$, we have

$$\phi_N(M; q) \equiv -\varepsilon(d)\mu(M) + 4(n_3 + n_5) \pmod{8}.$$

Moreover since

$$|H_1(M; \mathbf{Z})| = \pm \det A \equiv \pm 3^{n_3} 5^{n_5} 7^{n_7} \equiv \pm 3^{n_3} (-3)^{n_5} (-1)^{n_7} \pmod{8},$$

we obtain $\omega(|H_1(M; \mathbf{Z})|) = (-1)^{n_3 + n_5}$. Therefore we obtain the required formula. ■

Corollary 4.8. *Let $N = p^m$ with odd prime p and q an N -th root of unity.*

If M is a $\mathbf{Z}/p\mathbf{Z}$ -homology sphere, then

$$Z_N(M; q) = \left(\frac{r}{p}\right)^m$$

where $r = |H_1(M; \mathbf{Z})|$ and $\left(\frac{r}{p}\right)$ is Legendre's symbol.

Proof. Adding a splitted, unknotted component if necessary, we assume that $\det A$ is positive so that $r = \det A$. Let b, a_j 's, n_+ , and n_- be as in the notation of the definition of $\phi_{p^m}(A; q)$. Since M is a $\mathbf{Z}/p\mathbf{Z}$ -homology sphere, $(p, a_j) = 1$ and so $k_j = 0$ for any j . We also note that $r = \det A \equiv \prod a_j \pmod p$. Thus we have

$$\left(\frac{r}{p}\right) = \left(\frac{\prod a_j}{p}\right) = \prod \left(\frac{a_j}{p}\right) = \begin{cases} 1 & \text{if } n_- \text{ is even,} \\ -1 & \text{if } n_- \text{ is odd.} \end{cases}$$

For m even, we have $n_+ = n_- = 0$. Hence $Z_N(M; q) = 1$.

Next we consider the case that m is odd. In this case, $n_+ + n_- = n$, the size of A . So $n_+ = n - n_-$. We also have $\sigma(A) \equiv n \pmod 4$ since $\det A > 0$.

If $p \equiv 1 \pmod 4$, then by definition, we have

$$\begin{aligned} \phi_N(M; q) &= 2 \left(\left(\frac{b}{p}\right) - 1\right) n_+ - 2 \left(\left(\frac{b}{p}\right) + 1\right) n_- - \left(\left(\frac{b}{p}\right) - 1\right) \sigma(A) \\ &= 2 \left(\left(\frac{b}{p}\right) - 1\right) (n - \sigma(A)) - 4n_- \\ &\equiv 4n_- \pmod 8. \end{aligned}$$

If $p \equiv 3 \pmod 4$, then we also have

$$\begin{aligned} \phi_N(M; q) &= 2 \left(\frac{b}{p}\right) n_+ - 2 \left(\frac{b}{p}\right) n_- - 2 \left(\frac{b}{p}\right) \sigma(A) \\ &= 2 \left(\frac{b}{p}\right) (n - \sigma(A)) - 4 \left(\frac{b}{p}\right) n_- \\ &\equiv 4n_- \pmod 8. \end{aligned}$$

Therefore we obtain the value of $Z_N(M; q)$ as above, completing the proof. ■

5. Calculation for generators of linking pairings

Any linking pairing is a direct sum of the following linking pairings [36, 14]:

$$(p^{-k}r) (k \geq 1), \quad A^k(n) (k \geq 1), \quad E_0^k (k \geq 1), \quad \text{and} \quad E_1^k (k \geq 2),$$

where p is odd, prime integer, r is 1 or a fixed quadratic non-residue modulo p , and $n = 1 (k = 1), \pm 1 (k = 2), \pm 1$ or $\pm 3 (k \geq 3)$. Here we use the notation of

[14].

Since $Z_N(M; q)$ is an invariant of first Betti numbers and linking pairings (Proposition 2.5), and linking pairings split as above, we can calculate $Z_N(M; q)$ if we know them for 3-manifolds with the linking pairings above from Proposition 2.3. Note that the free part of the first homology affects $Z_N(M; q)$ only by absolute values (Theorem 3.2).

In the following, we denote $Z_N(M; q)$ by $Z_N(\Lambda; q)$ if the linking pairing on $H_1(M; \mathbf{Z})$ is isomorphic to Λ in the above.

Theorem 5.1. *Let p and p' be odd, prime integers ($p \neq p'$), and b' integers with $(p, b) = 1$ and $(p', b') = 1$, and d an odd integer. Put $q = \exp(2b\pi\sqrt{-1}/p^m)$, $q' = \exp(2b'\pi\sqrt{-1}/p'^m)$, $q'' = \exp(d\pi\sqrt{-1}/2^m)$, and $\zeta = \exp(\pi\sqrt{-1}/4)$. We also use the notations (4.1).*

(1) *The case $\Lambda = (p^{-k} r)$.*

$$Z_{p^m}((p^{-k} r); q'') = \begin{cases} 1 & \text{for } (*, 0, *, *) , \\ -\omega(p) \left(\frac{r}{p}\right) & \text{for } (*, 1, 0, 1) , \\ -\varepsilon(d) \omega(p) \left(\frac{r}{p}\right) \sqrt{-1} & \text{for } (*, 1, 0, 3) , \\ -\left(\frac{r}{p}\right) & \text{for } (*, 1, 1, 1) , \\ -\left(\frac{r}{p}\right) \sqrt{-1} & \text{for } (*, 1, 1, 3) . \end{cases}$$

$$Z_{p^m}((p^{-k} r); q) = \begin{cases} \sqrt{p^m} & \text{for } (+ \text{ or } 0, *, 0, *) , \\ \left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{p^m} & \text{for } (+ \text{ or } 0, *, 1, 1) , \\ -\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^m} & \text{for } (+ \text{ or } 0, *, 1, 3) , \\ \sqrt{p^k} & \text{for } (-, 0, *, *) , \\ \left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{p^k} & \text{for } (-, 1, *, 1) , \\ \left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^k} & \text{for } (-, 1, 0, 3) , \\ -\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^k} & \text{for } (-, 1, 1, 3) . \end{cases}$$

Here $(\cdot, \cdot, \cdot, \cdot)$ is (sign of $k - m, k \bmod 2, m \bmod 2, p \bmod 4$).

$$Z_{p^m}((p^{-k} r); q') = \left(\frac{p'}{p}\right)^{mk} .$$

(2) *The case $\Lambda = A^k(n), E_0^k$, or E_1^k .*

$$Z_{2^m}(A^k(1); q') = \begin{cases} \zeta^{-d} \sqrt{2^m} & \text{for } (+, *, 0), \\ \zeta^{-e(d)} \sqrt{2^m} & \text{for } (+, *, 1), \\ 0 & \text{for } (0, *, *), \\ \sqrt{2^k} & \text{for } (-, 0, *), \\ \omega(d) \sqrt{2^k} & \text{for } (-, 1, *). \end{cases}$$

$$Z_{2^m}(A^k(3); q') = \begin{cases} \zeta^{5d} \sqrt{2^m} & \text{for } (+, 0, 0), \\ \zeta^d \sqrt{2^m} & \text{for } (+, 1, 0), \\ \zeta^{e(d)} \sqrt{2^m} & \text{for } (+, 0, 1), \\ \zeta^{-3e(d)} \sqrt{2^m} & \text{for } (+, 1, 1), \\ 0 & \text{for } (0, *, *), \\ \sqrt{2^k} & \text{for } (-, 0, *), \\ \omega(d) \sqrt{2^k} & \text{for } (-, 1, *). \end{cases}$$

Here (\cdot, \cdot, \cdot) is (sign of $k-m, k \bmod 2, m \bmod 2$).

$$Z_{2^m}(A^k(-1); q'') = \overline{Z_{2^m}(A^k(1); q')} \quad (\text{complex conjugate}).$$

$$Z_{2^m}(A^k(-3); q'') = \overline{Z_{2^m}(A^k(3); q')}.$$

$$Z_{2^m}(E_0^k; q'') = \begin{cases} 2^m & \text{if } k \geq m, \\ 2^k & \text{if } k < m. \end{cases}$$

$$Z_{2^m}(E_1^k; q'') = \begin{cases} (-1)^{m+k} 2^m & \text{if } k \geq m, \\ 2^k & \text{if } k < m. \end{cases}$$

$$Z_{p^m}(A^k(n); q) = \begin{cases} -1 & \text{if } m \text{ and } k \text{ are odd, and } p \equiv \pm 3 \pmod{8}, \\ 1 & \text{otherwise.} \end{cases}$$

$$Z_{p^m}(E_0^k; q) = Z_{p^m}(E_1^k; q) = 1$$

Proof. For $(p^{-k} r)$, we consider the lens space $L(p^k, r)$. It can be obtained from a framed link with linking matrix of the form

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 1 & a_2 & 1 & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}.$$

Here the continued fraction

$$a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}}$$

is equal to p^k/r . See for example [32]. So we can calculate $Z_{p^m}((p^{-k} r); q)$ using Theorem 4.5. The value $Z_{p^m}((p^{-k} r); q')$ can be calculated using Corollary 4.8. $Z_{2^m}((p^{-k} r); q'')$ can be obtained from Corollary 4.7 and the fact

$$2\mu(L(\alpha, \beta)) \equiv 2(\alpha + 1) - 4(\beta | \alpha) \pmod{16},$$

where $(\beta | \alpha)$ is the Jacobi symbol [12, Theorem 8.14]. Note that our definition of the μ -invariant differs from that in [12].

For $A^k(1)$ and $A^k(3)$, we choose linking matrices of the form

$$(2^k) \quad \text{and} \quad (-1)^{k+1} \begin{pmatrix} (4^{m+1} - (-2)^k)/3 & 2^{m+1} \\ 2^{m+1} & 3 \end{pmatrix},$$

respectively. (Note that they are diagonal in $\mathbf{Z}/2^{2m+1}\mathbf{Z}$.) Then we can calculate $Z_{2^m}(A^k(n); q'')$ ($n=1, 3$) using Theorem 4.5. Since if the linking pairing for a 3-manifold M is $A^k(n)$, then that for $-M$ is $A^k(-n)$, the values $Z_{2^m}(A^k(n); q'')$ ($n=-1, -3$) are obtained from Proposition 2.1.

To calculate $Z_{2^m}(E_0^k; q'')$ ($m \neq k$), we use the relation (see [14])

$$A^k(1) \oplus 2A^k(-1) = A^k(-1) \oplus E_0^k.$$

Since $Z_{2^m}(A^k(-1); q'') \neq 0$ for $m \neq k$, we obtain $Z_{2^m}(E_0^k; q'')$ from Proposition 2.2.

For $m=k$, we can directly calculate it choosing $\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$ as a linking matrix for E_0^k .

Using the relations (see [14] again)

$$3A^k(1) = A^k(3) \oplus E_1^k \quad \text{and} \quad E_1^k \oplus A^{k+1}(1) = E_0^k \oplus A^{k+1}(-3),$$

we can obtain $Z_{2^m}(E_1^k; q'')$ for any m .

The values $Z_{p^m}(A^k(n); q)$, $Z_{p^m}(E_0^k; q)$ and $Z_{p^m}(E_1^k; q)$ are easily obtained from Corollary 4.7.

The proof is complete. ■

REMARK 5.2. The series $\{Z_N(\cdot; q)\}$ is *not* a complete invariant of linking pairings. For example $Z_N(32A^1(1) \oplus 16A^2(1); q) = Z_N(16A^1(1) \oplus 24A^2(1); q)$ for any N and q but $32A^1(1) \oplus 16A^2(1)$ is not equivalent to $16A^1(1) \oplus 24A^2(1)$.

From Theorem 5.1, we have another condition for $Z_N(M; q)$ to be zero.

Corollary 5.3. $Z_N(M; q) = 0$ if and only if there exists $x \in H_1(M; \mathbf{Z})$ of order 2^m with $\lambda(x, x) = c/2^m$, where $N = 2^m b$ with b odd, c is an odd integer, and λ is the linking pairing on $\text{Tor } H_1(M; \mathbf{Z})$.

Proof. From the above theorem and Proposition 2.2, $Z_N(M; q) = 0$ if and only if the linking pairing has a direct summand of the form $A^k(n)$. If $Z_N(M; q) = 0$ then the existence of an element x as in the statement of the corollary

follows easily. Conversely, suppose that there exists x as above. Then since the linking pairing restricted to the cyclic group generated by x is non-singular, it has $A^k(n)$ as a direct summand with $n \equiv c \pmod 8$ (see [36, Lemma (1)]). The proof is complete. ■

6. Invariants for links. For an oriented link L in S^3 (without framing) and an integer $s(\geq 2)$, one can construct the s -fold cyclic branched covering space branched along L associated with the kernel of a map $H_1(S^3 - L; \mathbf{Z}) \rightarrow \mathbf{Z}/s\mathbf{Z}$ sending each meridian to 1. Since it is a closed, oriented 3-manifold, we can define $Z_N(L; q, s)$ to be $Z_N(M(L, s); q)$, where $M(L, s)$ is the s -fold cyclic branched covering space as above. $Z_N(L; q, s)$ is an invariant of L for every s since $M(L, s)$ is uniquely determined by L and s .

A framed link description for $M(L, s)$ is given by S. Akubult and R. Kirby [1]. Denoting a Seifert matrix for L constructed from a connected Seifert surface by V , its linking matrix is given by $V \otimes B + {}^tV \otimes {}^tB$, where $B = (B_{ij})$ ($1 \leq i, j \leq s-1$) with $B_{ij} = 1$ for $1 \leq i \leq j \leq s-1$ and $B_{ij} = 0$ otherwise. So we have

Lemma 6.1.

$$Z_N(L; q, s) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} \sqrt{N}^{-g(s-1)} \sum_{l \in \langle \mathbf{Z}/N\mathbf{Z} \rangle^{g(s-1)}} q^{l|A|l},$$

where $A = V \otimes B + {}^tV \otimes {}^tB$ and g is the size of V .

Note that if $s=2$, $\sigma(A)$ is just $\sigma(L)$, the signature of L [29, 34].

In [4], E. Date, M. Jimbo, K. Miki, and T. Miwa define link invariants using generalized chiral Potts models. They are give as follows.

DEFINITION 6.2. [4]. *Let N be a positive odd integer, q a primitive N -th root of unity, and C an $(s-1) \times (s-1)$ integral matrix ($s > 1$). For an oriented link L with Seifert matrix V of size g , we put*

$$\tau(L; N, q, s, C) = \sqrt{N}^{-g(s-1)} \sum_{l \in \langle \mathbf{Z}/N\mathbf{Z} \rangle^{g(s-1)}} q^{l|(V \otimes C)l}.$$

Since ${}^t l(V \otimes C + {}^tV \otimes {}^tC) l = 2({}^t l(V \otimes C) l)$, we have

Proposition 6.3. *Let $q = \exp(2b\pi\sqrt{-1}/N)$ and $q' = \exp((N+1)b\pi\sqrt{-1}/N)$ with $(b, N) = 1$. Then*

$$Z_N(L; q', s) = \left(\frac{G_N(q')}{|G_N(q')|} \right)^{-\sigma(A)} \tau(L; N, q, s, B),$$

where $A = V \otimes B + {}^tV \otimes {}^tB$ and B is as above. Note that q' is also a primitive N -th root of unity because N is odd.

REMARK 6.4. For a positive even integer N and a primitive N -th root of unity q ,

$$\tau(L; N, q, s, C) = \sqrt{N}^{-g(s-1)} \sum_{i \in (\mathbb{Z}/N\mathbb{Z})^{g(s-1)}} q^{i(V \otimes C)}$$

is also an invariant of a link L . This follows from the fact that the above formula is invariant of S -equivalence class [3, 29, 34] of Seifert matrices for links. Proposition 6.3 also holds in this case. (q' is now a primitive $2N$ -th root of unity.)

The cyclotomic invariant $T_N(L)$ [19] is given by $\tau(L; N, \exp(2\pi\sqrt{-1}/N), 2, (1))$ for an integer greater than 1. (See also [9, 13].) So we have

Proposition 6.5. Put $q = \exp((N+1)\pi\sqrt{-1}/N)$. Then

$$T_N(L) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{\sigma(L)} Z_N(L; q, 2).$$

For relations of the cyclotomic invariants to the polynomial invariants for links, see [9, 19].

7. A family of quasitriangular Hopf algebras. We will give another description for $Z_N(M; q)$ using representations of some algebras. A Hopf algebra \mathcal{A} is an algebra over a field k with comultiplication $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, counit $\varepsilon: \mathcal{A} \rightarrow k$ and antipode $\gamma: \mathcal{A} \rightarrow \mathcal{A}$. Let R be an element in $\mathcal{A} \otimes \mathcal{A}$. The pair (\mathcal{A}, R) is called a quasitriangular Hopf algebra [6] if R is invertible in $\mathcal{A} \otimes \mathcal{A}$, $P \circ \Delta(a) = R \Delta(a) R^{-1}$ for any $a \in \mathcal{A}$, where P is the permutation operator ($P(x \otimes y) = y \otimes x$), and

$$\begin{aligned} (\Delta \otimes \text{id})(R) &= R_{13} R_{23} \\ (\text{id} \otimes \Delta)(R) &= R_{13} R_{12} \end{aligned}$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, and $R_{13} = \sum \alpha_i \otimes 1 \otimes \beta_i$ for $R = \sum \alpha_i \otimes \beta_i$.

Let r be a positive integer and q a primitive r -th root of unity. We define a quasitriangular Hopf algebra A_q over the field $\mathbb{Q}(q)$. The algebra A_q is generated by $1, K$, and K^{-1} with relation $K^r = 1$. A comultiplication, counit and antipode are defined by $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $\gamma(K) = K^{-1}$, respectively. Let R be $r^{-1} \sum_{i,j=0}^{r-1} q^{-ij} K^i \otimes K^j$. Then we have

Lemma 7.1. (A_q, R) is a quasitriangular Hopf algebra.

Proof. The inverse element of R is given by $r^{-1} \sum_{i,j=0}^{r-1} q^{ij} K^i \otimes K^j$ because

$$\begin{aligned} &R \cdot r^{-1} \sum_{i',j'} q^{i'j'} K^{i'} \otimes K^{j'} \\ &= r^{-2} \sum q^{i'j'-ij} K^{i+i'} \otimes K^{j+j'} \\ &= r^{-2} \sum_{i,i',k} q^{ik} \left(\sum_j q^{-(i+i')j} \right) K^{i+i'} \otimes K^k \quad (k = j+j') \\ &= r^{-1} \sum_k \left(\sum_i q^{ik} \right) \cdot 1 \otimes K^k \\ &= 1. \end{aligned}$$

Here the third and fourth equalities follow from Lemma 3.1.

Since A_q is commutative, we have $R\Delta(a)R^{-1}=\Delta(a)=P\circ\Delta(a)$. Moreover

$$\begin{aligned} R_{13}R_{23} &= r^{-2} \sum q^{-ij-i'j'} K^i \otimes K^{i'} \otimes K^{j+j'} \\ &= r^{-2} \sum q^{-i'k} (\sum q^{(i'-i)j}) K^i \otimes K^{i'} \otimes K^k, \quad (k = j+j') \\ &= r^{-1} \sum q^{-ik} K^i \otimes K^i \otimes K^k \\ &= (\Delta \otimes \text{id})(R). \end{aligned}$$

A similar calculation shows $(\text{id} \otimes \Delta)(R) = R_{13}R_{12}$. ■

Since A_q is commutative, all irreducible representation spaces are one-dimensional. We denote these representations by $\{V_j\}_{j=0,1,\dots,r-1}$, with the action $\rho_j(K)$ given by the multiplication by q^j . For representations $\rho_i: A_q \rightarrow \text{End}(V_i)$ and $\rho_j: A_q \rightarrow \text{End}(V_j)$, a tensor product representation is defined by $(\rho_i \otimes \rho_j) \circ \Delta: A_q \rightarrow \text{End}(V_i \otimes V_j)$. The action ρ_j^* on the dual space V_j^* induced from the antipode γ is given by the multiplication by q^{-j} . We can easily see that $(A_q, R, v, \{V_j\})$ is a modular Hopf algebra [31] putting $v = r^{-1} \sum_{i,j=0}^{r-1} q^{j(i-j)} K^i$. With this algebra $(A_q, R, v, \{V_j\})$, we can construct invariants of 3-manifolds according to [31]. We survey an outline of the procedure for constructing them.

Let L be a framed link and consider its diagram. We assume that its framing f_i of a component L_i is parallel to L_i in the plane. A coloring of L is an assignment of V_j to each component of L . Now we associate an operator Ω with each crossing of a colored framed link as follows.

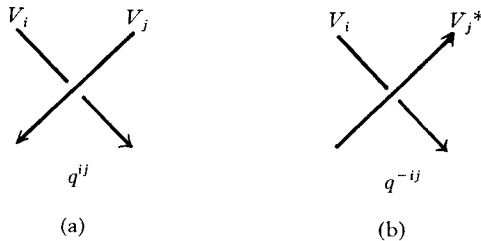


Figure 7.1.

If the crossing is as in Figure 7.1(a), then Ω is a homomorphism from $V_i \otimes V_j$ to $V_j \otimes V_i$ given by $x \otimes y \mapsto (P \circ ((\rho_i \otimes \rho_j)R))(x \otimes y)$. It follows that $\Omega(x \otimes y) = q^{ij}(y \otimes x)$ because

$$\begin{aligned} ((\rho_i \otimes \rho_j)R)(x \otimes y) &= r^{-1} \sum_{i',j'} q^{-i'j'} (\rho_i(K^{i'})x \otimes \rho_j(K^{j'})y) \\ &= r^{-1} \sum_{i'} q^{ii'} \sum_{j'} q^{(j-i')j'} (x \otimes y) \\ &= q^{ij}(x \otimes y), \end{aligned}$$

where the last equality follows from Lemma 3.1 again. If the crossing is as in Figure 7.1(b), then Ω is a homomorphism from $V_i \otimes V_j^*$ to $V_j^* \otimes V_i$ given by $P \circ ((\rho_i \otimes \rho_j^*))R$ and we see that $\Omega(x \otimes y^*) = q^{-ij}(y^* \otimes x)$. Similar calculations show that if the crossing is positive, then Ω is the multiplication by q^{ij} (and the interchanging of the coordinate) and if the crossing is negative, then Ω is the multiplication by q^{-ij} .

Then we can obtain an invariant of a 3-manifold as the sum of the products $\prod_{\text{positive crossings}} q^{ij} \prod_{\text{negative crossings}} q^{-i'j'}$ for all colorings after some normalization.

$Z_N(M; q)$ corresponds to this invariant putting $r=2N$ for N even and $r=N$ for N odd.

8. Operator invariants for 3-dimensional cobordism and invariants of Gocho

As in [31] we can extend the invariants $Z_N(M; q)$ to operator invariants of 3-dimensional cobordisms with non-empty parametrized boundaries, using the modular Hopf algebra structure in A_q described in §7. In this section, we define them by using linking matrices, and prove that invariants of T. Gocho [8] are essentially the absolute values of our invariants. See [31, §4] for the precise definition of 3-dimensional cobordisms with parametrized boundaries.

We denote by $G_g^T(G_g^B, \text{ resp.})$ a horizontal line segment with g arcs glued to the top (bottom, resp.), which is embedded in S^3 as described in Figures 8.1 and 8.2. Each arc has a framing (or parametrization) indicated by a thin line parallel to it in the plane.

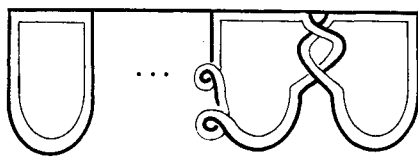


Figure 8.1.

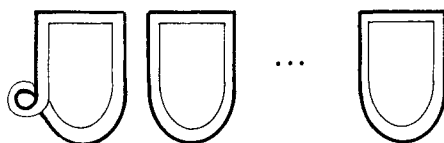


Figure 8.2.

Let $\hat{G}_g^T(\hat{G}_g^B, \text{ resp.})$ be a farmed link obtained by eliminating short segments between arcs from $G_g^T(G_g^B, \text{ resp.})$ as in Figures 8.3 and 8.4.

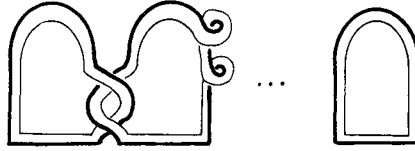


Figure 8.3.

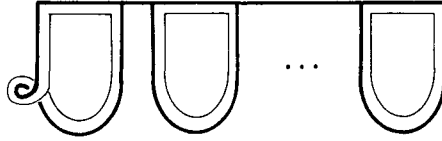


Figure 8.4.

Let (M, F', F) be a 3-dimensional cobordism with connected M whose parametrized boundaries are F' and F . For simplicity we assume that F' and F are connected surfaces of genus g' and g respectively. We can represent M by Dehn surgery on S^3 as follows. We consider graphs $G_{g'}^B$ and G_g^T , and a framed link L in S^3 , where L is located between $G_{g'}^B$ and G_g^T as shown in Figure 8.5. With suitably chosen L , we can put $M = M_L - (\text{int } N(G_{g'}^B) \cup \text{int } N(G_g^T))$, where M_L is a 3-manifold obtained by Dehn surgery in S^3 along L , and $N(G_{g'}^B)$ and $N(G_g^T)$ are tubular neighborhoods of $G_{g'}^B$ and G_g^T respectively.

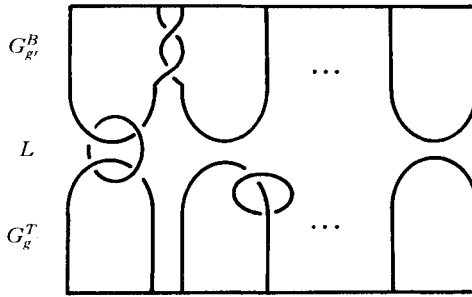


Figure 8.5.

Let V_g be an N^g -dimensional complex vector space with basis $\{e_h\}$, where N is an integer greater than 1 and $h \in (\mathbf{Z}/N\mathbf{Z})^g$. V_g^* is its dual with dual basis $\{e_h^*\}$. We define an operator invariant of M in $V_{g'}^* \otimes V_g \cong \text{Hom}(V_{g'}, V_g)$ by

$$Z_N(M; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} |G_N(q)|^{-n - (g'/2) - (g/2)} \sum_{\substack{h' \in (\mathbf{Z}/N\mathbf{Z})^{g'} \\ h \in (\mathbf{Z}/N\mathbf{Z})^g}} \left(\sum_{i \in (\mathbf{Z}/N\mathbf{Z})^n} q^{i \begin{pmatrix} h' \\ i \\ h \end{pmatrix} A \begin{pmatrix} h' \\ i \\ h \end{pmatrix}} \right) e_{h'}^* \otimes e_h,$$

where q and $G_N(q)$ are as in §1, A is the linking matrix of $\hat{G}_{g'}^B \cup L \cup \hat{G}_g^T$, and n

is the number of components of L . In a similar way as the proof of Theorem 1.3, we can show that this is a topological invariant of M as a 3-dimensional cobordism with parametrized boundary.

The following proposition is a corollary to [31, Theorem 4.5]. We give a direct proof using the formula above.

Proposition 8.1. *If a 3-dimensional cobordism (M, F_1, F_3) is a composition of two cobordisms (M_1, F_1, F_2) and (M_2, F_2, F_3) , then for some integer c*

$$Z_N(M; q) = \zeta^c Z_N(M_2; q) \circ Z_N(M_1; q),$$

where $Z_N(M_1; q) \in V_{g_1}^* \otimes V_{g_2} = \text{Hom}(V_{g_1}, V_{g_2})$, $Z_N(M_2; q) \in \text{Hom}(V_{g_2}, V_{g_3})$, g_i is the genus of F_i , and $\zeta = \exp(\pi\sqrt{-1}/4)$.

Proof. For simplicity, we assume that $F_1 = F_3 = \emptyset$. We present M_1 and M_2 by $L_1 \cup G_{g_2}^T$ and $G_{g_2}^B \cup L_2$ respectively, where $M_1 = M_{L_1} - \text{int } N(G_{g_2}^T)$ and $M_2 = M_{L_2} - \text{int } N(G_{g_2}^B)$. Then M is presented by a framed link $L_1 \cup L_0 \cup L_2$, where L_0 is a framed link obtained from $G_{g_2}^T$ and $G_{g_2}^B$ by gluing arcs as shown in Figure 8.6.

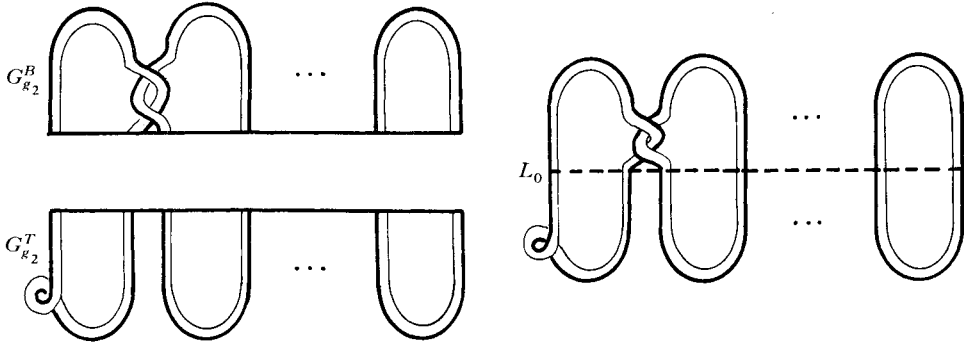


Figure 8.6.

Let A, A_1 , and A_2 be the linking matrices of $L_1 \cup L_0 \cup L_2, L_1 \cup \hat{G}_{g_2}^T$, and $\hat{G}_{g_2}^B \cup L_2$ respectively. We have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix},$$

where 0's are zero matrices with suitable sizes. Hence we have

$${}^t \begin{pmatrix} l_1 \\ h \\ l_2 \end{pmatrix} A \begin{pmatrix} l_1 \\ h \\ l_2 \end{pmatrix} = {}^t \begin{pmatrix} l_1 \\ h \end{pmatrix} A_1 \begin{pmatrix} l_1 \\ h \end{pmatrix} + {}^t \begin{pmatrix} h \\ l_2 \end{pmatrix} A_2 \begin{pmatrix} h \\ l_2 \end{pmatrix}.$$

It follows that $Z_N(M; q)$ is equal to $Z_N(M_2; q) \circ Z_N(M_1; q)$ with a scalar multiple

$(G_N(q)/|G_N(q)|)^{\sigma(A)-\sigma(A_1)-\sigma(A_2)}$. Since the phase of a Gaussian sum has a value of eighth root of unity, we obtain the required formula. ■

Let \mathfrak{M}_g be the mapping class group of a closed surface of genus g . With this proposition we obtain a representation of \mathfrak{M}_g to $PU(V_g)=U(V_g)/U(1)$ as follows. Let F be a closed surface with parametrization of genus g and $f: F \rightarrow F$ a homeomorphism. We denote by C_f the mapping cylinder of f , that is, $F \times [0, 1]$ with parametrization in $F \times \{1\}$ induced by f . For fixed N and q , we have a map $\mathfrak{M}_g \rightarrow PU(V_g), f \mapsto Z_N(C_f; q)$. By Proposition 8.1 this map becomes a representation.

In the case that N is even and $q = \exp(\pi\sqrt{-1}/N)$, this representation coincides with a representation constructed by T. Gocho [8]. Let N and q as above in the following of this section. By a geometric method based on $U(1)$ gauge theory, Gocho constructed a representation ρ_g of \mathfrak{M}_g to $PU(V_g)$ which factors $Sp(2g; \mathbf{Z}) \ni f_*: H_1(F; \mathbf{Z}) \rightarrow H_1(F; \mathbf{Z})$. The representation $\rho_g: Sp(2g; \mathbf{Z}) \rightarrow PU(V_g)$ is given by the next formulas.

$$\begin{aligned} \rho_g \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} e_h &= \sqrt{N}^{-g} \sum_{h'} q^{2^t h \cdot h'} e_{h'}, \\ \rho_g \begin{pmatrix} X & 0 \\ 0 & {}^t X^{-1} \end{pmatrix} e_h &= e^{t X^{-1} h}, \\ \rho_g \begin{pmatrix} 1_g & Y \\ 0 & 1_g \end{pmatrix} e_h &= q^{-t h Y h} e_h. \end{aligned}$$

Here $X \in GL(g; \mathbf{Z})$ and Y is a $g \times g$ symmetric integral matrix. Note that $\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & {}^t X^{-1} \end{pmatrix}$, and $\begin{pmatrix} 1_g & Y \\ 0 & 1_g \end{pmatrix}$ generate $Sp(2g; \mathbf{Z})$. We can check that this representation coincides with our representation by calculaing about generators of \mathfrak{M}_g . In [8], Gocho also defines a topological invariant of M by

$$W_N(M) = \sqrt{N}^{g-1} \langle \rho_g(f_*)e_0, e_0^* \rangle \in \mathbf{C}/U(1),$$

where M is presented by a Heegaard splitting $M = H_g \cup_f (-H_g)$ with H_g a handlebody of genus g . Noting that $W_N(S^3) = \sqrt{N}^{-1}, Z_N(H_g; q) = \sqrt{N}^{g/2} e_0$, and $Z_N(-H_g; q) = \sqrt{N}^{g/2} e_0^*$, we immediately have the next proposition.

Proposition 8.2. *Let N be even. Then we have*

$$\frac{W_N(M)}{W_N(S^3)} = |Z_N(M; \exp \frac{\pi\sqrt{-1}}{N})|,$$

where $W_N(M)$ is Gocho's invariant defined above.

9. Invariants of Dijkgraaf and Witten for $G = \mathbf{Z}/N\mathbf{Z}$.

In this section we will show relations between our invariants and invariants of R. Dijkgraaf and E. Witten.

Let G be $\mathbf{Z}/N\mathbf{Z}$. We choose a class $q \in H^3(BG, U(1))$. Since $H^3(BG, U(1)) \cong \mathbf{Z}/N\mathbf{Z}$ (see for example [11, Lemma 9.2]) for a classifying space BG for G , we regard q as a (not necessarily primitive) N -th root of unity with an inclusion $\mathbf{Z}/N\mathbf{Z} \rightarrow U(1)$. Let M be a closed orientable 3-manifold. In [5], Dijkgraaf and Witten defined invariants as the sum over all possible G bundles over M :

$$D_N(M; q) = \sum_{\gamma \in \text{Hom}(\pi_1(M), G)} \langle f_\gamma^* q, [M] \rangle \in \mathbf{C},$$

where $f_\gamma: M \rightarrow BG$ is a classifying map corresponding to γ and $\langle f_\gamma^* q, [M] \rangle \in U(1)$. We regard $U(1)$ as the set of units in \mathbf{C} and the sum is taken in \mathbf{C} .

Proposition 9.1. *Let N be a positive integer, K a divisor of N , and q an N^2 -th (primitive) root of unity. Then the following formulas hold.*

$$\begin{aligned} \text{For } N \text{ odd} \quad D_N(M; q^{N^K}) &= Z_{N^2/K}(M; q^K) Z_K(M; q^{-N^2/K}). \\ \text{For } N \text{ even} \quad D_N(M; q^{N^K}) &= Z_{N^2/2K}(M; q^K) Z_{2K}(M; q^{-N^2/4K}). \end{aligned}$$

Before we prove this proposition, we show some lemmas. Since $\text{Hom}(\pi_1(M), G) = \text{Hom}(H_1(M; \mathbf{Z}), \mathbf{Z}/N\mathbf{Z}) = H^1(M; \mathbf{Z}/N\mathbf{Z})$, we denote by $\bar{\gamma}$ the corresponding element to γ in $H^1(M; \mathbf{Z}/N\mathbf{Z})$.

Lemma 9.2.

$$\langle f_\gamma^* q, [M] \rangle = q^{\langle \bar{\gamma} \cup \delta^*(\bar{\gamma}), [M] \rangle}$$

where $\delta^*: H^1(M; \mathbf{Z}/N\mathbf{Z}) \rightarrow H^2(M; \mathbf{Z})$ is the connecting homomorphism with respect to an exact sequence $0 \rightarrow \mathbf{Z} \xrightarrow{N} \mathbf{Z} \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$ and $\cup: H^1(M; \mathbf{Z}/N\mathbf{Z}) \times H^2(M; \mathbf{Z}) \rightarrow H^3(M; \mathbf{Z}/N\mathbf{Z})$.

Proof. Let $\gamma' \in \text{Hom}(\pi_1(BG), G) = \text{Hom}(G, G)$ be the identity map which is the monodromy representation of a classifying space $EG \rightarrow BG$. We denote by $\bar{\gamma}'$ a corresponding element to γ' in $H^1(BG, G)$. Some calculations show that $\bar{\gamma}' \cup \delta^*(\bar{\gamma}')$ is a generator of $H^3(BG, G) \cong \mathbf{Z}/N\mathbf{Z} \cong H^3(BG, U(1))$, where $\delta^*: H^1(BG; \mathbf{Z}/N\mathbf{Z}) \rightarrow H^2(BG; \mathbf{Z})$ is the connecting homomorphism. Let q be $\exp(m \cdot 2\pi\sqrt{-1}/N) \in H^3(BG, U(1)) \subset U(1)$. Then $f_\gamma^* q = \exp(m(\bar{\gamma} \cup \delta^*(\bar{\gamma})) \cdot 2\pi\sqrt{-1}/N) \in H^3(M, U(1)) = U(1)$ because $\bar{\gamma} = f_\gamma^* \bar{\gamma}'$. Hence we have the required formula. ■

The following lemma is obtained in a similar way as a proof of Lemma 3.4.

Lemma 9.3. *Let $l \in \ker L_A \subset (\mathbf{Z}/N\mathbf{Z})^n$ be the corresponding element to $\bar{\gamma}$ under the isomorphism ι in Lemma 3.3. Then we have*

$$\langle \bar{\gamma} \cup \delta^*(\bar{\gamma}), [M] \rangle = \frac{1}{N} \langle \bar{l} A \bar{l} \in \mathbf{Z}/N\mathbf{Z} \rangle,$$

where $\bar{l} \in \mathbf{Z}^n$ is a lift of l and A is the linking matrix of the framed link.

Proof of Proposition 9.1. By Lemmas 3.3, 9.2, and 9.3, we have

$$D_N(M; q^{NK}) = \sum_{l \in \ker L_A} q^{K \langle \bar{l} A \bar{l} \rangle}$$

with $L_A: (\mathbf{Z}/N\mathbf{Z})^n \rightarrow (\mathbf{Z}/N\mathbf{Z})^n, l \mapsto Al$.

For N odd, we have

$$\begin{aligned} & Z_{N^2/K}(M; q^K) Z_K(M; q^{-N^2/K}) \\ &= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_{l_1 \in (\mathbf{Z}/N^2K^{-1}\mathbf{Z})^n} q^{K \langle l_1 A l_1 \rangle} \sum_{l_2 \in (\mathbf{Z}/K\mathbf{Z})^n} q^{-N^2K^{-1} \langle l_2 A l_2 \rangle} \\ &= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_{l_2} \sum_{l'_1} q^{K \langle l'_1 A l'_1 + 2N \langle l'_1 A l_2 \rangle} \quad (l_1 = l'_1 + NK^{-1}l_2) \\ &= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_h q^{K \langle h A h \rangle} \sum_{l_2, l_3} q^{2N \langle (Kl_3 + l_2) A h \rangle} \quad (l'_1 = h + Nl_3) \\ &= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} N^n \sum_{h \in \ker L_A} q^{K \langle h A h \rangle}, \end{aligned}$$

where $\Gamma = G_{N^2/K}(q^K) G_K(q^{-N^2/K})$. Similar calculations show $\Gamma = N$. Hence we obtain the required formula.

For N even the required formula is obtained in a similar way. ■

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